Transport Phenomena
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Lecture Number 21
Boundary Layers (Cont.)

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So we are going to start with an alternative treatment of the boundary layer as we have seen before that the analytical differential treatment of boundary layer is possible only for the simplest flow situation which is slow over a flat plate zero pressure gradient no body force and limited to laminar flow only. Now of course in real life you are most likely going to get turbulent flow over a curved surface in any of the applications that you can think of it's always it is you are not going to get a straight plate flow over a straight flat plate and the flow should, will be in laminar condition so in order to assess, in order to address this type of problems i have discussed before that instead of a differential approach, an integral approach would probably be better to start with.

And in integral approach unlike in differential approach we are not interested in obtaining let us say the velocity or the velocity gradient in the every point in the flow field we are more interested in finding out what would be these parameters, velocity or velocity gradient at crucial points where we need to know the value in order to predict for example what would be a drag force of a submerged, submerged object when there is flow over it. So we would like to know what is the velocity gradient at the liquid solid interface. So in order to use the integral approach which we understand is going to be approximate and we would see towards the end of this class how good these approximations are going to be in terms of predicting something which is very close to those cases for which analytic solution is available.

So whenever you propose anything new you must do, uh you must show that the approach you are proposing is going to provide values which are close to those which are already established. It would be better if you can match your proposal, proposed methodology with an, with the results from an analytic approach. So once you can, you establish the correctness of your method by comparing with established results, by benchmarking it with established results then you can, you can proceed to obtain more involved cases using the methodology proposed by you. So in the case of momentum approach or the momentum integral equation we would do the same.

But before we go into the derivation, conceptual derivation, I will not do all the steps of derivation in this, you are going to take a look at your textbook. In this case I am following Fox and McDonald, so you take the look at the derivation I would only explain the important conceptual text, conceptual steps of the derivation and the rest you can look and if there is any question I would be happy to answer them. But before we start there has to be, there is a fundamental relation which you must have studied in fluid mechanics, in undergraduate fluid mechanics which I would invoke in here and instead of deriving is since you already know about it, I would once again explain the significance of each term in that equation.

So this equation is an integral approach, this equation is macroscopic balance and when we say macroscopic balance it could be a balance of mass, it could be a balance of momentum, or it could be balance of anything so to say. So we would first see what that equation is and then start using that equation in order to obtain what would be the profile what would be the thickness of the boundary layer as a function of actual distance because that is our ultimate objective. We would like to know what, how does delta which is the boundary layer thickness vary with x, the actual distance and is it possible to use that information to obtain what would be the velocity gradient at the solid liquid interface because if we know the velocity gradient at the solid liquid interface then it would be possible for us to obtain the value of the shear, the expression of the sheared stress and sheared stress integrated over the flow area over the entire flow area would give us the drag experienced by a moving object in a fluid.

So that's our, our goal but in order to do that we must first establish what is the macroscopic balance equations or rather starting with the macroscopic balance equation we will slowly move on to the momentum integral equations and we would see applications of momentum integral equations for situations in which the answers are known to us may be from Blasius solution or we would also use the momentum integral approach for solving situations in which no analytical solution is possible. But the first and important step is benchmarking. Before we go to that point let's start with the macroscopic balance equation which I have written over here. If you would look at the macroscopic balance equation

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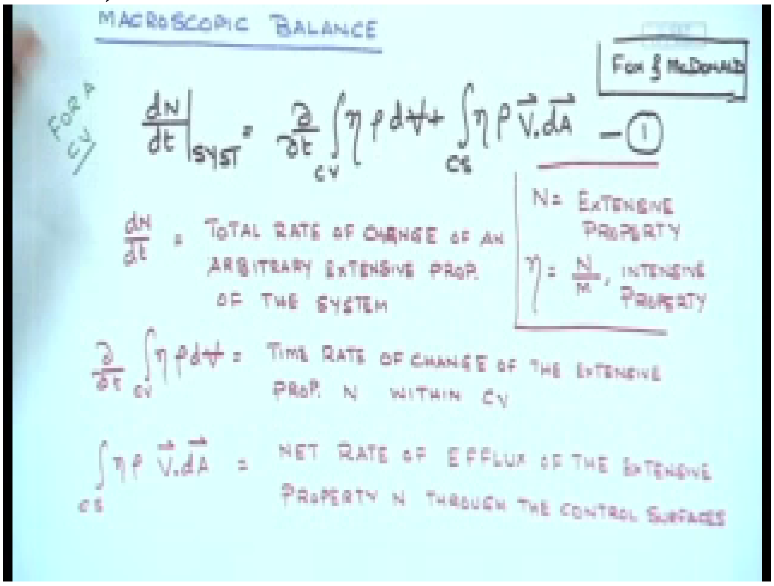

the one that is, in this case I am following the, as I said

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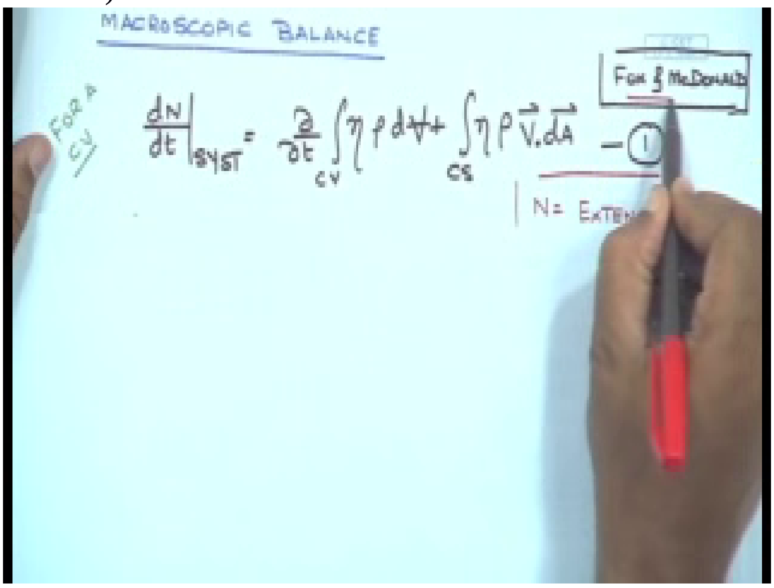

the treatment provided in Fox and McDonald, so for any control volume this d N d t of the system is equal to these two terms. So I would slowly go through each of the steps what they are and then I will explain them.

