Transport Phenomena.
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Lecture-52.
Boundary Layer Similarity.

So we would continue with our discussion on the convection transfer equation in this class. And what we are planning to do here is to look at the 3 equations that describe momentum, heat and mass transfer inside the respective boundary layers and we would like to see what are the similarities between these 3 equations, what are the boundary conditions, are those conditions, boundary conditions similar and if we can express these 3 equations along with their boundary conditions in such a way that the equations become identical as far as their forms are concerned, as well as the boundary conditions will also be the same. So can we do that?

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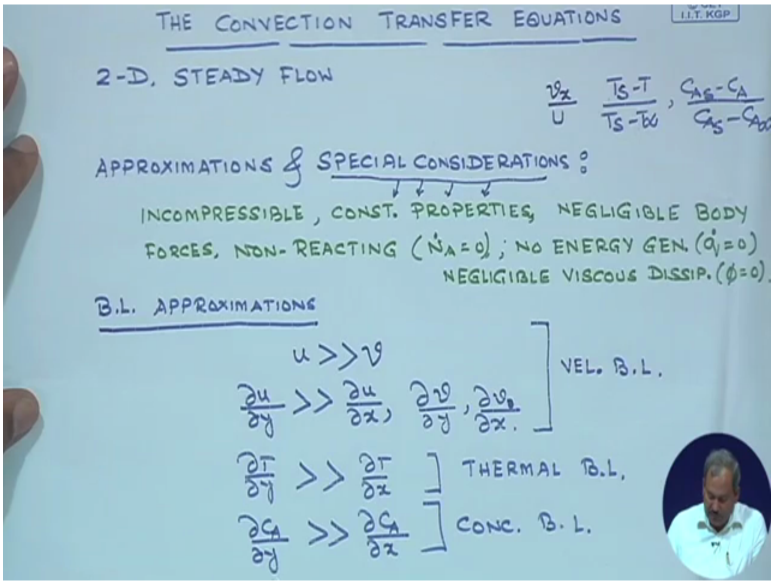

The 1st of what we have done in order to achieve that was the identified 1st what are the 3 equations and what are the dimensionless parameters which arise automatically in those equations. So we started with a two-dimensional steady-state flow assumption and the approximations that we have made are incompressible flow with constant properties, the body force, effect of body force is insignificant, it is non-reacting system and since it is non-reacting system, no species is produced or destroyed, so therefore N dot A would be equal to 0.

And there is no energy generation by whatever means inside the system, therefore Q dot, the heat generation per unit volume is also 0. And since it is mostly low velocity force convection system, so the net, the viscous dissipation which arises only when the velocity gradient of the velocity itself is very high, so this is also negligible and therefore the dissipation function Phi that we have defined in case of energy transfer would also be 0. The approximation that we have used for velocity boundary layer is that the axial component of velocity is very large compared to the component of velocity which is perpendicular to the direction of flow.

The gradient of velocity, the gradient of axial velocity with distance from the solid wall is significantly higher compared to the gradient of the velocity with respect to X, the gradient of the V component of velocity with respect to Y and that of with respect to that of X. For thermal boundary layer, we understand that since for all these cases, these arise, these conditions arise since the thickness of the boundary layer is very thin, so therefore the change in velocity over the, over the vertical distance from the wall will overshadow the change in temperature with respect to the axial positions.

And similarly the change in concentration of species A over the thickness of the boundary layer will be greater, significantly greater than the concentration, species concentration changes with respect to the axial distance. So these equations together form the concentration boundary layer which would help us in resolving the corresponding equations by cancelling some of the terms which are not going to be relevant here.

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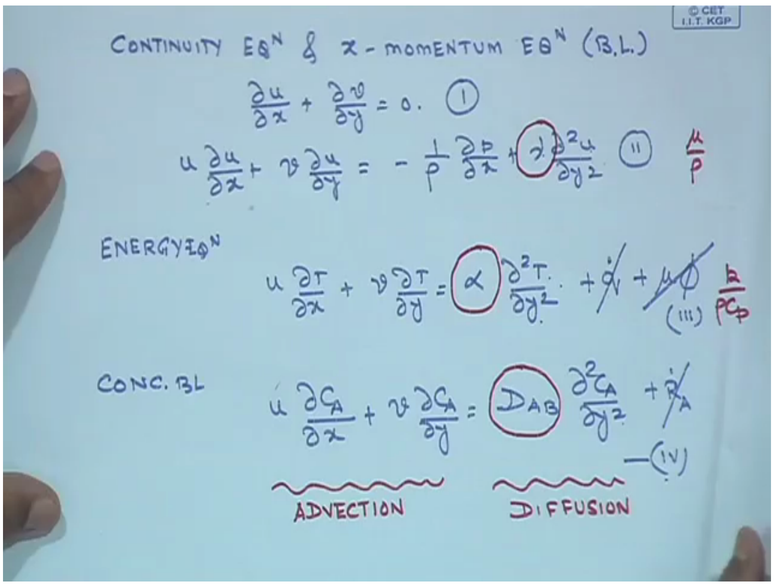

