Transport Phenomena
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Module No 5
Lecture 23
Boundary Layers (Contd.)

So we would continue with the development of momentum integral equation which I am spending so much time on net because conceptually you should be very clear in terms of the different contributions of momentum through the control surfaces in a flow field because once these concepts are clear, you should be able to handle problems which are which are slightly out of the ordinary and you would be able to use your concepts in for in those problems to obtain solutions even more complex problems than what are discussed in some of the textbooks.

So what we have shown here is that starting with the the momentum equation and with the with the imposition of the conditions that these these are steady two-dimensional flows, constant property, we identified what are the forces on the left-hand side of the momentum equation. The forces on the left-hand side of the momentum equation, the constituent of body force and surface forces. We will resume that there are no body forces present in the system.

So if I write the equation for the X component, what I have is FSX is equal to since it is steady-state, the 1st term on the right-hand side would be 0, the del del T term would be 0 and what I have is the net flux of momentum into the control volume because of flow. So FSX when then we have we have drawn drawn a control volume ABCD, ABCD where BC is located very close to the edge of the boundary layer and AD is located very close to the solid liquid interface, very close to the solid play.

So what I have is MF, the momentum flow through a surface AB, BC, CD and AD. We have also identified that the when when we talk about the forces, the forces since only the surface forces will remain on the left-hand side of the equation. What could be the surface forces? We realise that these these 2 surfaces AB and CD, they are being perpendicular to the direction of flow, they are exposed to pressure force.

This surface AD which is parallel to the flow direction, it will not have any force in the X direction, any force due to pressure in the X direction. Whereas the surface at the top, BC which is slightly curved will have an X contribution of of force. Since the surface is curved, the force which is acting on this curved surface, multiplied by the projection of this curved surface in the X direction would give me the force which is force due to pressure on surface BC.

And we have also understood that on surface BC if you in think of look at the figure, the surface BC being very close to the edge of the boundary layer, will be devoid of, will not contain any velocity gradient. It does not contain a velocity gradient, then the shear stress would be equal to 0. So on 3 surfaces, 2 sides and the top surface, pressure forces will act. On the bottom surface, there would be no pressure force, only a shear force would act.

If we if I assume that the pressure at the left at at the left edge of that the flow is coming in, if the pressure here is P and the thickness is Delta, the depth is Z, DZ, then the force on this surface AB would simply be equal to P times Delta times DZ. Let us assume that the that the surface at the other end, the vertical surface at the other end, CD has a pressure equal to P + DPDX evaluated at X multiplied by DX which is nothing but the Taylor series expansion of whatever be the pressure at this point.

So this is the pressure. What is the thickness over here? Is Delta + D delta. So the force acting on surface CD would be pressure which is the Taylor series expansion of pressure over here, the area would be delta + D Delta multiplied by DZ. So those are the 2 pressure terms which we have identified. The curved surface, the projection of this curved surface in the X direction is D delta. The depth of it is DZ. So therefore the area which, of area which was contribute to an X component of pressure force is simply going to be D delta times DZ.

And the pressure which is acting on surface BC can be approximated as the arithmetic average of the pressure at B and dad at sea. Pressure at B is C, pressure at at C is simply the Taylor series expansion. So the average, arithmetic average pressure on surface BC multiplied by D delta DZ give us the X component of pressure force acting on surface BC. The only other thing which remains is what is the shear force on surface AD, the surface AD which is located very close to the solid surface.

Now we are expressing it, the force on surface AD as - of Tao W times the area. What is the area? The length times the depth, DX times DZ. So - Tao W times DX DZ would give us the shear force acting, the X component of the shear force acting on surface AD. Why this Tao W and why this -? Tao W is introduced because Tao W is the engineering parameter that we would like to know.

What is the shear stress expressed by the moving fluid on the solid object is the 2nd criteria, 2nd point of our interest? Our 1st interest is to obtain delta as a function of X. The 2nd, probably more important is to find out what is the wall shear stress, what is the shear stress experienced by the solid object when a when there is a relative motion between the fluid above it and the plate below?

So Tao W is the force experienced by the solid but then we think of the control volume of the fluid we need to find out what is the force on the control volume. If the force experienced by the plate is Tao W, then the force experienced by the fluid above it must be equal to - Tao W. So the force experienced by the control volume due to shear in the X direction would simply be equal to - of Tao W multiplied by DXDZ.

