Transport Phenomena
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Lecture Number 18
Boundary Layers (Cont.)

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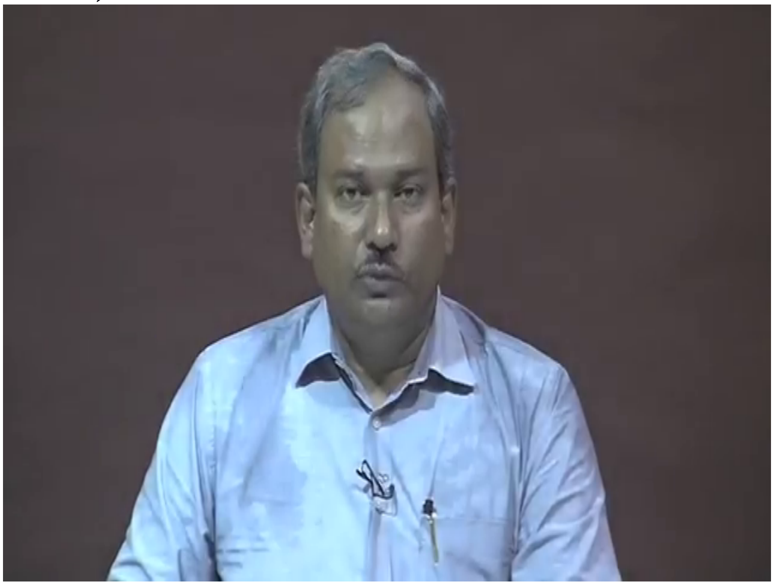

So what we have seen in our previous treatment of boundary layers, we were restricting ourselves to a flat flow over a flat plate and when the flow was laminar. What we understood that it's the effect of the flat plate will become less and less at certain point inside the fluid after a certain depth the flow becomes equivalent to that of the freestream flow. So if we have a flat plate and the fluid is approaching it with a constant velocity which is also known as the approach velocity then close to the solid layer the effect, the velocity will vary from the no-slip condition which is zero velocity on the flat plate to asymptotically it will merge smoothly to the flow outside of the thin layer which is known as the boundary layer in which the effect of viscous forces are important.

So beyond the thickness of the boundary layer the velocity will remain constant and for a flat flow over a flat plate laminar flow over a flat plate, this velocity, the constant velocity beyond the boundary layer which is known as the freestream velocity, the velocity which is free from the viscous effects, its known as the freestream velocity and for the special case of flow over the flat plate the approach velocity and the freestream velocity, these two are going to be equal. Inside the thin boundary layer, the effect of viscous forces will be predominant. Now close to the surface the effect of the viscous forces would be more and as we move away from the flat plate, flat solid force the effect of the viscous forces will be progressively smaller and beyond the boundary layer the flow can be treated as inviscid where there would be no effect of viscosity and therefore the velocity profile in the flow outside of the boundary layer can just resemble, just a flat profile.

So approach velocity and the freestream velocity outside of the boundary layer for flow over a flat plate, they are equal inside the boundary layer the flow is two dimensional, the effect of viscous forces are going to be important, beyond it the effect of viscous forces would not be applicable. So truly speaking Euler's equation which is for an inviscid flow is valid for the region outside of the boundary layer where as the Navier–Stokes equation which is for a viscous fluid, which is for equation of motion of a viscous fluid of constant mu and Newtonian fluid, the Navier–Stokes equation would be valid for flow inside the boundary layer.

And we have already discussed the historical significance of the boundary layer concept and how does that help in correlating the theory and the experiments while designing the motion of ships in sea. The thickness of this boundary layer is arbitrarily defined as the point at which the velocity of the fluid is equal to 99 percent of the freestream velocity. So it is very close to the freestream velocity and the number is defined as the 99 percent of the freestream velocity. So if we call small u as the velocity inside the boundary layer which since the flow is two-dimensional this u is going to be a function both of x where x is action direction or direction of flow and y where y is the distance from the, vertical distance from the top plate.

So u inside the boundary layer is a function both of x and y where as outside of the boundary layer it's a constant, the small u will simply becomes equal to u infinity u suffix infinity where u infinity is the freestream velocity so how does this small u, the velocity inside the boundary layer changes with distance from the solid plate is something which is of importance and you will subsequently see why it is so but by the definition of boundary layer thickness, small u divided by capital U, the local velocity inside the boundary layer and the freestream velocity outside of the boundary layer, the ratio of these is equal to point nine nine which is known where the value of y where this condition is reached is called the boundary layer thickness or the disturbance thickness.

Now we can understand since the velocity near the edge of the boundary is very small so a small change in I mean it is difficult to demarcate the exact location at which the velocity becomes ninety nine percent of freestream velocity so this definition though it gives a pictorial view of the thickness of the boundary layer is difficult to obtain experimentally with sufficient accuracy. This there is definitely, therefore there is a a need some thickness which unlike the previous one which is differential in nature unlike the differential nature of boundary layer thickness, it would be an integral nature and the integrand would be such that the integrand nearly at the, at or near the edge of the boundary layer.

So we have, in the previous class we have also seen two integral thicknesses defined in such a way that when expressed as a series a summation of a series the later terms which are the terms which represent the region close to the edge of the boundary layer, their effect on the total summation is negligible. So these two thicknesses are known as the momentum thickness and the displacement thickness. So displacement thickness when talk in terms of the reduction in mass flow rate because of the presence of the boundary layer and momentum thickness when we talk in terms of the reduction in momentum of the actual mass which is flowing through the boundary layer.

So this is what we have done in our previous class. Now we would like to see if it is possible to obtain an analytic solution or close to an analytic solution for flow inside a boundary layer. We understand that this is a complicated situation because inside this region, the velocity is the function of actual distance, the velocity is the function of distance, the vertical distance from the solid wall, the effect of viscosity is important and therefore the equation we need to start with is the equation of motion or the Navier–Stokes equation for two dimensional flow for two dimensional flow inside the boundary layer and there are various terms which would lead to complicacy whenever we try to analytically obtain a solution for such a case.

So we would see that some of the approximations which are quite common in the treatment of boundary layer, collectively they are called as the boundary layer approximations. So what are those approximations, the logical basis on which these approximations are made and how they can simplify a very complicated problem to something which is so we would start this class with the boundary layer approximations for laminar flow over a flat plate. And whenever there is a flow over a flat plate under laminar flow conditions it's, it's approximated as a zero pressure gradient flow and if we assume the plate to be horizontal there would be no effect of body forces.

