Transport Phenomena.
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Lecture-04.
Example of Shell Momentum Balance.

So we were discussing about the modelling of the flow of a falling film. I will once again tell you the salient features of the problem that we are trying to model. It is liquid which is in contact with an inclined plane and it is falling while maintaining constant thickness of delta on the solid plate. The coordinate system is drawn in such a way that in the direction of flow it is Z, across the depth of the fluid it is X and this is the Y direction. So the velocity changes only in the X direction, it does not change in the Z direction or it does not change in the Y direction.

So we can, we are imagining a shell of some thickness delta X in the flow field which is aligned in the flow field, the length of this shell is L and the width of this shell is W, these 2 are constants. So there would be one face which is perpendicular to the flow, so if this is my imaginary shell, this face is going to be, it is going to be perpendicular to the flow, so across this face, mass is going to enter the control volume. Across this face mass is going to leave the control volume, there would be no mass flow rate across this face, this face or these 2 faces, the bottom and the top faces.

So convective flow of fluid can only take place through these 2 faces. So we have written the expressions for the amount of flow that comes in, the amount of the mass that comes in and the momentum that comes in. Since the velocity is not a function of Z, so therefore these 2 terms in and out by convection will cancel each other. We also understand the velocity at this point and the velocity at this point are different because this is my X direction and somewhere over here I have my solid plate.

So the velocity over here is more as compared to the velocity over here because as I approach towards the solid plate, the velocity decreases and on the solid plate because of mostly condition it just becomes equal to 0. So since there is a variation of velocity from Newton’s law of viscosity, from the concept of viscosity I know that there exists a shear stress on the top surface and on the bottom surface.

If the fluid is Newtonian, that means if it follows Newton’s law, you can simply express the shear stress as mu, the viscosity of the liquid multiplied by the, by the variation, by the gradient of velocity at this point. So you have VZ wearing with X, so the gradient is simply going to be d VZ dX evaluating it at X and over here the gradient expressions will remain the same, only thing is you are evaluating it as X + delta X. So the molecular transport of momentum is simply going to be mu, velocity gradient multiplied by the area.

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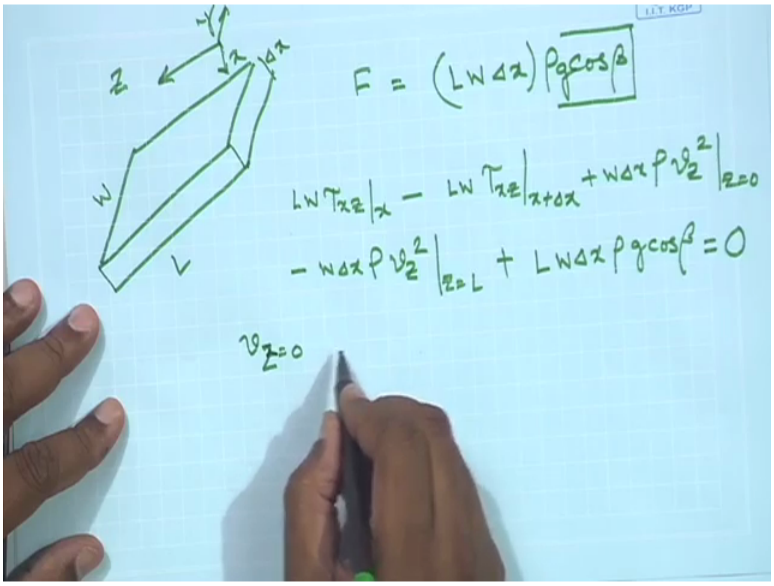

What is this area, it has a length equal to L and width equal to W. So the area on which the molecular transport that that is taking place is L times W. So that is what I have written over here, is L times W is the area and this is the Tao XZ evaluated at X and Tao XZ evaluated at X + delta X. So this is essentially the molecular transport of momentum. Okay. So I have a convective part and a molecular part which are acting on it. There is also a gravity force which is acting on this surface, so on this control volume.

This is again my control volume of thickness delta X, length L and the width is W. So the mass of the liquid which is contained in the volume, which is LW delta X, this is the volume of the fluid inside the control volume and multiplied with rho in order to get the mass of the control volume and cos Theta which is, G cos Theta which is the component of the body force. So this and total it gives me whatever be the force which is acting on the system.