So now we have all the terms present in order to to be used in the equation of motion. All the forces terms, 3 pressure and one shear and on the right-hand side we have all the net all the net momentum flow, the momentum that comes into the control volume because of flow. So all the terms are in place. You plug in, put in the term of the equations and then you simplify.

I will not do the simplification process in here because no additional concepts are involved, it is only an algebraic manipulation, algebraic manipulation of these 6 or 7 terms in order to arrive at the final, final form. So that part is done in your text but since nothing new in terms of conceptual education is required, I will not do that in the class, you can take a look at your textbook, Fox and McDonald where this has been done in great detail.

So I am going, what I am going to do is I am simply going to write the final form of the simplified equation when all these considerations are put into place. And what we are going to get out of this is known as the momentum integral equation.

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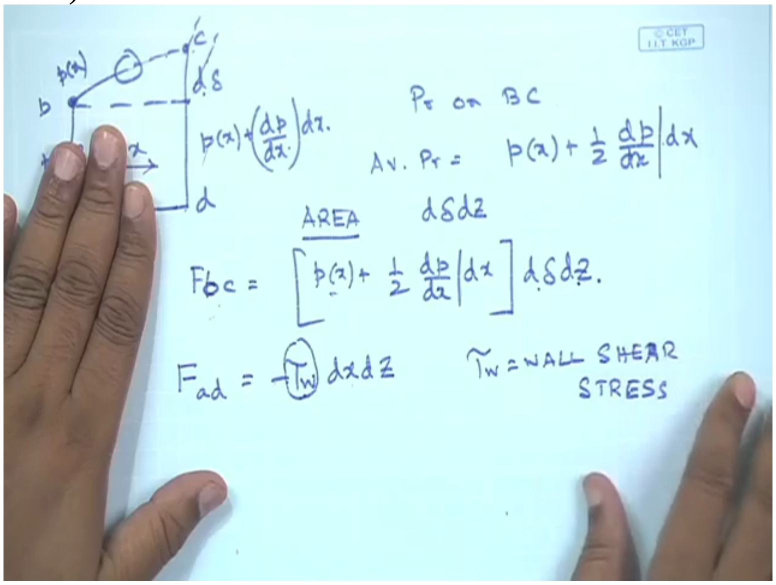

So I think conceptually it is clear how it is to be done. The only thing that remains is to to put those terms, simplify them, rearrange them and get the final form of the equation that I am going to right now.

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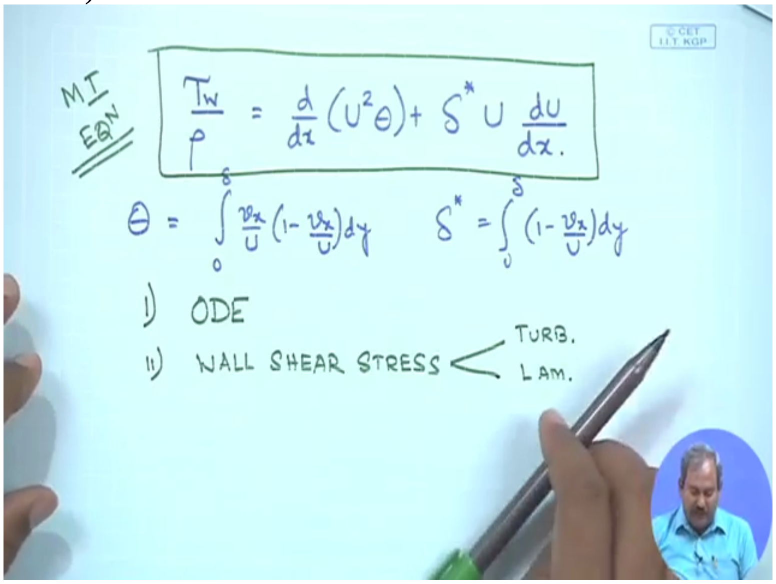

What you get is Tao W by rho is equal to DDX of U square theta + delta star UDU DX. this theta I think you will recall is the momentum thickness which is defined as VX by U 1 - VX by U times DY and delta star is the displacement thickness which was defined in the following form.

So this is the all important momentum integral equation. So what is then momentum what are the salient features of momentum integral equation?

That it relates the wall shear stress with the momentum thickness with the displacement thickness and variation of the velocity and variation of the free stream velocity over X. So points to note here is 1st of all this gives an ODE. Unlike the analytic approach we have used before, we have seen before, this equation gives rise to an ordinary differential equation and not a partial differential equation which is a huge improvement over of which is a huge improvement of our previous approach that we delivers an ordinary differential equation.

