**Transport Phenomena.**
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**Lecture-26.**
**Turbulent Boundary Layers (continued).**

So we are going to see based on our previous discussion how this one seventh power law can be used in turbulent boundary layers in order to obtain the expression for the growth of turbulent boundary layer and we know the shortcomings of the one $7^{th}$ power law. The major shortcoming is that even though it is successful in expressing the data points near the Central line, it does not meet the, it is not very successful in predicting the data close, in a region close to that of the wall.

More importantly the profile, the one $7^{th}$ power law profile predicts an infinite shear stress at the wall which is obviously not true. So when we think of using moment integral equation as I mentioned there is the left-hand side which contains tao W and there is a right-hand side which is essentially an integration of the profile or the moment integral thickness which is an integration of the profile. As long as we have integration, we are integrating a moderately successful expression for velocity the errors introduced are not significant.

At the moment we differentiate the profile as would be the case to evaluate the expression of the value for tao W, the errors would be, the errors would be significant and it simply cannot be used. So we must think of other ways to express tao W in turbulent flow so that we can use one $7^{th}$ power law for the right-hand side of the expression but not on the left-hand side which contains the wall shear stress. And the wall shear stress using one $7^{th}$ power law is absolutely not possible.

So we would start with our boundary layers, boundary layer treatment in turbulent flow with the with the, with the Naviar Stokes equation, Z component for a horizontal pipe and we know, we have just derived in the previous class is that the pressure difference, how it is related to the wall shear stress. The relation between the wall shear stress and the pressure gradient for the case of flow in a horizontal pipe would be our starting point to evaluate the left-hand side of the momentum integral equation which we are going to do now.

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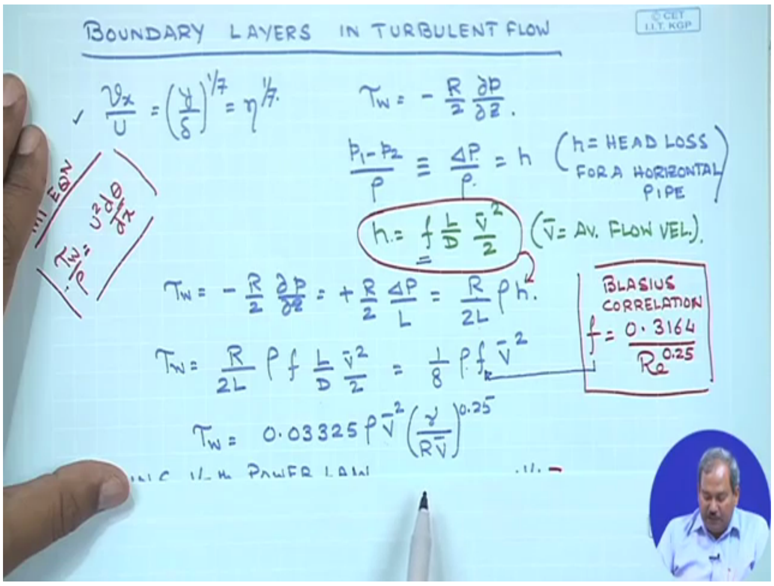


So the 1st thing that we have is the one 7th power law profile that we intend to use but we still do not know how to use it on the left-hand side. So we start with the expression which we have obtained before, the wall shear stress is equal to - N R which is the radius of the pipe by 2 dell P dell Z. And it has been defined, it is just a definition that P1, the upstream pressure - P2 which is the downstream pressure, the difference in this pressure by rho is denoted, is defined as delta P by rho and from our study of fluid mechanics, this delta P by rho is also known as the head loss and commonly expressed as H.

So for a horizontal pipe, this delta P by rho is simply going to be equal to H where H is the head loss. We also understand that this head loss using the friction factor formula again from the fundamental fluid mechanics, H is simply F L by d V square by 2 where V, sorry V bar square by 2 where V bar is the average flow velocity. So we, one would then able to express tao W, the wall shear stress as - R by 2 dp dZ - R by 2 dP dZ and then instead of dP, I am going to bring in delta P to therefore since dell p is equal to P 2 - P1 and delta P is defined as P1 - P2, so this becomes +, so it is delta P by L.

