Transport Phenomena
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Lecture Number 19
Boundary Layers (Cont.)

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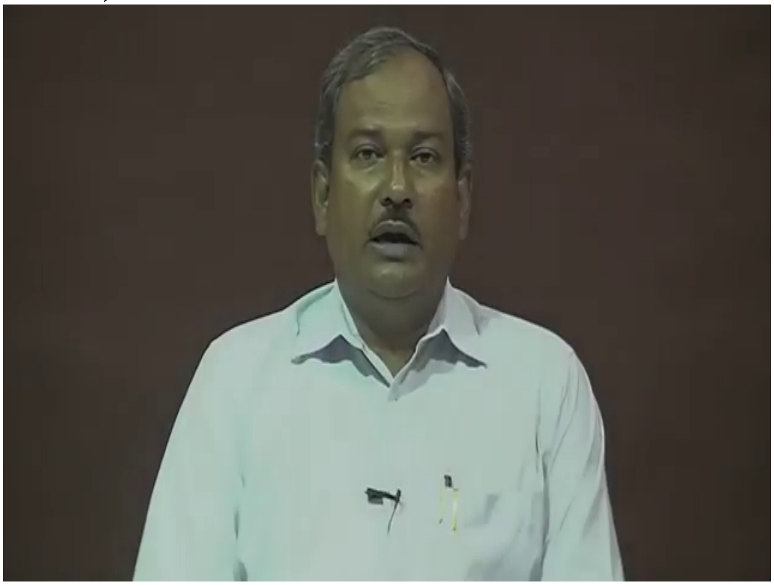

In continuation with what we have done in the previous class, we were looking at how to analyze the growth of a boundary layer on a flat plate in laminar flow. The objective of that exercise was to obtain delta, the thickness of the boundary layer as a function of actual distance that is the distance along the length of the plate. And we have also seen that inside the boundary layer the flow is two-dimensional. Outside of the boundary layer, the flow is inviscid in nature such that Euler's equation, Bernoulli's equation is applicable. However inside the boundary layer, the complete Navier–Stokes equation has to be solved in order to obtain the profile of the thickness of the boundary layer as a function of x where x denotes the actual distance.

We, from our basic understanding of the physics of the process, we have done some analysis based on which is going to be a significant term in Navier–Stokes equation and which is not. First of all we have assumed that it is a zero pressure gradient flow and since the plate is horizontal there would be no effect of gravity. So the, and it’s a two dimensional flow where the velocity v x, velocity components v x and v y would be functions of both x and y. The plate is wide in the z direction therefore the z dependence of velocity does not appear in the, in our analysis of Navier–Stokes equation. So we have four terms in the Navier–Stokes equation. The first two on the left hand side as we have seen before, these two terms,

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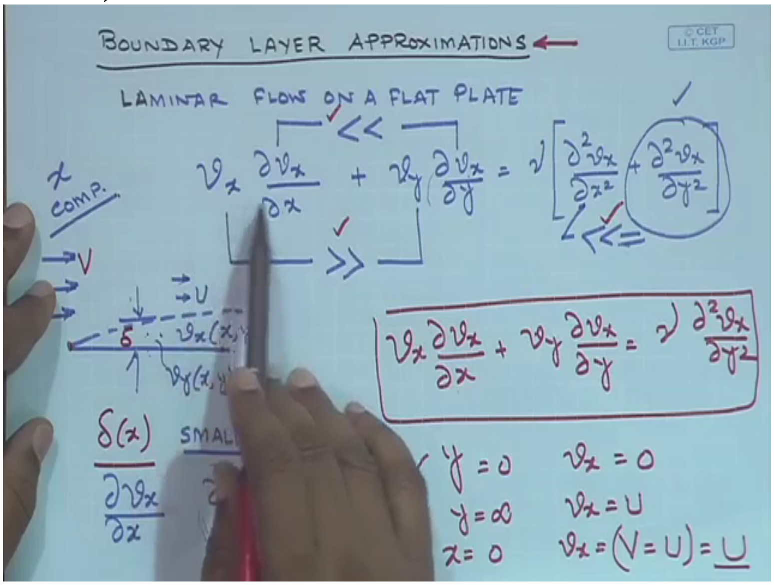

essentially they refer to the convective momentum and this is the conductive or molecular transport of momentum And using the boundary layer approximations what we have seen is that v x is relatively large compared to v y because the principal motion is in the x direction. However since the thickness of boundary layer which is denoted by, the dimension is denoted by y, since the gradient of the variation in velocity with respect to y is large in comparison to the gradient of velocity with respect to x, none of these two terms can be neglected, can be equated to zero based on sample heuristical analysis. However if we come to the right hand side, this denotes the molecular transport of momentum in the y direction and this is molecular transport of momentum in the x direction. Since the gradient of v x with respect to y is large, it is expected that the transport of momentum, molecular momentum in the y direction would be much more than the transport of molecular momentum in the x direction since the gradient of velocity in the x direction is small.

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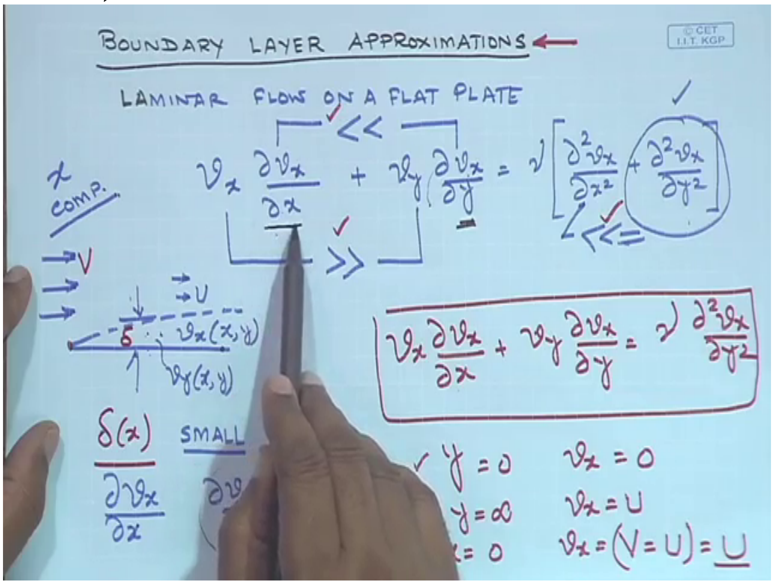

So based on our idea about the gradient of the x component of velocity in the y direction and in the x direction, it is safe to say that the gradient

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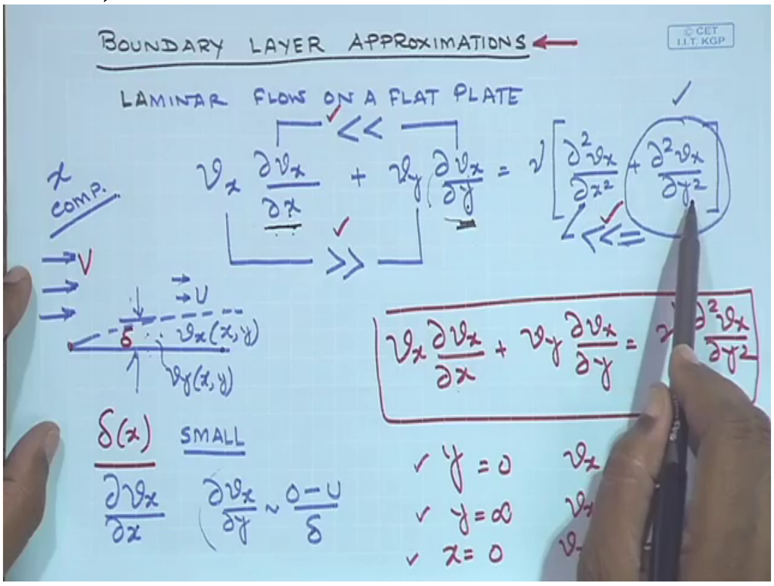

of the molecular transport of momentum in the y direction would be much more than the molecular transport of momentum in the x direction which leads us to the governing equation and we understood that there would be no slip at y equals zero and at the edge of boundary layer or beyond the boundary layer when y mathematically speaking when y tends to infinity, the velocity would simply be equal to the freestream velocity, that is the velocity outside of the boundary layer and at the x equals to zero, v x, that is actual component of velocity in the x direction would simply be equal to u where u is the approach velocity. And we also understand that for a flat plate, v, the approach velocity would be equal to u, the freestream velocity so therefore we can simply write v x equals u. In the next step we have

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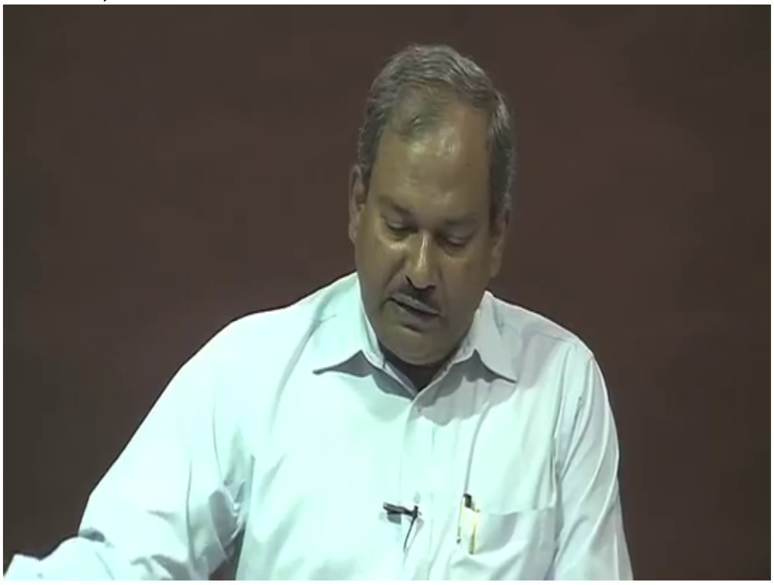

proposed

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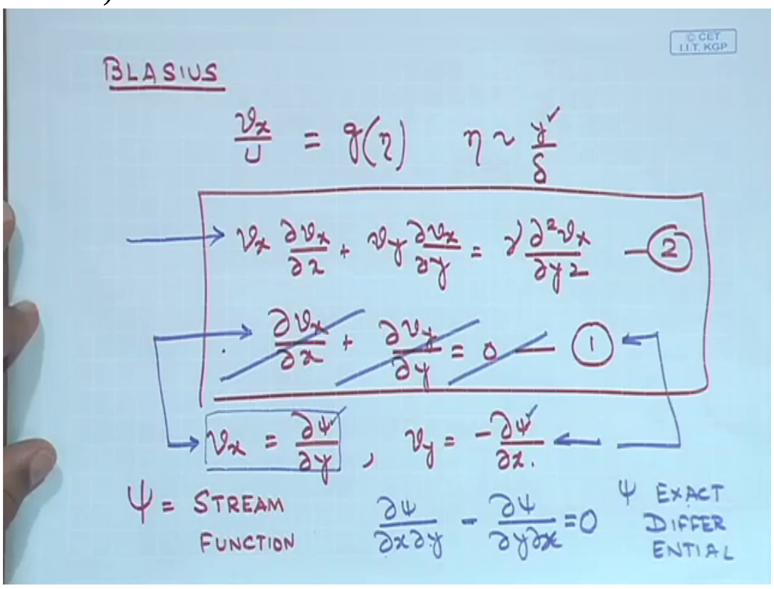

the solution is by Blasius where he reasoned that the dimensionless velocity profile when plotted against the dimensionless distance from the solid wall would be similar for all cases. And this eta which is the dimensionless distance from the solid wall is expressed as y by delta where delta is a local thickness of the, of the boundary layer. However we understand here that this delta is a

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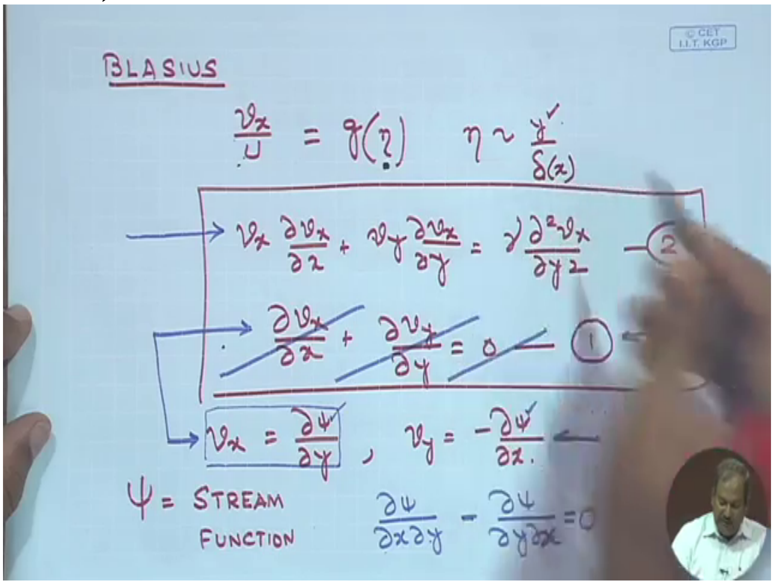

function of x as we progress in the x direction, the delta keeps on increasing. So therefore his reasoning, Blasius reasoning is that, which is supported by experimental data, that dimensionless velocity profile would be similar when plotted against the dimensionless distance from the solid wall. So with this, then we have two equations to deal with. The first one is equation of continuity and the second is equation of motion. If we introduce a stream function, then because of the properties of the stream function the first equation, the equation of continuity gets automatically satisfied and we are left only with the equation of motion. This