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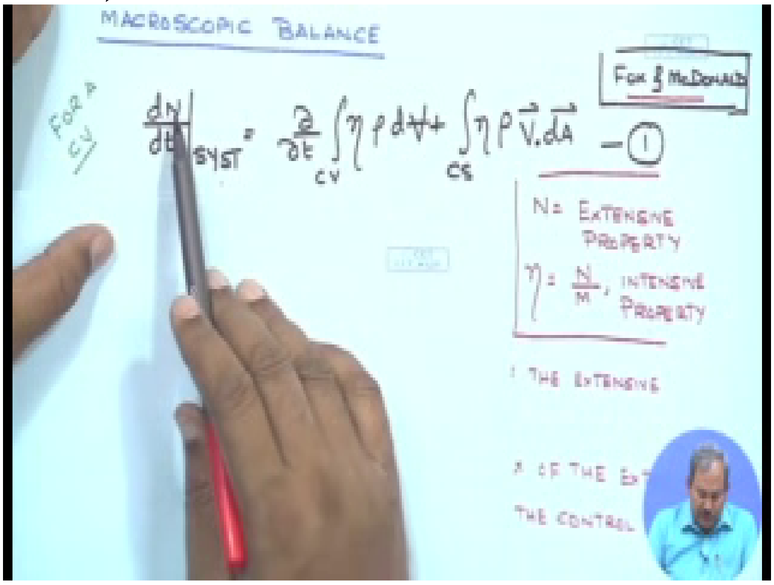

N is any extensive property of the system. N could be the mass of the system, N could be the momentum of the system. So N is any extensive property of the system where as eta which is N by mass, m stands for the mass of the system so eta is equal to N by m is the corresponding intensive property so N is the extensive property and eta is the corresponding intensive property. Now let's also go through what is the significance of each of these terms. So first of all d N d t, the

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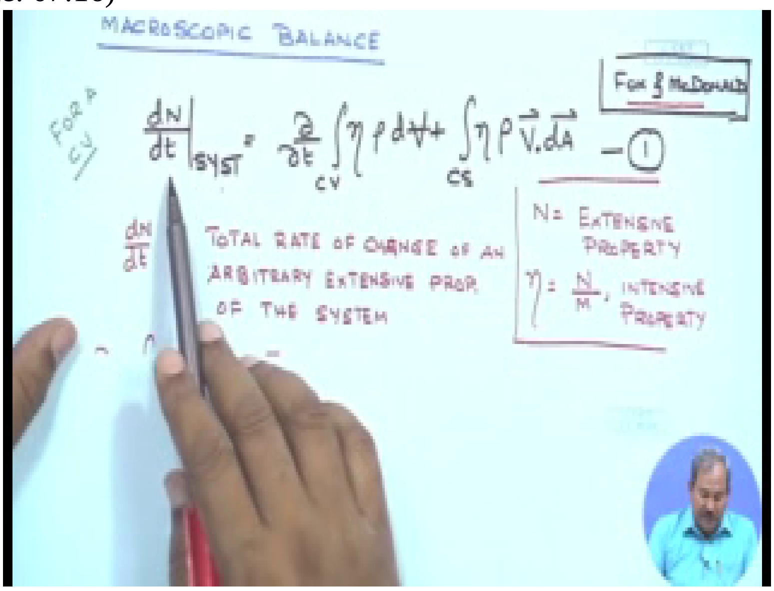

d N d t system is the total rate of change of an arbitrary extensive property of the system. So this arbitrary extensive property can be of several things and we would give examples, we would see examples, rather

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use of this N, one in the case of

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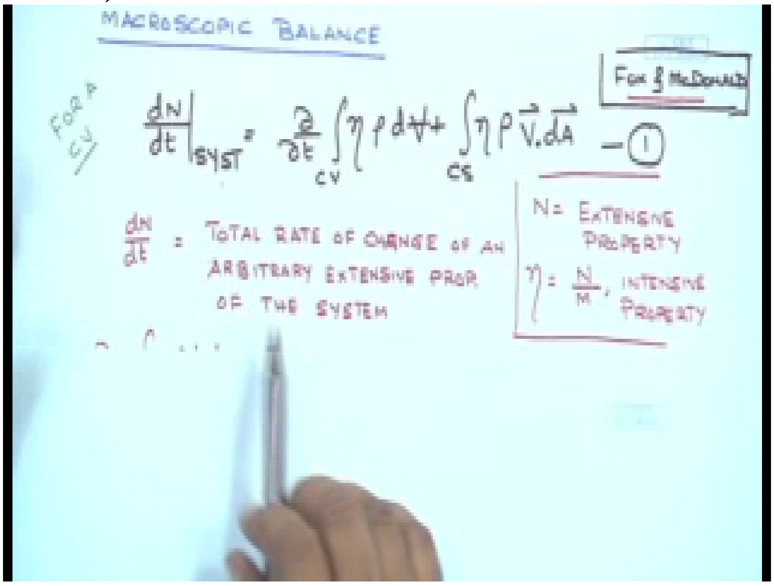

mass and the second is in the case

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of momentum. So this equation relates the change in the extensive property of the system represented by N to eta which is the corresponding intensive property obtained by dividing the extensive property with the mass of the system. So the change in the extensive property of the system is going to be the, the algebra is the result, is going to be the algebraic sum of two different quantities. If you assume that this is,

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let's say this is the control volume then you can have extensive property coming into the system from all the sides. So through the control surfaces, through the control surfaces, the extensive property can come into the control volume and there could be something, a process which would change the total, total content of the extensive property inside the control volume. So the system,

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the extensive property change of the system is the sum of the time rate of change of int, of the extensive property inside the control volume and the amount of extensive property which comes into the control volume through the control surfaces. So that is essentially the physical statement of the law, of the rule I have just written.

So now let's go back to the, to the slide once again and see what they are. So your d N d t is

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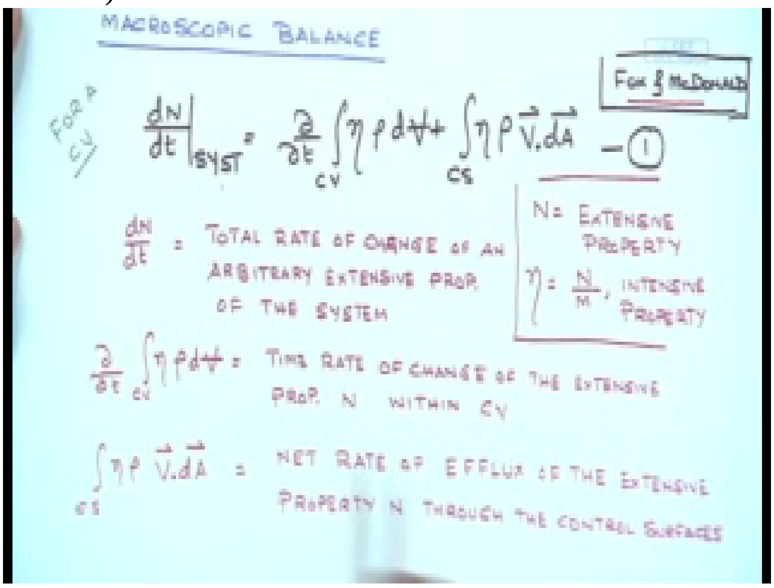

the total rate of change of an arbitrary extensive property; del del t of eta rho d v when we have the intensive property and this rho times d v where v is the volume, this is essentially the mass so if we integrate this quantity over the entire control volume what you are getting, and take the time derivative of that, what you are getting is time rate of change of the extensive property N leaving the control volume. Again look at it once again eta is essentially the intensive property, rho d v is the mass so if we integrate eta over the entire control volume what you get over here is the extensive property contained within the control volume and you are taking the time derivative of that. So the significance of this term is the time rate of change of the extensive property N within the control volume.