So the continuity equation, continuity equation for the X transport of momentum, continuity equation and the X momentum equation would simply be these advection terms, this is a pressure gradient, axial pressure gradient term and then we have a diffusive transport of moment in the Y direction. The energy equation would simply, again be the advection terms on this side, the property which is Alpha, the thermal diffusivity and its variation, it is Dell 2T by Dell Y square, there is no Q dot and Phi is 0. For the concentration boundary layer, we get the similar expressions, so we see that the, all the changes in concentration or that of temperature are associated with a velocity, so therefore these convection terms together they are called advection terms, so there are advection terms on the left and diffusion terms on the right.

So we have, these are the advection terms in the transport equations where these are the diffusive transport of momentum, energy and species. And the corresponding properties are the kinematic viscosity which is mu by rho, the thermal diffusivity which is K by rho CP and the mass diffusivity which is DAB. So with this with this starting point, I think we are now in a position to look at what the boundary layer similarity.

And in order to do the boundary layer similarity, we are 1^{st} going to normalise the convection equations that we have written over here, so we would like to express them in terms of dimensionless forms and the dimensionless para, the dimensionless, the nondimensionalizing parameters are X star by this is the length of the plate over which length of the surface, it is length scale, actual length scale. Y star is Y by L, U star is U by V, where V is the approach velocity.

So this V is the approach velocity, and V star is similarly defined, the dimensionless temperature is T - the temperature of the substrate, substrate divided by T infinity - TS where T infinity the temperature of the bulk fluid and in a similar way CA star, that is the dimensionless concentration gradient is also expressed as CA which is the species concentration and which we understand is going to be, can be function of both, would be function of X and Y or in dimensionless form, X star and Y star. So CA - CA infinity, CA - CAS divided by CA infinity - CAS.

The reason that equation, all these equations are nondimensionalized is that when you nondimensionalize, not only everything, the equation become, the equations become more stable, they vary between the limits of 0 to 1 as in the case of let say velocity, where the velocity is going to be 0 or at the solid liquid interface and it is going to be equal to the free stream velocity. So the range of dimensionless velocities will always be from 0 to 1 and similarly the range of temperature gradient, temperature in concentration would also vary between 0 to 1.

There are other advantages of nondimensionalizing any governing equation is that in some, at some, for some cases you would see that certain numbers would appear. The collection of numbers would appear, which are dimensionless and which relate to, which essentially specify the process, they tell us something about the dynamics of the process, or they tell us thing about the nature of that, nature of the process, the importance of each of the terms, different mechanisms which are present in any process. So it is always advisable to nondimensionalize a set of governing equations to get more insights into the physics of the problem.

So that is what we have done and then based on the nondimensionalizing parameters that I have just described, the previous 3 transport equations, the equation of momentum, the equation of, the thermal energy equation and the species balance equation in the momentum thermal and the mass transfer boundary layer, they are nondimensionalized. And what we get is the following form of the equation.

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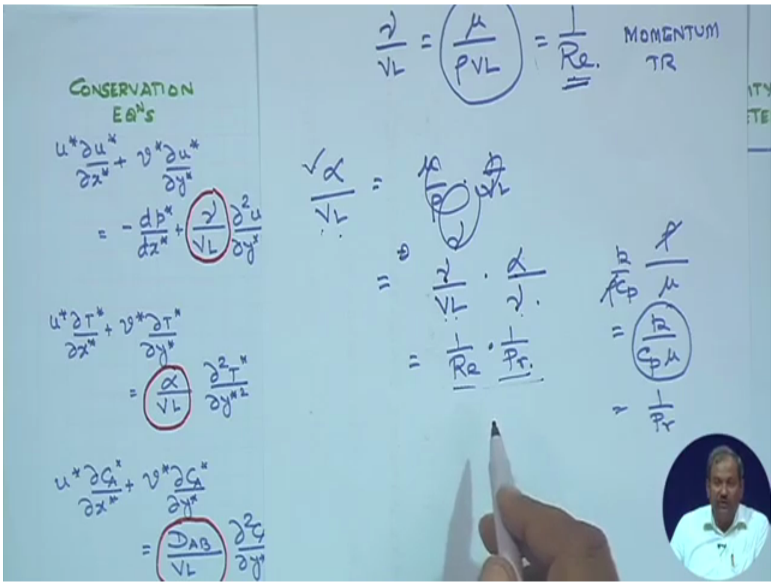