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equation of motion

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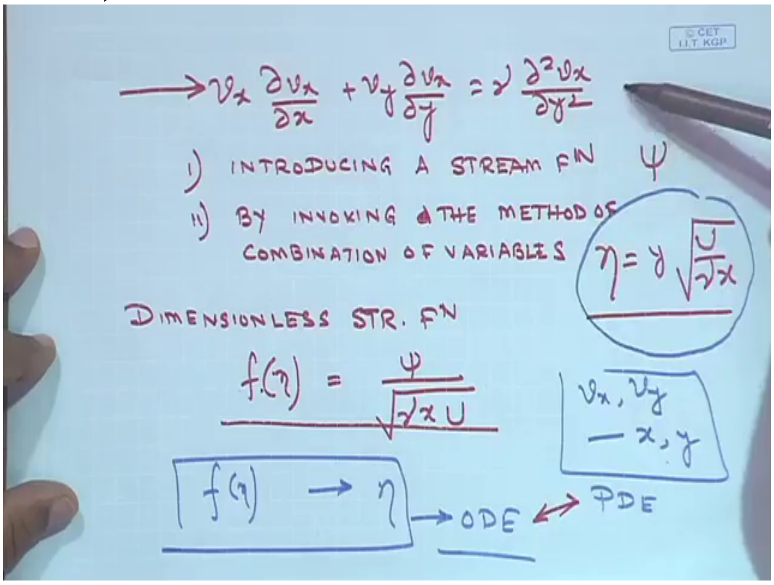

and using the method of combination of variables which we have discussed in the previous class, we are hoping that this p d e can be transformed to an o d e and that's essentially the method of combination of variables and we would see whether it would work here in this specific case or not. And since we are expressing everything in dimensionless form, so instead of a stream function, we are introducing a dimensionless stream function denoted by f and we understand that this f is a function of eta, the dimensionless distance from the solid wall and this is how the dimensionless, the dimensionless stream function is expressed. So we do not deal with v x, v y any more, we rather deal with chi which is a stream function since chi we know what would be the expression for v x if we know chi and since it is dimensionless equation we also do substitute chi with a dimensionless stream function.

So therefore f eta is a function of eta and we are, we are hoping that this p d e can be transformed to an o d e when instead of v x del v x del x v y del v x del y and so on. We introduce the dimensionless stream function and instead of x and y as the independent variable we will bring eta as an independent variable. So if we are correct, then our final equation would contain f and eta. So if it contains f and eta and if you understand that f is a function only of eta then the governing equation between f and eta will simply be an o d e. So this is

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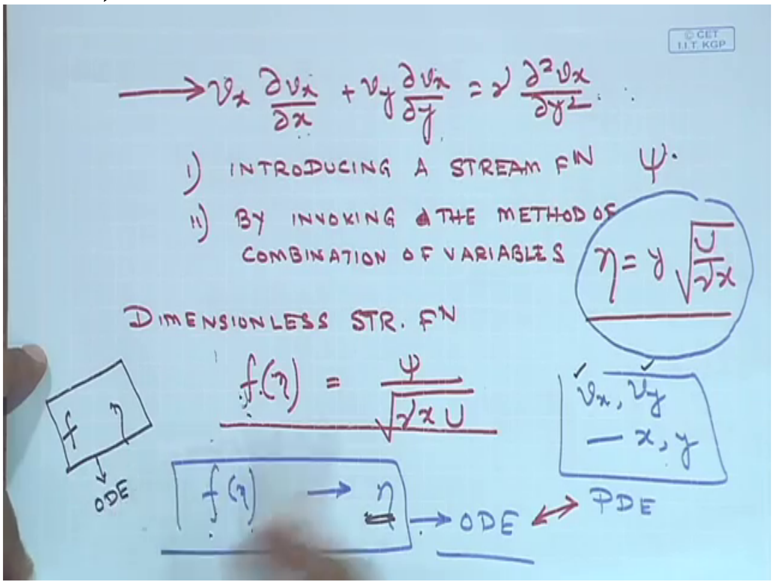

what we have, we have

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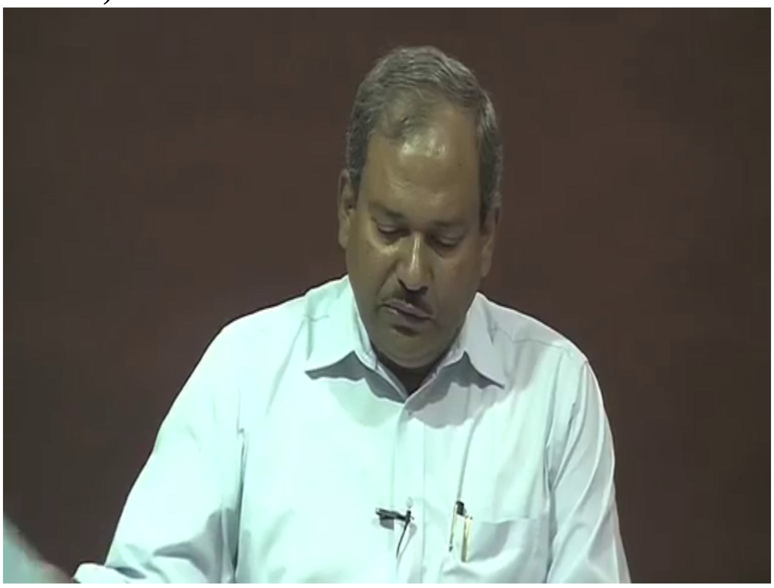

done in the previous class and based

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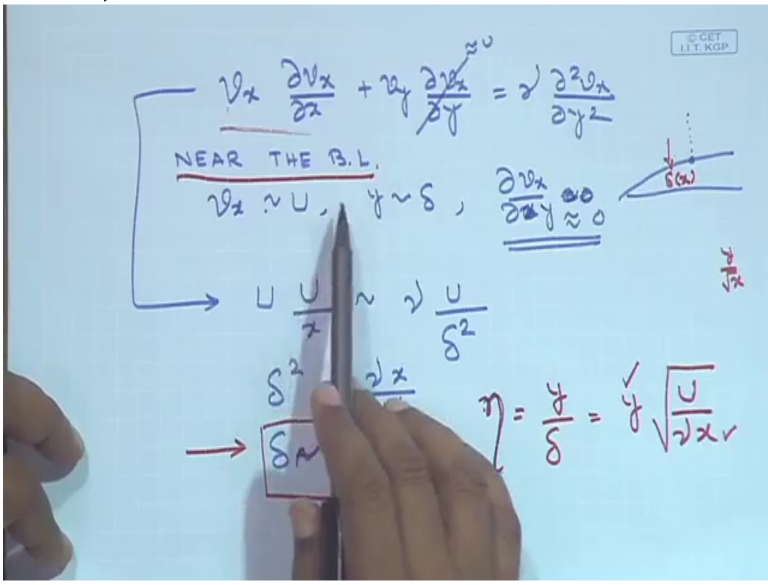

on an approximate solution of the equation near the edge of the boundary layer where we have seen what would be the orders of magnitude of these various terms, we also have defined that the dimensionless distance

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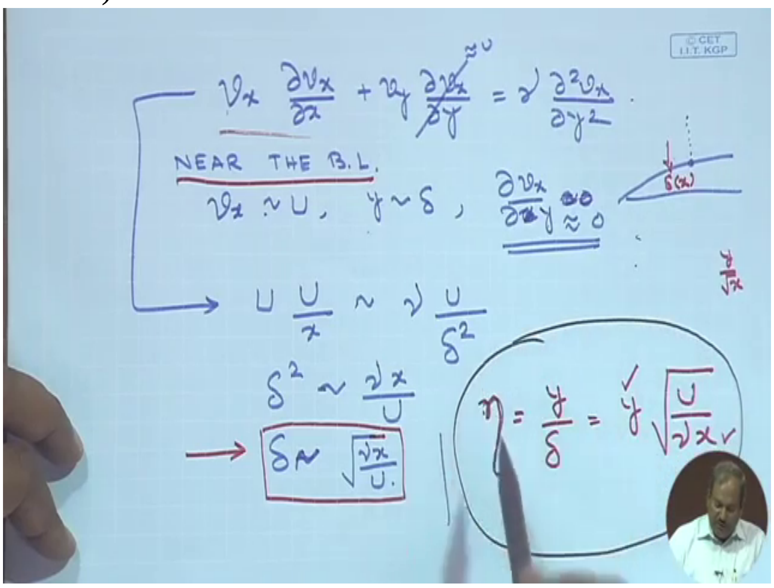

or rather the combination variable which contains both y and x should be of the form where eta equals y root u by nu x, this is the kinematic viscosity. So with this and with our definition f and eta we would see whether this equation can now be transformed to an o d e. So we

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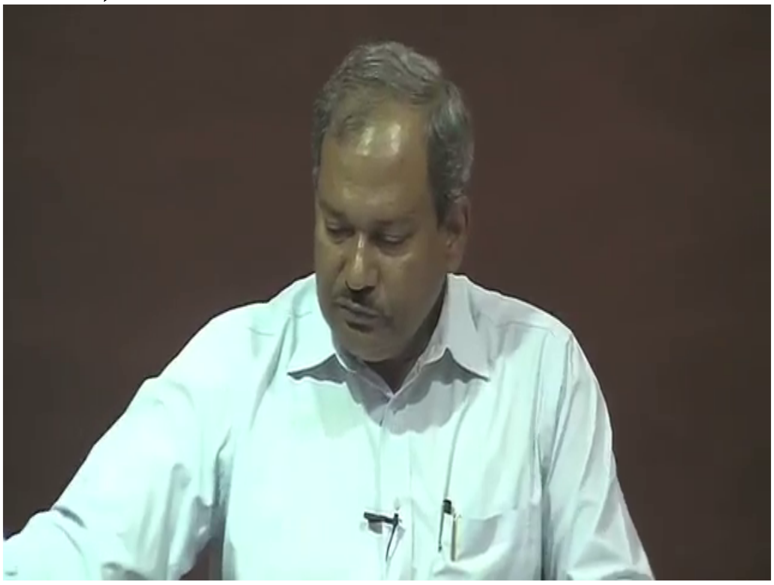

will start from that point and what we will first do is using the definition of v x which is del chi by del y and I would write here how we have defined eta, this is how we have defined eta. And I would show you just one example and the governing equation is v x del v x del x plus v y del v x del y equals kinematic viscosity del square v x by del y square. So this is the governing equation. We need to to substitute

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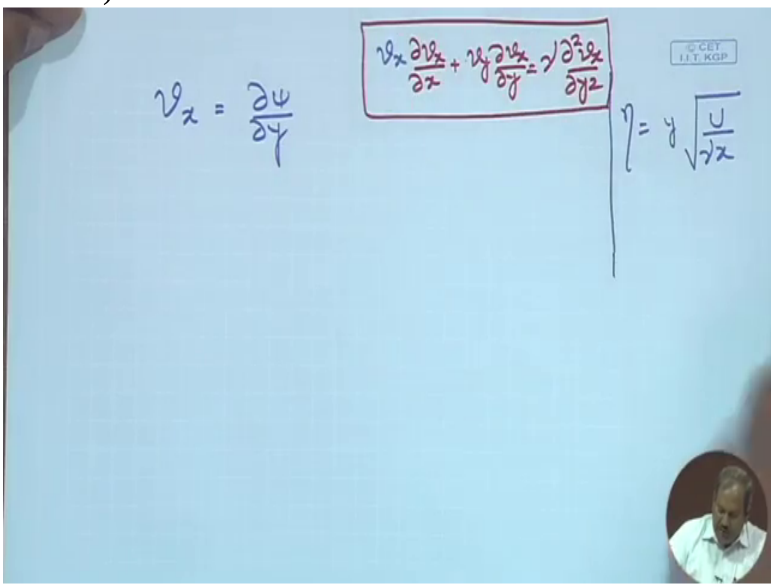

each of these terms in terms of chi or in other, in dimensionless form in terms of eta So I will just show you one, two examples of how to get the expression for v x and v y and the expressions for del v x del x and del v x del y and del square v x by del y square are given in the textbook and for this part, I am following the book of Fox and McDonald. So if you see the book of Fox and McDonald, Fluid Mechanics which is one of the textbooks for this course, the complete derivation