If you consider gravity is the only body force which is present, so the Navier–Stokes equation which we are going to write will, will be for a situation which is zero pressure gradient since it's flow over a flat plate and it's, it’s a situation at which the effect of body forces are not present, are negligible. So we are going to write the Navier–Stokes equation for flow inside a boundary layer which would look something like this. We all know that the,

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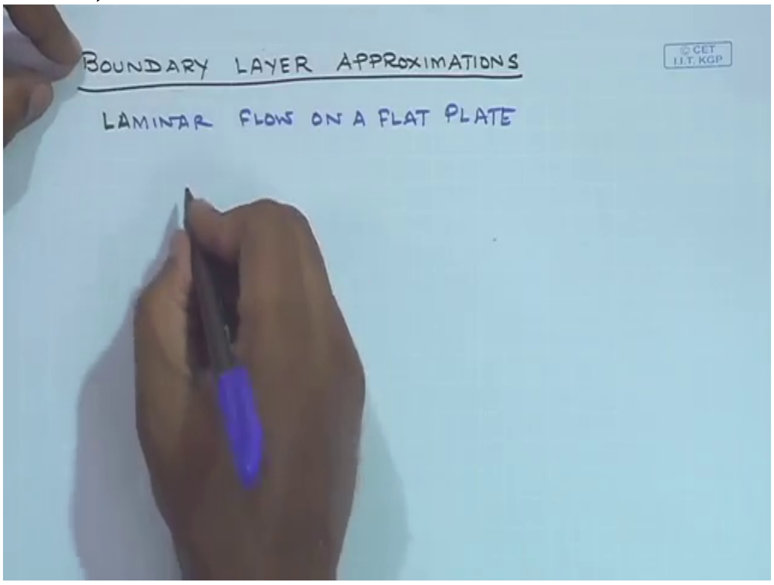

from our statement of Navier–Stokes equation, if we write the x component and we are writing the x component since this is the principle direction of motion, plus v y del v del y nu, which is nothing but mu by rho, the kinematic viscosity, del square v x by del x square plus del square v x by del y square. And there is, as I mentioned before, there would be no effect of, there is no pressure gradient or body force term in this. Now here we are going to make

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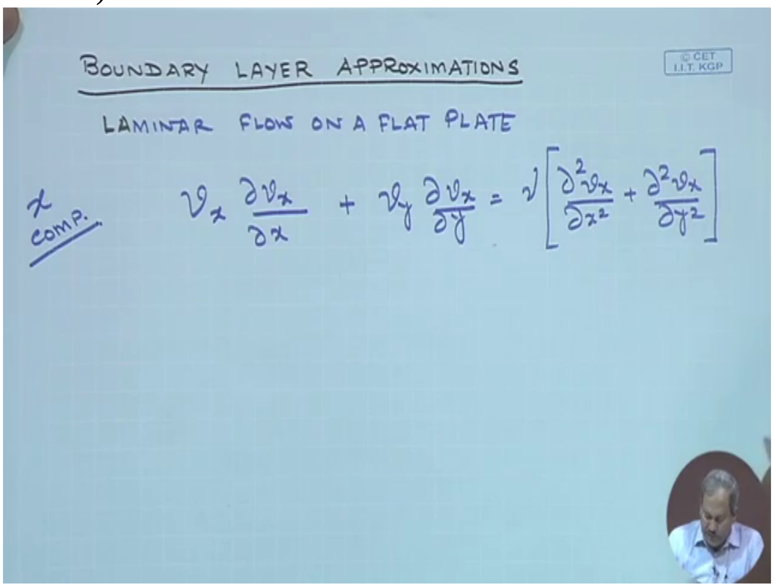

a quick uh mental calculation of the significance of each of these terms Now we are talking about a flow where the flow and this is the boundary layer and here I have v x which I understand is a function of both x and y. And for this boundary layer to grow I also have a v y which could also be a function of both x and y. But if we see, if we just analyze this you can simply see that this v x is going to be very large as compared to v y. So v x, it's the principal direction, the component in the principal direction

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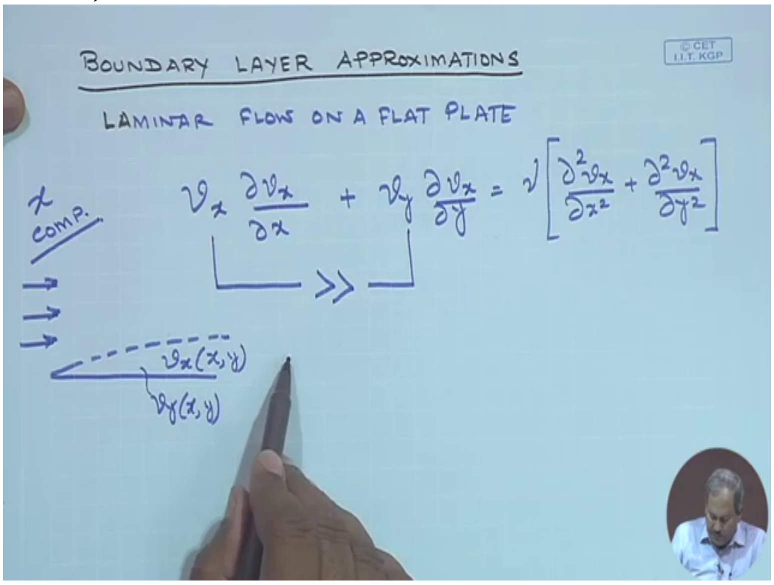

of motion must be higher, significantly higher than v 1. But the thickness, the boundary layer thickness at any point is delta and we understand that this delta is a function of x. As x increases, delta keeps on increasing. However the value of delta is quite small. It is the thickness of the boundary layer is generally small, so the variation of v x with x and variation of v x with y, if we consider these two terms, v x varies from zero over here at the solid liquid interface equal to u where u is a freestream velocity. So v x changes from zero to u over a small distance delta so therefore del v x del y is going to be quite large as compared to del v x del x. del v x del x simply tells you the variation in velocity between these two points over a certain distance x so the reason, the analysis, or the understanding here clearly tells us that this del v x del y v x del y

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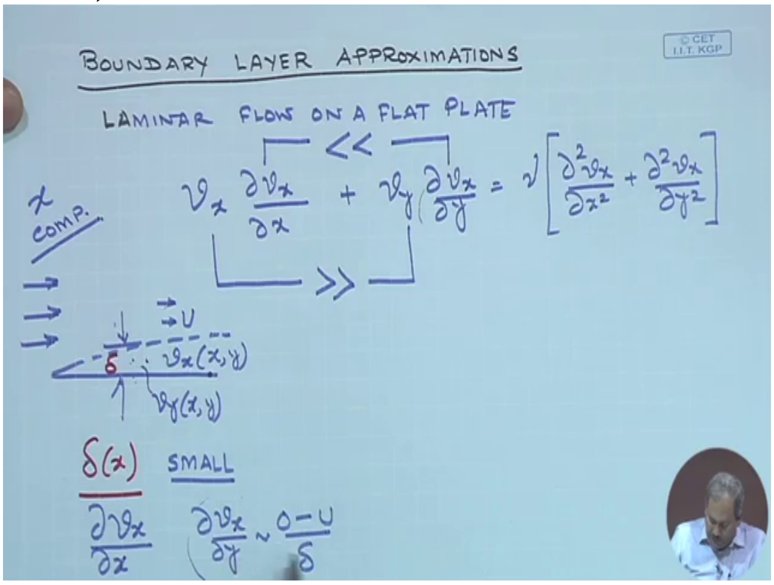

because of the small value of delta is going to be significant larger than del v x del x or in the other words, the gradient in velocity in the vertical direction is going to quite large as compared to the axial velocity gradient. So as you see from here, since v x is large as compared to v y and del v x del x is large in comparison to del v x del x, the product of these two, we cannot make any