The 2nd one that you would you would see is that the wall shear stress appears in this expression and we have not made any suggestion that whether it is going to be what is the relation between Tao W and the velocity gradient. So it is open to Newtonian as well as non-Newtonian type of flows and since the relation between Tao W and the wall shear stress are not specified, it can be used for laminar flow as well as turbulent flow.

So the expression of without since the specification specification of the wall shear stress is not not there, therefore it is equally valid for turbulent flow as well as for laminar flow okay. So this is the equation which we are going to going to deal with in in our subsequent analysis. So but (()) (13:26) I would just like to work with me for a few more minutes and see how this equation can simplify of what we know about flow in a boundary layer.

The analysis, this is the gyration of this maybe (())(13:46) but what you get at the end of it is a compact expression. And any time you propose something, you have to benchmark it, you have to prove that what you are saying is correct. And how do you prove that? You must compare it with the results that are already known to us. So what is the result which is known to us on which we have sufficient, we know it to a sufficient confidence that the result is correct.

We have an analytic solution followed by a numerical solution for a very special case where there is flow over a flat plate with 0 pressure gradient, a Newtonian fluid, laminar flow. And we know what is the express what is the expression for delta as a function of X. We also know what is the functional form of the frictional coefficient for such a case.

So the 1st thing one should do is apply this momentum integral equation to that problem and see if you are predicting results which are close to that of the result for flow over a flat plate 0 pressure gradient steady laminar case. So that is what we are going to do and while doing so, it would be it would be clear to us how to handle this momentum integral equation. And I can assure you that by the end of this by the end of this chapter, you will all the exports of losing momentum integral equation in much more complicated cases.

So always start from the basic, think about the assumptions which are used, think about the if you do not assumed at which point you should start in order to obtain a solution for the case. But right now, let us see how we can use this equation for the simplest possible case.

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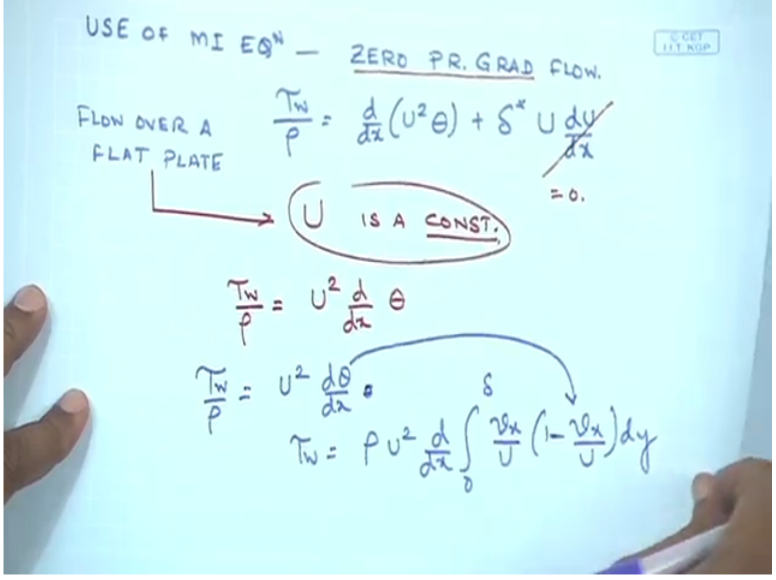

So we start with the simplest case where we have the…so use of MI equation 0 pressure gradient flow and we have Tao WU square theta + del Star U DUDX. So we are using it for flow over a flat plate. The moment we say flow over a flat plate, and it is a 0 pressure gradient flow, we understand that U is a constant. So for flow over a flat plate as we have seen before and for 0 pressure gradient flow, U is a constant.

The moment U is a constant, this equation will simply revert to Tao W by rho. And since U is a constant, U can be taken out and DDX of theta and this term would be equal to 0. So for for the situation where we have a flow over a flat plate, 0 pressure gradient, this would be the form of the equation and Tao W by rho would be equal to U square times D theta DX equals Tao W is rho U square DDX of 0 to delta VX by U 1 - VX by U times DY. So this is simply the definition of the momentum thickness.