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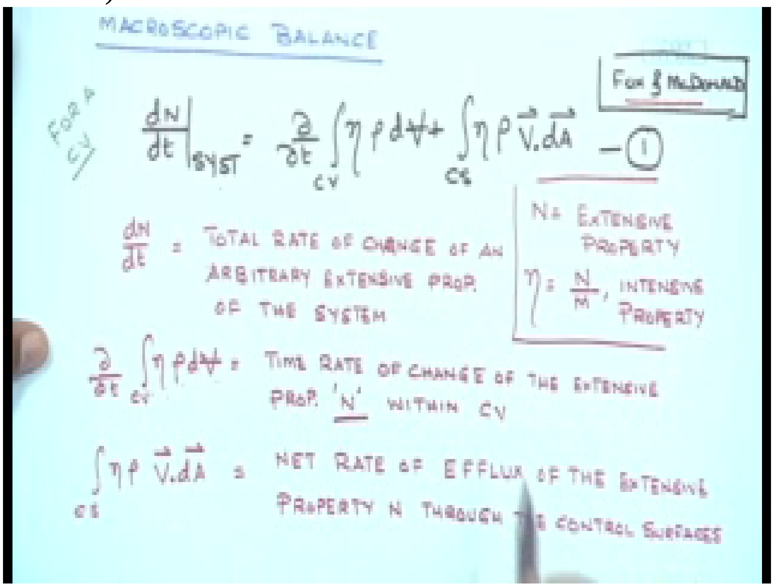

Now let's take a look at the third, the second term on the right hand side. First of all, eta is the intensive property, rho is the mass of the system and v dot d a is, is essentially telling you how much of the liquid, how much of the, how much of the extensive property is coming in through the control surfaces. So v dot d a multiplied by rho is going to give you the extensive property, the extensive property, the entire thing is going to give you the extensive property which is coming through the control surfaces, one more time. This v times d a multiplied by rho and multiplied by eta, this term rho times v dot d a is the mass of the, mass which is coming in through the control surfaces. So when I integrate rho v d a over all possible control surfaces, what I get is the mass which is coming, net mass through some surfaces the mass may be entering,

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through some surfaces mass may be going out, Ok. When you integrate over all possible control surfaces, rho times v dot d a, when you integrate through, over all control surfaces that define the control volume, what you get is the total amount of mass which is coming into the system, coming into the control volume. So when we integrate over the control surfaces eta times rho v dot d a then you are essentially finding out what is the net inflow of the extensive property N into the control volume through the control surfaces. So by means of convection or by any means when something, the extensive property crosses the control surface you essentially have the net efflux, that is net of inflow and outflow of the extensive property to the control volume. So in the control volume, two things are happening. One is the total amount of the extensive property contained in the control volume may be changing if it is an unsteady state process, Ok and secondly some amount of extensive property is coming in through the control surfaces and, and when you integrate over all such possible surfaces you find out the net addition of extensive property to the control volume.

Very quickly what, I am sure you know what are control volumes and control surfaces but I will just give you a quick update on this, just remind you what it is. So the control surface has no mass, Ok. So you can think of it as if it is this page,

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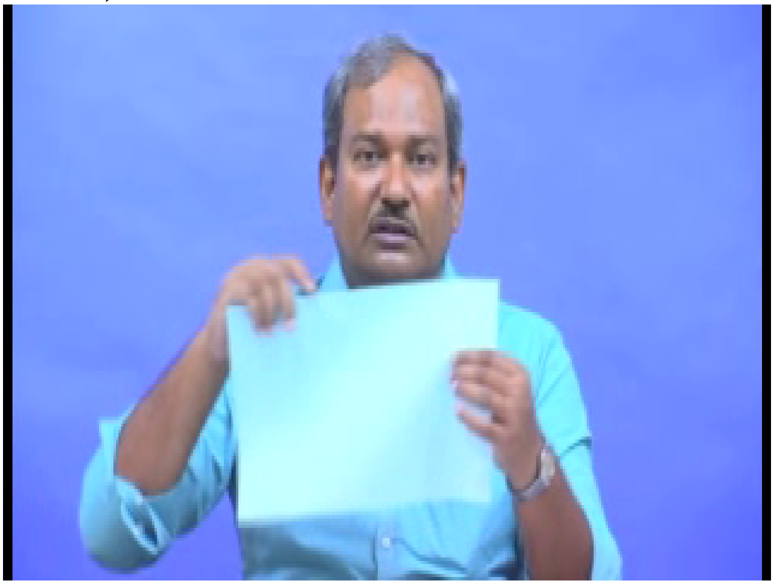

if you assume that it has no mass and therefore the conservation equation for the control surface would simply be mass, energy or anything in must be equal to out. So for a control surfaces, control surface in is always going to be out. So a control surface doesn't have a mass of its own. The control surfaces are only used to define a control volume. So control surfaces, the control surfaces define the control volume which has a mass of its own, Ok. So the conservation equation which takes the form in equals out for the case of the control surfaces will change to the more conventional one that is, in minus out plus generation plus or minus generation is equal to accumulation.

So a control volume, for a control volume, the full form of the conservation equation has to be used in minus out plus or minus generation is equal to accumulation while for a control surface, since it does not have a mass of its own, in of anything, in of mass will always going to be equal to out of mass. So when we think of control surfaces back to the our previous equation, through the control surfaces, some extensive property whatever be that extensive property, that extensive property may enter the control volume or may leave the control volume. So when we integrate these quantities over all possible control surfaces that define the control volume, what you are getting is the net inflow sometimes it is also called the efflux, the efflux of the extensive property N to the control volume. So the second term of the equation that I write over here essentially

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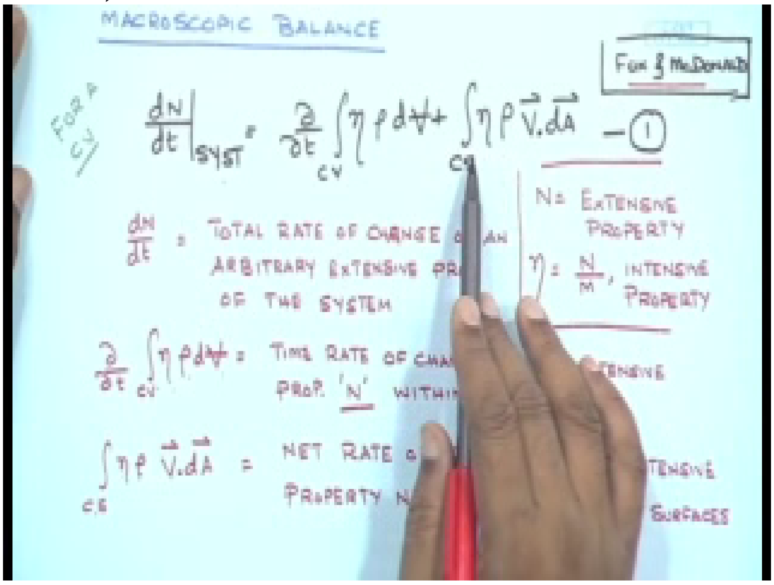

tells me this is the net rate of efflux of the extensive property N through the control surfaces. So what I have then, the first term refers to the system, the second two terms refer to the control volume. So it is time rate of change of the extensive property within the control volume and this is the net rate of the efflux of the extensive property N through the

