Course on Transport Phenomena
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Module No 5
Lecture 24
Boundary Layers (Contd.)

Just a quick recap of what we have done in previous classes. We started with the microscopic balance in which the extensive property of the system, how would that change inside the control volume as a result of unsteady state effect and as a result of convective motion of fluid coming into the control volume.

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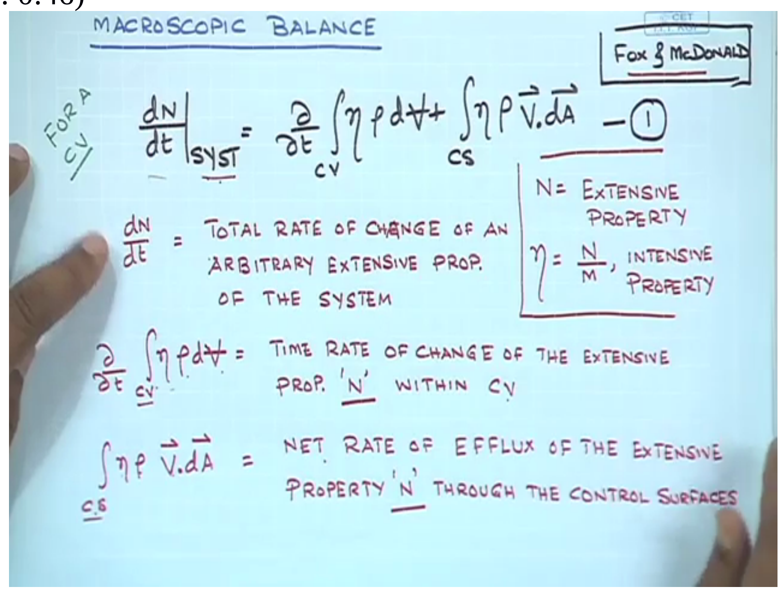

Based on that we have started with an equation which looks something like this that for a control volume, this is the expression with which which was the starting point of all our previous discussions where the DNDT is the total rate of change of an arbitrary extensive property of the system. The 2nd term is the time rate of change of the extensive property n. It could be anything within the control volume.

And the corresponding intensive property is Eeta. So Eeta is nothing but N by M where M is the mass. So Eeta is the intensive property corresponding to the extensive property denoted by capital N. And the last term, this term on the right-hand side, gives us the net rate of eflux that is the algebraic sum of inflow and outflow of the extensive property N through the control surfaces.

So with this equation we we took this N in the 1st case to be the mass of the system and if N is the mass of the system, then obviously the corresponding intensive property would simply be equal to 1.

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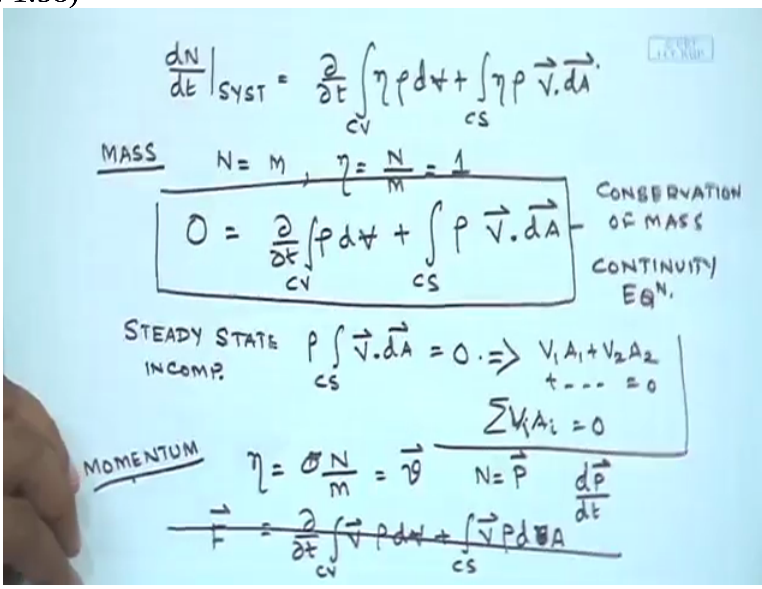

And the equation which we have obtained out of that is nothing but the is is nothing but the for for the for this case what we get is the conservation of mass, equation of conservation of mass or the integral form of the continuity equation. In the 2nd 2nd part, we take the extensive property to be equal to momentum such that capital N is now the momentum. So therefore the corresponding intensive property would simply be equal to the velocity of the velocity of the control volume.