Now this again, this is my Z direction, this is my X direction and somewhere over here is my Y direction. So this gives me the body force which is existing, which is acting on the control, on the imaginary con imaginary Uhh control volume. So if I write the momentum balance equation now, it simply becomes LW Tao XZ at X which is molecular transport of momentum in - LW Tao XZ at X + delta X which is transport, molecular transport of momentum out of the control volume + W dell X rho VZ square XZ equal to 0 which is the convective transport of momentum into the system and this is convective transport of momentum out of the system + sum of all forces acting on it and the only force here is the gravity which is this.

Since is at steady-state, all these forces, the algebraic sum of all these forces must be equal to 0. So I have defined a control volume, identified all the sources of momentum that comes that that can come into the system and I have also identified the only relevant component of force present in the system. I identify that it is the body force which is present in the system, there is no surface force. Because it is a freely falling film, so the pressure on both sides are the same.

So I have a complete equation, now if you see this equation, this equation has delta X in it. So it is an expression which which is which is valid over this control volume defined by W, L and delta X. What I am going to do next is cancel the term which are not relevant, which cancelled out each other, in this case conductive flow of momentum, and divide both sides by dell X and take in the limit when dell X approaches 0. And you would see that this would lead to the definition on the 1st derivative and from this difference equation I would be able to obtain the differential equation.

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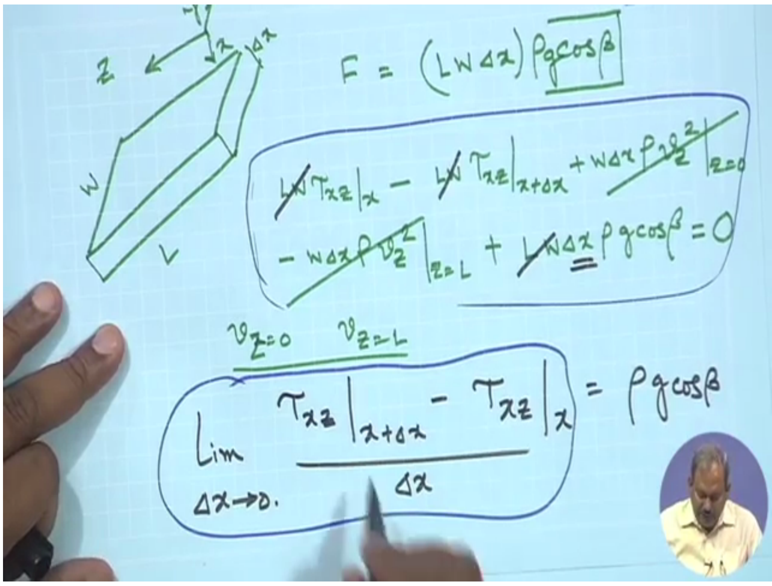

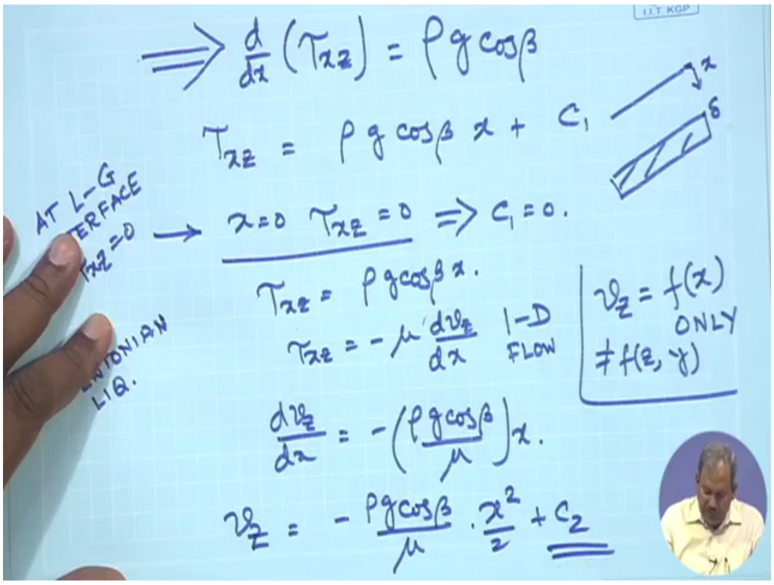

So let us see what happens when we divide both sides by dell X. We identify here that V at X equals 0, sorry V at Z equal to 0 is equal to V at Z equal to L. So this term and this term would cancel out and then I divide both sides by dell X. So what I am going to get then is and cancel out the LW from here as well. And I bring this dell X from here over this point and I take the limit went dell X tends to 0. So this entire thing, this equation that I have written is a statement of the physics of the process.