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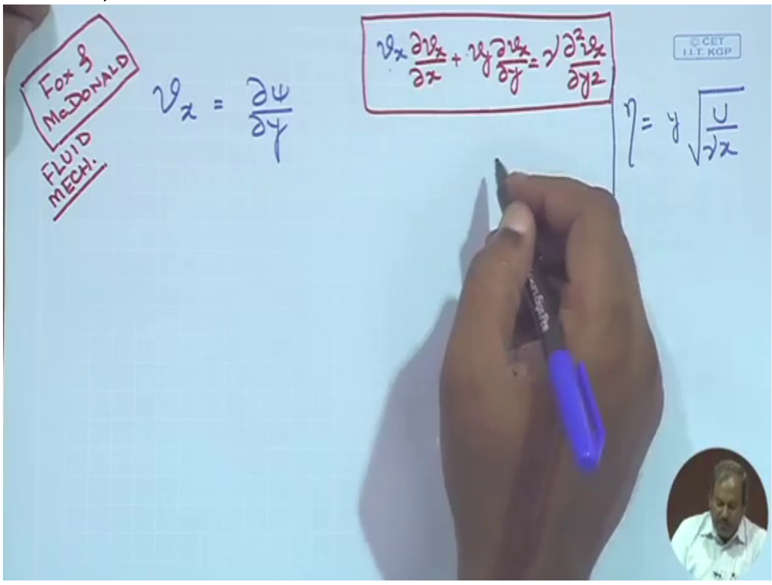

with all its particulars would be,

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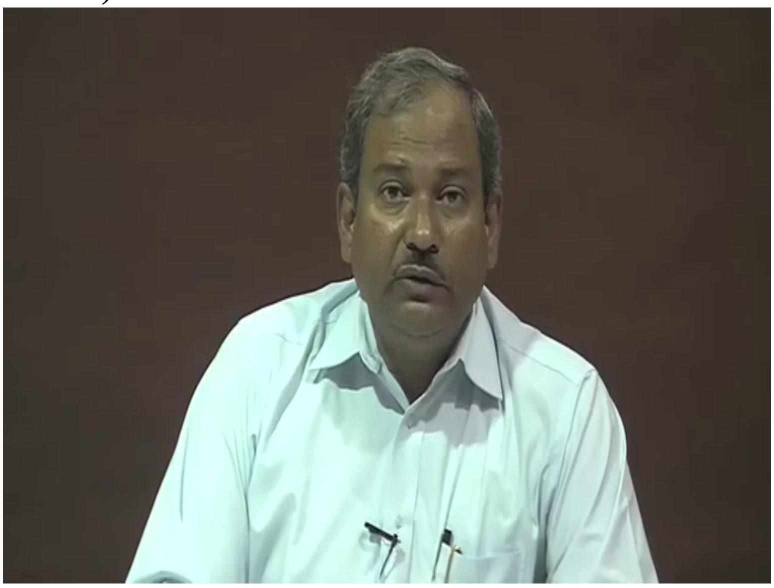

would be clear to all of you So we are trying to right now find out what is going to be the expressions for v x, v y and all other terms in the equation of motion in terms of chi, the dimensionless dependent variable and eta, the dimensionless independent variable. The expression of eta we have already obtained based on the order of magnitude analysis. So I am going to show you how to get the expression for v x and v y, the other terms of the equation of motion can be seen from the textbook Fox and McDonald. So let's see how we do v x in this case. So from definition, we can simply write this and if we introduce the f is defined as chi by root over nu x u. So if we do this,

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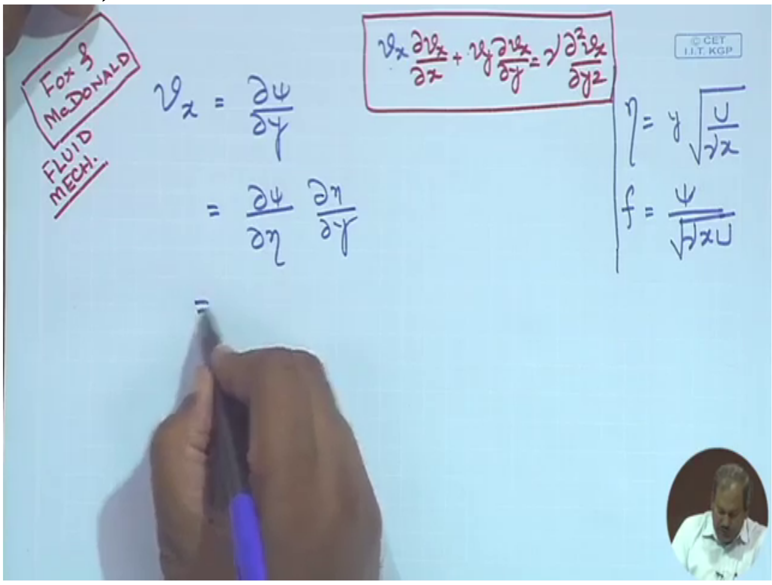

del chi by del y, if we transform that chi to f it would be simply be equals nu x u times d f by d eta. See that since f is a function only of eta, the combination variable, I am getting rid of the partial sign and I am simply getting the ordinary differential equation, ordinary differential form in this. So it is d f by d eta and not del f by del eta and if I find out what is del eta by del y from over here, I would simply get as u by nu x. So this directly follows from our definition of eta. So therefore

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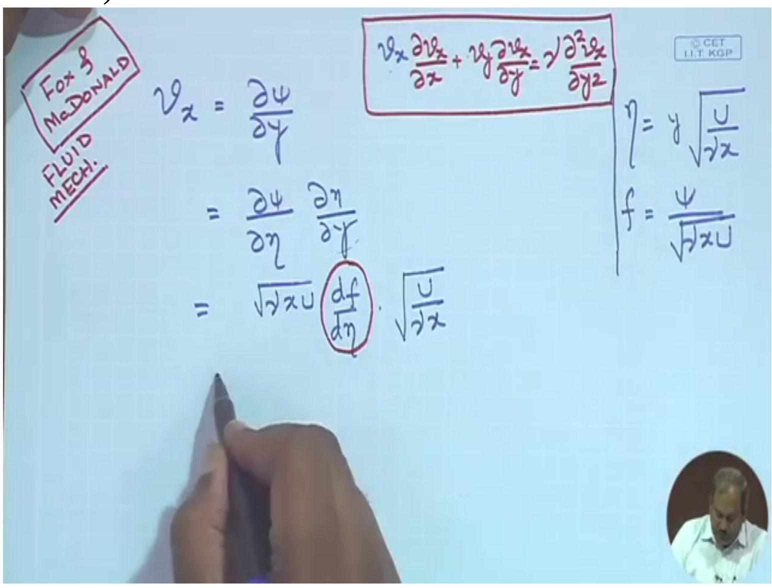

this expression is u times d f d eta so my v x therefore equals u times d f d eta this can subsequently be substituted in here. Similarly when we

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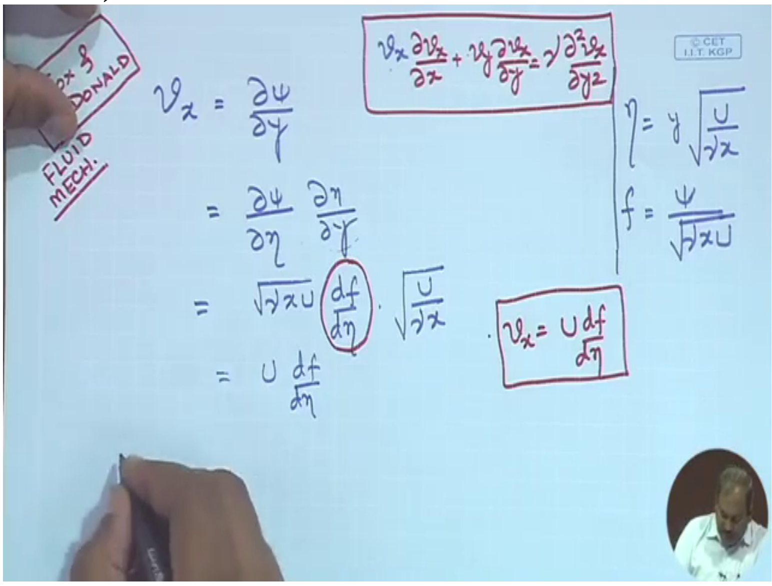

start with v y which is by definition of stream function del chi by del x, I can write it as minus del del x instead of chi I am going to write here as nu x u which can then

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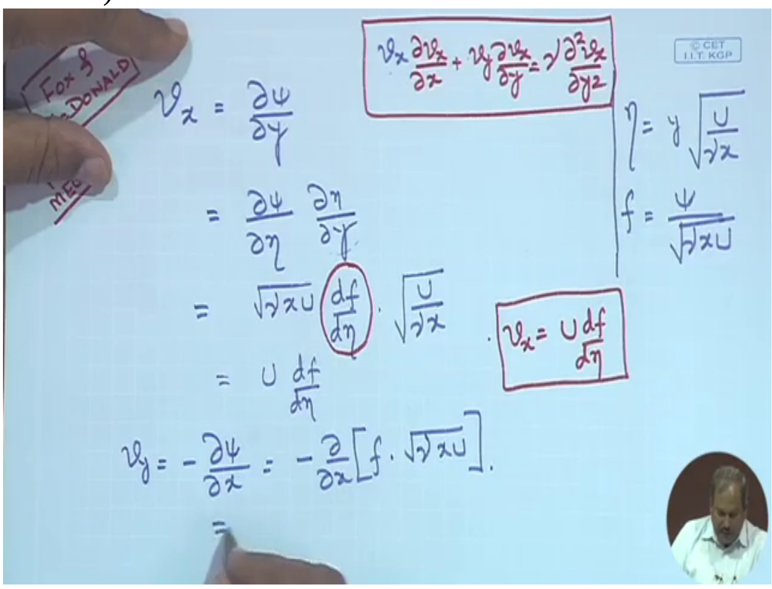

be expanded as, so this is just an expansion,

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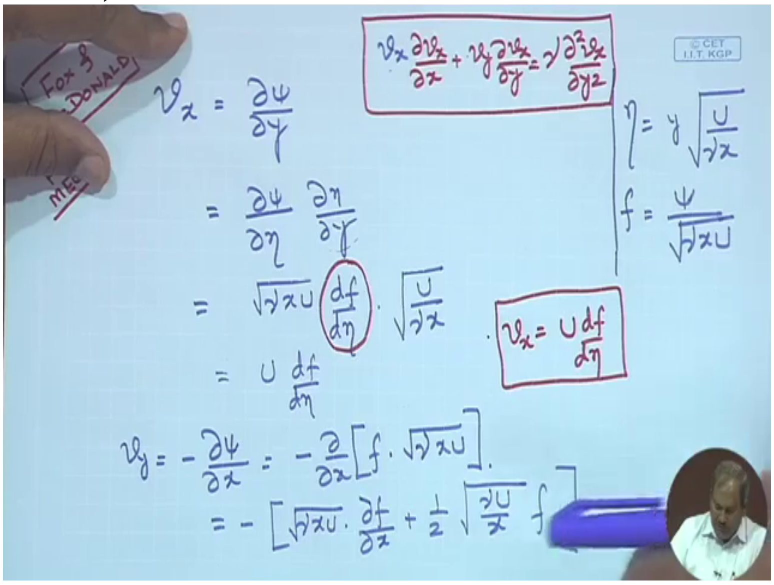

and when we continue with the derivation

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of this, what we would get u i from the previous page, it is simply just a substitution

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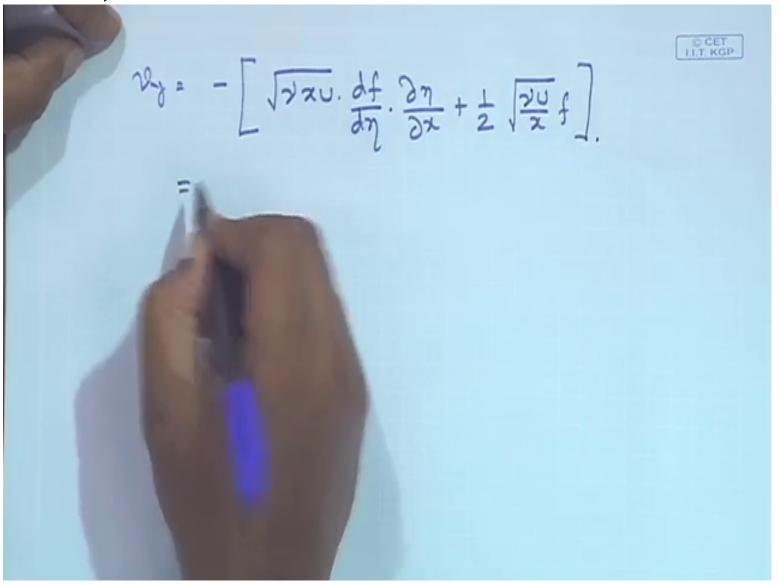

of the independent variables, so instead of del f del x I simply write d f d eta since f is a function only of eta. I will simply write the final steps since it's given in a nice form in the equation, in the textbook, so this is the expression for v y and from

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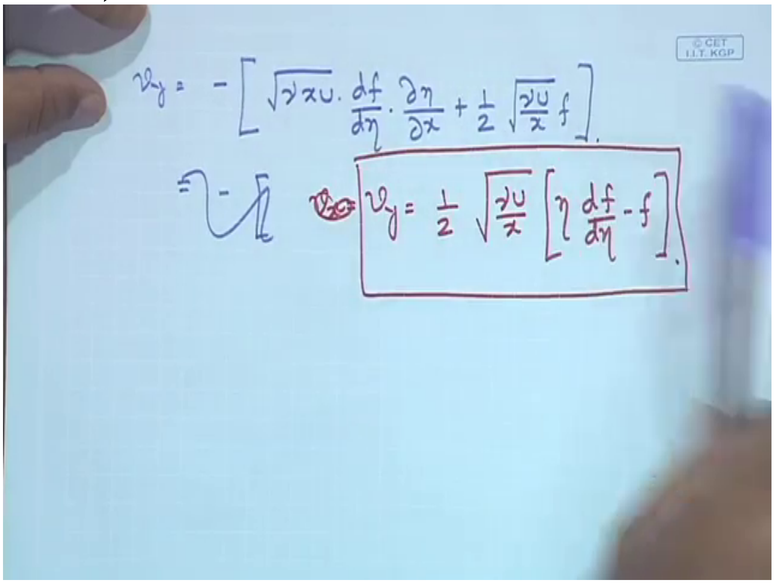

previous slide we have seen v x is equal to u times d f d eta. The expressions for and the governing

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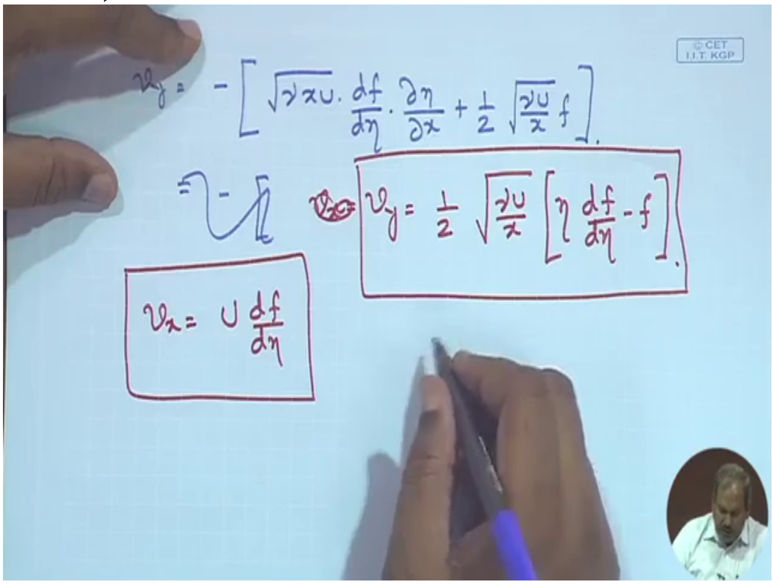

equation as we have seen before is del v x del x, so I can substitute the expression for v y in here, the expressions for v x in here, all these, the remaining terms can also be evaluated which I am not showing in in here but it is given in the text.