So if N is equal to P where P denotes the momentum, therefore the corresponding intensive property would be V and the DNDT which now becomes equal to DPDT would simply be the time rate of change of momentum which is nothing but the force acting on the system. So here also, instead of Eeta, the we we are going to put V the velocity and therefore the this is going to be our steady-state term and this is the net eflux of momentum through the control surfaces inside into the control volume.

So DNDT system is when the limiting condition when the system and the control volume coincide and in that case, DNDT would simply refer to the forces acting on the control volume.

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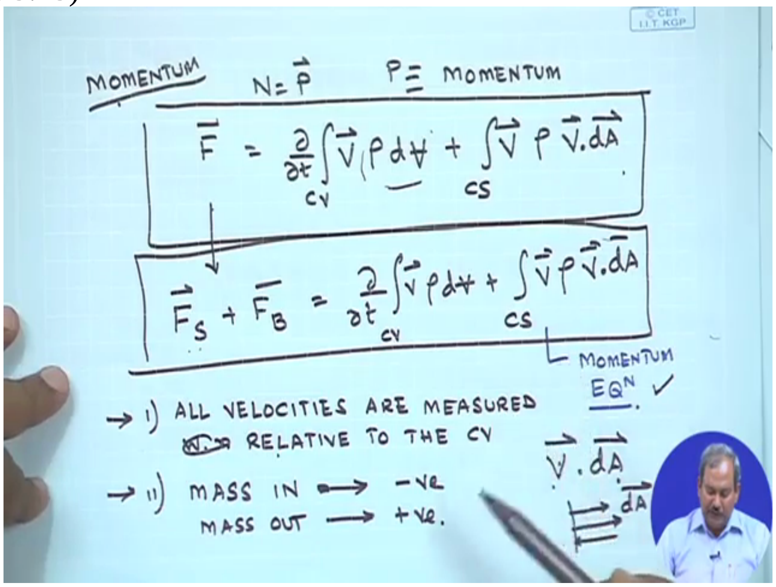

So when we have forces acting on the control volume, the relation that they have opted is the force acting on the control volume is equal to the time rate of change of momentum of the control volume and the net momentum eflux into the control volume. And forces are identified as either the surface force or the body force. The common examples of surface forces are the pressure and shear wherever an example of body force could be the gravity force in acting in the system.

So this equation is the, is known as the momentum equation or rather the integral form of the equation of motion. The points to note here are that that all low 50s that we refer here are measured relative to the control volume and by convention, mass in is always taken to be negative because of the dot product of the velocity and the area vector where the area vector is always directed perpendicular to or towards perpendicular to the surface.And mass out would would be would be positive.

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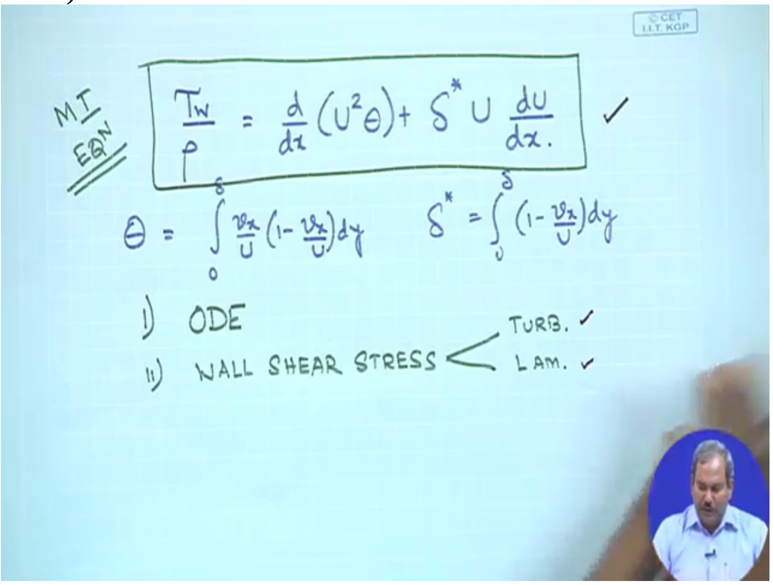