And delta P is simply equal to rho H, so it is R by2 L times rho H. The expression of H can now be substituted in here, so that is the arrow points to that, so the only unknown here is what are we going to do about the friction factor. If you remember your fluid mechanics, the Moody diagram and so on, the friction factor for a smooth pipe can be expressed in terms of Blasius correlation, this is again I would request you to take a look at fundamental textbooks, basic textbooks in fluid mechanics which deals with the friction factor and the Blasius correlation gives F, the friction factor in as a function of Reynolds number to the power 0.25.

So my tao W now becomes equal to R by 2 L, instead of rho H, I bring in, bring in this f rho f L by 2 V square by 2 and this is simply equal to 1 by 8 rho F V square and this expression for F is substituted at this point and what we have then is tao W equals this. Now if, the beauty of this is, if you if you carefully look at it, then the expression for tao W in this case does not involve the gradient of a velocity, rather it expresses it in terms of velocity.

So the problem of using the one $7^{th}$ power law to evaluate tao W is taken care of, is somewhat taken care of since we are not using the gradient of velocity but we are simply using velocity by our incorporation of the head loss and by our incorporation of the Blasius correlation. So now we are going to use one $7^{th}$ power law, we understand that is at this point that this involves some approximation, we are introducing some errors into our analysis but as long as we are aware of the point where errors can creep in and later on we can we can always go and find out if this error is acceptable or not we, we are we should be fine, the alternative is impossible.

We have seen how difficult it is to use the analytical approach, the differential approach for the simplest possible case of flow over a flat plate. And we have also seen the difficulties associated with having and universal velocity profile, a semi-empirical approach was the best thing that we could come up with divides the flow region into 3 different parts viscous sublayer, transition region and the turbulent course. And an universal velocity profile equally valid in all these 3 subregions was simply not present.

The other alternative is one $7^{th}$ power law with its inherent limitations. So in order to evaluate the left-hand side of the momentum integral equation tao W that contains a velocity gradient, we know that we cannot use one $7^{th}$ power law, so therefore we go to a head loss formula and the concept of friction factor and the empirical expression of friction factor in turbulent flow which is the Blaisus correlation, plug them all together and we get an expression for tao W in terms of velocity or its square but not in terms of velocity gradient.

So we at this point of time, we think that we have no other alternative but to use the one $7^{th}$ power law into the expression for tao W but since we are using the one $7^{th}$ power law expression, not its differential form, not its derivative, we are somewhat safe. But keep in mind that this is one sorts of error that one approximation that we are making in using 17 power law in the expression of Tao W.

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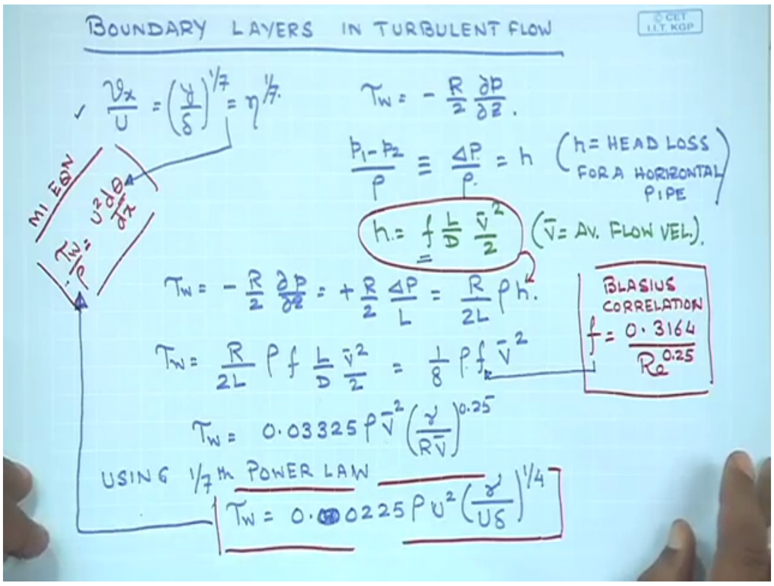