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judgment about which one is going to be significant So the each term contains two terms and the nature of the variation or the magnitude of these terms are such that you need to keep both of these terms in the Navier–Stokes equation. There is no way you can say that the, let's say the first term is significant, the second is not and so on. So therefore you cannot make any judgment and both the terms are to be kept into this equation. Now when we come to this part over here,

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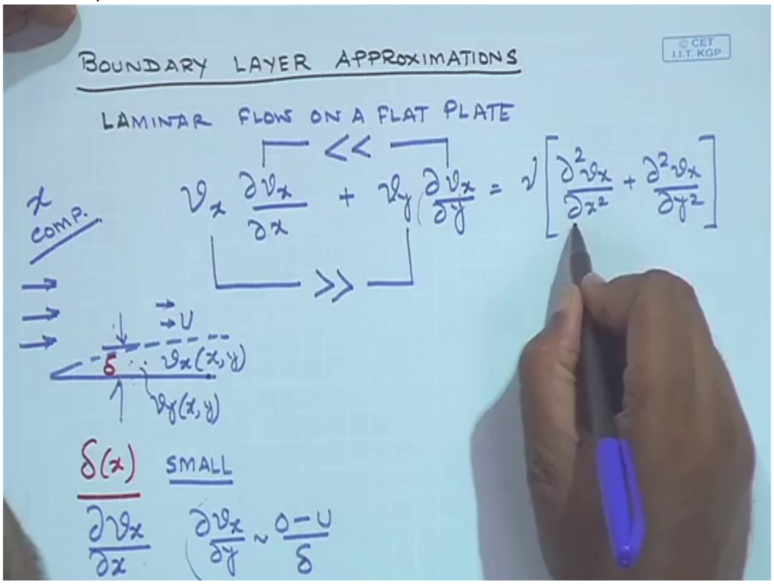

here we see that these are essentially telling me about the viscous transport in the x direction and viscous transport in the y direction, gradient of viscous transport in y direction. And from our discussion over here, we understand that this term is going to predominate and therefore

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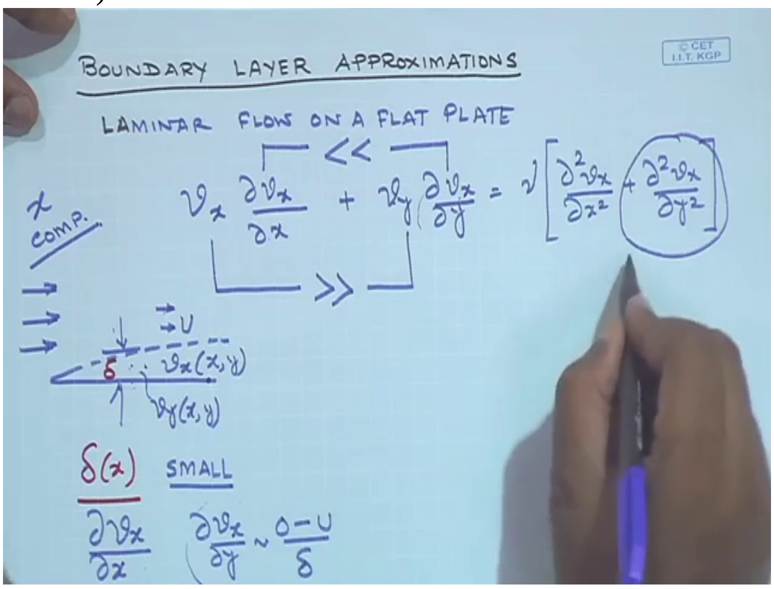

this is going to be quite large as compared to the this term so the second term is going to remain in the Navier–Stokes equation and the first term can simply be neglected.

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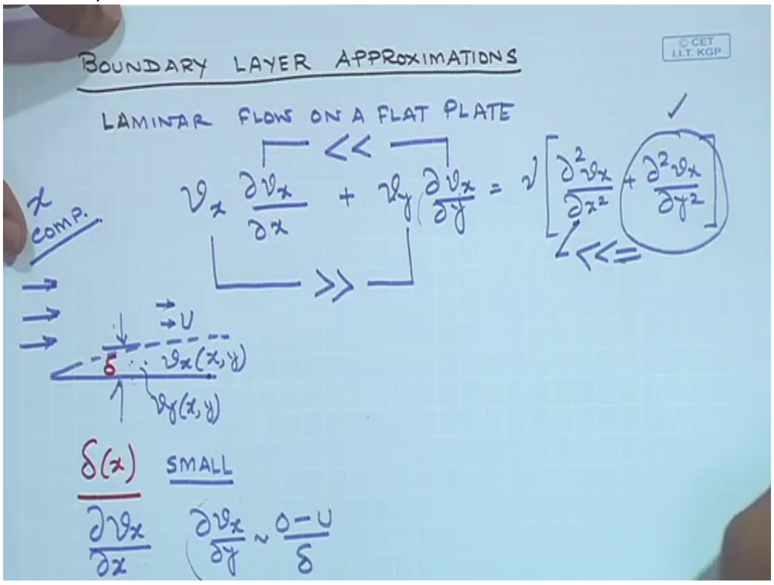

So the final form of Navier–Stokes equation for flow for inside a boundary layer on a flat plate can be written as this. And whatever I have discussed so far in terms of the magnitudes the relative magnitudes of these terms

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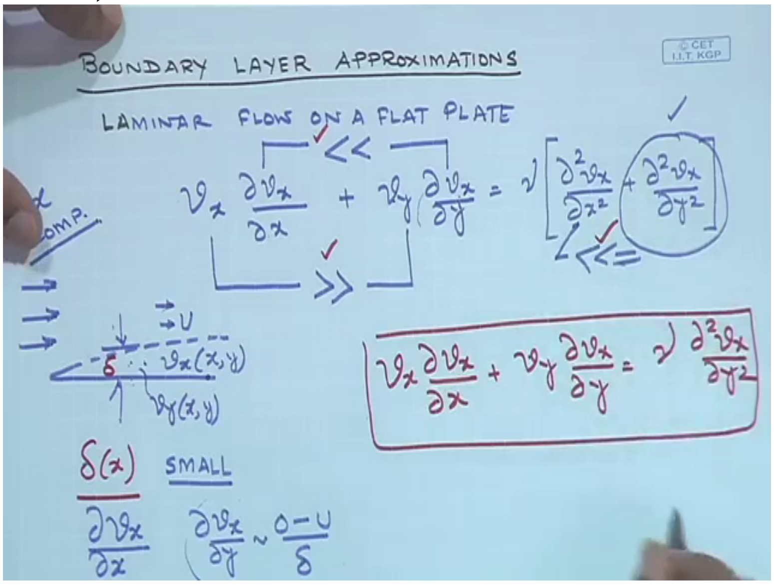

and their inter-relation, they are collectively known as the boundary layer approximations or the boundary layer assumptions so one has to solve this equation for flow inside a boundary layer with appropriate boundary conditions. And what we need is we need two conditions on y and the two conditions on y are at y equal to zero which is at