Starting with this and some more simplifications we have ultimately obtained an ordinary differential equation and where U is the free stream velocity, theta is the momentum thickness that we have earlier derived to be equal to this. And Del star is the is the thickness more is the is the displacement thickness which is of data previously in this form. So the duty of this equation is that 1st of all it is an ODE and it does not say anything about Tao W, the nature of Tao W. So whether it is laminar flow or turbulent flow.

So therefore this equation the momentum integral equation in its present form is equally valid both for turbulent flow and for laminar flow. And since we have the variation of the free stream velocity incorporated into the equation, therefore this equation is also valid for cases in which the it is a curved surface such that free stream velocity is also a function of the axial position. So this is a very general equation and we have seen the use of this equation in in solving some of the problems that we have we have done before.

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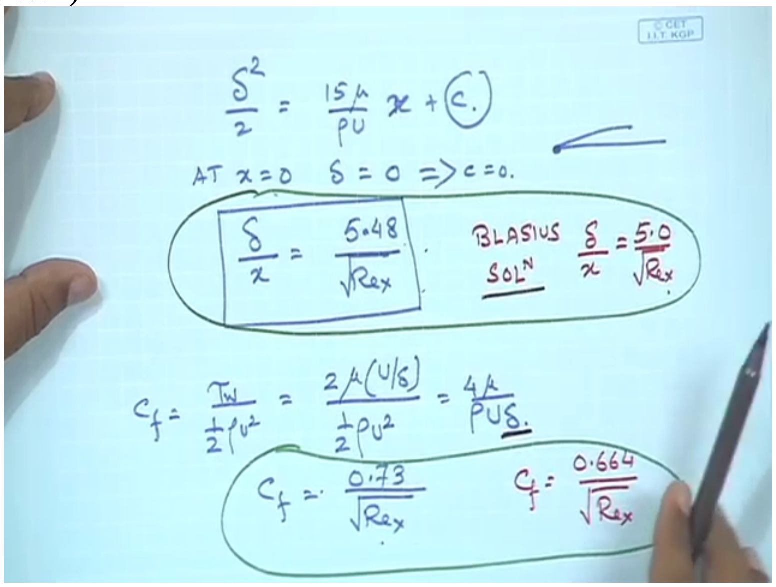

And the final final form for the case of flow over a flat plate that we have obtained in the previous class where delta by X, the growth of the boundary layer as a function of axial position and as a function of other parameters including the impose condition of the approach velocity which is embedded into Reynolds number and the properties of the fluids also incorporated into the Reynolds number, they are of this form.

If you recollect the exact solution for flow over a flat plate which is the exact solution which is also known as the Blasius solution, the form of the equation was exactly the same, only difference is the variation in the value of the constant. In the case of exact solution, it was 5 whereas in the case of the momentum integral equation, it was 5.48. Similarly we have defined what is the shear stress coefficient defined as Tao W divided by half to U Square where U is the free stream velocity.

So we following this and incorporating the the expression of Delta from here, we have obtained the expression for the sheer stress coefficient to be equals 0.73 by REX and our result from the right solution of Blasius was the same form. Only difference is in the value of the constant. So we are getting within we can predict the growth of the boundary layer within an accuracy of + or - 10% in comparison with the Blasius solution.

And similarly, the coefficient of friction, the friction coefficient is also very close to the fiction coefficient obtained from the exact analytical solution. Now considering the problems which one encounters in solving a partial differential equation even for the simplest possible case of flow over a flat plate, the approach of momentum integral equation is therefore this quite helpful.