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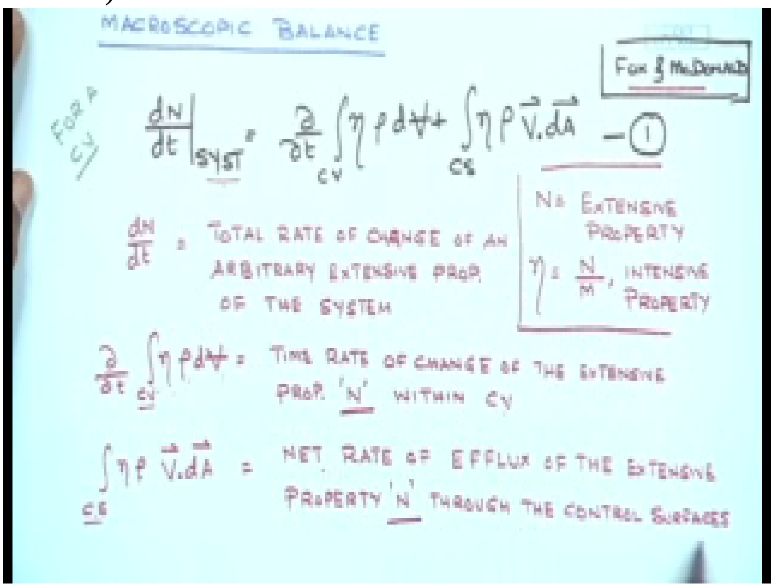

control surfaces. So one is an unsteady effect, the second one is due to the flow of the extensive property through the control surfaces. As a result of these two, the total content of the extensive property N in the system will change. So that is essentially the conservation equation for the extensive property N and this is the stepping stone for integral analysis of fluid motion. So with this now we will go back, we will

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see what are the effects, what are the different control, what are the different types of N

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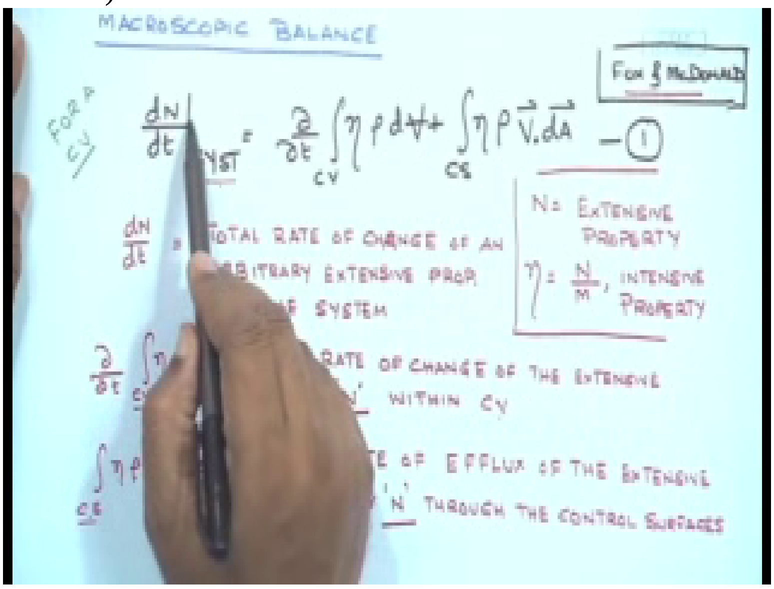

we can think of and what would give rise to, in terms of the equations that we already know. So I would start with, I would write this equation one more time d N d t system is del del t of eta rho d v plus over the control surface eta rho v dot d a. Now if we think of mass,

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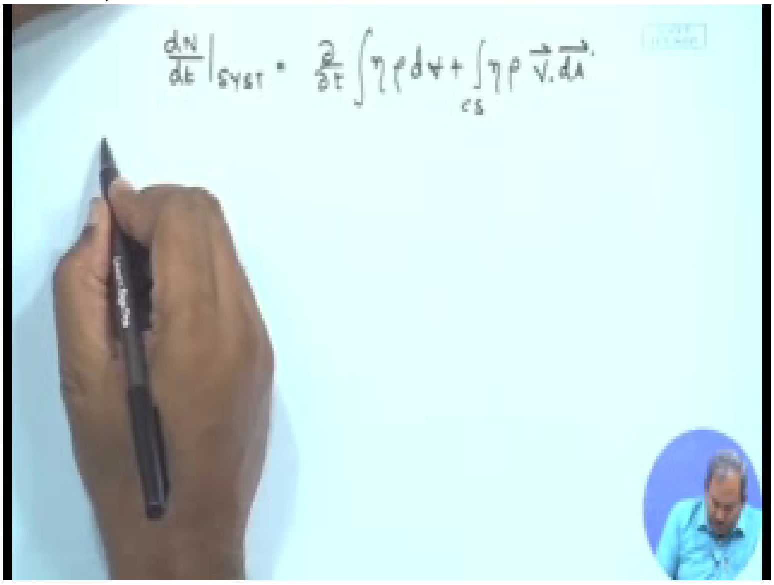

if we think of mass, then N, the extensive property is simply going to be mass and eta which is defined as the extensive property per unit mass will have a value equal to 1. And since we, from the conservation

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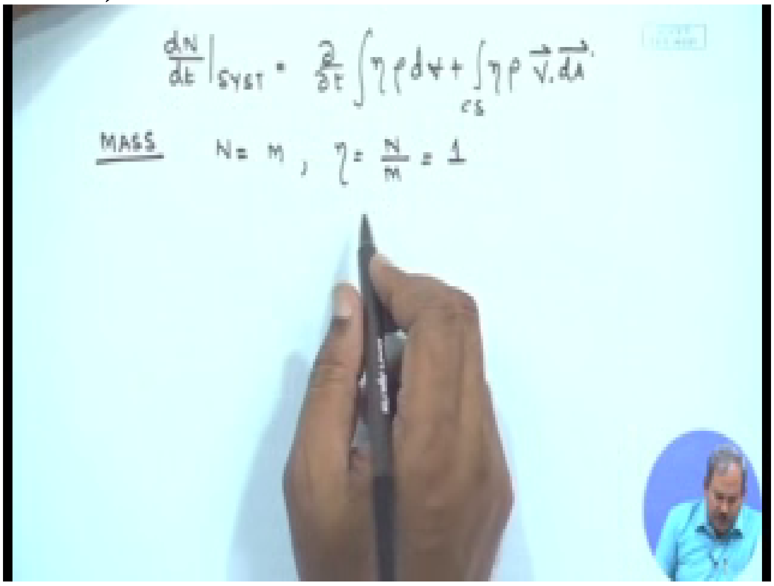

one can write that the left hand side of this is going to be equal to zero due to mass conservation and this one is going to be del del d of rho d v plus integration over control surfaces eta is going to be 1, rho v dot d a. So this is nothing but the statement of the conservation of mass and more commonly in fluid mechanics it is known as the continuity equation. So this is the continuity equation and for the case of the steady state, this equation tells me that this term would be zero and therefore integration c s rho is , and if this is incompressible fluid then rho can be taken out, v dot d a is going to be equal to zero since