And what I have done over here is the definition of the 1st derivative. So what you get out of this is simply d dX of Tao XZ equal to rho G cos beta. So what you can see here is that from this difference equation I get the definition of the 1st derivative and when I use the definition of the 1st derivative, what I get is a differential equation that governs that essentially give me some idea of what is, how the shear stress is changing and how it is balanced by the body forces.

So I integrate it once which would give rise to rho G cos beta X less even where C1 is the constant of integration and I understand that at liquid gas interface, Tao XZ must be equal to 0. So this is the plate, this is my film, here is X and this is X equal to 0 and this is X equal to delta. What it tells me is that at X equal to 0, Tao XZ is 0, it is direct correlation, direct relation from direct observation from liquid gas interface, the shear stress will be 0, which we have explained before.

So starting with the governing equation is using the appropriate boundary condition should give me a compact form, analytical form of velocity and that is what we would like to do here. If this is 0, then this should give me C1 to be equal to 0 and your Tao XZ would simply be rho G cos beta times X. If we assume that this is a Newtonian fluid, Newtonian liquid, then what I am going to get is Tao XZ is equal to - mu d VZ by TX, I can write the normal ordinary derivatives, ordinary derivative because my V Z is a function of X only.

My VZ is not a function of Z or a function of Y. So this is truly a 1-d one-dimensional flow. Had this been a two-dimensional flow or a three-dimensional flow, then the system would be more complicated and you will not be able to solve it in a simple way. But this simple approach essentially gives us an idea about the, about this modelling process, a simple modelling process where wherein by identifying the shell, identifying the contributions of momentum, convective convective, diffusive or molecular transport and the body forces would give rise to a governing equation that can be solved to obtain a compact equation for the velocity as in this case.

So we start with this. So when you are substituting in here, what you are going to get is d VZ by dX is - rho G cos beta by mu times X. You integrate it once again, you will get VZ rho G cos beta by mu times X square by 2 + C2, where C2 is another integration constant.

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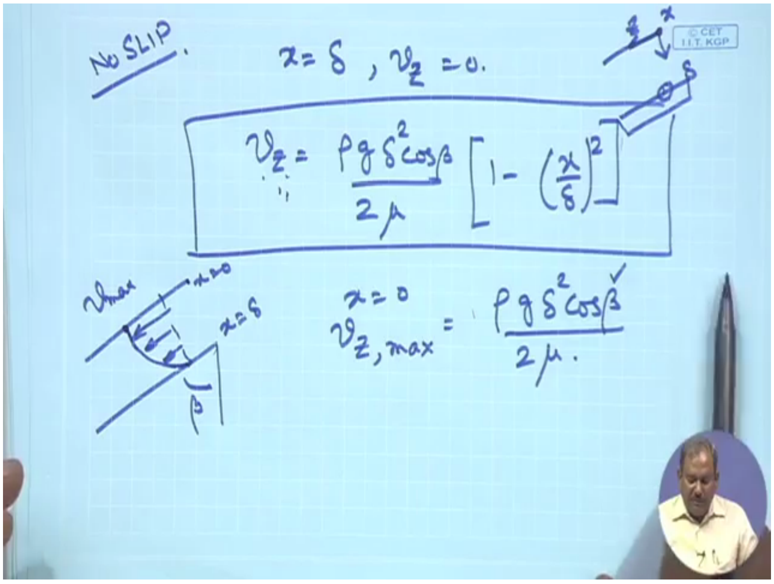

Now in order to solve C2, we use the 2nd boundary condition which is no slip. And what does no slip tells us a is that at X equals delta, that means at the solid liquid interface, so this is my X and this is delta, this is Z, so at X equal to delta, that means on this on this solid liquid interface, my velocity must be equal to 0 which is known as closely condition. So when you put that in here and in you solve for C2 and you substitute it into this equation was again, what you get is VZ as rho G del square cos beta by 2 mu times 1 - X by delta whole square.