And we would see later on that since this equation takes into account the possibility of having turbulence was not in the system, the range of applicability of this equation is much more and it is quite easy to handle in comparison to the exact analytical method. So what additionally we have done is that we we we have we are going to we have seen how to how to assume a possible velocity profile inside a boundary layer for the case of laminar flow.

And they were expressed for example in the previous problem as a polynomial A + B Eeta + C Eeta Square where Eeta is the dimensionless distance from the wall defined as Y by delta where Y is the distance from the wall and delta is the local value of the boundary layer thickness. So if VX by U velocity inside the boundary layer divided by the free stream velocity.

So that is the dimensionless velocity defined as VX by U if it is expressed as capital A + B Eeta + C Eeta Square then you then using the boundary conditions, the relevant boundary conditions and the relevant boundary conditions are, no slip at the solid liquid interface that means VX by U is 0 when Eeta is 0, secondly velocity approaches the asymptotically velocity approaches the free stream velocity at the edge of the boundary layer and at the edge of the boundary layer, Eeta is simply equal to 1 since Y is equal to delta.

So at Eeta equals to 1, that means at the edge of the boundary layer, the value of, dimensionless value of the velocity would be equal to 0 and also, the gradient of the velocity at that point, that is at Eeta equals to 1, would also be equal to 0 since the velocity approaches the free stream velocity asymptotically. So using these conditions, we were we were successful in evaluating the values of A, B and C, the constants A, B and C.

And we have obtained an expression for the velocity. The expression for velocity when substituted in the momentum integral equation, give us the results that I will I have just described that is the and we could compare, we could benchmark the utility, the accuracy of momentum integral equation by comparing it with the results which are which were analytically obtained and we compare the expression of Delta and we compared the extortions of CF.

In both cases, they come out to be very close to each other. The reason for that is since boundary layer is extremely thin, and if we could correctly identify what would be the boundary conditions at the solid liquid interface, and at the edge of the gondolier, then what happens in between is probably going to have very small effect on the overall growth pattern of the boundary layer and the value of the fiction coefficient.

So this is the reason why momentum integral equation is successful in in providing an expression correct expression for delta and for correct expression for CF. So these are reasonably accurate, within 10 percent accuracy considering the amount of the amount of the less amount of effort that we have to we have to provide, we have to, we have to sub, we have to do in in terms of getting the solution, this is a welcome change.

So from now onwards, we will restrict ourselves to the solution of boundary layer phenomena using momentum integral equation. Before I move onto the treatment of the turbulent boundary layer which was not possible using any analytic method, we would in this class we would 1st solve, try to see how to fall one more problem where the profile of the boundary layer, the profile of velocity inside the boundary layer is provided.

And with that velocity profile, we would again see how close we are to the actual value of the boundary layer. So the 1st problem that we would do in this tutorial class is, given a velocity profile, can we find out what is the what is the what is the expression for the growth of the boundary layer and what is the expression for the fiction coefficient normally denoted by CF. So in this problem might have been mentioned that the the velocity profile inside the boundary layer is simply linear.

So that is the simplest possible variation. So it waves with a value equal to 0 on the solid surface to a value equal to V at the edge of the boundary where this velocity is essentially the free stream velocity. So if I express it into express the non-dimensional velocity as VX by capital U where capital U is the free stream velocity than VX by U has a value equal to 0 at Eeta equals 0. And

VX by U would be 1 when Eeta is Eeta is equal to 1. That means at the edge of the boundary layer.

Remember that Eeta is defined as Y by delta where delta is the local film thickness. Therefore the simplest possible profile that we can think of is VX by U is simply equal to Eeta. The functional form of velocity variation inside the boundary layer is VX by U is equal to Eeta. And we would quickly see how how this how this can be used with the help of momentum integral equation to obtain an expression for delta, the thickness of the boundary layer and an expression for CF.

So we start with our analysis of the argument, with the assumed profile that VX by U is simply equal is Eeta and see from there how to proceed in order to obtain the expressions of delta and CF.