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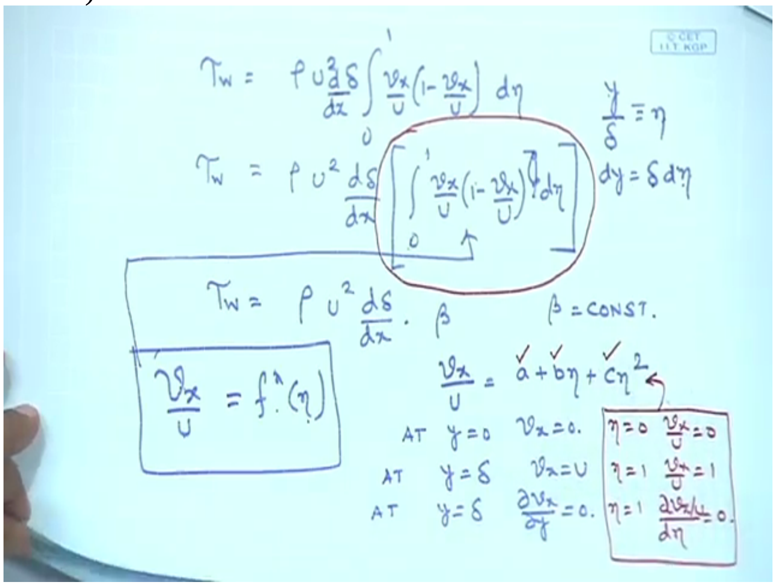

So we have it up to this and then we continue. And we will say that Y by delta is equal to eeta where it since the velocity in the previous previous equation the velocity is dimensionless. Since we are dividing velocity, local velocity by the free stream velocity which is a constant, so we are dividing Y by delta in order to, dividing Y by delta in order to bring in a new variable which we call as eeta and therefore we name it from 0 to delta.

Since we are dividing it by delta, it would simply be equal to 0 to 1 0 to 1 the VX by U will remain unchanged 1 - VX by U. And from DY it simply would be D eeta and therefore D eeta and I bring the delta outside where I have the DDX as well. Is the step is are the steps clear? 1st of all from the previous expression, I see that the velocity is dimensionless. So it was Y there, so I am going to change Y to a dimensionless variable eeta.

So I divide both sides, I divided by delta. This is just a definition. So therefore DY would simply be delta times D eeta, del ties the local thickness of the boundary layer. So to the previous equation, I substitute instead of DY, I substitute delta times D Eeta. So delta times D Eeta suited that can now be taken outside and I would clearly write it, rho U square D delta DX0 to 1, let us put it inside a bracket VX by U times 1 - VX by U times D Eeta.

Now if you look at the 3rd bracket, this one in here, it is a definite integral. If it is a definite integral, then what we are going to get out of this is a constant. So your Tao W would simply be equal to rho U Square D del DX times beta where beta is simply a constant. This beta is the this one. Now therefore in order to do this, one must know what is VX by U as a function of Eeta.

If I can find, if I can some means, if I have some means to find out the variation, the dimensionless velocity as a function of Eeta, then by substituting this functional form in here, should be able to obtain the numerical values of beta. And I am integrating it from 0 to 1, I am integrating it over a fixed interval. So what I would get out of this is just a constant. So the only job that one has to do you know in in in in solving a problem with momentum integral equation is to provide is to suggest what would be DX by U?

What would be the profile of BX you in terms of Eeta. Now one can choose a linear profile, a parabolic profile, a cubic profile or or some other profile. Surprisingly, you would see that the results are going to be very close to the one that we have obtained from the blacious solution. Why? That we will discuss later.

But yet let us start with some assumed profile of VX by U in terms of Eeta the dimensionless distance. Let us 1st assume it to be a parabolic profile. So VX by U is equal to A + B Eeta + C Eeta square where A, B and C are constants. Whenever you propose such a profile, you must evaluate, you must have a way to evaluate A, B and C. So how to evaluate that? In order to evaluate that, you should know VX by U or the variation of VX by U at different values of Eeta.

Remember again, Eeta is Y by delta and Y is the distance from the solid plate over which the flow takes place. So what is the condition on the solid plate? VX is 0, no slip condition. So you can say that Eeta equals 0, VX by U is equal to 0 1st condition. But you need to know to obtain what is going to be (())(23:36) to obtain the values of the other 2 constants. What are the what is the other boundary that you can think of where is the edge of the boundary layer.

And what happens at the edge of the boundary layer? What is going to be the value of VX by U at the edge of the boundary layer? It must be equal to U at the edge of the boundary layer. And secondly, another factor that that another another characteristic of the profile that you know is that is going to happen is that the gradient of the velocity with respect to Y disappears when Y is equal to delta.