So we come to this point that the tao W is therefore the equal to this expression where once you plug in the expression for F and if you use the power law profile, you get the final expression of Tao W as this is where this capital U is the centreline velocity, this is the kinematic viscosity, this is the thickness of the boundary layer and now this expression of tao W can be used as the left-hand side of the momentum integral equation. So we are safe with this. And in the left-hand side and the right-hand side, I can I can directly plug-in the expression for VX by U in the one $7^{th}$ power law form.

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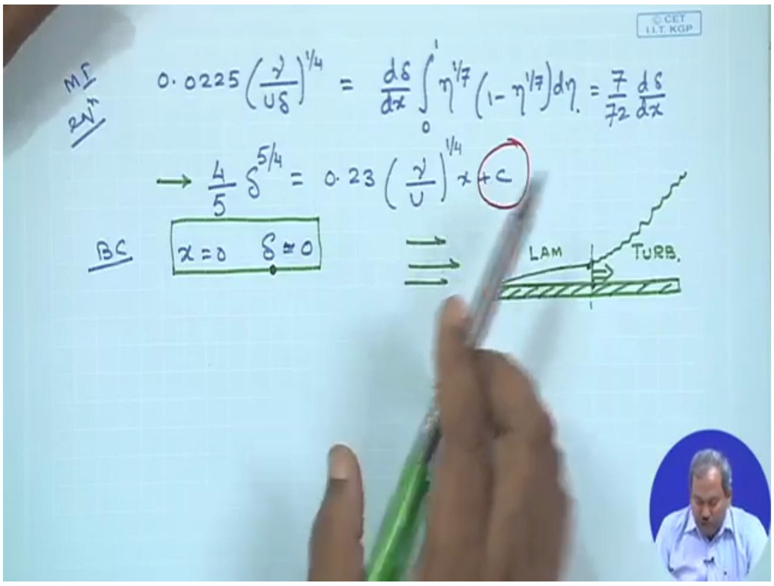


So once I write it once again, the exact form would look something like this. The left-hand side which we have obtained previously is equal to d del dX, the ordinary differential equation 0 to 1, I am integrating over the entire floor area Eta which is the dimensionless distance from the solid wall and 1 - Eta to the power one $7^{th}$ d Eta. So the MI equation for a turbulent flow takes this form where I am integrating the one $7^{th}$ power law and I have used Blasius correlation to obtain the expression for tao W in this case. So once you perform the integration, this would become 7 by 72, 72 d dell dX.

You integrate it once, the ordinary differential equation and what you get is delta to the power 5 by 4 is 0.23 kinematic viscosity by the Central line velocity to the power one $4^{th}$ X + C. So the problem still remains is how do you get the integration constant C. Remember previously what we have done for the case of laminar flow is that at X equal to 0, delta is equal to 0. This is the boundary condition that we have used for the case of laminar flow. But are we justified in saying that X is, at X equal to 0 delta equal to 0?

Because if you see what happens for flow over a 5, over over a plate when you have the formation of, when you have the flow coming towards it, the boundary layer starts to grow, it stays laminar and at certain point it starts to become turbulent and then it rises rapidly. So this part is laminar, this part is turbulent. So therefore the governing equation that we have obtained is valid for this region.

On this on this side I still have laminar flow, so I normally I would not be able to use this boundary condition that at X equal to 0, delta equals 0, that means at the at the at the starting point, the thickness of the boundary layer even for a turbulent flow is equal to 0, though we know that the turbulent flow starts from this point onwards where the boundary layer thickness, there is the boundary layer thickness, delta is not equal to 0. So here, here in we are introducing another approximation.