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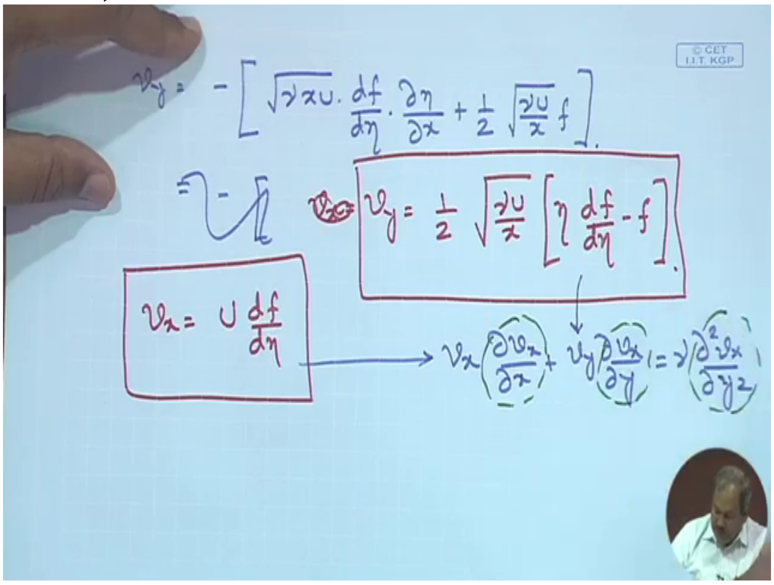

So I will leave that out and what it turns out, what it would result in is this as my governing equation. So this is going to be the governing equation.

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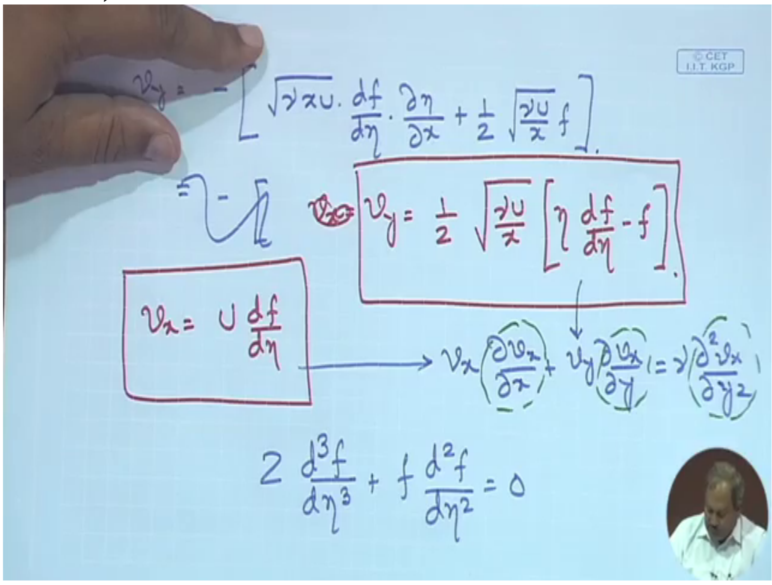

The major thing that one should first see is this is an O D E.

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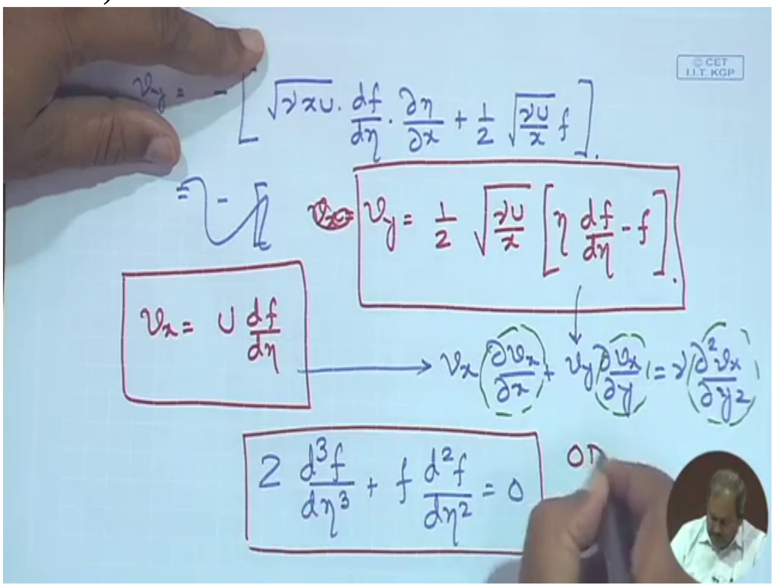

So we are successful in transforming this P D E to O D E by invoking the

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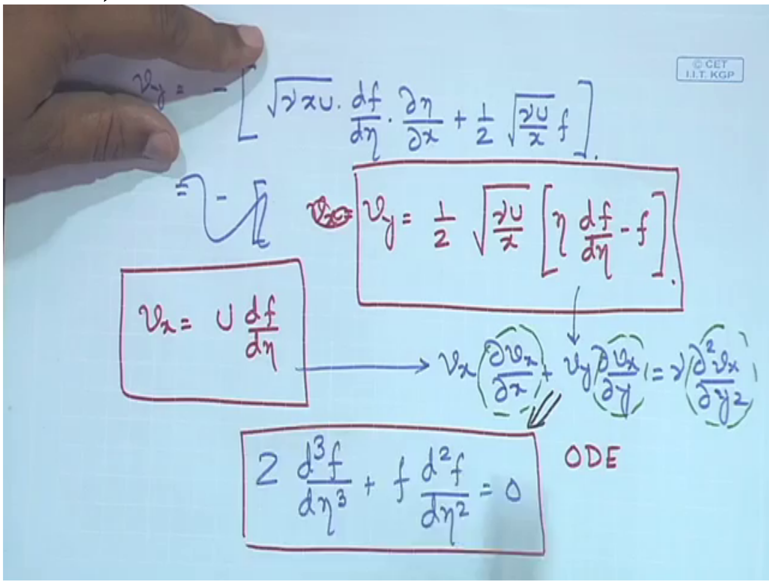

definition, by invoking a stream function by using, through a use of a combination variable eta that contains both x and y and a judicious selection of the, judicious selection of the, of the expression of eta

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from an order of magnitude analysis enables us to convert the p d e to an o d e. However if you look at the P D E, look at the O D E, it's non-linear O D E of higher order terms. So even though we could get an expression, a neat governing equation, ordinary differential equation for the growth of boundary layer for the simplest possible case which is flow over a flat plate, it is still impossible, not possible to obtain a closed form solution. So even though we have the governing equation, we have the, we know what are the boundary conditions, a solution to this is not possible using analytical methods. So numerical methods will have to be used, was used by, by a researcher named as Howarth and what he has presented is, he solved this equation numerically and presented a table containing, in the column, the first column is values of eta, corresponding values of f, the next column contains the value of f prime which is d f d eta, f double prime and so on.

So looking at the table, the numerical solution of the governing equation with appropriate boundary conditions you generate the results table. But the results table in itself is quite informative and it would give us, give us a form, a compact form of the growth of the boundary layer. Or in other words, the first thing that we started our, this exercise is to obtain, to obtain the relationship of delta as a function of x. So this table can now be used to obtain this functional form of delta in terms of x, that's what we would see next. So we are, we are starting with this equation, the governing equation. We also know that eta is defined as, and we have seen the expressions of v f to be equals d f d eta and the expressions for v y to be equals half, these are the expressions that we have seen. So we will use this later on.

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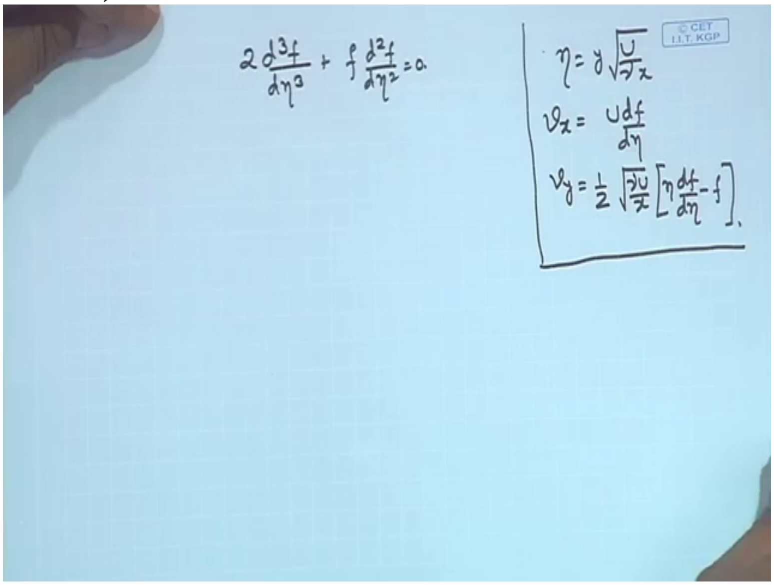

We also understand that the boundary conditions to solve this equation are, at eta equals zero, that means eta equals zero means y equals zero, at y equals zero we understand that v x is going to be equal to zero. So if v x is zero, then d f d eta must be equal to zero. I will repeat it once again. At eta equals zero, the first condition, at the liquid solid interface that means at y equals zero, the velocity in the x direction due to no-slip condition must be equal to zero. So if velocity in the x direction is zero at eta equals zero which essentially tells me that d f d eta would be equal to zero. So d f d eta would be zero, that's my boundary condition. I also understand

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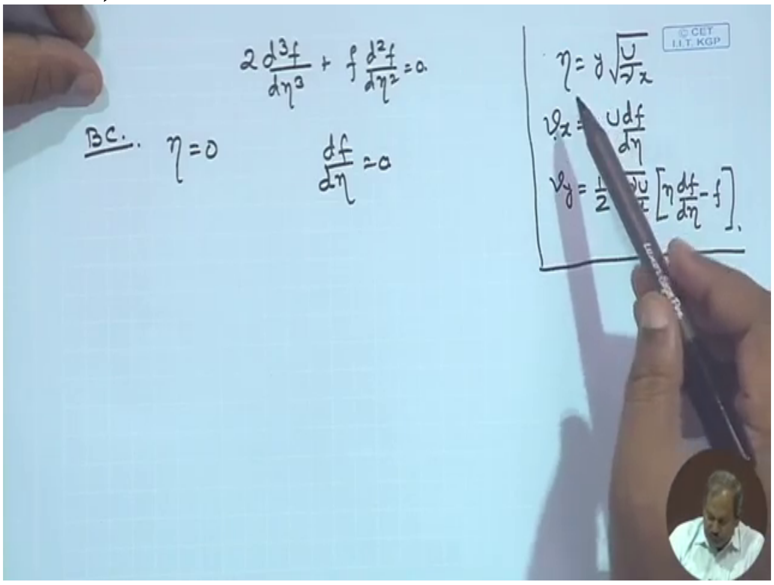

that due to the same no-slip condition, not only v x would be zero at eta equals zero, v y would also have to be zero at eta equals zero. So when you have a no-slip condition, both components of velocity, that is v x and v y would be zero.