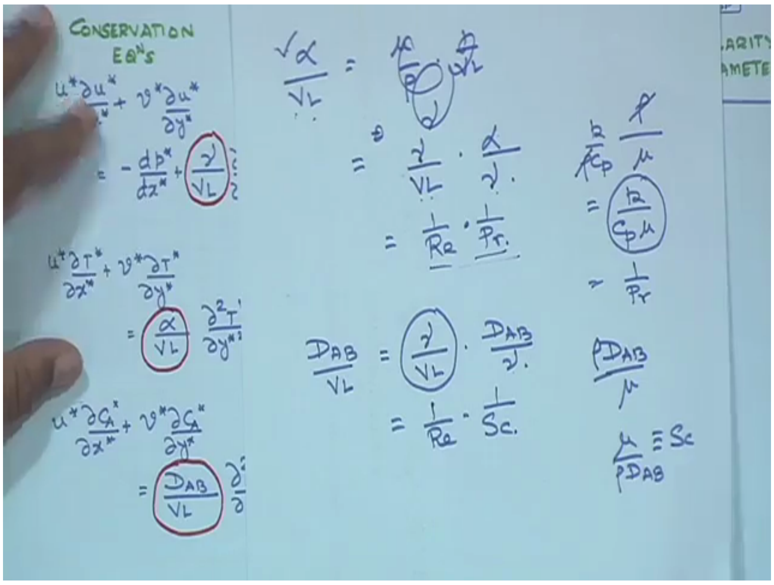

So the equation that you are going to get, so the conservation equation would take this form where U star, V star, X Star, Y star, this we have already defined. So this is the pressure gradient term, this is the energy equation, again the same thing, same, as before, but in dimensionless form. And here we have the species transfer equation in this. Now note the terms that I have circled in red, all these are dimensionless numbers, these are the so-called similarity parameters would tell us a specific characteristic of the transport processes.

For example this nu by VL can be expressed as, nu by VL is nothing but mu by rho VL. So if you see this, this is nothing but 1 by Reynolds number. So the Reynolds number therefore appears in any relation or correlation that you can think of in the case of momentum transfer. So momentum transfer will most likely will have the Reynolds number, the corresponding Reynolds number would be, the corresponding dimensionless number will be Reynolds number. So this nu by VL is nothing but the Reynolds number.

Similarly if you look at, if you examine Alpha by VL, Alpha is Alpha by VL is nothing but mu by rho times 1 by VL. So this can be expressed as the kinematic viscosity, the kinematic viscosity , sorry this, I will explain it in a different way. As this is nu by VL times Alpha by nu. So this is the, this is the thermal diffusivity, this is the velocity and its length, so I bring in the nu in here and divide it by nu at this point and what we have here is then as like this, this becomes 1 by Reynolds number. And this is, Alpha is K by rho CP and this is mu by rho, so rho and rho cancels, what you get is K by CP mu.

And we understand that K by CP mu or rather CP mu by K is Prandlt number, so this is 1 by Prandlt number. So this entire term what appears over here in the energy equation is nothing but 1 by Reynolds number times Prandlt number. So the circled terms which has to be dimensionless, since all other terms in this equation are dimensionless, this can be rearranged, this is the thermal diffusivity, this can be rearranged as nu which is the kinematic viscosity by VL times Alpha which is the thermal diffusivity by nu. And they are nothing but Reynolds, combination of Reynolds and Prandlt number.

So when we deal with heat transfer, you would expect that these 2 numbers would automatically appear in any relation or correlation that you can think of for heat transfer. So you see there is a simple process of dimension, nondimensionalizing the governing equation tells you so many things. That if you have to fit a data experimental data with the other parameters of the system, your natural choice for dimensionless number ease something which is being predicted, which is being projected by the governing equation itself to be Reynolds and Prandlt number.

Similarly few look at the mass transfer, mass transfer boundary layer, so by simple intuition you can say and you can verify that by looking at the red circles that I have drawn across, I have drawn around the bunch of dimensionless numbers, dimensionless quantities. The Reynolds number would obviously be there in mass transfer as well, however the Prandlt number is to be replaced by its equivalent in mass transfer. So the equivalent of Prandlt number is mass, in mass, Prandlt number in mass transfer is Schmidt number.