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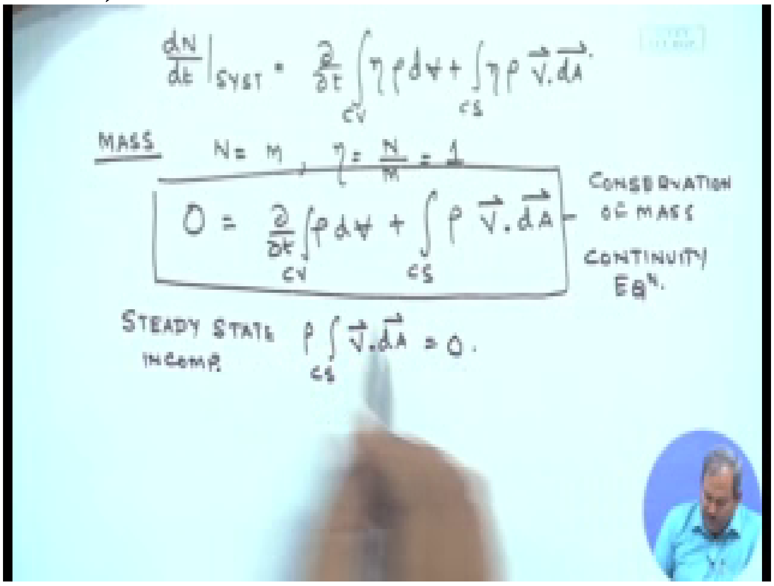

it's a steady state, this term would not be there and this would lead to the, the more common equation that you know is that v 1 a 1 plus v 2 a 2 plus is going to be equal to zero. So

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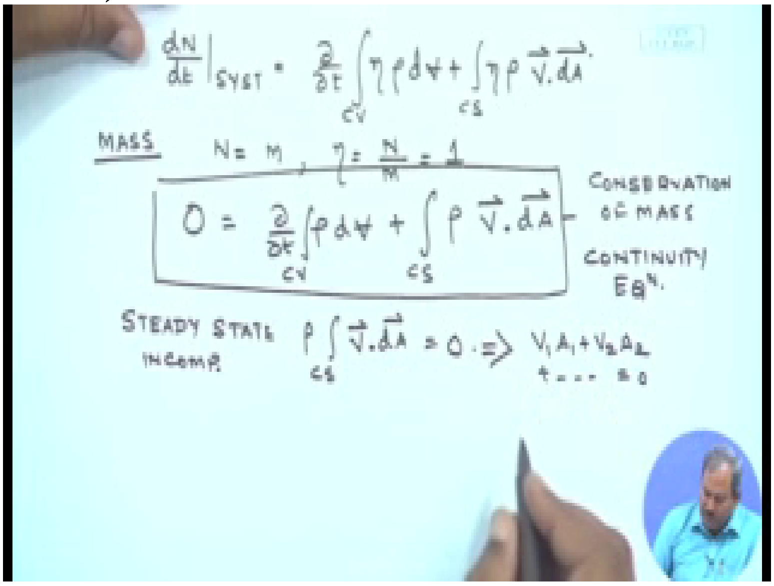

summation v i a i would be equal to zero. So this form of, if continuity equation you have used, I am sure you have used somewhere or the other. So when you take the N to be

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equal to m then the corresponding intensive property is one and what you get out of it is simply the equation of continuity. Now we would see what happens if N is, N is going to be equal to momentum. So if N is momentum, mass times velocity then the corresponding intensive property must be equal to velocity. Since intensive property is equal to extensive property per unit mass. So if my, if I write this equation for momentum what I get is my eta would simply be equals extensive property per unit mass. So this is going to be equal to the velocity ok and

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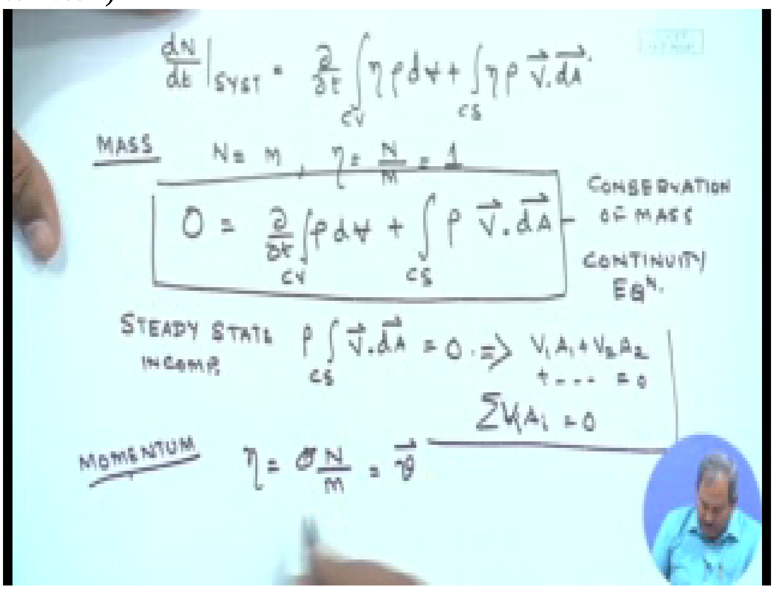

what is going to be the left hand side, it is d d t of momentum, time rate of change of momentum and what is, if we, so the corresponding extensive property I call it as p where p is the momentum and therefore the left hand side simply becomes d p d t and this d p d t is nothing the time rate of change of momentum so this

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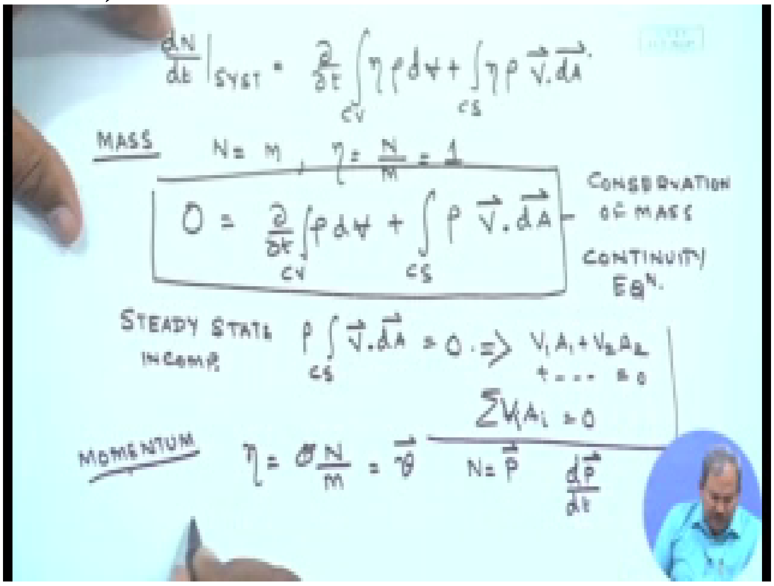

would simply give me that this is the force acting on the control volume and this would simply be equals del del t of v, the velocity this is, which is, which was eta before, rho d v plus this is over the control volume and this is over the control surface v rho d, sorry v rho d a. I will write it again.