The approximation that we are introducing is that at X equals 0, delta is equal to 0. So we will use the same condition as before as in the case of laminar flow. There are certain situations in which the errors introduced by this assumption is not going to be significant. If we have a turbulent flow turbulence promoter placed at X equals 0, there are certain situations in which it is better if we have turbulent flow from the very beginning, a quick rapid growth of boundary layer.

It is it is advisable to have this sort of a boundary layer and for those applications where artificially we create localised turbulence at the point of $1^{st}$ contact of the liquid with the solid plate, the assumption that at X equal to 0, delta is 0 or in other words the flow, the boundary layer starts as a turbulent boundary layer is somewhat justified. So the, it is still an assumption and it still introduces some error but it creates a nice compact form and it precludes the requirement that you need to know what is the value of delta when the flow starts, when the flow becomes turbulent.

So for those cases in which you have turbulence promoter creating localised turbulent, mimicking a situation close to that of turbulent flow from the very beginning, the use of X equals 0, delta equals 0 is justified. But keep in mind that this is another source of error that we are intentionally putting into our development inaudible to keep the final form simple to use, okay. So we will still then use the condition, boundary condition that at X equals 0, delta is equal to 0.

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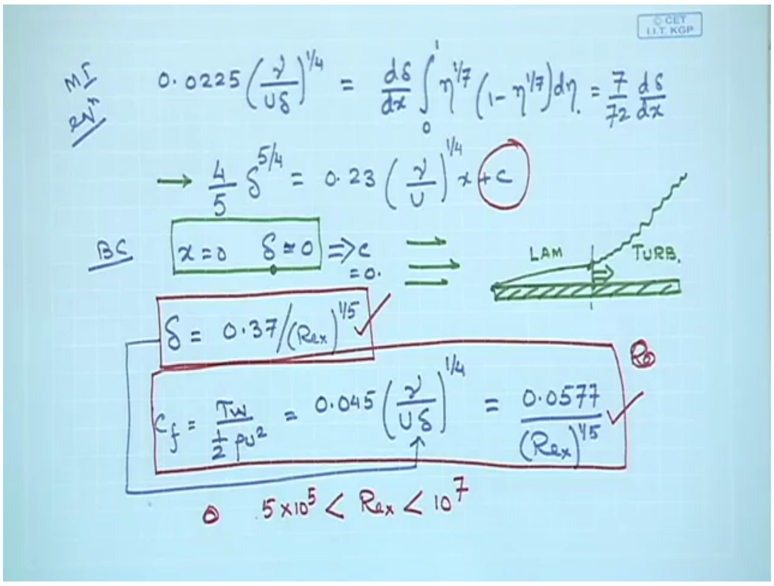


So when we do that the expression that you would get for delta, so this would give C to be equal to 0 and delta would simply be equal to 0.37 by REX to the power 1 by 5. And similarly he would be able to obtain what is skin friction coefficient defined as before as Tao W by half rho U square which would be equal to 0.045 nu by U delta to the power one 4$^{th}$. You can substitute the value of delta in here and therefore he would obtain the value of CS to be equals 0.0577 by RE, Reynolds number to the power 1 by 5.

So we have, now we have 2 expressions for the growth of boundary layer in turbulent flow and the expression for CS, the friction factor in turbulent flow, so this this expression and this expression. And the validaty of these 2 expressions are, for the Reynolds number which is which is going, the validity would be, Reynolds number should be more than 5 into 10 to the power 5 and less than 10 to the power 7. The important point here is not only we have obtained symbol 2 expressions quite easily using certain approximations which we are aware of, the expressions for delta and CF.

But the proof of the efficiency, the proof of the utility of these 2 expressions would be when you compare them with the experimental data. Astonishingly the data and the prediction of these 2 correlation, they are within 3 to 4 percent errors. So there is only 3 to 4 percent errors when you use any of these expressions to predict either the thickness of the boundary layer in turbulent flow or the skin friction coefficient in turbulent flow in the boundary, in the in the in the developing flow inside the boundary layer when the conditions are turbulent.