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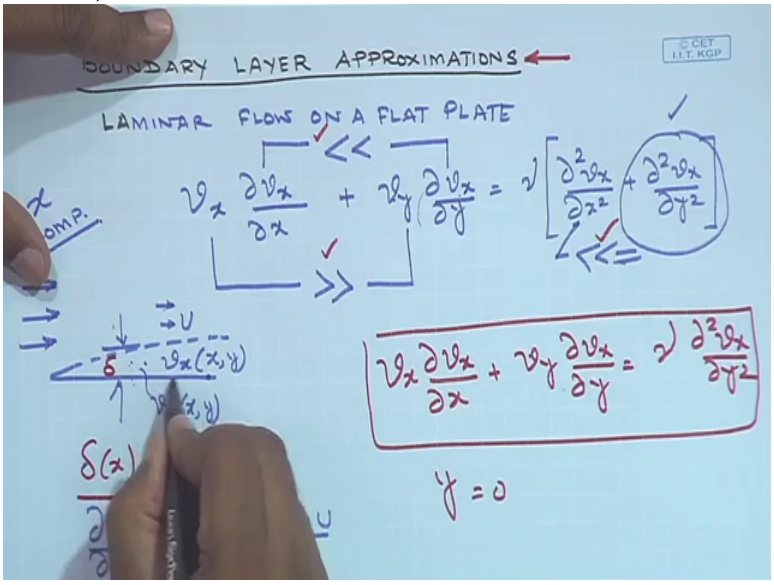

this point your simply v x at no-slip condition would be zero and at

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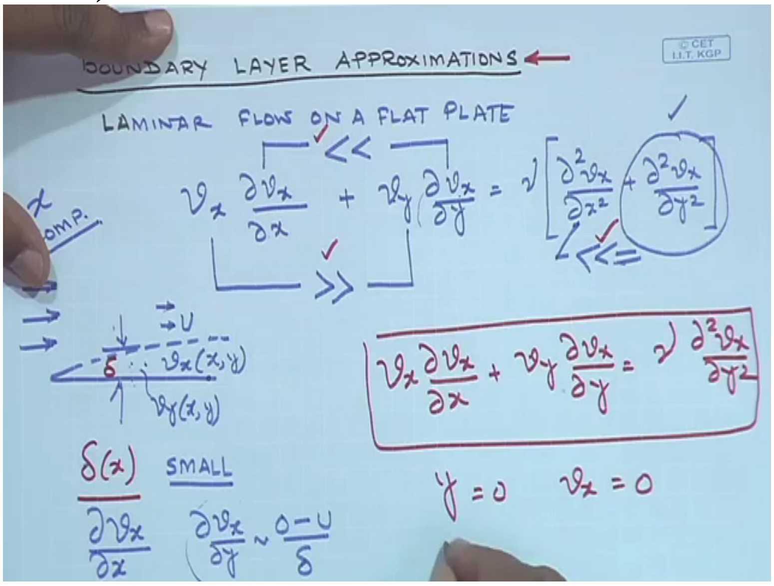

y equals infinity that means at

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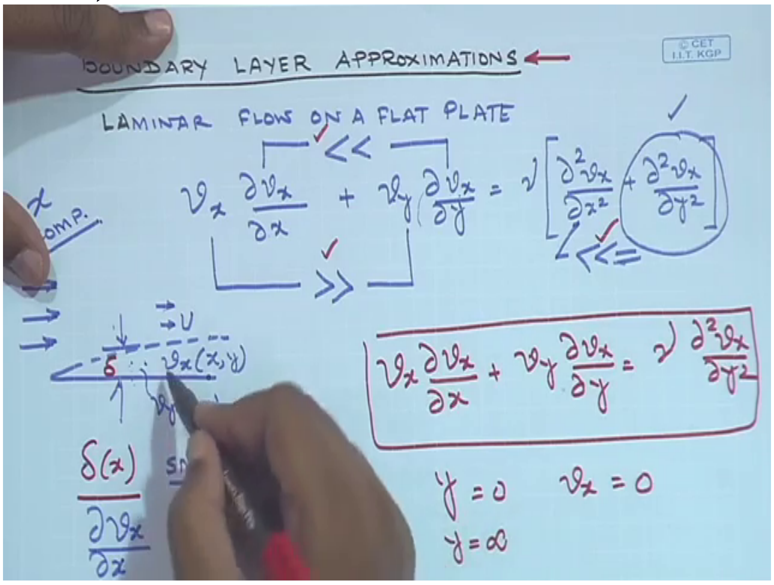

a distance far from the wall the v x would simply be equal to u where u is the freestream velocity, Ok.

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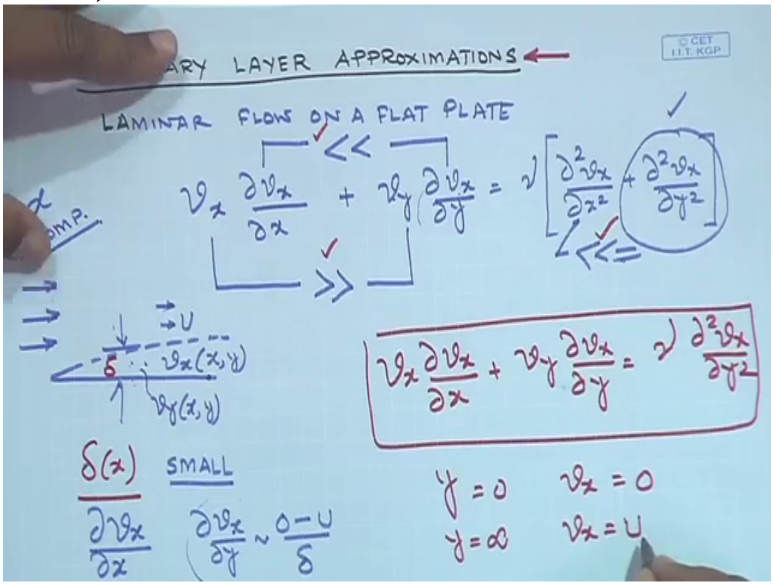

So these are the two conditions what we can say and the other condition, the initial condition is at x equals zero, v x would simply be equals capital v where capital v is the velocity, the approach velocity and we understand for special case of flow over a flat plate, this v would simply be equal to u so the three conditions that are needed to solve this equation are at y equal to zero no-slip condition at y equals infinity at far from the flat plate, the velocity is equal to freestream velocity and at the beginning at x equal to zero, the velocity for the flat plate is