When Y is equal to delta, Eeta is equal to 1. So at Eeta is equal to 1, VX by U is equal to 1 and del del Y del del Y of VX by you is going to be equal to 0 since the velocity gradient disappears at the boundary layer. Let me write it and it would be more clear to you. What I am proposing is that VX by U is equal to A + B Eeta + C Eeta square.

So I therefore need boundary conditions, so what I am saying is that at Y equals 0, VX is 0, at Y equals delta, VX is equal to U and at Y equals delta, del VX by del Y is equal to 0. So you can convert them to in terms of Eeta as well. So what it essentially tells is that Eeta to equals 0, VX by U is equal to 0. At Eeta equals 1, VX by U is equal to 1 and at Eeta equals 1, DVX by U by D Eeta is equal to 0.

So these are the conditions which are to be used with this equation to obtain the the values of A, B and C.

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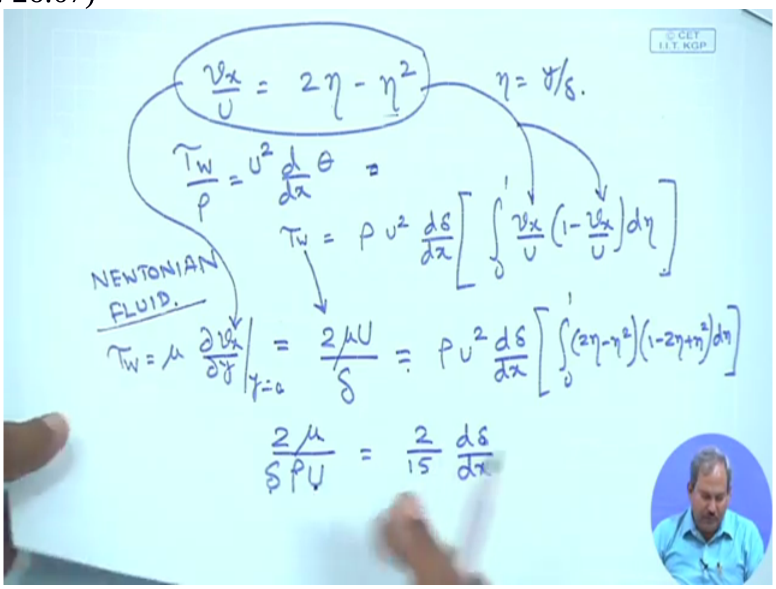

So when you when you evaluate the values of A, B and C is using these fundamental physical relations, what you get is VX by U to be 2 Eeta - Eeta square. So where Eeta is equal to Y by delta. So now it becomes pretty straightforward for the flow of 0 pressure gradient flow Tao W by rho is D DX of rather U Square is outside times theta and which would again leads to when I express it in terms of dimensionless quantities, Tao W is equal to rho U Square D delta DX 0 to 1 VX by U1 - VX by U times D Eeta.

For assumed profile, I know what is VX by U. I put it over here and I put it over here as well. When I and I assume that it is a Newtonian fluid. If it is a Newtonian fluid, then Tao W is simply going to be mew del VX del Y at Y equals 0. When you use this over here, you would simply get is to be equal to 2 mew U by delta. So your tao W has now become equal to 2 mew R U by delta, what you have is rho U Square over here D del DX and when you perform the integration over here from 0 to 1, this becomes 2 Eeta - Eeta square and this becomes ones - 2 Eeta + Eeta square times D Eeta.

Perform the integration. I will not do it over here but what you would get is simply 2 mew by delta rho is equal to 2 by 15 D del DX. Remember since this is a definite integral, you are just going to get a constant and which is which turns out to be equal to 2 by 15. So your choice of Newtonian fluid and your choice of any arbitrary profile has given rise to a compact, ordinary differential equation connecting delta with X in terms of the properties mew and rho, the unknown delta and the and the and the flow, the free flow, the the velocity at the outside of the boundary layer.