So with all these approximations, you are still able to fit the, still able to predict the results within 3 percent of the, with 3 percent with 3 percent error only. That shows the utility of momentum integral equations and even with all these approximations, the efficiency or the importance of these 2 expressions in expressing turbulent flow, the growth of the boundary layer and the friction coefficient. So that is the beauty of the approach that we have used so far.

One more point I would like you to appreciate is that delta here depends on REX Reynolds to the power - 1 by 5 which currently identifies the rapid growth of boundary layer with X once the flow becomes turbulent. So the figure that I have drawn over here as the growth of the boundary layer, it is going to be very slow, gradual as long as the flow is laminar and then it starts rapidly growing by the flow becomes turbulent. So turbulent boundary layer develops more rapidly than the laminar boundary layer and the agreement with experimental results shows the use of momentum integral equation as an effective method.

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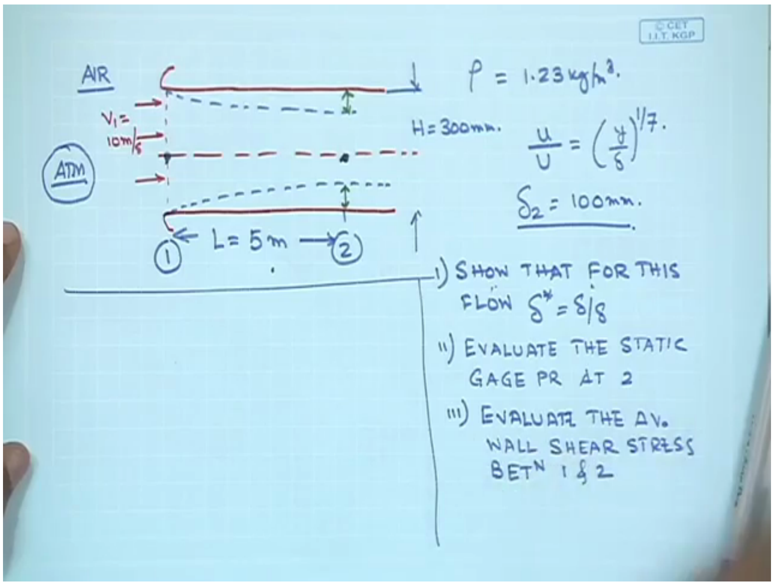


So these 2 are the take-home points from the exercise that we have done so far. So next what I would do is I will I will try to show you the use of these concepts in the form of a problem which we will solve, which I will I will solve and discuss with you. So the problem that I am going to solve over here is essentially deals with a pipe like this, a duct which is a horizontal duct, the intersection is well rounded, this is the Central line and the, at the entry point, the velocity is 10 metre per second.

There are 2 sections, section 1 and at some point downstream it is section 2, it is air which is flowing, flowing into this and the rho, density of air is 1.23 KG per metre cube. The duct height, the height of the duct which I denote by H is equal to 300 millimetre. And the conditions are such that the turbulent boundary layer starts to grow from the very beginning itself. The flow is not fully developed and it can be assumed that the velocity profile in the boundary layer which forms both from the top and from the bottom is given by Y by delta to the power 1/7.

The inlet flow is uniform at 10 metre per second at section 1, at section 2 the boundary layer thickness on each of the wall of the channel, delta 2 is equal to 100 millimetres. So the duct size is 300 millimetres but at section 2, the boundary layer thickness at, at the top and at the bottom is equal to 100 millimetres. The 1st thing you have to do is show that for this flow delta star, the displacement thickness is equal to delta by 8 and the 2nd part of the problem is evaluate the static gauge pressure at 2.

And 3rd is evaluate the wall shear stress, the average wall shear stress between 1 and 2, where this length is equal to 5 metres. So you can see that the flow is going in outside over here we just have the atmosphere present, so there 1st thing that we need to realise is that there must be a suction present which pulls the air into the duct. So you have atmospheric pressure over here, at this point the pressure must be lower than atmosphere, and as you move over here, since the flow is taking place in this direction, this pressure is going to be lower than this pressure and the gauge pressure here therefore is going to be is going to be negative.