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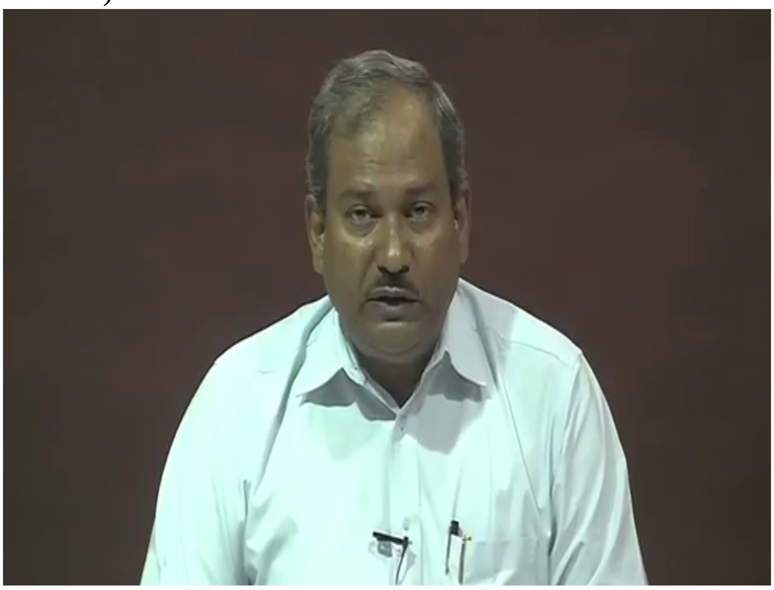

So if both v x and v y to be zero, then I, from the expression of v x I have obtained that the necessary boundary condition is f prime or rather d f d eta to be equal to zero. Now let's look at the expression of v y to see what would be the no-slip condition on v y would give me the value of either f, f prime or f double prime to be used as a boundary condition to solve this problem. So let's look at the v y.

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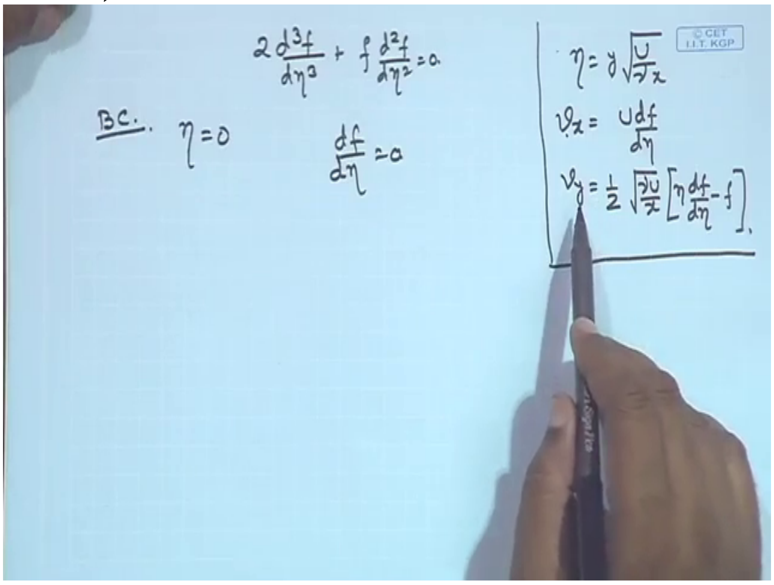

At eta equals zero that means at y equals zero, I understand that this is zero since my v x is zero which I have already obtained. So at v y to be zero, this, since this is zero, this term must also be zero. So it's simply going to be

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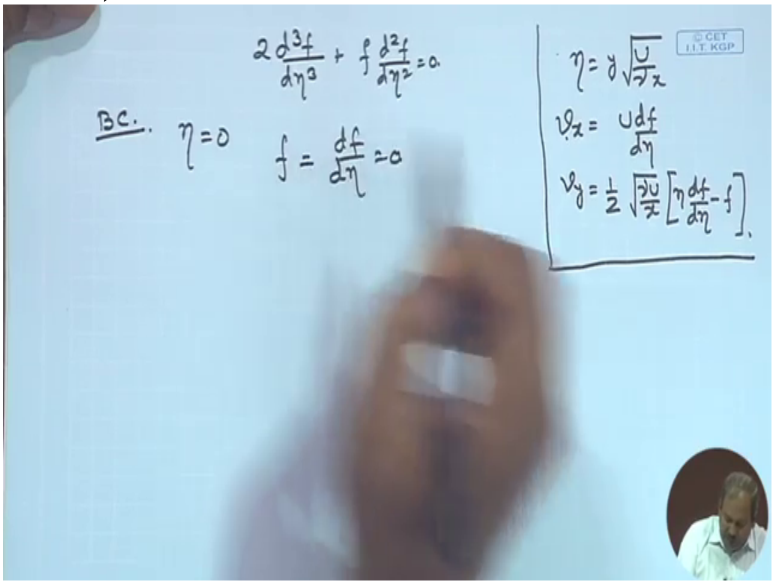

not only the gradient of f with respect to eta is zero at eta equals zero which directly can be seen from the expression of v x, the requirement that v y will also have to be zero and since d f d eta is already zero it must also be zero. So the boundary condition, no-slip boundary condition would be f equals zero and d f d eta is also equals zero. And the other boundary condition, when eta is infinity, when eta is infinity that means y to be very large and when y is very large, v x approaches or v x becomes equal to u. So at a point when eta tends to infinity which physically represents a point far from the solid plate, the velocity of the, velocity of the fluid, velocity of the fluid at that point would simply be equal, would simply be equal

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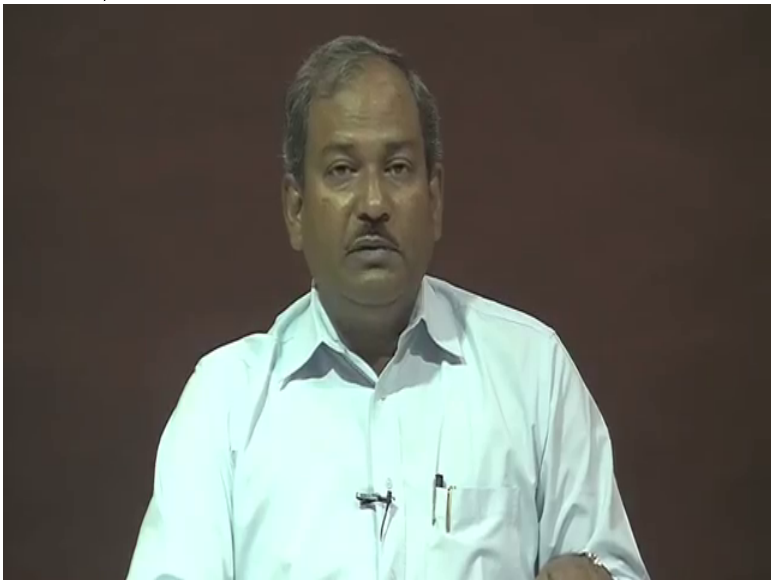

to the freestream velocity So the form of the equation for v x would tell you that the d f d eta since v x is equal to u times d f d eta, since v x and u are equal at that point, you simply have d f d eta to be equal to one which would constitute the other boundary condition to solve the problem, to solve the O D E. So for this third order O D E, now we have three conditions. One is the no-slip, that is eta equals zero, both f and f prime are equal to zero and at the other end when we are far from the flat plate, since v x is equal to u, I am simply going to get f prime is equal to one. So these are three boundary conditions which Howarth has used in order to obtain a numerical solution of the governing differential equation. So at eta equals infinity, f prime is going to be equal to one.

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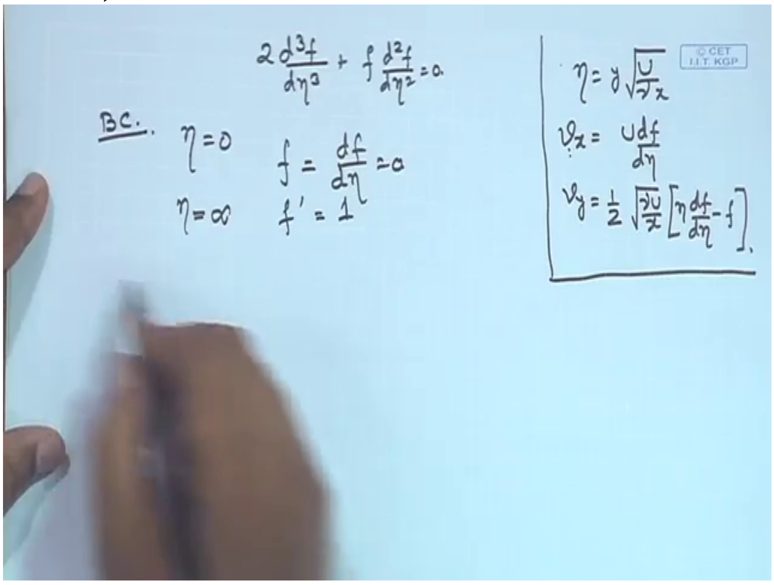

Now the table that he has provided is eta then f then f prime, f double prime. I will

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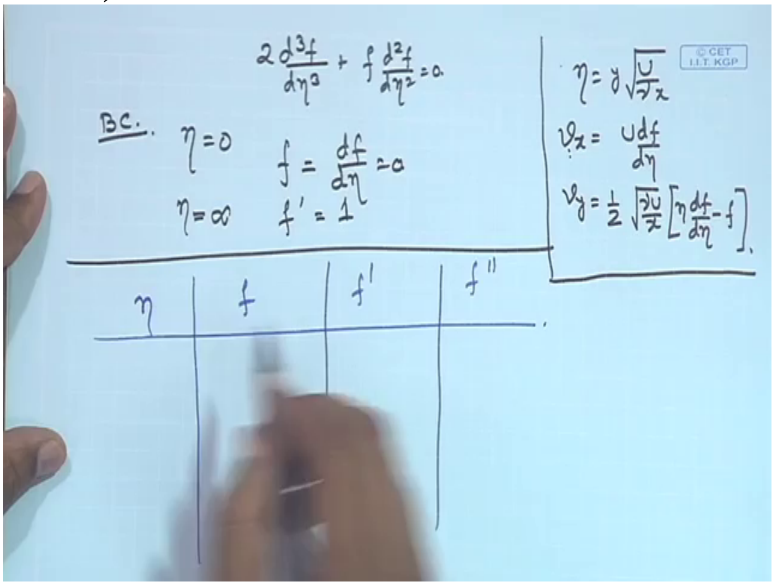

only list some of the values in here.

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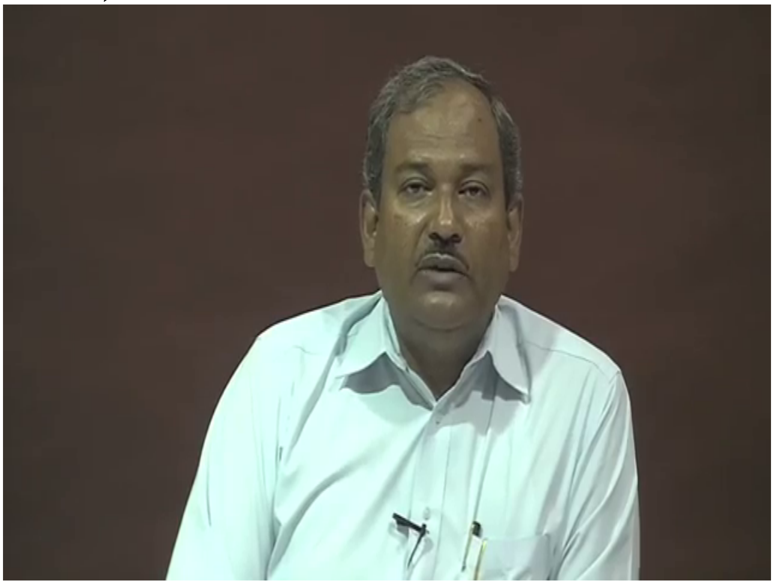

The rest you should be able to see in your textbook and I would only write those terms which are going to be relevant for our subsequent analysis. But he has, he has given a detailed list of values of f, f double prime and f, f prime and f double prime for different values of eta. But I would list only some of the values, 4 or 5 values of this table.

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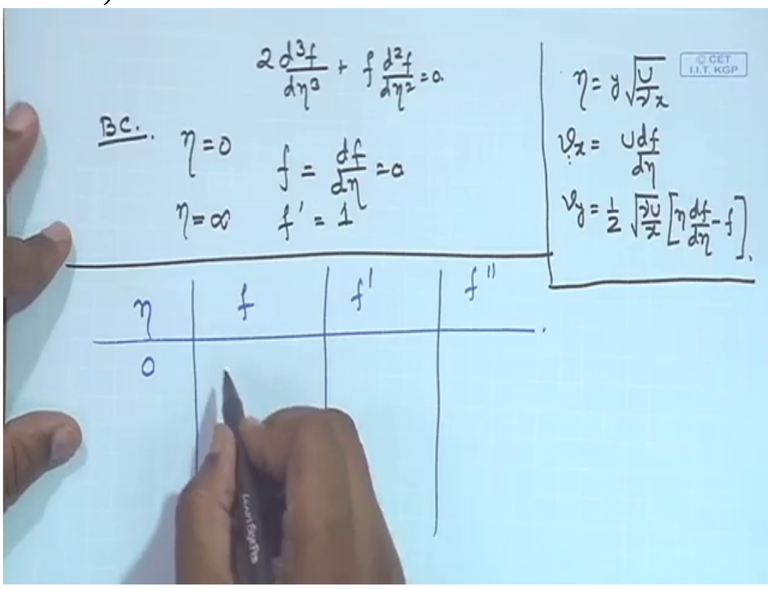