Now I have what we have set out to do from the very beginning. I have an expression for velocity and this expression for velocity contains several parameters. So if we if we think intuitively, the velocity is going to be maximum at the liquid air interface and the velocity is going to be minimum, 0 in this case at the liquid solid interface and in between the velocity varies.

When you look at the functional form of the velocity variation, you would see that the variation is parabolic in nature, so therefore if we plot the velocity, it is going to be something like this. So the velocity varies from 0 to some V max at X equals 0, this is X equals delta. So the velocity which is the maximum over here and in between it follows a parabolic path. So at X equals 0, you will get your VZ to be the max which should be rho G del square cos beta by 2 mu.

Whenever you solve an equation, model a system and solve an equation, you should always try to see if it is consistent with the physics of the process. So let us see what happens if you increase beta, that means if you increase the angle of inclination, if you increase Beta, the velocity has to increase, with this clear over here. If for a constant system geometry, if you decrease the value of new, the velocity increases. So that means a lesser viscous fluid will flow faster along an incline in compared to a thicker Uhh to a more more dense, more dense or more viscous fluid.

So that is also there. And further you are from the solid plate, the velocity increases. So this gives you some idea of what is going to be the velocity at maximum, the velocity and the maximum velocity in the system. But in many cases as engineer, you are not interested in to know what is the maximum velocity or what how how the velocity varies between the solid plate up to the liquid interface, liquid gas interface, you are more interested to know what is the average velocity.

So in order to obtain the average velocity, there are different ways by which you can average to find out the velocity. The most logic, most common and logical way to do this averaging is if you average across the flow cross-section, because across this flow cross-section, across this velocity varies from 0 over here to VZ max at this point. So this is the cross-section in which the velocity varies.

So you really need to find out an area average of the velocity, area average of the point velocity in order in order to find what is the average velocity. So all the average velocity that we are going to differ from now on will be area average velocity whereas the area across which I am averaging is perpendicular to the principal direction of flow. So here the principal direction of flow is in the Z direction, so it is it is an area which is perpendicular to the Z direction, so it is the Z face across which I am doing the integration with my understanding that the velocity varies with X, it does not vary with Y.

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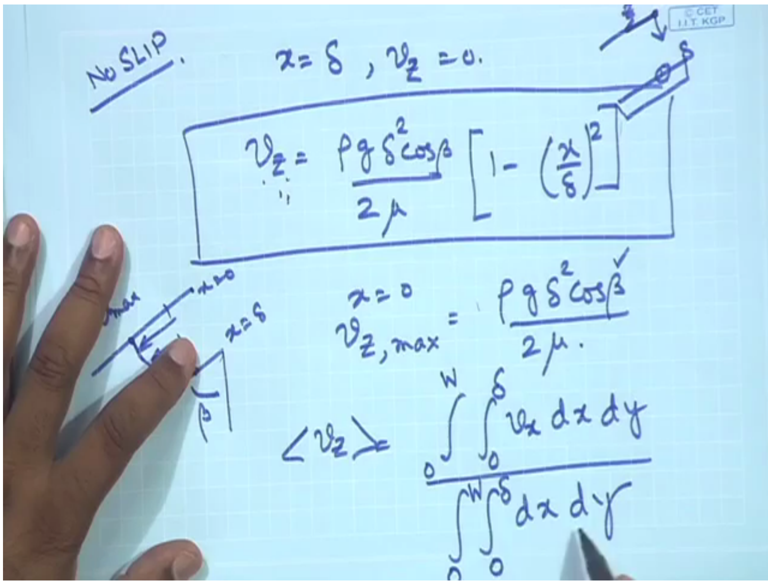