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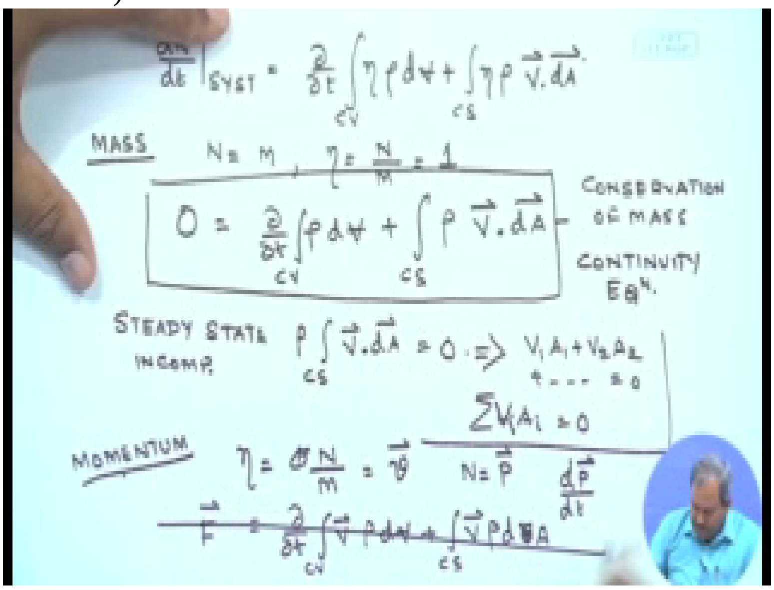

For the case of momentum, what we get is N equals p where p is the momentum. So therefore the left hand side of the equation is d p d t and

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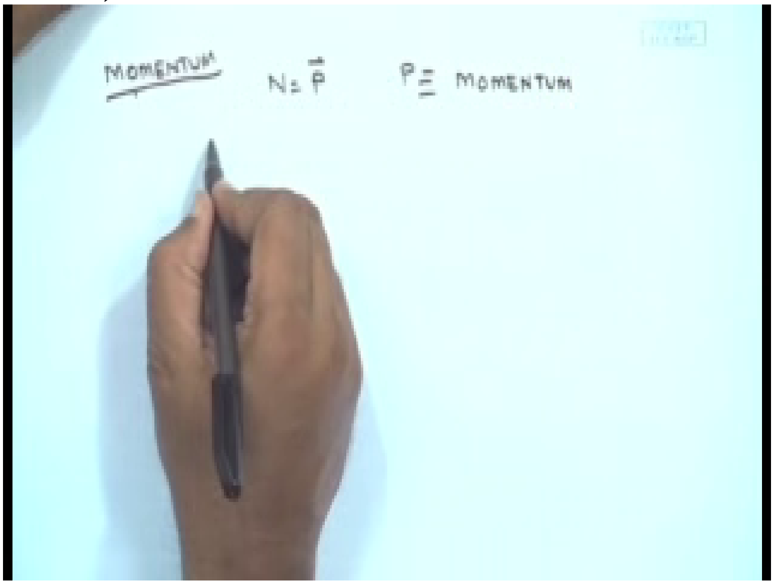

d p d t is essentially the time rate of change of momentum which is, which is the force in, if this is the force then the left hand side would be del del t integration over the control volume; instead of eta the corresponding intensive property I have v now, rho d v plus efflux through the control surfaces, intensive property then rho v dot d a. So this is what I have here is the momentum equation. Ok, let's think about

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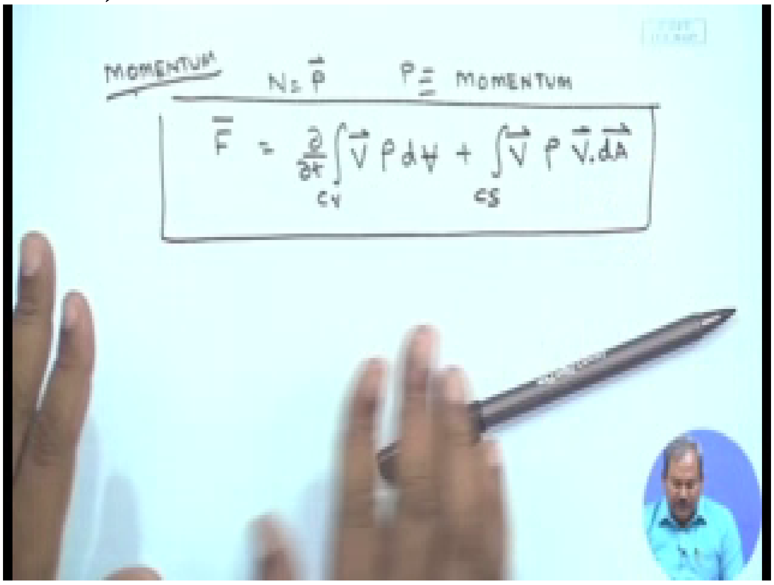

the significance of it once again. What I have on the left hand side is d d t of p where p is the momentum. So time rate of change of momentum is essentially the force acting on the control volume. On the right hand side I have del del t instead of eta, I have the velocity and over here also instead of eta I have the velocity. So this is rho times d v is the mass, mass times velocity is, is the momentum. So time rate, del del t of that, essentially the amount of momentum, the momentum accumulation of the control volume. So if it is a steady state then this del del, this term would be zero. If this is not, this is the time rate of change of momentum inside the control volume. I come back to over here. What this tells me is rho v d a is the mass which is entering to the control volume through the control surfaces and once you have, once you multiply it with v, this is the momentum which is either coming in or going out of the control volume through the control surfaces. So this term signifies the net addition of momentum through the control surfaces because of flow.

So the force on the control volume is essentially can be expressed in this form starting with the macroscopic balance equation which we have already provided before. So this gives us a neat handle on the equation, on the momentum equation which we would use for the case of boundary layers. Now the force can also be divided into two forces, one, one is the surface forces, the example of surface forces is pressure, the example of another surface force is shear stress. So the force acting on the control volume can be divided into two different types of forces, the first one is surface force; the second one is body force. Examples of surface force would be pressure and shear, the example, the common example of body force would be gravity. So the left hand side of the equation that I have just written can be written as the sum of the total force can be the sum of the surface force and the body force. So the form that I would finally use for this is, this is the final form of the momentum equation

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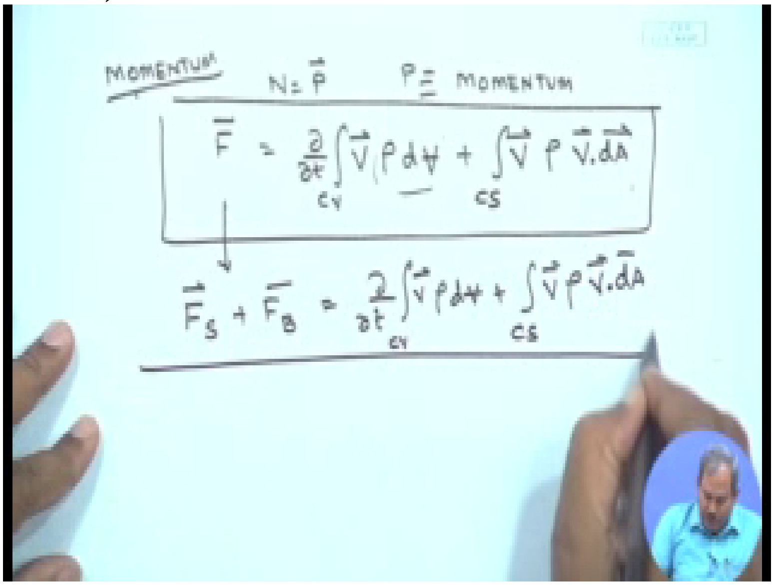

that we would use in all our subsequent analysis.