So the circled one, the circle term that I have over here as DAB it, it can simply be expressed as nu, the kinematic viscosity by VL times DAB by nu, so this as we have seen for from here is nothing but the 1 by Reynolds number and this one, DAB by nu is, DAB by mu by rho, so nu is mu by rho. So mu by rho DAB is nothing but the definition of Schmidt number. So therefore this is going to be 1 by RE times SC.

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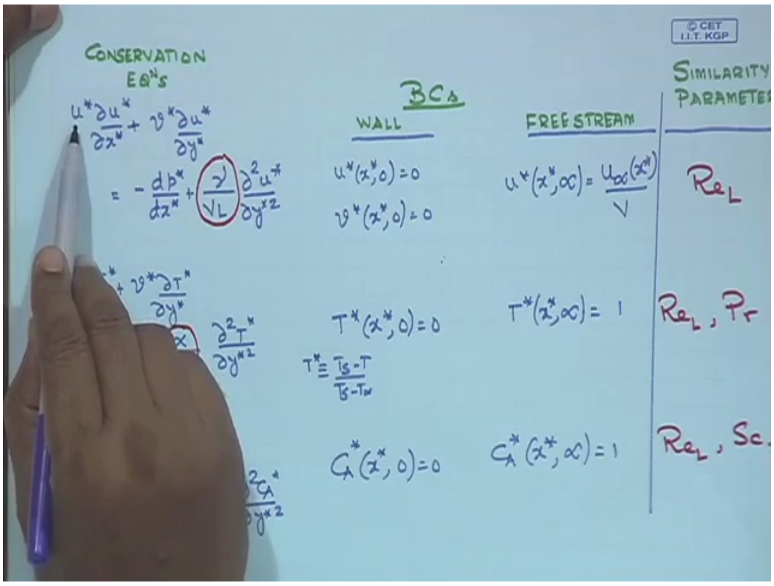

So the 3, the 3 terms of that I have circled with red over here, the so-called similarity parameters which I can write for this case is going to be Reynolds number based on the length of the surface, length scale and for the thermal one, the thermal boundary layer, it is going to be Reynolds number and Prandlt number and for the case of mass transfer, it is going to be Reynolds number and Schmidt number. So these are the various similarities parameter that appear in the conservation equation when you nondimensionalize them. Now let us look at the boundary conditions that we have in here.

Let us 1^{st} think about what we have in the case on the wall. Due to no slip condition, U star at Y star corresponding to 0 which is essentially on the surface, U star would be 0 and V star would also be equal to 0. At the free stream, the velocity, the dimensionless axial component of the velocity would be equal to U infinity, divided by V and , so this is the approach velocity and this is the free stream velocity and for the special case of flow over a flat plate, U, U infinity would be equal to V and therefore U star X star infinity for the case of a flat plate will be equal to1.

But in order to keep the system, in order to keep the maintain the generality of the solution that we are going to get, I have U infinity and V at this point not, may not, they may not be equal to each other. So the dimensionless velocity at the free stream would simply be the ratio of the free stream velocity, divided by the approach velocity. Similarly when we look at the expression, look at the dimensionless temperature difference, dimensionless temperature difference, dimensionless temperature difference as the free stream which is defined as.

So T star is defined as TS - T by Ts - T infinity. So it would simply, it would simply be equal to 0 at the free stream because this T is going to be equal to TS, sorry at the at the wall and at the free stream its value is going to be 1 since the numerator and the denominator would cancel, would cancel each other. Similarly when we have the CA star, that is the dimensionless concentration at the free stream, this is the definition of CA star which we have used as CA - CAS by CA infinity - CS, so at the common the surface of the solid, CA would simply be equal to CAS and therefore CAX star 0 is 0 and CA star at X star infinite, that is at a point far from the solid plate, its value is going to be equal to1.

So these are the 3 equations, the corresponding boundary conditions and the values of the and the expressions for the sunlight the parameters that appears in the, that can be, that appear when you nondimensionalize the system. So we wanted to see how can we say that the solution of these 3 equations would be identical. If we can somehow establish that the solution without actually solving them, the dissolution of these equations would be identical, based, because of some special, special form of the boundary conditions and some values of the dimensionless numbers, then we have established some sort of analogy between them.

So that would be the next task which we are going to do. So the 1st thing that we are going to do is we are going to write what are the functional forms, functional form of the solutions. We were not going to actually solve them but we are going to look at the functional form of the solutions. So if you see the 1st equation, if I if I could solve U star, what would U star be a function of? So U star if you see this equations, U star would definitely be a function of the independent variables which are X star and Y star, it is also quite be function of what kind of pressure gradient you have present, you have in the system.