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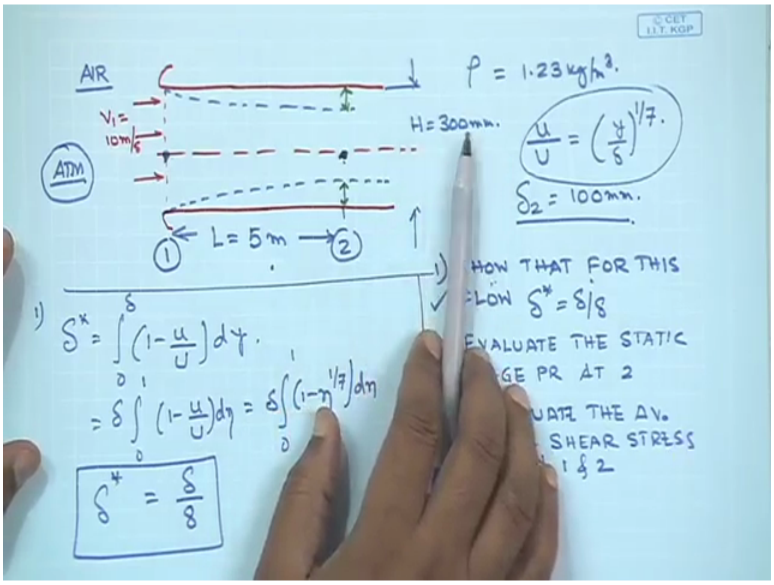


Here is a gauge pressure is negative and at this point the gauge pressure is going to be more negative. But let us 1st try to see how the 1st part can be done. The definition of delta star when I do, do the 1st definition of delta star is simply 0 to delta 1 - small u by U Times dY, or if you express it in dimensionless form, definitely going to be 1 - u by U Times d Eta. And since 1/7 power law is provided over here, I can simply write it as delta 0 to 1, 1 - Eta 2 to the power 1/7 d Eta and when you when you perform this integration, you should be able to see that delta star would turn out to be equal to delta by 8.

So the 1st part therefore is straightforward. But your you also have to keep in mind is that if I would like to find out what is the pressure at this point, I am going to use Bernoulli’s equation. And the Bernoulli’s equation can only be used if the Bernoulli’s equation can only be used if the flow is inviscid in nature. But we have a situation here where there is a viscous flow taking place along the sides, the boundary layer is growing. So the concept of displacement thickness will play a critical role in this.

Now if you refer back to our discussion of what is displacement thickness, it is the distance by which the solid plate will have to be raised in an inviscid flow situation so as to get the same amount of reduction in mass flow rate which is there due to the presence of the boundary layer. So if this problem is to be considered as an inviscid flow problem so that we can use Bernoulli's equation, we need to realise that at the entry point, the distance between the 2 plates is equal to 300 millimetres or H.

But at location 2, if I incorporate the concept of displacement thickness, the distance between the 2 plates has to be reduced by delta farm delta star from the top and delta star from the bottom. So to transform this problem into an inviscid flow problem, the entry, at the entry point, the distance between the 2 plates and the area would be equal into 300 millimetres multiplied by whatever be the width. Whereas at location 2, the distance between the 2 plates is going to be H - 2 delta star where delta star is the displacement thickness multiplied by W.

Only when we make this this change, we can use Bernoulli's equation for a situation in which we have frictional forces present and in which the flow is still developing. If you understand this concept, then the rest of the problem is simple. So on one hand I have growing boundary layer, viscous flow, the moment I use the boundary, the concept of displacement thickness, I simply reduce the flow area by a distance equal to the displacement thickness twice because boundary layer, about really grows from the top and from the bottom, so here is the floor area is H times W, here the flow area is H - 2 delta star times W.