So this turns to be zero. This is zero and this would be point 3 3 2. And

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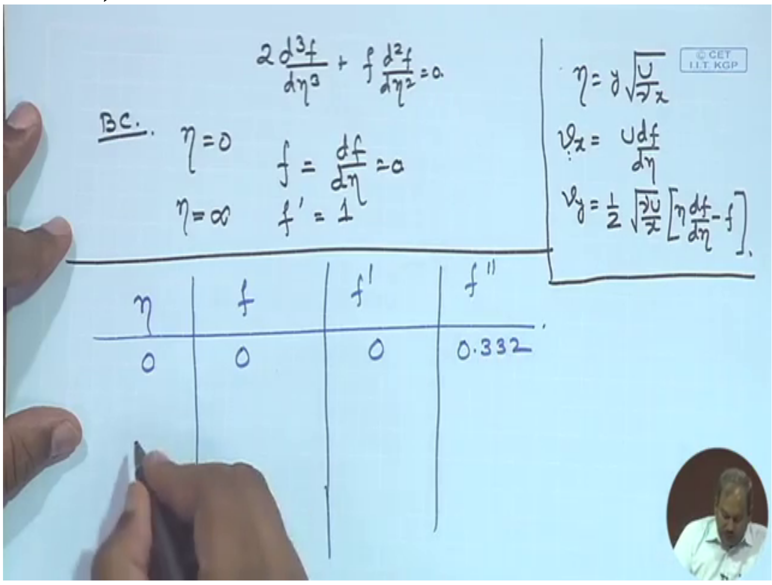

5 point 0, 3 point 2 8 3 2 9, 0 point 9 9 1 5 5 and 0 point 0 1 5 9 1 and some other values

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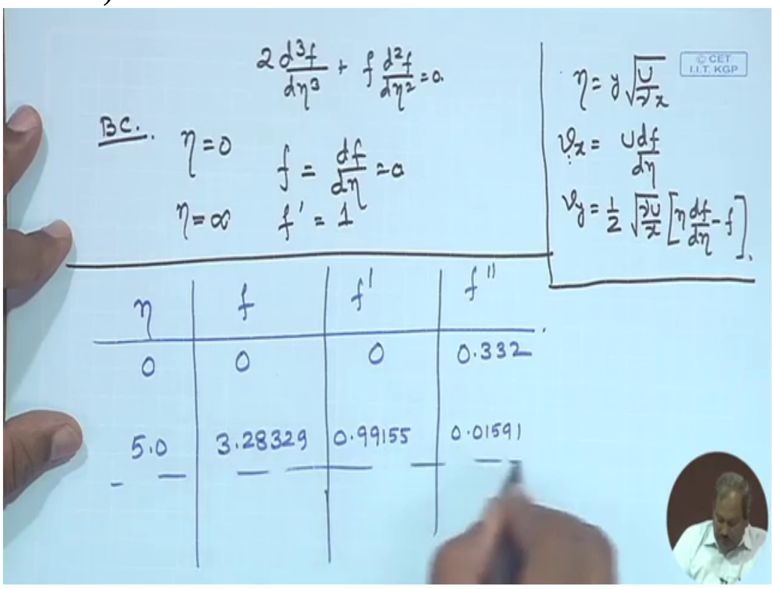

which we would subsequently use in a tutorial class Almost 1, this is almost 1, Ok.

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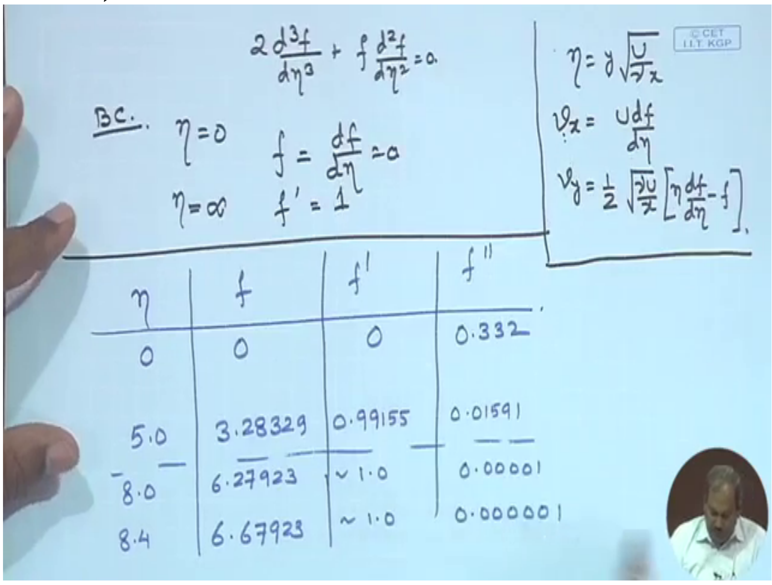

Let's look at these values. These are from the numerical solution of Howarth. So if you look at

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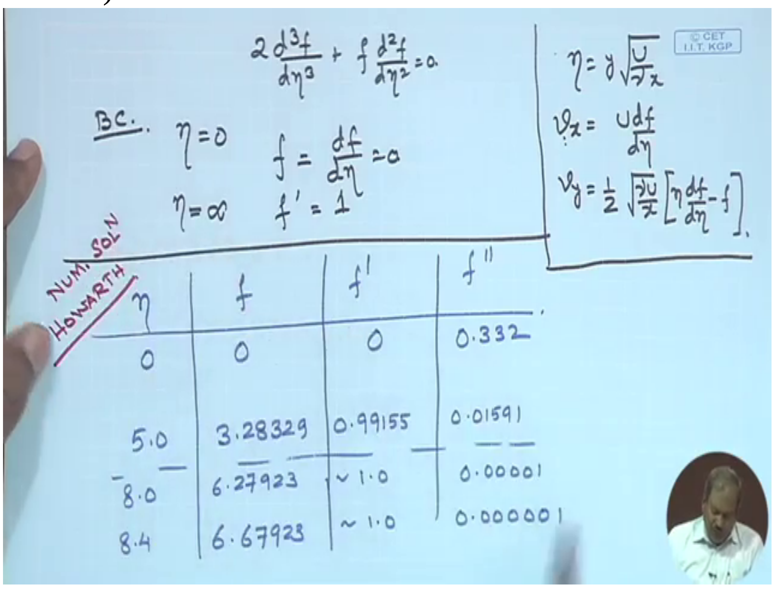

the definition of v x, you understand that v and v x to be equal to u, f prime has to be equal to 1. So the definition of v x

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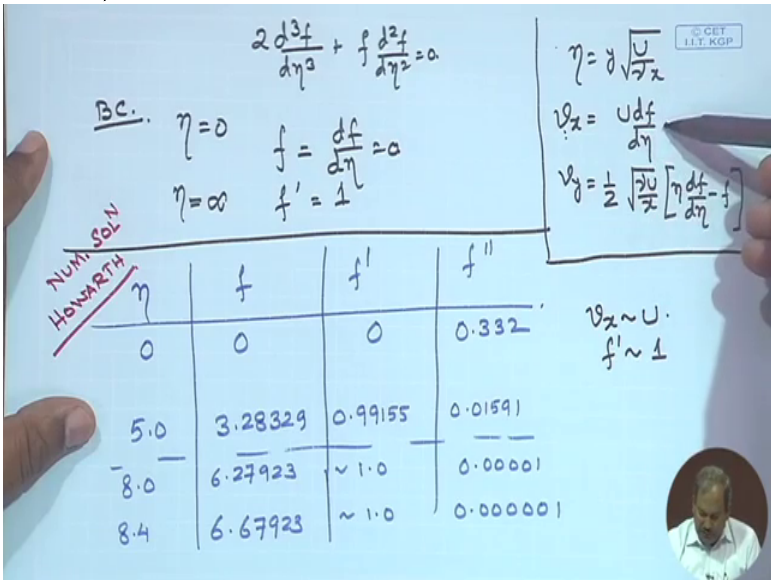

tells you that in order for the velocity to approach the freestream velocity, to be equal to the freestream velocity, the value of d f d eta must be equal to 1 Now if you recall the definition of

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the edge of the boundary layer or the thickness of the boundary layer, we have seen that it is at point at which the velocity reaches 99 percent of the freestream velocity. So let's see where we have this 99 percent. It's at this point. So at this point, the f prime is about 99 percent, so therefore v x by u is equal to point 9 9 and therefore it reaches 99 percent of the freestream velocity. And when does it happen; when eta is equal to phi. So I will write that eta equals phi where f prime is equal to 1 can be written as

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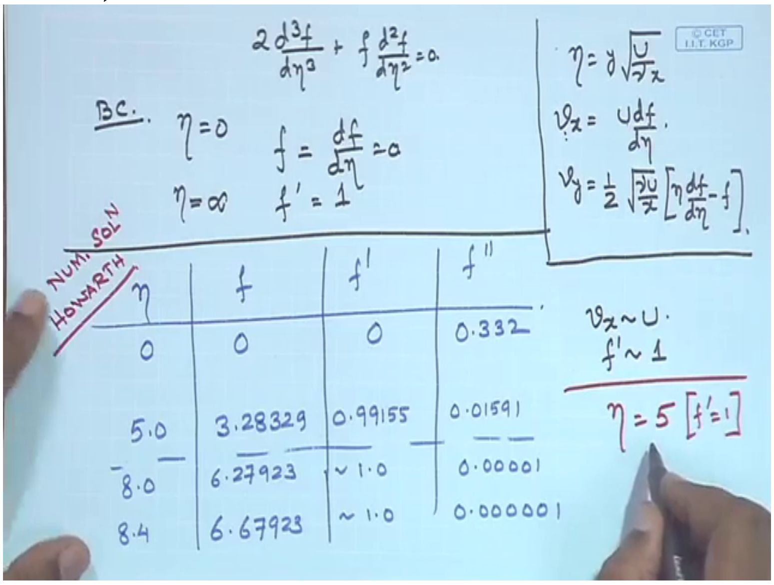

the edge of the boundary layer, can be taken as the edge of the boundary layer; so let's see how does that help us. We understand that by definition eta is y root over u by nu x and eta

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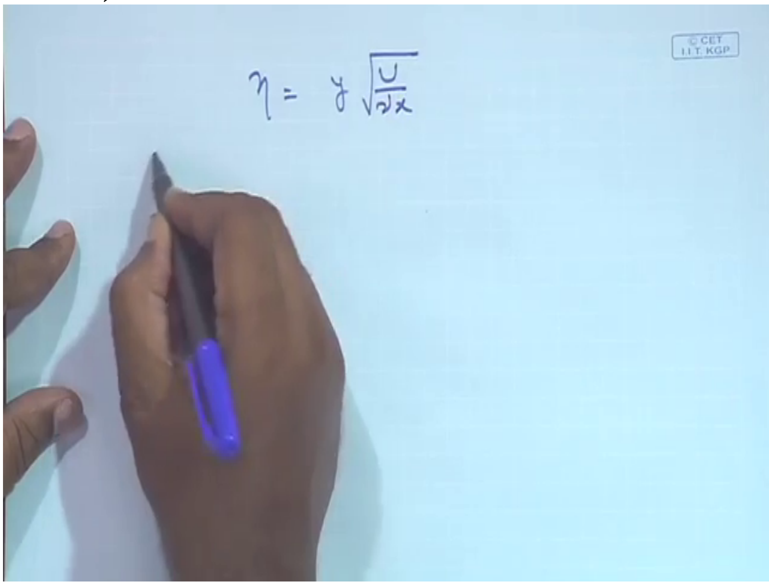

equals 5 can be taken as the edge of the boundary layer. So if that is so then at that point, when eta is 5 point 0, this y would simply become equal to delta where delta is the thickness of the boundary layer. Once again, for value of eta we have seen f prime is equal to point 9 9 1, that is the velocity v x by u is equal to point 9 9. So velocity reaches 99 percent of the freestream velocity and at that point y must be equal to, y must be equal to delta.

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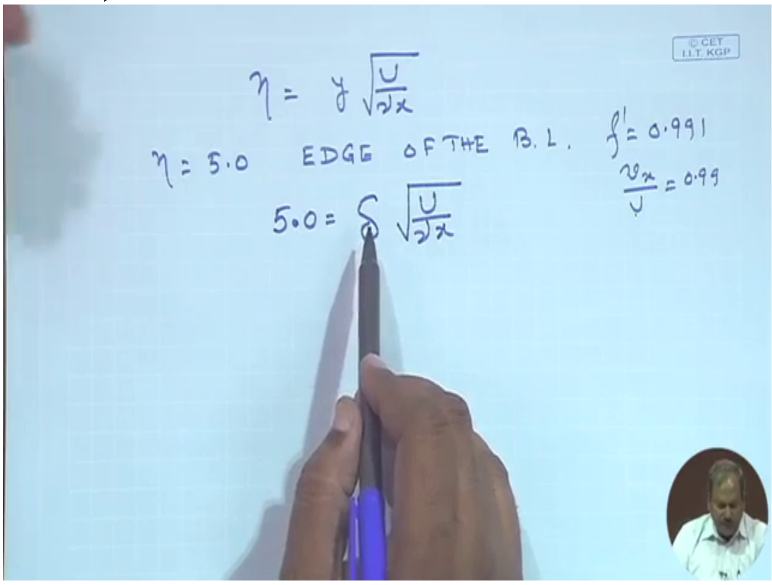

So you have an expression for delta from here as 5 point 0 by root over u by nu x which can be slightly modified, slightly changed. This is r e x, it's just a reorganization of the terms and nothing else.