So U star is going to be function of the pressure gradient DP star DX star and definitely U star is going to be function of the dimensionless parameter that has appeared in the governing equation. So the functional form of U star would contain, we still do not know what that functional form is but it should contain, if we could derive that, we know that it would contain X Star Y star, DP star and the Reynolds number.

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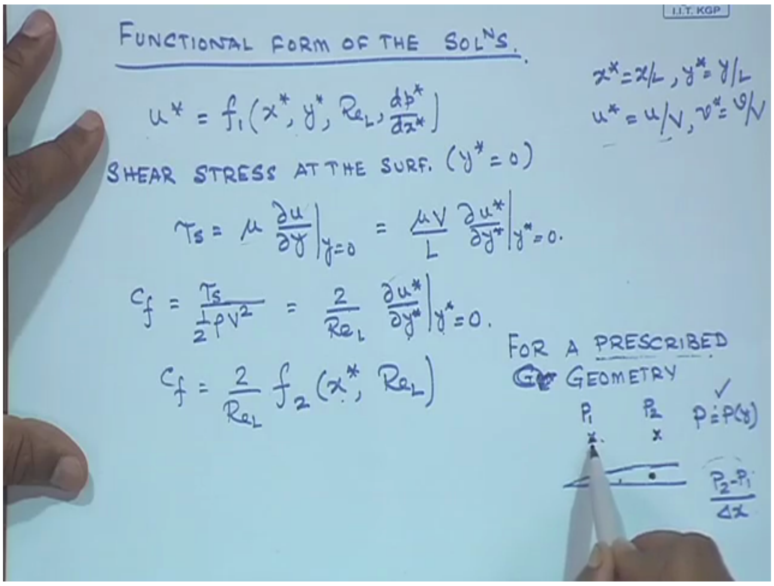

So the functional form of U star, if I call it as function 1, would you not know what that function one is but we do know now that it should contain X star, Y star, Reynolds number and DP star DX star. So this is, this is obvious when you look at the form of the equation. So we, but we are more interested not in velocity but on the shear stress at the surface. So the shear stress at the surface, that means I am specifying the value of Y star, which is at the surface means it is going to be Y star equal to 0. So this form, the functional form of that, that Tao S would simply be equal to mu, if it is an internal fluid, Dell U by Dell Y at Y equal to 0, this is expression for shear stress.

And if you nondimensionalize this, again the nondimensionalizing parameters are X star equals X by L, Y star equals Y by L, U star is U by V, similarly V star is v by V and this so on. So, this, if I nondimensionalize this part, then it would simply be equal to mu times V by L Dell U star by Dell Y star at Y star is equal to 0. And we know that the friction coefficient, the engineering parameter of interest is defined as Tao S by half rho V square, okay. So if this is the case, when you bring in this Tao is some expression of Tao is in here in CF, it would simply be equals 2 by Reynolds number Dell U star by Dell Y star at Y star is equal to 0 okay.

So I already know of what is the functional form of U star. So this CF is going to be equal to, if I define the functional form of this as F 2, it is going to be a function of X star and REL. I will go through it once again. What I have done is I identified from the equation that is given that we have that we have seen is U star is going to be function of the 2 independent variables namely X star at Y star, it is going to be function of the pressure gradient present in the system which is DP star DX star and it should also be a function of the Reynolds number of the flow. So my U star should is some function that I call as F1 and the F1 should contain the variables X star, Y star, the imposed condition DP star DX star and the dimensionless number which is Reynolds number.

Then I decide to find out what would be the expression for friction coefficient. We know that friction coefficient is defined as Tao S which is the wall shear stress divided by half rho U square. And what is Tao S, for a Newtonian fluid Tao S is simply equal to mu times Dell U Delta Y at Y equal to 0, the velocity gradient at Y equal to 0. So this velocity gradient, Dell U Dell Y, I express them in dimensionless form. So what I get for the case, for Tao S is simply this form where everything is expressed in dimensionless form and mu V by L, the V and L appear because I have divided Y by L and Y by L and therefore I have to divide these of L as well and U is simply nondimensionalized if you look over here as U by V. So it appears in this way.

So CF which is Tao S by this, would simply be equal to email to buy REL Dell U square by Dell Y star at Y star equal to 0. If you look at this expression, since I have specified the value of Y star to be equal to 0, therefore the function that I am going to write for F2 should not, will not contain Y star. Because I am evaluating this at a fixed value of Y star, therefore I have fixed Y star, so therefore this is no longer a variable in the expression, therefore F2 should not contain Y star. But I did not say anything about X star, so my X star appears over here. I did not specify what is Reynolds number, so Reynolds number appears in the functional form.