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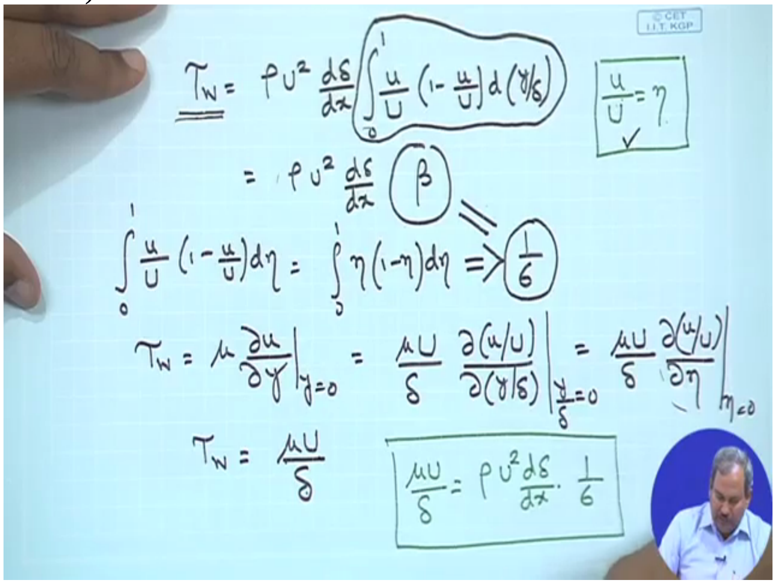

So we start with the bound the momentum integral equation which is Tao W is rho U Square D del DX from integration from 0 to 1 U by capital U, the velocity inside the boundary layer and the free stream velocity, 1 - small U by capital U DY by delta. And the note here that it has been given that u by U is simply equal to Eeta, that is linear velocity profile is prescribed for this condition.

So this can be now written as rho U Square D del UX and beta where this beta is essentially, this different integral, so beta is going to have just a numerical value. So since u by U, the velocity profile is given so we can find out what is going to be the different integral which is a very simple step, D Eeta so this is from 0 to 1 Eeta 1 - Eeta D Eeta. And this would once you integrate, this would result in the numerical value equal to 1 by 6. So this beta is therefore equal to 1 by 6 aince we have assumed linear velocity profile.

And also if the fluid is Newtonian, then all sheer stress can simply be written as Newton following Newton’s law of viscosity, Del Y at Y goes 0. So this is for a Newtonian fluid. And if I express everything in terms of the dimensionless velocity which is this and the Y is transformed to Eeta than what I would get is simply the dimensionless velocity and the dimensionless distance, everything is evaluated at Y by delta to be equal to 0.

In other words, this now becomes mew U by delta and Del u by capital U by Del Eeta at Eeta equals 0. Now since the velocity profile is provided to to me, the velocity is provided, so this is simply going to be equal to 1 and therefore the expression for Tao W would be equals mew U times delta. So what I have then is I have found out what is going to be the left-hand side of the momentum integral equation mew U by delta.

And the right-hand side of the momentum integral equation which is simply going to be 1 by 6 rho U Square D del DX. So this therefore is simply an ordinary differential equation and the final form of the ordinary differential equation would simply be equals times 1 by 6. This equation can now be integrated.

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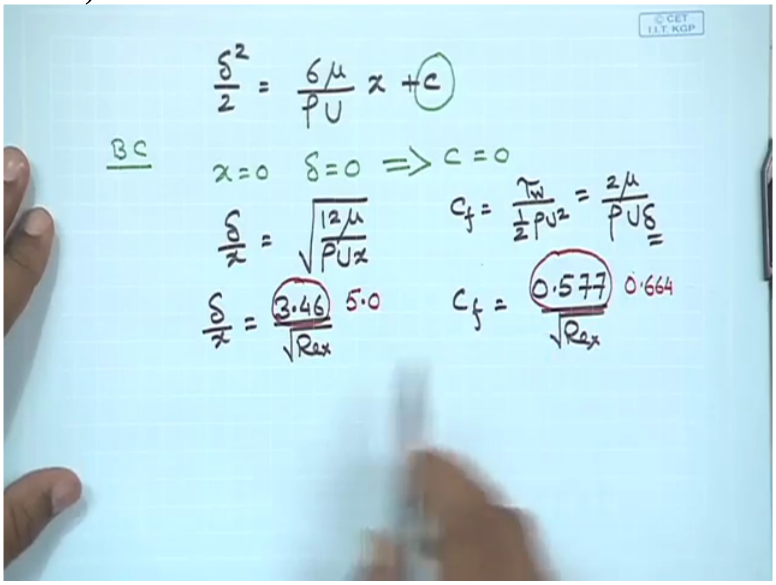