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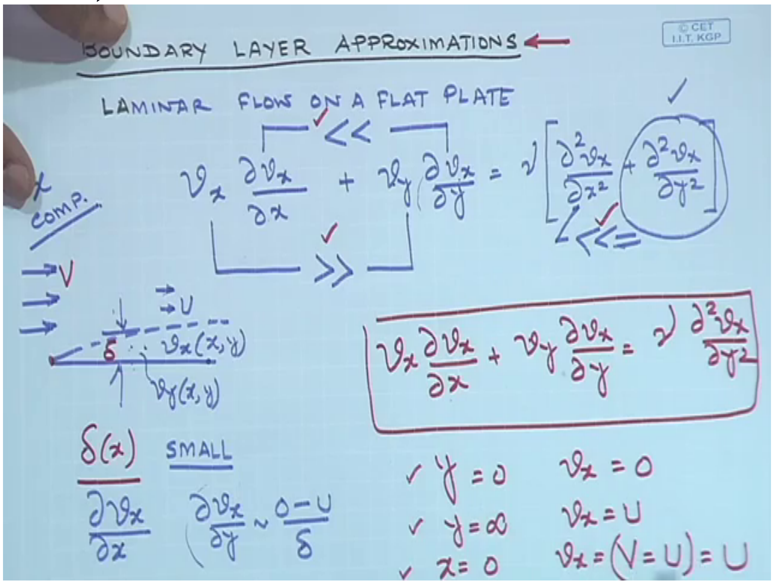

equal to the freestream velocity; so this equation has now to be solved for, for the, using the three boundary conditions Now whenever we come across

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such an equation, the equation was first solved by Blasius and it's also known as the Blasius solution. What Blasius

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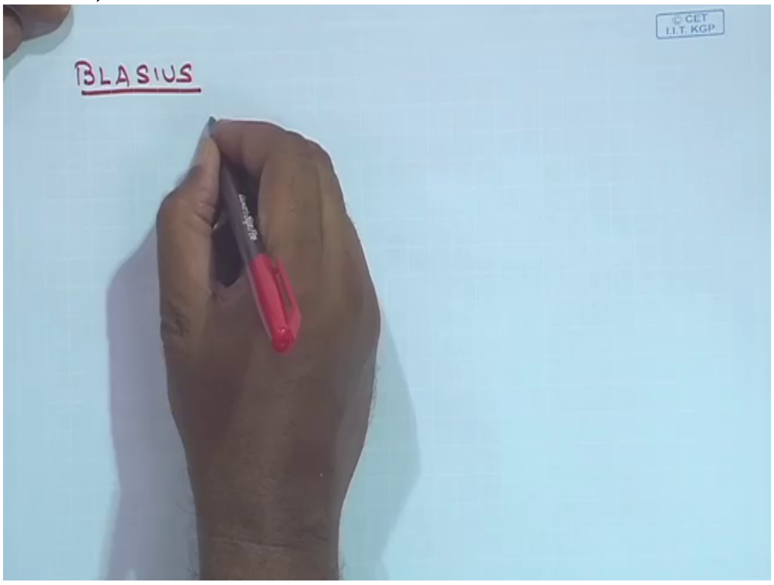

reasoned that the velocity profile, the dimensionless velocity profile should be similar for all values of x when plotted versus non-dimensional distance from the wall. So he has reasoned it's only an assumption that v x by u would be a function would be function of eta where eta is y by delta y is the distance from the wall, delta is the boundary layer thickness at that point. So the dimensionless velocity is going to be a

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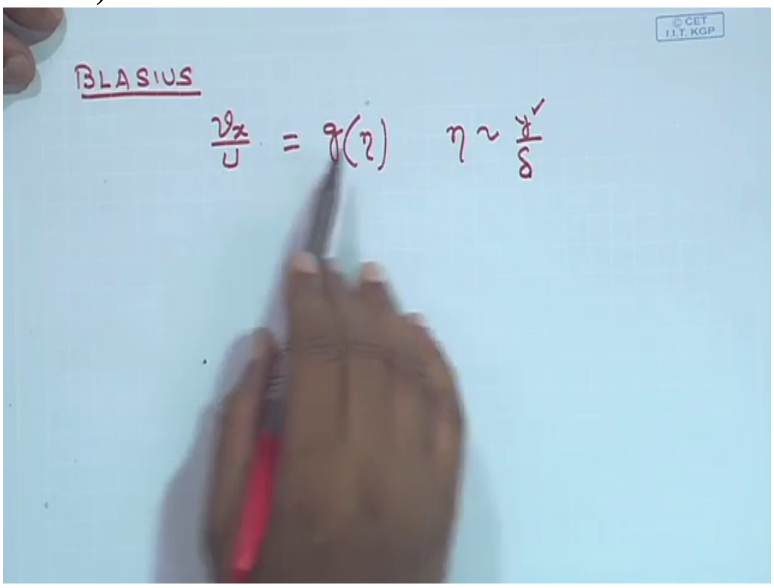

function of dimensionless distance from the solid wall, which, which is logical, because if you,

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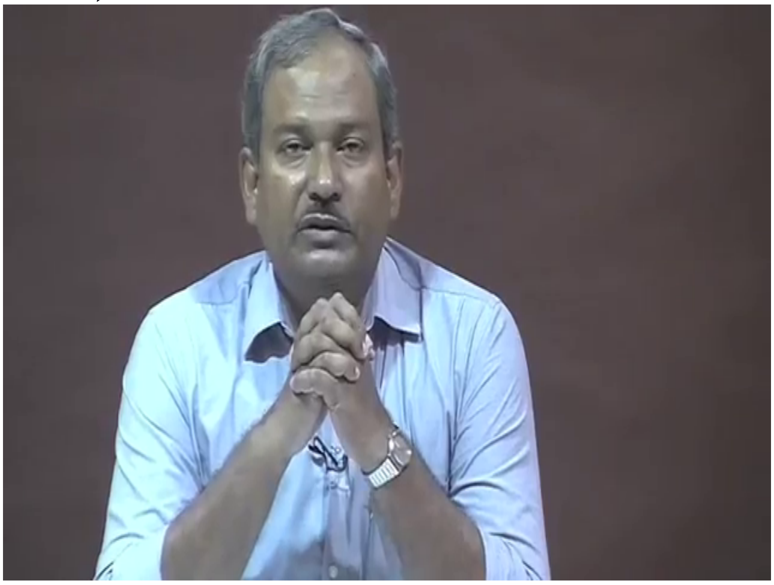

you think of the velocity, the dimensionless velocity profile it is definitely going to be a function of distance from the solid wall. But since the velocity is non-dimensionalized we need to non-dimensionalize the distance from the solid wall as well. So the, the only dimension which is physically significant in a direction perpendicular to that of the flow has to be, the thickness of the boundary layer. So eta which is the independent variable in this case is defined as some sort of equal to y by delta where delta is the dimensionless thickness. But if you go to the previous slide where I have written down the equation to be solved for the case of flow inside the boundary layer this is not the only