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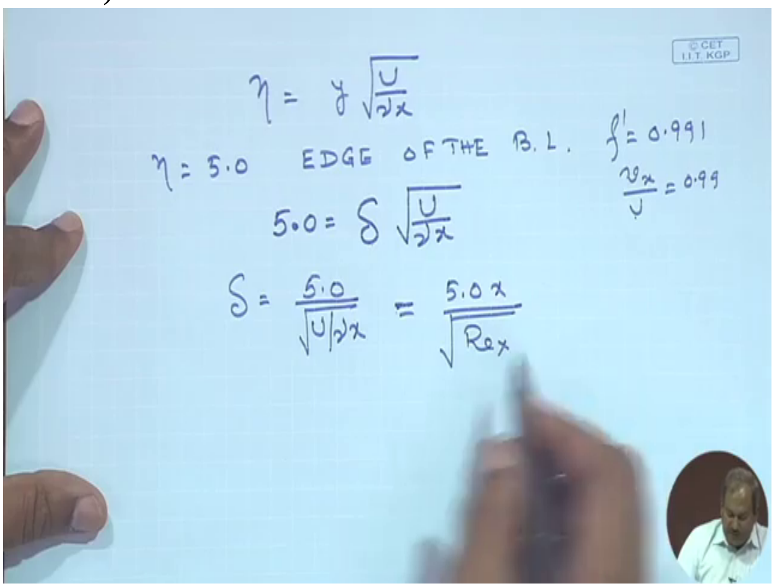

So what you get is delta to be equal to 5 point x by root over r e x. So this term, this expression

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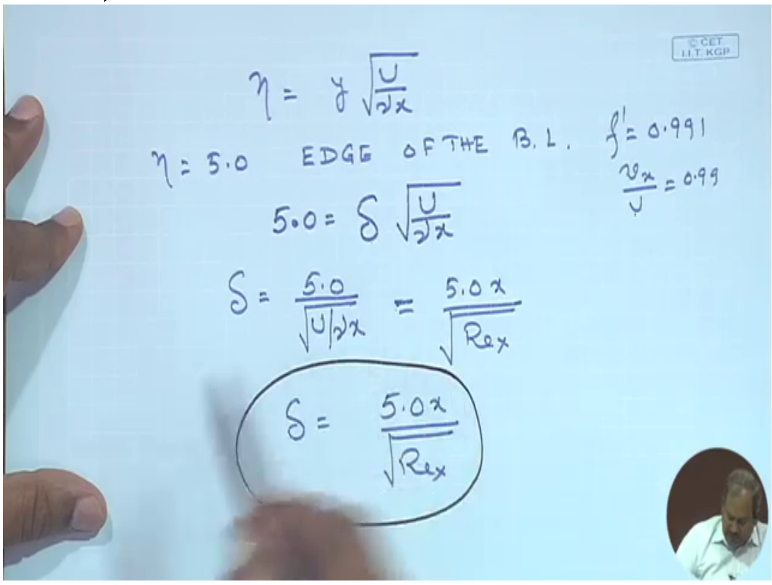

tells you what is the thickness of the boundary layer at a given axial position So if you have the growth of the boundary layer, if you fix the actual position which is x, you know exactly from this closed form equation what is going to be the delta at that point. So we have, we have achieved one of our goals, that is to obtain the thickness of the boundary layer at any given axial position.

Let us quickly

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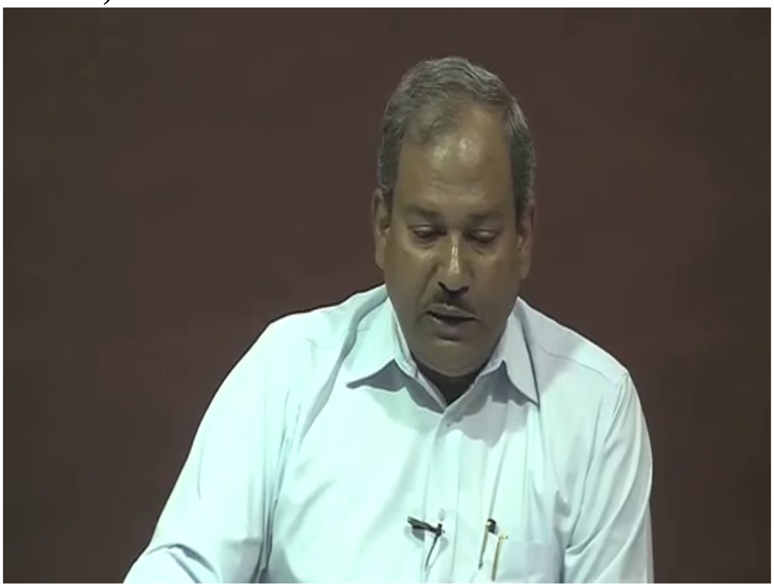

go through, look at some other important parameter which the engineers would like to have which is to obtain, what is the shear stress exerted by the moving fluid on the plate or in other words what is the wall shear stress. Can we use the numerical solution of Howarth to obtain such a closed form solution for the wall shear stress as well? So we quickly do that. The wall shear stress, wall shear stress is simply, which I denote by tau w is mu del v x del y at y equals zero which can be written

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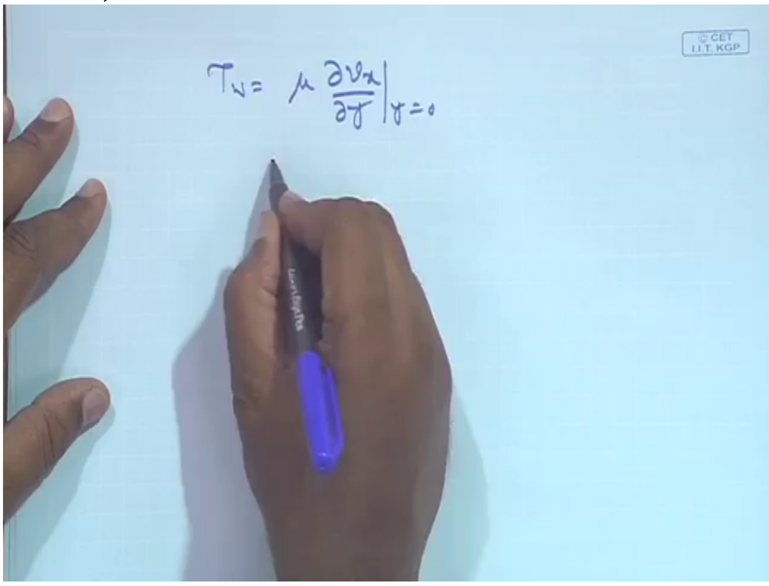

as del del y Instead of v x it is del chi by del y at y equals zero. Now if I bring in

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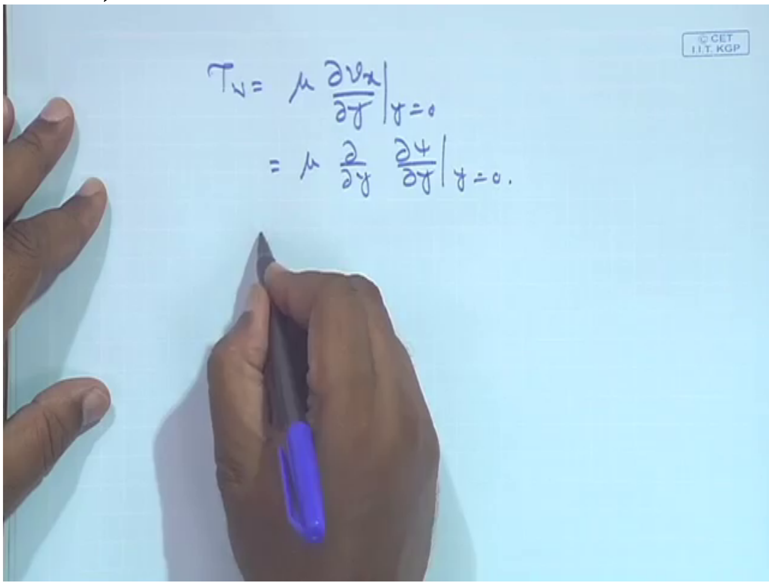

instead of y, the concept of eta, the combination variable, it is del del y of u d f d eta at eta equals zero. So after

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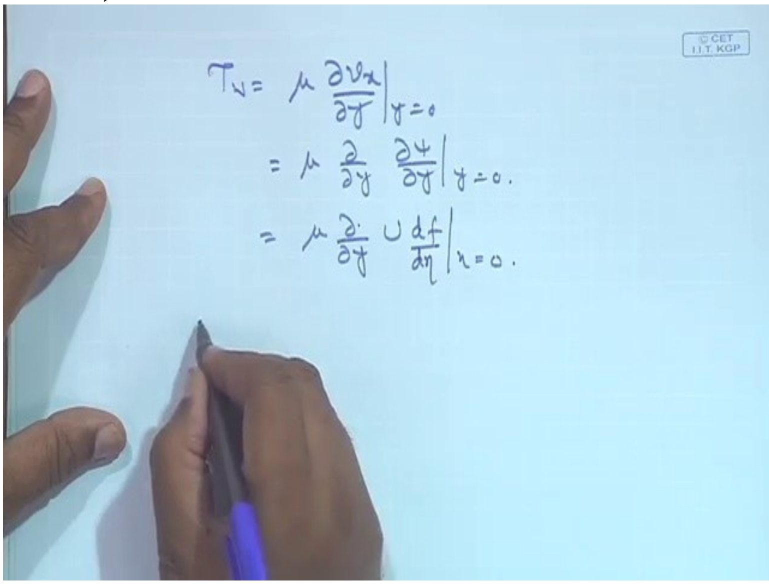

a bit of substitution which you would be able to see in your text, it would simply be equals mu times u root over u by mu x times d 2 f by d eta square at eta equals zero. So this is the expression for wall shear

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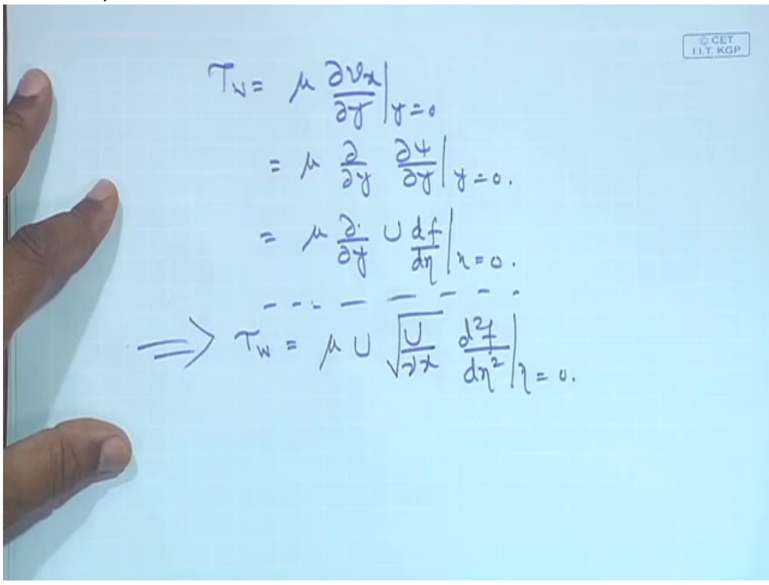

stress And the only thing unknown here is what is the, what is the number d 2 f by d eta square

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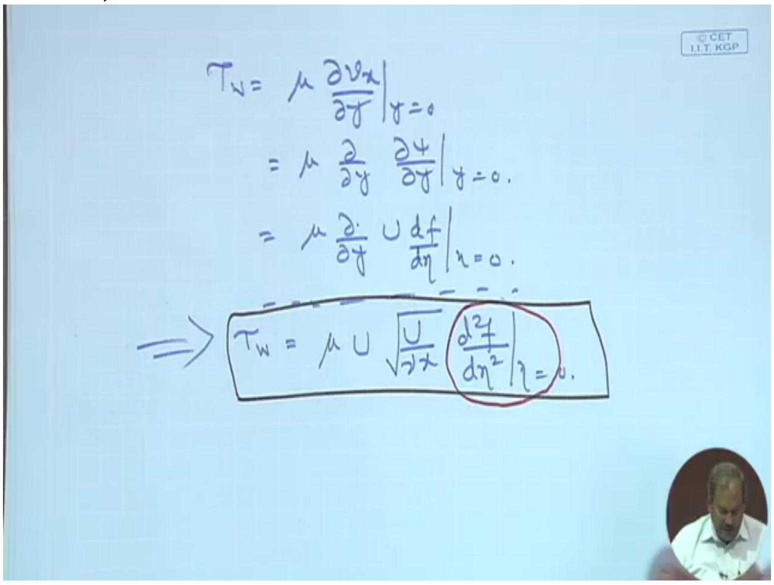

at eta equals zero. I bring in this, so d 2 f

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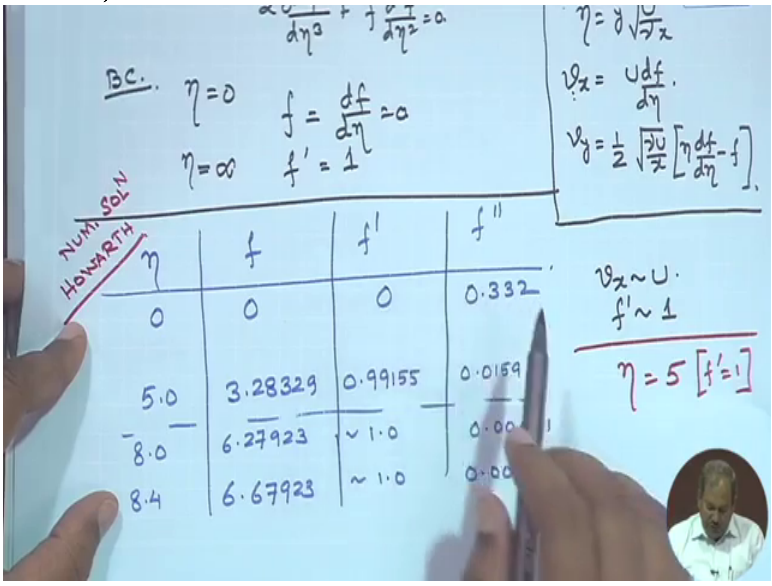

by d eta square which is nothing but f double prime, for the value of eta to be, at eta equal to zero, f double prime is simply equals point 3 3 2. So the numerical value of this is simply equals point 3 3 2 from the numerical solution of Howarth. So I will bring in this point 3 3 2 over here and what would I get at, since at eta equals zero f double prime to be equals point 3 3 2, what you would get,