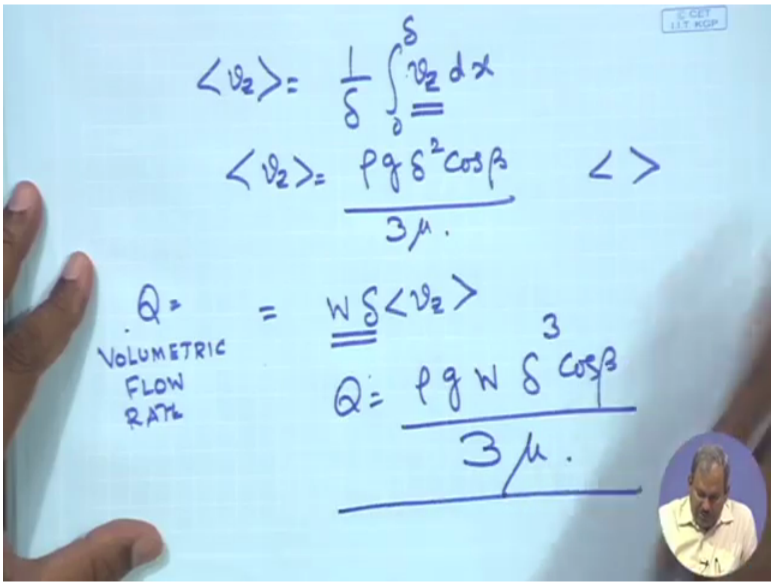

So even though I am using a double integration, in order to obtain, in order to average it across the Z face, across Y it does not very, across X it does very. So the expression for the average velocity mathematically would be VZ dX dY and here also it is dX dY, that is the area and if you see that X varies from 0 to delta, which is the thickness of the film, whereas Y is the width of the film, so it varies from 0 to W. So it is from 0 to W for Y and 0 to delta for X.

And when you do this integration, of course the Y would, YB, since dX does not depend on Y, so you can take this out and your VZ would simply be equal to 1 by delta, this is from 0 to delta VZ dX. And the expression of Z you are going to plug it in from the previous one and what you would get is rho G del square cos beta by 3 new. And once you have the average velocity, this denotes the average, average velocity, and then you would also like to know what is the film thick, what is a volumetric flow rate.

So this is the volumetric flow rate and the volumetric flow rate would simply be W delta times VZ, so this is the cross-sectional area through which this is taking place. So so your Q becomes rho GW dell cube cos beta by 3 mu. So you can you can see that starting with the very fundamental concept that momentum in, rate of momentum in - rate of momentum out + sum of all forces acting on the system at steady state must be equal to 0.

When you use that concept and define a control volume, the smaller size would be denoted by the direction in which the flow is changing. In this case flow is changing with the X direction, so therefore you have delta X as one of the dimensions of the control volume. And you write that, you also identify that we are dealing with the simplest possible case which is one-dimensional flow, so velocity is a function of X, it is not a function of Z or is function of Y.

You divide both sides by dell X, use the definition of 1^{st} derivative and what you get is a differential equation. Use solve differential equation with boundary conditions, no slip and no shear, you get a compact expression for velocity. Once you have the expression for velocity, then you would be able to obtain the average velocity and the total volumetric flow rate. So that can be obtained. One part I have not discussed, Uhh we have not done as it is, what is the force exerted by the moving fluid on the solid?

If you think you can clearly see that the force exerted by the fluid on the plate, the force is in the direction perpendicular to the motion, so the motion is in this place but the force gets transmitted, transferred from the liquid to the solid in this direction. And since it is moving, it would try to take the plate along with it. So in order to keep the plate stationary you must apply force in the reverse direction because the fluid flow would try to take the Uhh take the plate along with it.

So what is the force, what is the Genesis of the force, it must be due to the viscosity. So the viscosity of the liquid, the shear stress out of this viscosity is essentially shear stress exerted on the top area of the solid plate is the one which is the force exerted by the liquid on the plate. So what is the area on which it is acting on, it must be equal to the length times width of the plate, LW.

What is the shear stress, it is tao, Uhh the shear stress is simply going to be equal to Tao XZ. So the shear stress is Tao XZ and it is acting on an area equal to W Times L. So if I need to find out the total force being given, total force exerted by the fluid on the solid plate, I need to integrate this Tao XZ over dZ and dW. And the bound, the limits on Z and L, the limits on Z must be equal to 0 to L and the limits on W would be, limits on Y would be 0 to W.

So if I can perform, if I do perform this this one, I would be able to get an expression for what is the force exerted by the fluid on the solid plate. And the solid plate in turn gives the same thing back to the, back to the fluid. So let us figure out what is, let us find out what is the force exerted on the plate by the fluid.