And once you integrate this equation, what you would get would be Del Square by 2 equals 6 mew by rho times capital U times X + C where this C is the constant of integration. And C can be evaluated through the use of the boundary condition that at X equals 0, that means at the start of the plate, X equal 0, delta the thickness of the boundary layer must be equal to 0. So using this boundary condition, you would get the value of C to be equal to 0.

And does this delta by X would simply be equal to 12 mew by rho UX or delta by X to be equals 3.46 by root over REX. Similarly if you find out the expression for CF by definition which is equal to Tao W by half rho U Square, this would be to mew by rho U delta. And when you put the expression for delta from here to this, to this, what you would get is CF equals 0.577 by root over REX. Compare this with the Blasius solution where the form but remain the same, except the constant would be 5.0.

Again the form here but remain the same, the constant is going to be equals to 0.664. So here you see that even very simple approach, the simplest possible velocity profile if you assume that, you still are not too far off from the analytical result which was detained by solving a partial differential equation numerically and using the concepts to obtain the exact form of the boundary layer growth or that of CF.

So this again underscores the utility of momentum integral equation in solving any problem. So far we are limiting ourselves to laminar boundary layer and this is going to be the method of choice for all solutions from now on. I would also solve one more problem regarding the use, involving the use of momentum boundary use of momentum integral equation.

And this problem is essentially a flow, again flow over a flat plate and they are not only you have to obtain what is what is going here also profile would be provided. If the velocity profile is not provided, you can resume polynomial and find out what is going to be the what is going to be the variation of the velocity as a function of Eeta, the dimensionless distance from the wall.

In the in this specific problem, it was as to what is the maximum value of the boundary layer thickness and what is, where do you get the minimum value of wall shear stress. So if I have flow over a flat plate, at which point you are going to get the maximum thickness of the boundary layer and secondly where are you going to get the minimum value of the wall shear stress. See the 1st question is obvious, if there is a flow over a flat plate and since the boundary layer keeps on growing, you would get the maximum in the boundary layer thickness at the endpoint of the plate.

So when you reach to since the boundary layer keeps on growing, when you reach the end of the plate, the thickness of the boundary layer is prior is going to be the maximum. So that is an obvious answer. But we would still see if we get whether if we get the same value. But the question of shear stress is not that straightforward. Where all, at which location the shear stress is going to be minimum? So that is the 2nd part of the problem.

And the 3rd part of the problem is, find out what is the total force exerted by the fluid on the solid plate. In other words, in order to maintain the solid plate at a static position in a flowing fluid, what force must be exerted? So that is the 3rd problem, 3rd 3rd part of the same problem. So given the profile, find out the maximum value of boundary layer thickness, find the location and magnitude of the minimum wall shear stress and an expression for the total force experienced by the solid plate due to the motion of the fluid. That is the problem which we are going to solve now.

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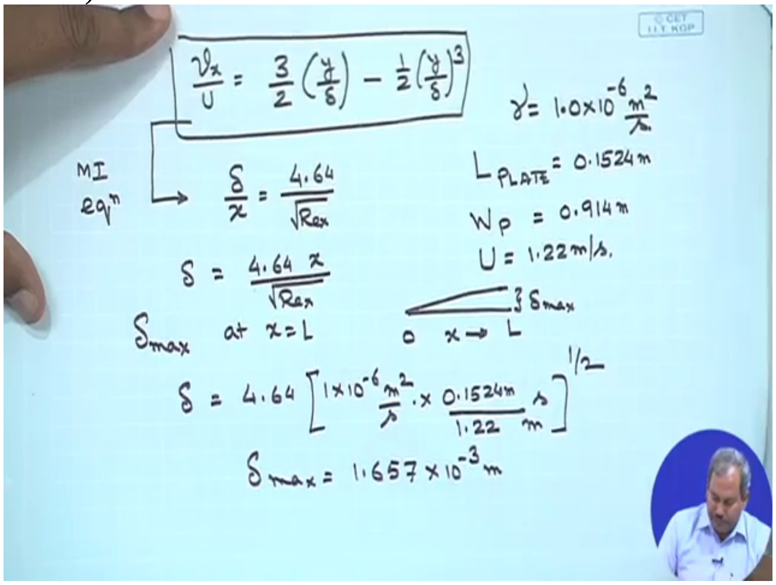