Y star did not, I specified, so Y star does not appear in here is but at the same time DP star DX star does not appear in this expression and I so for that I write a specialised condition that for a prescribe geometry, where the geometry is known to me, I am aware of what would be the geometry. So if this is a surface, whatever be the shape of the surface, this is the boundary layer, I can find out what is DP DX by analysing the flow outside of the boundary layer. And since my pressure is a function of Y only, so if I can calculate what is P1 and what is P 2, then if I can calculate P2 - P1 by Delta X where the distance between them is Delta X.

So that means DP DX can be calculated based on inviscid flow theory outside of the boundary layer. We realise that inside the boundary layer the flow is viscous, so Naviar Stokes equation will have to be solved in order to obtain what is the pressure difference, pressure gradient between these 2 points. But I need not do that because understand my purchase a function of Y only, so whatever be the value of DP DX at this point outside of the boundary layer would be the value of DP DX inside the boundary layer. And since the flow outside boundary layer is inviscid, so Bernoulli's equation or some such equation can be used to obtain what is P2 - P1 by Delta X.

So if I prescribe geometry, if I know what kind of geometry I am dealing with, then I have an inviscid flow theory that can independently give me what is the pressure difference, what is the pressure gradient. So as long as the geometry is known, the pressure gradient between 2 points in the inviscid flow field can be obtained easily without having to solve Naviar Stokes equation. So the moment the geometry is prescribed, the pressure gradient is known to me. And since the pressure gradient is known to me, therefore the pressure gradient does not appear in the functional form F2 that I have written.

So F2 does not contain the, the F2 does not contain the Y star since I have specified the value of Y star to be at 0. And F2 will not contain any D P DX if I prescribe, if up Priory I tell what kind of a geometry we are experiencing, we are encountering during the flow, so D P DX is known to me. Therefore the expression for CF should only contain the F2 should only contain the X star, the distance and Reynolds number.

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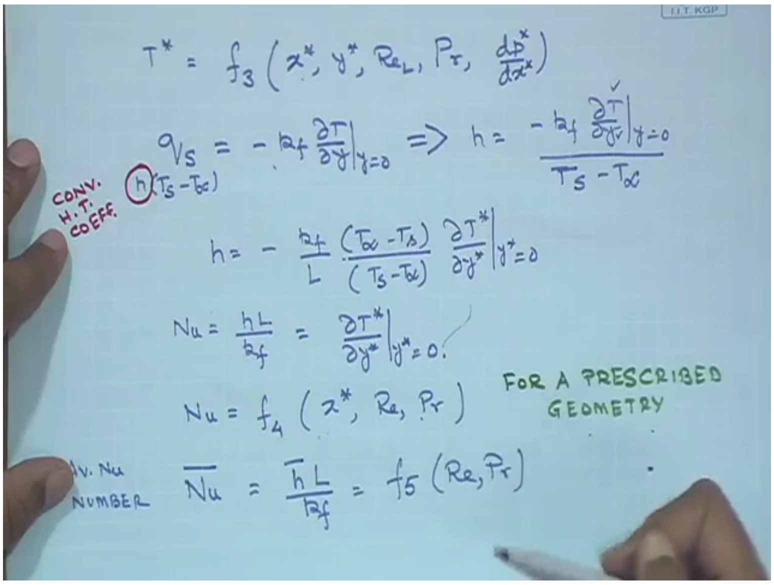

When in a similar way I am going to find out what my T star would contain. So let us say that T star is some function and I am looking at this equation, I am trying to see what would be the functional form formerly, not the exact form, what would be the functional form of T star. So T star must contain the independent variables X star, Y star and this parameter which is Reynolds number. And since U and V are involved, so T star would contain all the functions, all the functions, all the parameters which are functions, which are used in expressing U star and V star.

So therefore my T star would be function of X star, Y star, Reynolds number, Reynolds number, DP DX, DP star, D X star, all these, all these were there for U star. Now since in the in the equation that I have, T star is a function of U star, V star, etc. and so on, therefore my expression for T star would contain all these which was there in the expression of U star. So my F3 would contain X star, Y star, Reynolds number and DP DX. If you see this equation, the governing equation one more time, you would see that T star would contain not only the functions of corresponding to U star, it should also contain what are the similarity parameters.

So Alpha by VL is nothing but Reynolds times, 1 by Reynolds times Prandlt number. So it will only be a function of Reynolds number, it would also be a function of Prandlt number. So therefore the temperature expression would contain Prandlt number as well. So that is the only difference which we have in the functional, in the functional relationship of T star. We still do not know what F3 is, how X star is connected with T star or how Reynolds number is connected with T star and so on. But in functional form this is the form that we have. And as before, as in the case of momentum transfer we wanted to find out what is the shear stress and shear stress coefficient.