There is one more,

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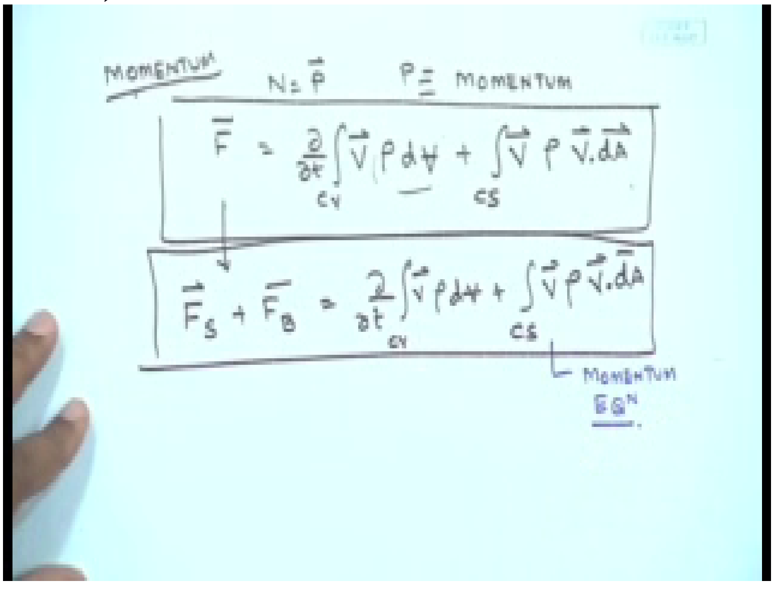

two points I would like to mention are that all velocities that we talk about are measured with respect to the control volume, relative to the

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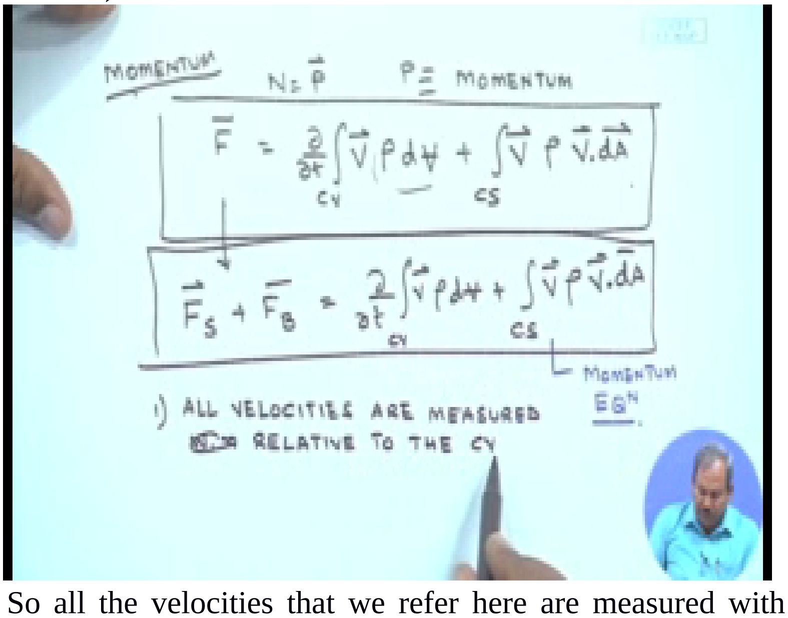

control volume. So all the velocities that we refer here are measured with respect to, are relative to the control volume and second by, by convention mass in is negative, mass out of the control, out would be positive which is, which is clear because

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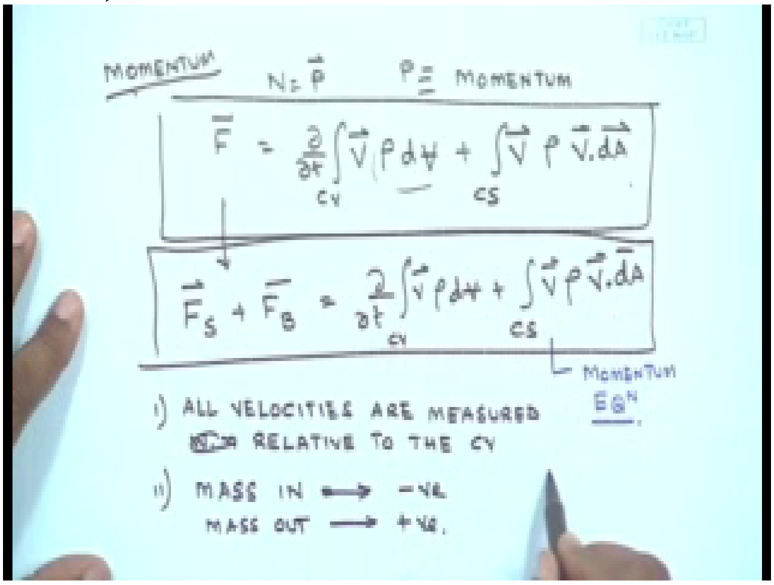

we have the product, the dot product of the velocity vector and the area vector. If this is the control surface the area vector will always point out, point to the outer side, outwards. Now if you have velocity in, which is coming in through the control surface then obviously the product is going to be negative where as if you have the velocity which is in the

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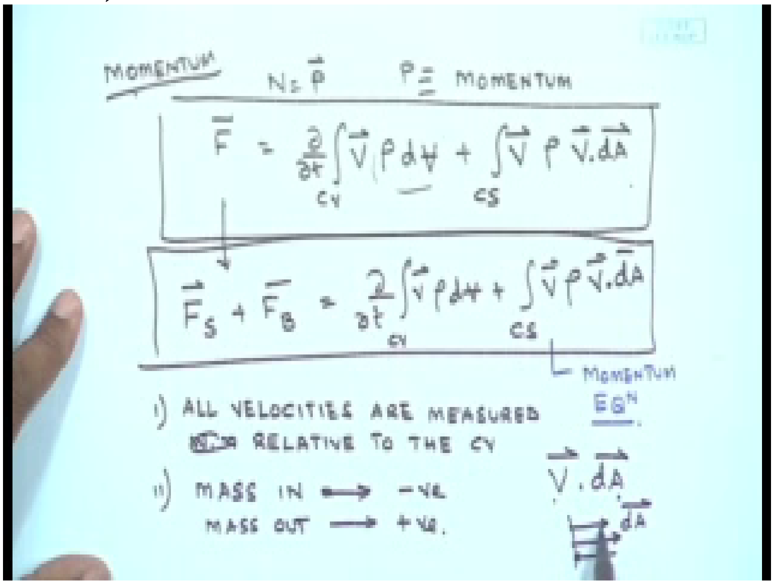

same direction as that of the area vector, in that case the dot product of the velocity vector and area vector would be positive. So whenever we evaluate this term, whenever we expand this term for all the control surfaces, let us say control volume consists of, control volume is defined by four

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or six surfaces. Through three of the surfaces the mass is coming in and through three, the other three mass is going out. So while writing the formula for total amount of extensive property that comes in to the control volume with the flow rate, anything that comes in will be treated as negative and anything that goes out will be treated as positive and which, which takes place because we express the amount of mass which is coming in to the control volume by the dot product of the velocity vector, the area, and the area vector and multiplying the whole thing with rho which is the density which is a scalar. So and since the area vector is always pointing outward, normal to the control surface, therefore in is negative and out is positive so please do remember that while solving the problems that we will use this convention, we will use this concept throughout our, throughout the rest of our treatment of boundary layers.