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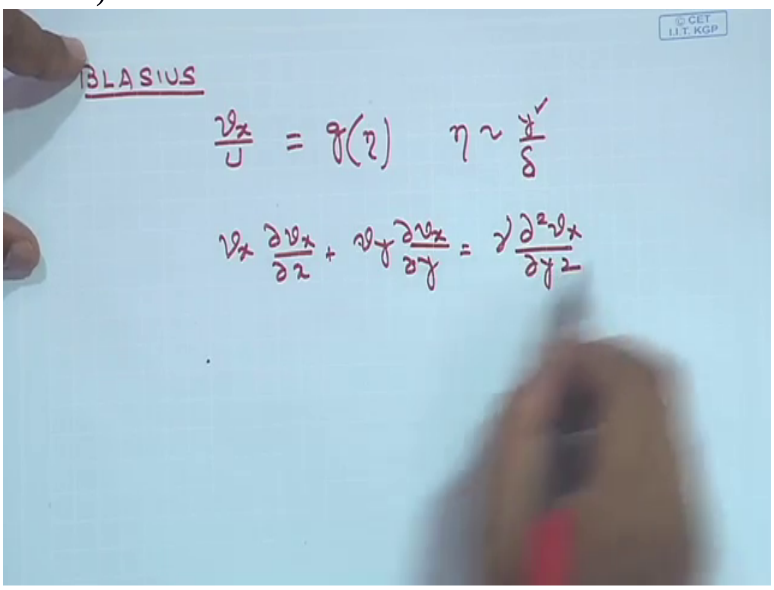

equation, this I call as equation two, this is not the only equation that needs to solved. Whatever be the solution that must also satisfy the equation of continuity which being the equation of conservation of mass must

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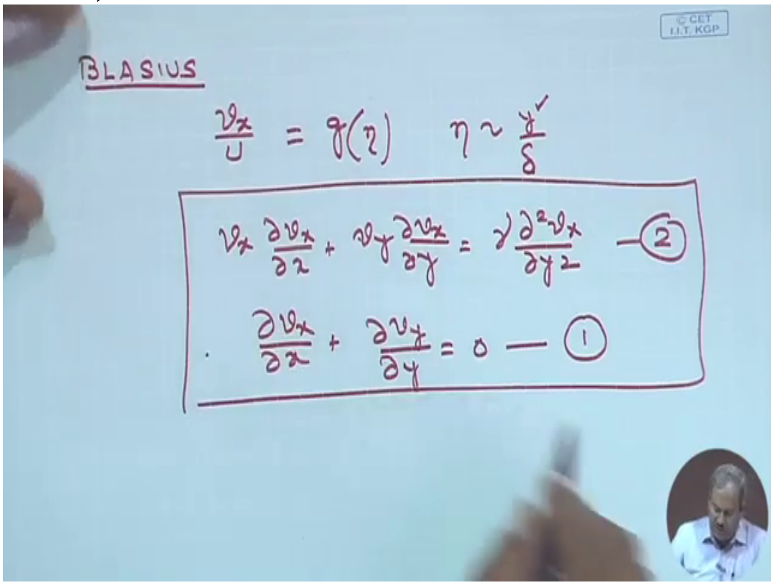

always be satisfied. So the two equations which are

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to be satisfied for describing flow inside a boundary layer, the first one being equation of continuity, conservation that has to be uh respected at all times and the second one is the equation of motion or the reduced form of equation of motion using the approximations that I have described commonly known as the boundary layer approximations. So I have two equations to deal with. Now it is sometimes advisable that instead of using two equations, if I could reduce it to one equation then it would be probably be, obviously it would be easier to handle. So I need to do something such that is it possible to do something such that these two equations would reduce to only one equation, such that I need to solve one equation instead of the two equations that I have right now.

And the way to do that is to define something which would automatically ensure that the equation of continuity is satisfied and the way to do that is to do that is to introduce the concept of stream function which from your fluid mechanics you already know that the velocities, in this case the velocities in the x direction and the y direction can be expressed in terms of a stream function which describes the flow and you also probably remember that what are the properties of the stream function like the distance between the stream functions essentially denote what is the volumetric flow rate between, between these two, two stream functions can never cross each other and the tangent to the streamline is, essentially gives you the direction of the velocity at the specific point. But anyway, we have, we know that velocities can be expressed in terms of stream function. So we, we understand that v x by definition would be del chi by del y where chi is

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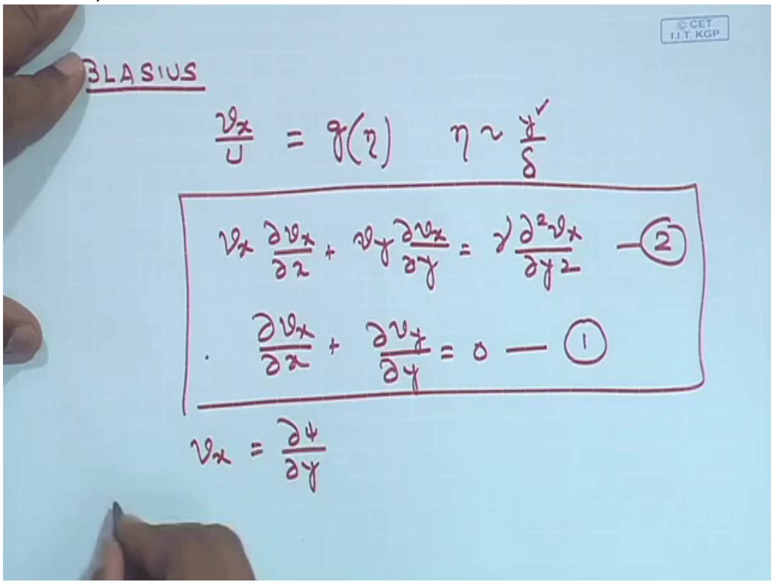

the stream function And v y

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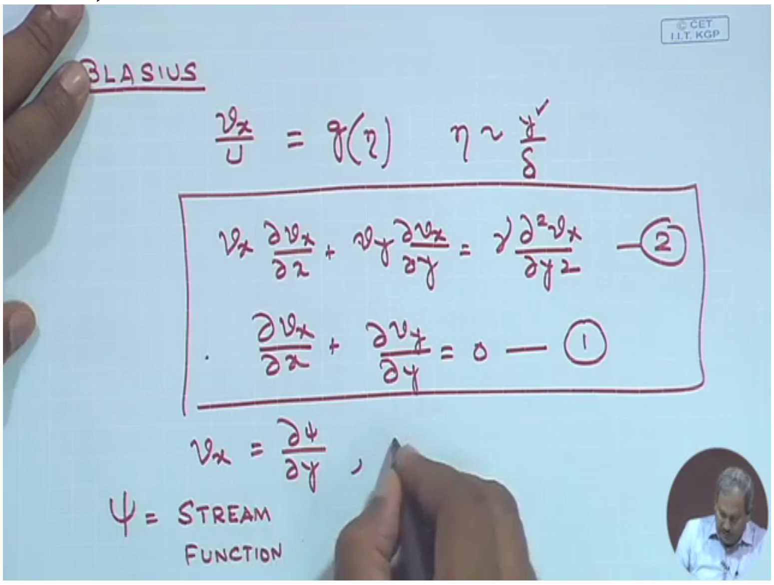

would be minus del chi by del x. So if you look at these two definitions of v x and v y and if you plug them into equation 1, what you see is it is going to be