So in the case of heat transfer it is going to be, whatever be the heat that is lost, whatever be the heat transferred from the solid surface which would simply be equal to KF Dell T Dell Y at Y equals 0, that is at the interface between the solid and the liquid and we can, we know that this heat which is lost from the solid surface is equal to H times TS - T infinity where this H is an engineering parameter of interest which is nothing but the convective heat transfer coefficient. So this is Newton’s law and at the, at the solid liquid interface, the Newton’s law of cooling, the convective heat transfer is equal to the conductive heat transfer and therefore this defines, this equation defines what is the convective heat transfer, that we have described before.

So this would simply give you H equals - KF Dell T Dell Y at Y equals 0 divided by TS - T infinity. So that is the classical definition of the heat, convective heat transfer coefficient which is the conductive heat flux divided by the temperature difference which comes from the equality of convection and conduction at the liquid vapour interface. So this, I am going to nondimensionalize this T, Y, etc. So H would simply be equal to - K F, just a nondimensionalization, nothing else, T infinity - TS by what I have over here is TS - T infinity and this would simply be equals Dell T star Dell Y star at Y star equal to 0.

So nothing new in here, I am simply expressing this T and Y in dimensionless forms. So these 2 will cancel the - sign would also disappear because of the nature of these. So what you have then HL by KF, that we know as Nusselt number, HL by KF, the same way we have obtained CF, the expression of C F in the case of momentum transfer, we are getting what is Nusselt number in here which you would see is Dell T star by Dell Y star at Y star is equal to 0. So the correct classical definition of Nusselt number would be, it is the dimensionless temperature gradient at the solid-fluid interface.

And now you can see why Nusselt number is equal to Dell T star by Dell Y star where T star is the dimensionless temperature and Y star is the dimensionless distance from the solid wall. So when you find out Dell T star by Dell Y star, the way we have defined T star and Y star, you precisely get what is the, what is nothing but the expression for the Nusselt number. So we will look at this and we are trying to see what is Nusselt, the functional form of Nusselt number. So let us call it as F4, so if you see the functional form of Nusselt number would contain X star, Reynolds number and Prandlt number.

Since we have specified the value of YY star equal to 0, so Y star does not appear in the functional form and since we have, we know what is this, so this is for prescribed geometry and since the prescribed, since the geometry is known to me, so therefore DP DX can be obtained independently and that is why it does not appear in this expression and therefore Nusselt number is equal to X star, RE, PR. If you find out what is the average value of Nusselt number, so if this is the average Nusselt number, the Nusselt number that I have obtained over here is here H is the local heat transfer convective heat transfer coefficient.

If I have to find out the average value of Nusselt number, I will simply integrate this expression over the entire length of the substrate and therefore that would simply, is denoted by the average value, the bar over here denotes the length average value of HL by KF and which if I call it as F 5, function 5, it will only be a function of Reynolds and Prandlt number. So since I have integrated over this X, therefore it, the X does not appear in my functional relationship anymore. So this is a, this is a fundamentally important result which simply tells you that the average value of Nusselt number will only be a function of Reynolds number and Prandlt number. Now we are trying to recall all the expressions of convective heat transfer, the relations and correlations of convective heat transfer that you have come across in your heat transfer, in your study of heat transfer.

So in all the expressions that you can think of, where or are functions of Reynolds and Prandlt number. So starting with (())(35:08) or any question for convective heat flow, the relations are always expressed in terms of Reynolds number and Prandlt number. And now you know why that expression of Nusselt number should always contain the Prandlt number and Reynolds number. So since we have done this for heat transfer, I can very quickly do it for mass transfer as well.

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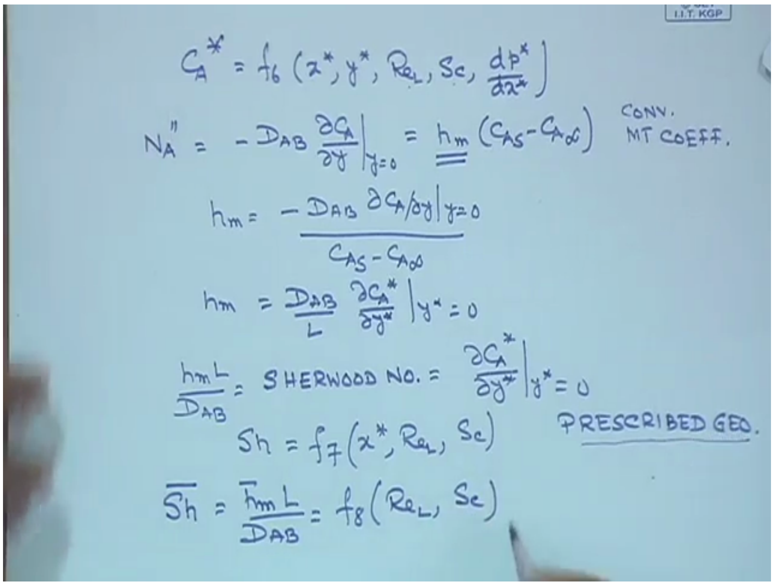