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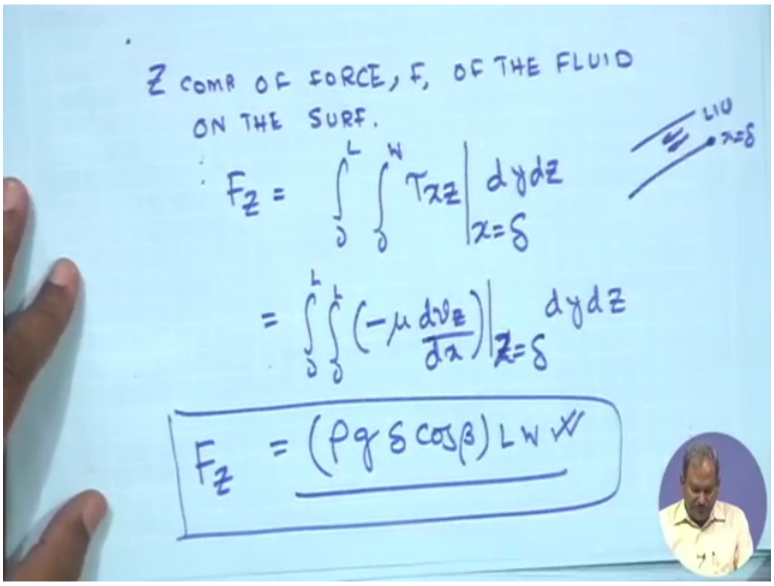

So Z component of force, let us call it as F of the fluid on the surface, it is in the Z direction from 0 to L, from 0 to W tao XZ, remember that we have to evaluate this shear stress at at X equal to delta because that is where it is acting. This is the liquid and this is that X equal to delta dY dZ. So this is the same as for tao - mu Times d VZ dX at Z equal to, sorry at X equals delta dY dZ, it would be equal to rho G delta cos beta L times W. This is the force in the X direction.

So this is interesting, what you see here is that this expression is nothing but the Z component of the weight of the entire fluid, entire fluid present in the film on the plate. So this gives you and there is no mu in here. So this entire thing tells you that the force exerted, force of the fluid on the surface is simply the width of the fluid contained in the film. So that is another interesting result. Whenever we talk about Uhh this kind of simple modelling, we must also be aware of the limits imposed by our simplified treatment.

The 1st thing is it is only valid for, it is not valid for very fast flow, it is only valid for laminar flow. If you do have turbulent flow, if the liquid is moving at a very high velocity, then you are going to get waves at the top and which you would probably not be able to use the concept that it is 0 shear at liquid vapour interface and so on. And I am also, we have also assumed that it is one-dimensional flow with straight streamlines. If it is two-dimensional flow or a three-dimensional flow, then this simple analysis would not be, you would not be able to able to use.

And finally whether or not it is laminar flow or turbulent flow, you would be able to obtain that, you would be able to get an idea by calculating what is the Reynolds number for the flow. And for certain range of Reynolds number, lower values of Reynolds number, this analysis is perfectly valid. And this is an ideal example to show how it can be done. The other complexity that we did not consider, let us say that solid plate on which the inclined plate on which the flow takes place is at an elevated temperature.

If it is at an elevated temperature, then the viscosity that we have used to express the shear stress using Newton's law of viscosity, that tao is equal to mu Times the velocity gradient, that mu would also be a function of X. So mu near the plate, since the temperature is more would be lower as compared to the mu near the top. So this variation in the physical property, the transport properties of the system can cause additional problems and you will not be able to obtain such a closed form simplified solution.

If you also have, if you have the plate at a higher temperature, then that could set in, that will set in heat transfer across the film as well. So not only the Thermo physical property would vary, the temperature would vary, as a result of which the flow field, the velocity field would be different than what we have done here. So this is just a tiny baby step that we took in this class towards understanding fluid flow, towards trying to see how the shell momentum balance in the simplest possible term can be used to find out what is the velocity field.

But the real-life is much more complex. As we progress in this course, we would try to get, we would try to model, simulate situation is which are closer to reality and this course should teach all of you the tools to analyse such real-life problems ultimately.