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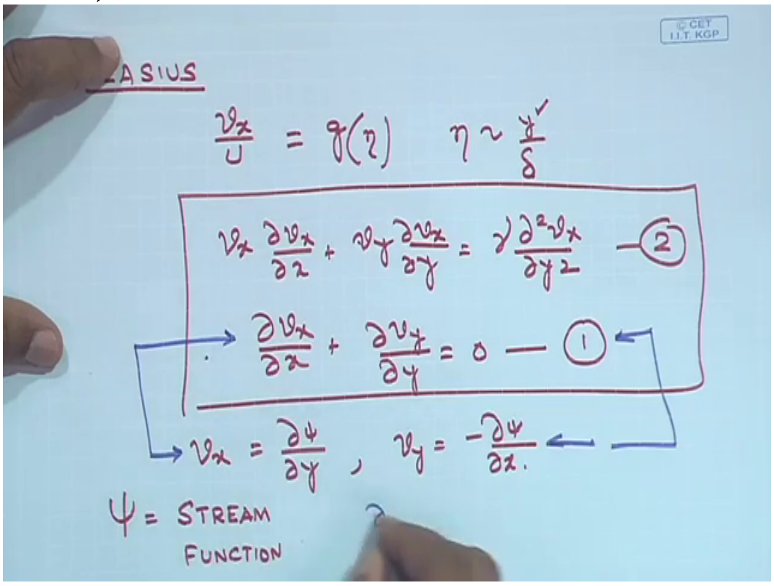

del chi del x del y and this one is going to be del chi del y del x. And chi being an exact differential, the order of the, order of the derivative is unimportant so therefore the result of this would be equal to zero,

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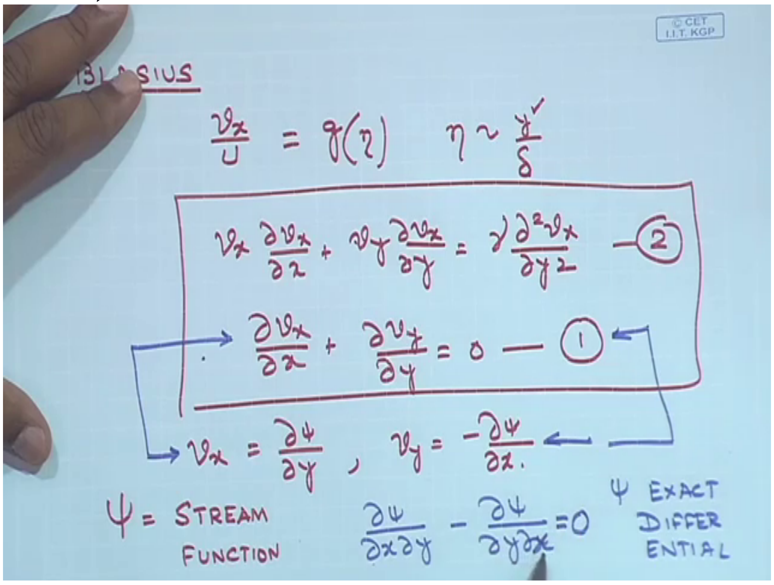

so since the order of differentiation is unimportant, irrelevant for the case of exact differential, so del chi by del x del y and del chi by del y del x are essentially equal and since we have a minus n here, the equation of continuity automatically gets satisfied the moment we start to express v x in terms of

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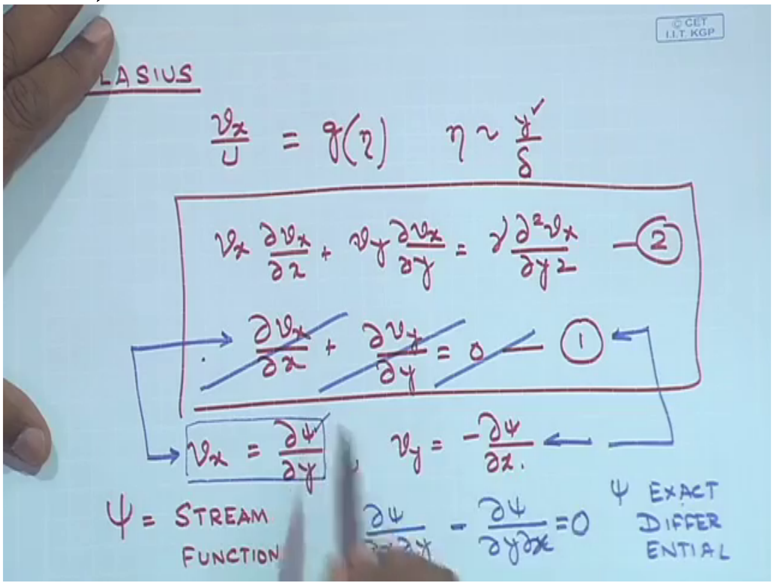

a stream function, v x and v y in terms of stream functions So now we do not have

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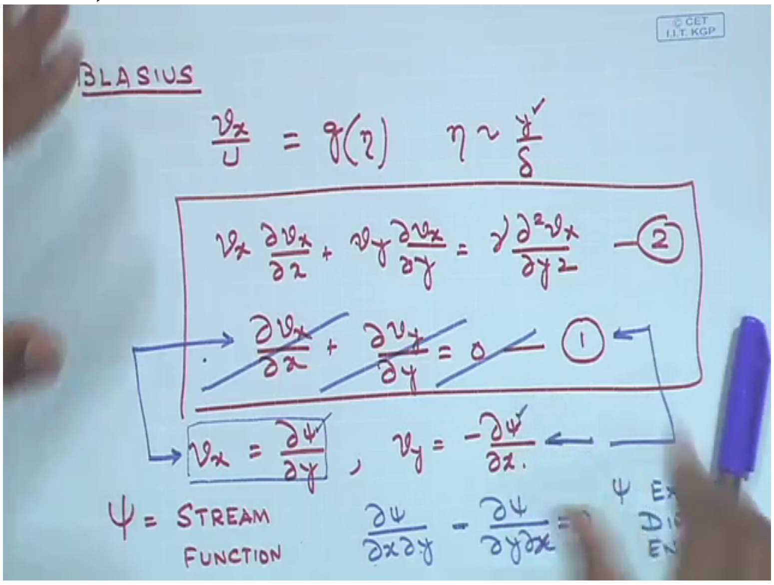

the equation of continuity any more It's automatically satisfied so we just have one equation