So here also it has been mentioned that the velocity profile which is VX by U is of this form 3 by 2Y by delta - half Y by delta cube. So this is the profile which has been provided to us and using the methodologies already prescribed here, the moment integral equation, it would be easy for you to obtain but I would still request you to do it on your own, you would see that delta by X but turn out to be 4.64 by root over REX.

The kinematic viscosity for the fluid is given as 1.0 into 10 to the power - 6 metre square per second. And the length of the plate length of the plate is 0.1524 metres and with of the plate is 0.914 metres. And the velocity, free stream velocity is provided as 1.22 metre per second. So this is free stream velocity and since it is flow over a flat plate, this is also equal to the approach velocity.

So since delta by X is given in this form, so you would you can you can see that the expression is simply going to be X by root over REX and the delta Max would simply be equal to at X equals L, that means if I have flow over this than of goes the Delta Max is going to be at X equals L. So X equals 0 to L, so therefore Delta Max would be equal to at X equals L. So when you plug in the values, delta would be 4.64 in 1 into 10 to the power - 6 which is the kinematic viscosity metre square per second.

The length is simply going to be 0.1524 meter. And the velocity is, free stream velocity is 1.22 metre per second, square root of that and your Delta Max would turn out to be 1.657 into 10 to the power - 3 metres or roughly it is close to 2 millimetres.

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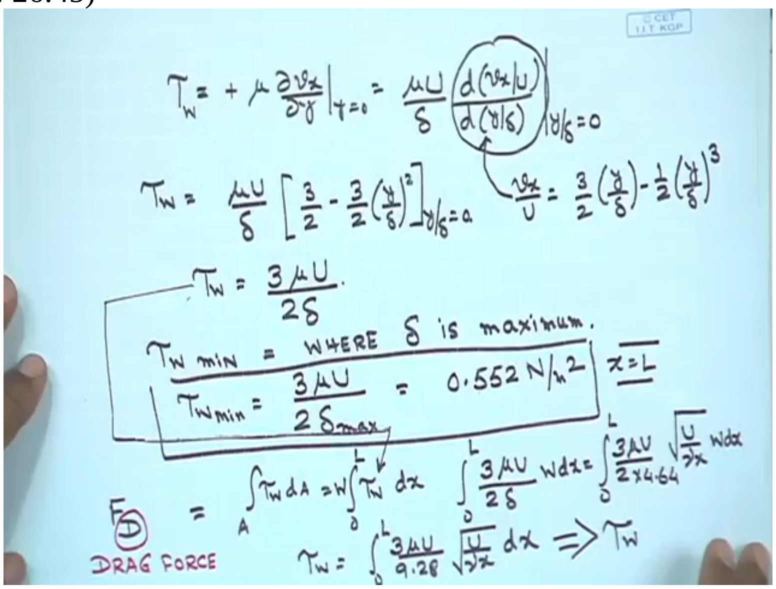

The wall shear stress on the other hand, the expression for that, the wall shear stress, so this is why it is going to be + mew times Del VX Del Y at Y equals 0. When you transform it to in terms of the dimensionless quantities, it would simply be mew U by Delta DVX by U DY by Delta and it is evaluated more wall shear stress is evaluated at a point where Y by Delta is equal to 0.

So therefore your Tao W would be equal to mew U by delta and for this, we plug in the expression of VX by U that has already been provided to be 3 by 2Y by Delta - half Y by Delta cube. So when I plug this in over here and take the, take the differential, what I would get is 3 by 2 - 3 by 2Y by Delta Square, everything is evaluated at Y by Delta to be equal to 0. So this Tao W therefore turn out to be 3 mew U by twice delta.