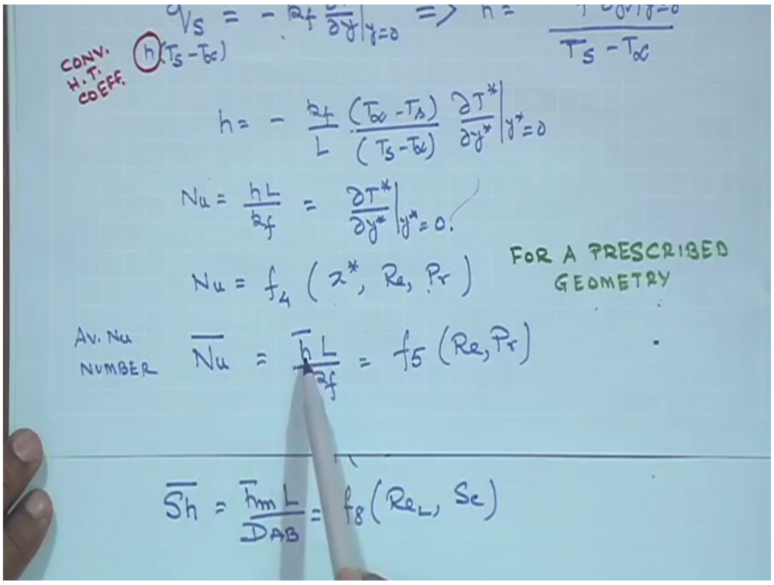

So the mass transfer CA star and then again I am looking at the expression which, looking at the relation which is this, that I have written, the CA Star would be a function of X, Y, when else, Reynolds, Reynolds, Schmidt and DP DX, the same way it was done for the temperature. So it is good to be a function, let us call it as F6, X star, Y star, Reynolds number, Schmidt number and DP star by DX star, exactly the same way, so I am not spending any more time. And we are more interested in finding out what is the flux, mass flux which is - DAB del CA Dell Y at Y equal to 0 which the same way as in convective heat transfer would be CAS - CA infinity and this HM is the convective mass transfer coefficient.

So your definition of HM would therefore be - DAB Dell CA Dell Y at Y equals 0 divided by CAS - CA infinity directly from this and which when you nondimensionalize, nondimensionalized this HM would be DAB by L Dell CA star by Dell Y star at Y star equals 0. And you bring this to other side, so it becomes HM L by DAB which in mass transfer terms is known as the Sherwood number which has the same as Nusselt number in heat transfer would be simply Dell C A star by Dell Y star at Y star equal to 0. So Sherwood number is something but the dimensionless concentration gradient at the interface.

So your Sherwood number would simply be a function, I call it as F7 of X star REL and SC and same again for a specific prescribed geometry. So if you look at this, Dell CA Dell Y would not contain any Y, it should contain Reynolds number, Schmidt number and since the geometry is specified, so DP star by DY star would also not appear over here. And the length average value of Sherwood number which I call it as S H bar which is the length average value of the convective mass transfer coefficient by DAB would be a function of Reynolds number and Schmidt number.

So if you compare these 2 expressions, average value of the Nusselt number is average value of the convective heat transfer coefficients L by KF is a function of Reynolds and Prandlt number, here you see, using the same logic we obtained the average value of the Sherwood number is equal to the average value of convective mass transfer coefficient and in functional form it should only contain Reynolds and Schmidt number. So therefore we are slowly coming to a point very star to see a picture is going to emerge, is slowly emerging where we can see that all these are coming into place and they are becoming similar, the expression of Nusselt number, the expression of Sherwood number and the expression for the friction coefficient, they all start to look like the same.

So they are converging to a point. So what we need to do, what we need to identify in the coming 2 classes is what are the special conditions which we must identify, which we must specify so that they, all these relations at some point of time would become identical. And therefore any relation of mass transfer can be interchangeably used as the relation for heat transfer, provided we substitute the similarity parameters relevant in heat transfer by the similarity parameters relevant in mass transfer. So that is what we are going to, we will establish in the next class.