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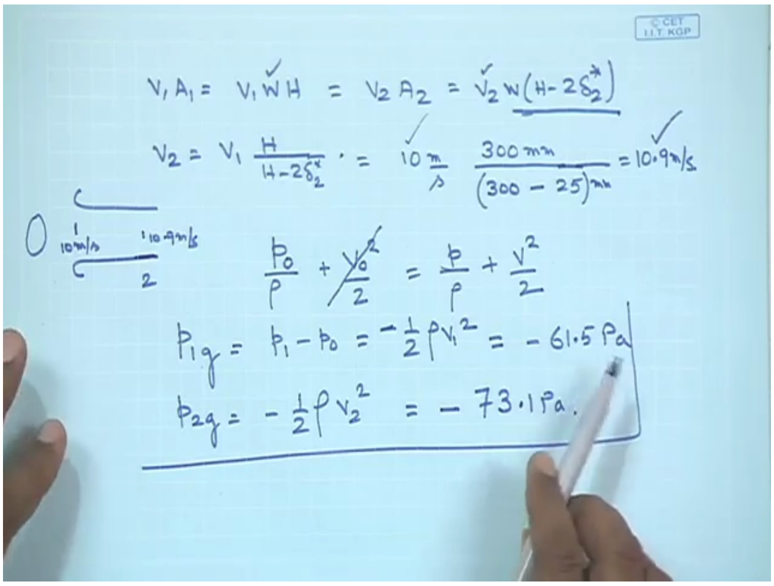

Only when we make this change, now I use Bernoulli's equation between a point which is outside of the duct, at the inlet of the duct and at location 2 of the duct. Okay. So with that behind us, with that understood, we can then find what is going to be the, 1st of all from the continuity equation we can write V1 A1 which is equal to B1 times W into H, W is the width which remains unchanged, constant. It should be equal to V2 times A2 and A2 now becomes V2 times W and as I was explaining it must be equal to 2 delta star, this is the displacement thickness.

So your V2 must be equal to V1 times H by H - 2 delta star, 2 delta 2 star and your V1 is 10 metre per second, H is 300 millimetres and your boundary layer disturbance thickness is delta by 8, so it should be 25 millimetres. This is going to be 10.9 metre per second. So at the entry point the velocity is 10 metre per second, at the location 2, it is going to be 10.9 metre per second, when we, since the flow area has now been reduced, the flow area has now been reduced by this amount.

So if the flow area has been reduced, the velocity has to increase. So if you use Bernoulli's equation now, then for a point outside where it is P0, the atmospheric pressure and since the flow is in the in the gain in over here, there is no velocity, it is still a must be a, so V0 would be 0, this is going to be P by rho + V square by 2. So the gauge pressure at location P1 would simply be P1 - P0 from this which would be equal to - half rho V1 square and p2 G would simply be - half rho V2 square, where we know the value of V1 and the value of and the value of V2.

So when you put in these values, this P1G should be should come out to be - 61.5 Pascals and P2 would be equal to -73.1 pascals. So this use of the, precondition for the use of the Bernoulli's equation is that we must transform this viscous flow to an inviscid flow problem. And I can change viscous flow to inviscid flow only when we use the concept of, concept of displacement thickness. The moment we use the concept of displacement thickness, the area available for flow has now reduced by the, by an amount equal to twice delta 2 star from the top and the bottom.

And since the flow area is reduced, the velocity has to increase. So we calculate what is the velocity at these 2 points. And outside of the duct, the air is still with no velocity, the pressure is 1 atmosphere. So using Bernoulli's equation once between the outside air and at location 1 we calculate what is the gauge pressure at location 1 since we know the velocity. And then we use Bernoulli's equation for the outside atmosphere and location 2 where we know what is the velocity, the increased value of velocity.

And there we would see the gauge pressure would be something negative at 1 and even more negative at 2 each essentially drives the fluid from location 1 to location 2. So this is a nice example of the use of displacement thickness. And in the next class since I would quickly go through the solution of the 3rd part of the problem which essentially tells us to find out what is the average shear stress for this condition between location 1 and 2, which I will take up in the next class.