So Tao W minimum would be at a point where Del ties maximum. And when you, so Tao W minimum would be equal to 3 mew U by 2 delta Max and when you plug in the values of delta delta Max mew U and delta, the numerical value of this, you are going to be equal to 0.552

newton per metre square. So therefore the minimum value of the wall shear stress will be 0.55 to Newton per metre square and it would take place where X is equal to L.

So the at the edge of the boundary layer, at the edge of the boundary layer, you get the at the at the end of the plate, the location at the end of the plate would provide you with the maximum value of the of the of the boundary thickness and also at this point, you would get the least value of wall shear stress. So wall shear stress is a monotonically decreasing function of distance and delta is a monotonically increasing function of X.

So since the X, since Tao varies from point-to-point, therefore the total force exerted by the fluid on the plate will also vary with position. So to obtain the total force, since we know that shear stress is equal to force per unit area, so the total force on the plate would be provided if we integrate Tao W over the entire area. So the force by the fluid on the plate F if we if we denote it by F would simply be integration Tao W DA.

Now Tao W is not a function of W, the width of the plate. So if I take the W outside it would simply be integration from 0 to L, Tao W times DX. So that would be the expression for force exerted by the fluid on the solid plate. So which if we do it here, so the force FD and the term D here stands for the drag force. So this is a force due to the fiction of the moving fluid over the stationary plate commonly denoted by the drag, commonly denoted by drag force.

The expression would of that would be the integral over the surface Tao W times DA. So if I express it, since we understand that Tao W does not depend on W, so it is Tao W times DX and the W can be taken outside. And what you do here is, you are going to put the expression of Tao W in here and therefore you would get from 0 to L 3 mew U by 2 delta times WDX 0 to L 3 mew U 2.

And the expression of Delta from the previous previous page, where the expression of Delta was 4.64 by root over IX, if we put put it in there, what we would get is it to 4.64 root over U by mew X times W times DX. So Tao W, the final expression would be integration 0 to L 3 mew U by 9.28 root over U by mew X times DX. so the rest is simple.

You can simply do the integration over the entire length of the plate from 0 to L and obtain an expression for the force on force on the plate and the final expression for Tao W can be obtained from there which I again leave leave it for you to find out what would be the final expression for Tao W. So this again underscores the advantage of the advantage of using the momentum integral equation.

Now I think we are confident enough like we would be able to try to solve turbulent flow inside a boundary layer. Now the moment we introduce the concept of turbulent flow, the moment we allow turbulence present in the system, there are few things which we have to keep in mind. 1st of all, the expression of Tao, the shear stress to be if it is a Newtonian, to be equals - mew times velocity gradient will no longer hold because in turbulent flow, the transport of momentum is not only carried, not only by the molecular motion, it is the actual physical motion of packets of fluid having different velocities from one point to the other.

So the momentum transfer will no longer remain a molecular phenomenon. It will also have, it will also involve the formation of EDs or packets of fluid which will move with a specific momentum and therefore transport momentum in between layers in a fashion that we did not encounter or we did not envisage in our treatment of the laminar transport of molecular momentum or shear stress. In other words, the shear stress in turbulent flow would be much more, more significant as compared to the laminar flow.

So we have to keep in mind the molecular transport as well as the convective transport of momentum. The moment we bring in the concept of ADs then we would see that the stresses that we have encountered in the laminar flow will have to be modified by incorporating additional terms in the Navier Stoke's equation collectively known as Reynold stresses which depend on the fluctuating, locally fluctuating component of velocity as a result of turbulence present in the system.

So it is a very complicated case whenever we talk about turbulence, mathematically treating I mean it is it is possibly easier to visualise what happens in turbulent flow. But whenever you try to explain it mathematically, it becomes very complex.

So here again, the use of momentum integral equation and the approximations that we would use would definitely be helpful in obtaining closed form expressions of shear stress and growth of boundary layer, the expression for drag force and so on in turbulent flow. That is what we are going to do in the next class.