So what we have done here, we haven't gone into the boundary layer as yet. What we have done so far is we have started with a macroscopic balance equation. The macroscopic balance equation tallies the total amount of the extensive property, any arbitrary extensive property of the system with that of the change in the extensive property inside the control volume and the net efflux of the extensive property to the control volume. Now in the limiting case, when the system and the control volume coincides, what we have then is the, the, we have then use the extensive property, let us say the mass, where extensive property is the mass and corresponding intensive property is equal to 1,we have obtained the conservation equation.

In the second case, and if it is steady state then the first term on the right hand side, the del del t integration c v that term would disappear and what you would get is the conservation equation, the continuity equation that we are more familiar with, which is v 1 a 1 plus v 2 a 2 plus v 3 a 3 up to v N a N would be equal to zero. The positives and negative signs are to be incorporated by, by thinking whether it is a flow in or flow out. If flow in, it is going to be negative, if flow out it is going to be positive. Next, we have assumed this extensive property to be the momentum of the system, so mass times velocity.

The moment it is momentum, the left hand side of the original equation, the d d t of N, N being the momentum this becomes the force, time rate of change of momentum and the force can be classified either into body force or into surface force and in the right hand side the we have, the, the time rate, the change of the extensive property inside the control volume which at steady state would be equal to zero and the right hand side would be the net efflux of the, of the, of momentum into the control volume. So we have the final equation is then

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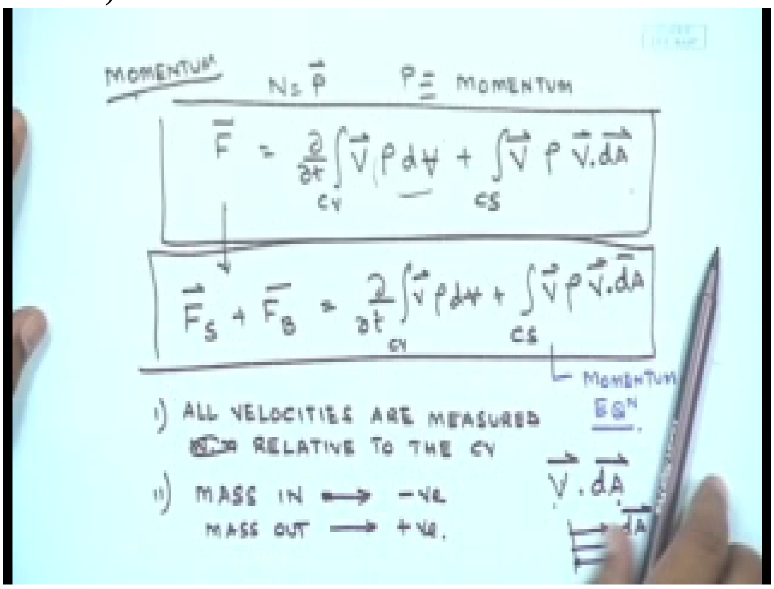

is the surface force, the body force, the time rate of change which in the case of steady state would be zero and this is v, rho, v dot d a so this essentially my momentum equation that I am going to use in the next, in the next

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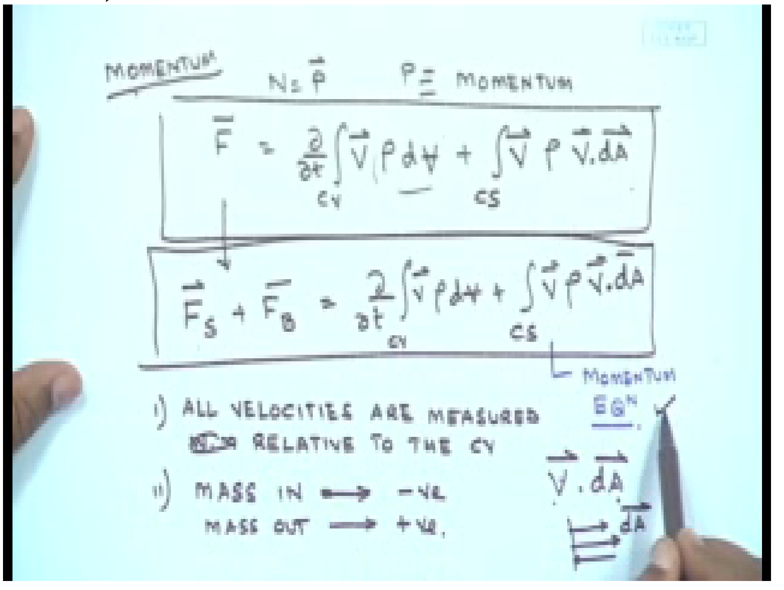

I am going to use this for analyzing the boundary layer. Two special points,

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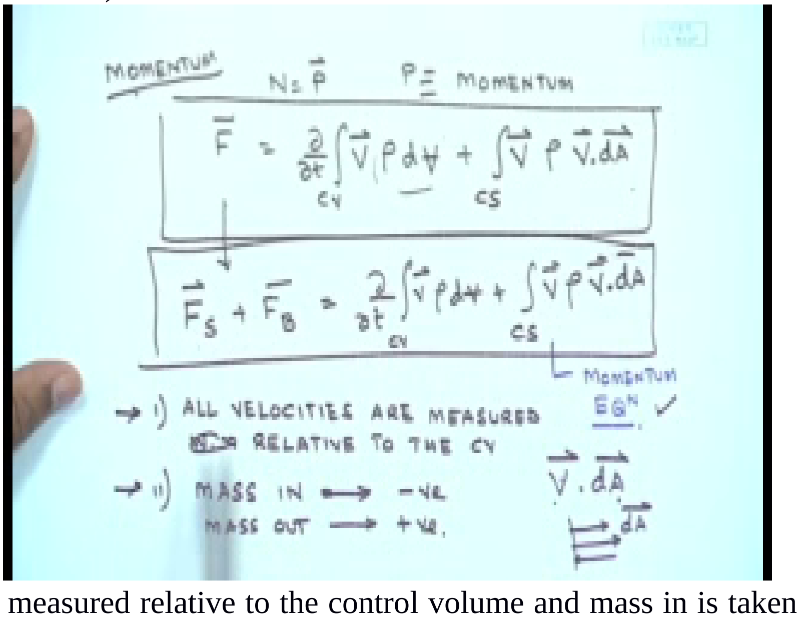

all velocities are measured relative to the control volume and mass in is taken to be negative and mass out is going to be positive and in the next part, segment we will use this equation, these two equations, this equation as well as the conservation equation for solution of boundary layers uh where the restrictions of zero pressure gradient,

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the restrictions of laminar flow et cetera would not have to be present.