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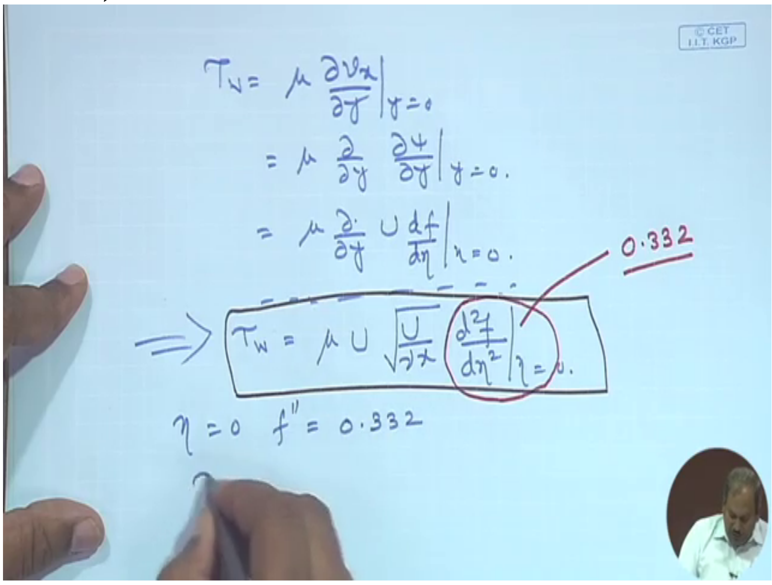

tau w to be equals point 3 3 2 u times root over rho mu u by x equals point 3 3 2 rho u square by root over r e x.

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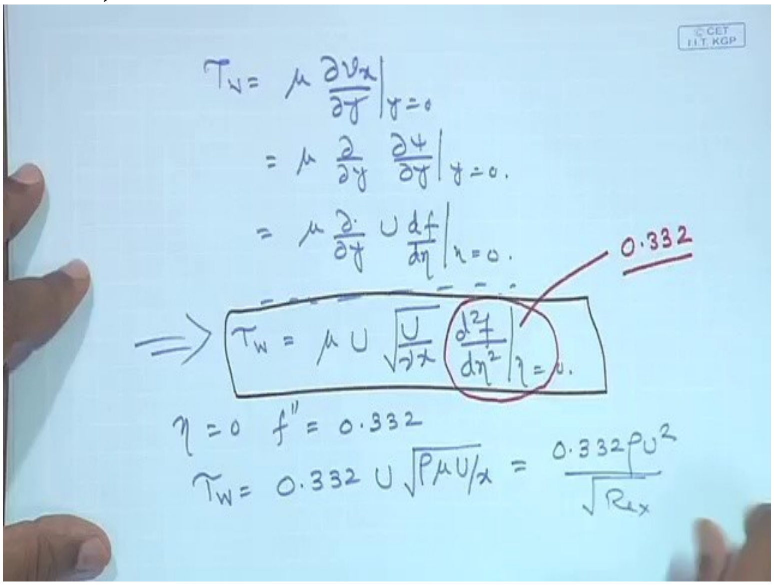

So once again I write here the expression of tau w to be equals point 3 3 2 rho u square where u is the freestream velocity by r e x, that is another compact expression for

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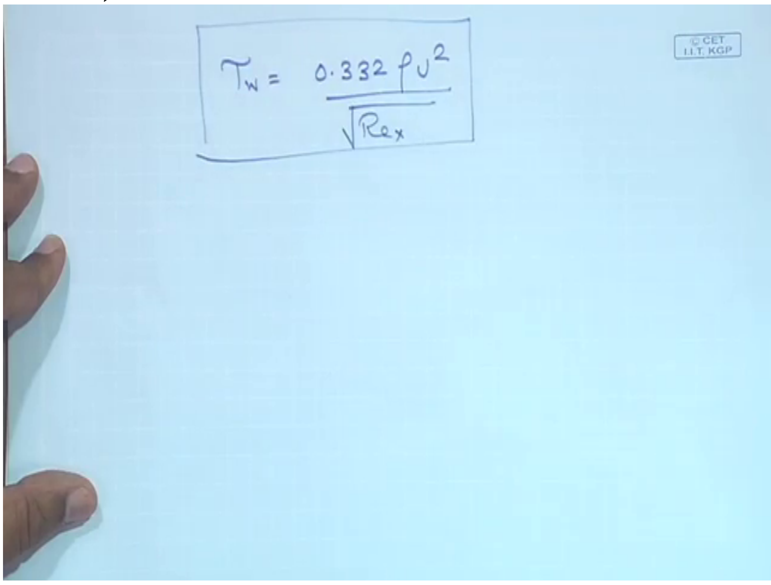

the wall shear stress and sometimes we define a shear stress coefficient which is traditionally denoted by c f.

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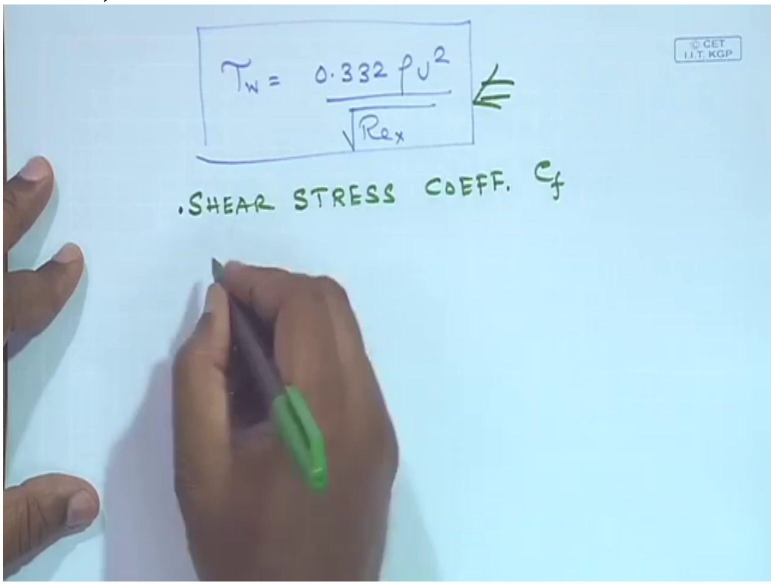

By definition, c f is wall shear stress by the dynamic pressure and therefore if you bring in the expression of tau w in here, what you would get is point 6 6 4 by root over r e x. So the three equations, three conditions, three expressions that we have obtained

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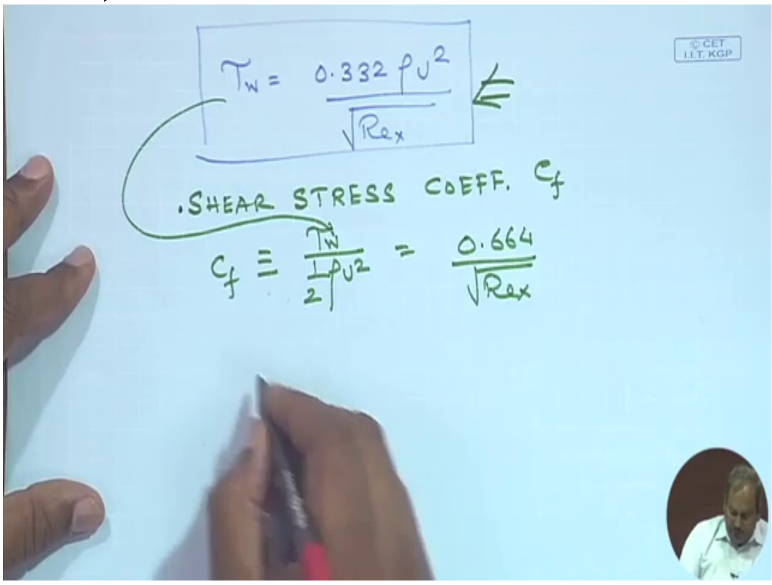

are through this exercise, are these three. One is the first is an expression of

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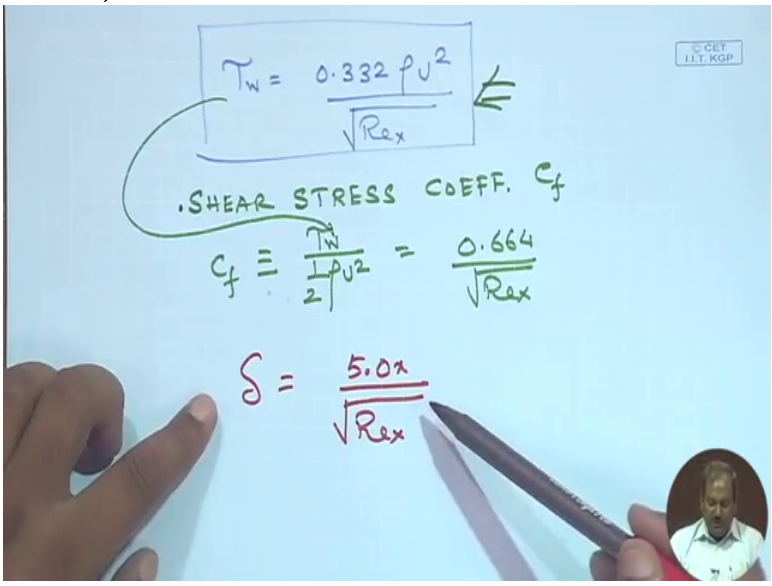

the boundary layer thickness The second is an expression of the wall shear stress and the third which is just by definition more commonly known as the shear stress coefficient, these two are equivalent. These two are equivalent, an expression for the shear stress coefficient which is point 6 6 4 by root over r e x. So one can see then that through the

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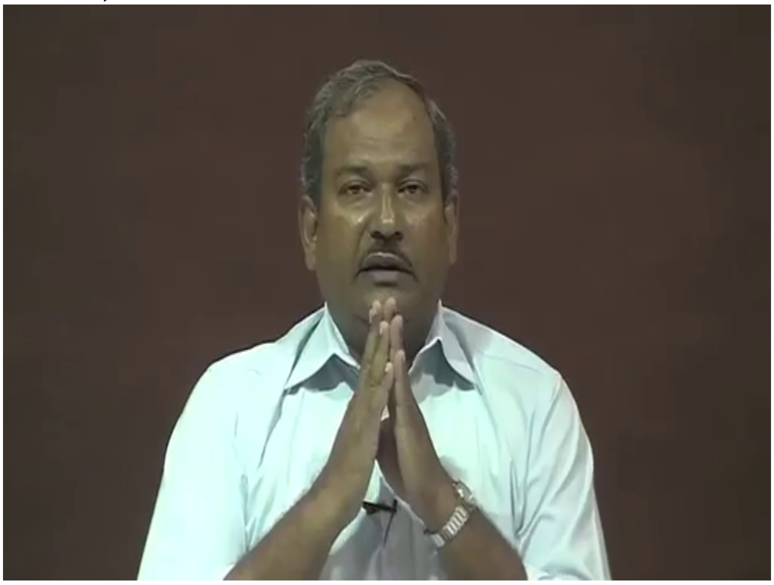

use of Navier–Stokes equation, boundary layer approximation, identification of the appropriate boundary conditions, evaluation of a combination variable which combines both y and x into eta through an order of magnitude analysis, introduction of stream function and dimensionless stream function, all these complicated steps are necessary to convert the Navier–Stokes equation inside the boundary layer that too for two dimensional flow inside the boundary layer to an o d e but even that o d e is, is non-linear o d e and it is higher order as well.

So analytical solution was possible and one had to use numerical solution techniques to, to obtain the results which is for any value of eta, what are the values of f, f prime and f double prime. And since we know how would, how would the velocity v x, velocity in the x direction behave when we reach the edge of the boundary layer and what would be the gradient of velocity at y equals zero, that is on the solid plate, we would get, we finally got expressions of delta as a function of x and tau w or c f, the friction coefficient as a function of x and other flow parameters, the physical property of the fluid and so on. So for the simplest possible case, it is complicated, Ok.

So this method can be used for flow over a flat plate but this approach is, cannot be used for any complicated geometries. This is only limited to laminar flow. So if the fluid is, if the fluid is undergoing, or in turbulent flow this expression cannot be used. This is for a zero pressure gradient flow. If you have a pressure gradient present in the system because of the, since it is not a flat plate, then approach cannot be used. So we have a solution but the solution is for the simplest possible case and it cannot be termed as a general solution or easy to use approach in solving the boundary layer parameters for any type of flow on any geometry or any type of surface, any geometric surface.

So there has to be a generalized method which is easy to use and is not restricted by all these constraints. So what we would do in the next few classes after I solve one problem, a tutorial class on this is to, is to show you a generalized approach in which it would be far more easier to handle situations which are not so-called the ideal systems, flow over a flat plate. It would be approximate but it would still allow us to compute these numbers, the growth of the boundary layer, the value of the wall shear stress and so on in a much more effective and easy to use way. So that is what we would do in the next class.