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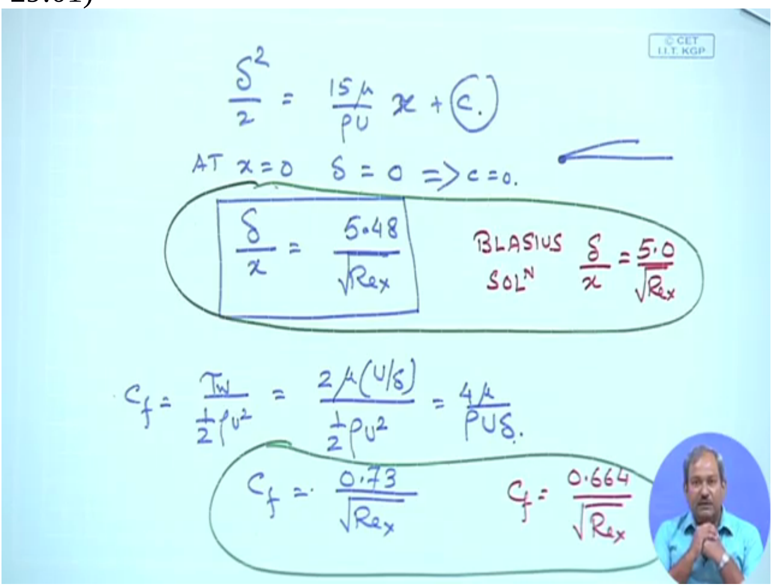

So integrate it once and what you get is del Square by 2 is equal to 15 mew by rho U times X + C where C is the constant of integration. So at X equal to 0, at the edge, at just the beginning of the flat plate, at X equal to 0, delta the boundary layer thickness is 0 which would give rise to C equal to 0. So therefore when you substitute it, is equal to 5.48 by root over REX. So this is the thickness of the boundary layer as a as a function of axial distance, thermo physical properties and so on.

And if you look at the expression of CF now which is friction factor, Tao W by half rho U Square which would be 2 mew times U by delta divided by half rho U Square in the it and would be 4 mew Y rho U delta. And when you substitute the efficient for delta in here, you get CF equal to 0.73 by root over REX. So these quick solutions give you the growth of the boundary layer and what is the frictional coefficient.

If you do not remember, I am going to use what we have done in Blasius solution, what we have obtained in Blasius solution, delta bikes was 5.0 by root over IEX. And from Blasius solution we have obtained 0.664 by root over REX. That is the beauty of it. See the form of the 2 equations are identical between a method which is integral, which is easy to use, which does not assume most of the things that are required in Blasius solution was the solution is so simple and you get a you get a relation of delta which was only 10 percent different from that of the exact isolation of Blasius, exact solution of Blasius coupled with the numerical solution by Howard.

And this is 0.73 and this is 0.664. So if we just look at these 2 things together, we will highlight the utility of momentum integral equation. That means you are getting 2 cases, you are getting 2 cases. One is so complicated to use, you are working with PDEs, you are working with numerical solutions and the other, you are working with an ODE which is versatile, which is easy to use and at the end both are giving you almost same results, same form and almost same value numerical values.

So definitely momentum integral equation is the method of choice for solving flows flow for boundary layer problems. And later on we would see how this equation momentum integral equation can be used for other type of geometries as well. But at the end of the day, still the question persists, why was momentum integral equation successful in getting a result which is within 10 percent of the 10 percent of the accurate result from by Blasius.

What is the what is the secret of the momentum integral equation? The secret lies in the fact is that we are dealing with a very thin boundary layer. And in a very thin boundary layer, if you can correctly identify what is the boundary condition on the solid plate which is now shear and what is the what is the condition at the edge of the plate where is the velocity is equal to the free stream velocity and the gradient of the velocity is 0.

So you identify 3 boundary conditions at the edge of the boundary layer and on the surface and the entire thing is very thin. So if you if in a thin flow domain, your 3 boundary conditions are correctly specified, then no matter whether you take is as A + B Eeta + C Eeta square or A + B Eeta Q or any other form, the chances are that it is almost sure that you are going to get results which are to be very close to the differential approach, the differential approach, the velocities, et cetera are going to be validated every point.

In the integral approach, it is not it is not accurate but it is so easy to use. Since momentum integral equation is is is in order of magnitude simpler to use and it is not restricted to the type of low, unsteady flow, presence of a pressure gradient. So it is a flexible equation to begin with and depending on the complexity of the problem that you are dealing with, you have to, you have to solve sequentially.

But at the end of the day, it is an ordinary differential equation unlike the Blasius solution case. So momentum integral equation is therefore the method of choice for handling most of the boundary layer problems. So in our subsequent classes we would see the use of momentum integral equations for the most complicated case, for turbulent flow as well and see how good they can represent or predict the experimental results and that would underscore the utility of momentum integral equation even more.