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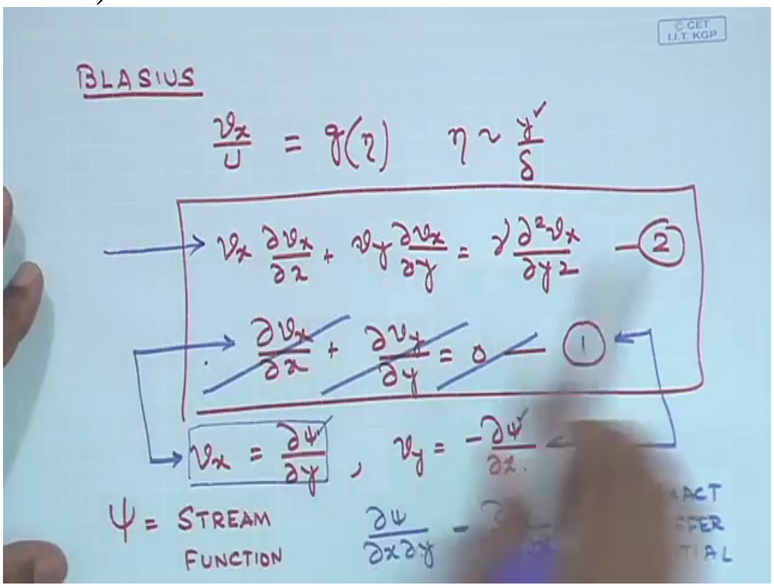

to deal with which is the

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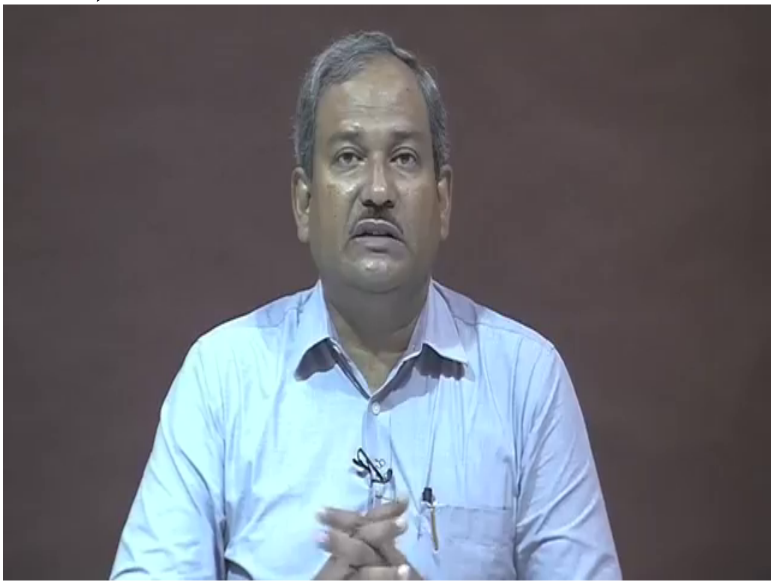

equation of continuity, which is the equation of motion So bringing in the concept of stream function allows me to substantially reduce the complexity of the problem. The next, it is still a partial differential equation, v x and v y are functions of both x and y. x and y are both independent variables. So is it possible somehow to club these two independent variables and define a new independent variable which is, which is a result of x and y in a certain form related to the new dimensionless variable and then express v x, v y and chi in terms of the new variable. I will repeat this once again. What we see here is we are dealing with the partial differential equation in the form of equation of motion. It would be even simpler if we can revert this partial differential equation to an ordinary differential equation.

And since v x and v y are functions of x and y, two independent variables, if I can club two independent variables in a specific way such that I end up with only one independent variable and v x and v y are now not functions of x and y but functions of the new independent variable. Now in that case, I then have, instead of a partial differential equation I have an ordinary differential equation where v x and v y are functions of only one variable which is some sort of the combination of the two independent variables x and y. So the method of converting a partial differential equation to an ordinary differential equation by combining the independent variables in a specific way is known as the method of combination of variables. So we will see how the method of combination of variables can give a solution to this specific problem. And we would first, our aim is to obtain an expression, a relation between x and y. So I start with this equation again and try to see what is going to happen if I do an order of magnitude analysis of this equation near the boundary layer.

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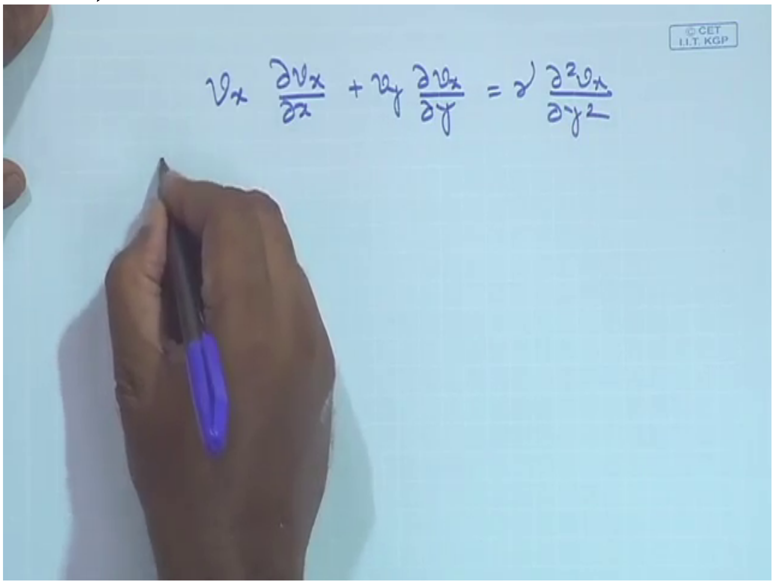

So what is going to happen near the boundary layer, I am going to make a judgment on that. Near the boundary layer, v x is approximately going to be equal to u, y is essentially equal to delta and since the velocity does not change, v x does not change beyond the boundary layer, del v x del x sorry del v x del y is going to be equal to zero. So if this is the boundary layer, v x does not vary with y beyond the boundary layer. So at the edge of the boundary layer the variation in axial component of velocity with y is approximately equal to zero. So if we, if we see this then what we, what we are going to do is

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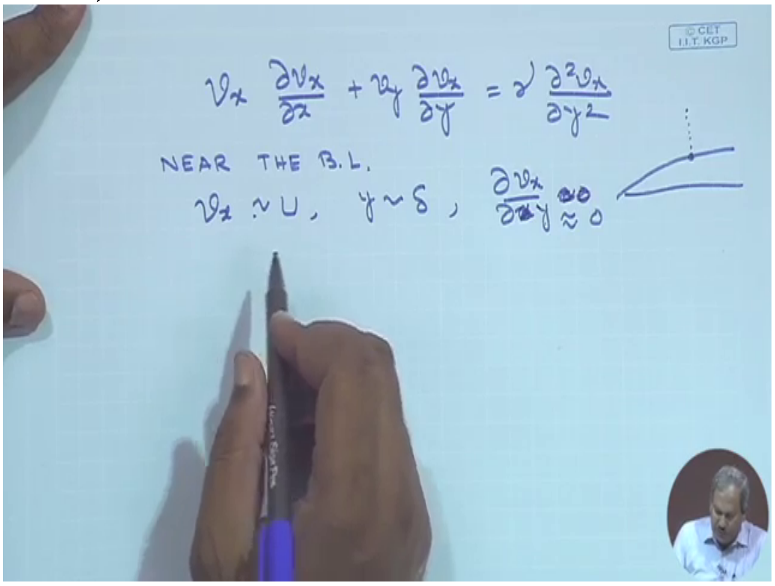

I am going to write this equation, the approximate form of this equation is as u instead of v x u by x which would be equal to this this part is going to be approximately equal to zero since I have written over here and this is going to be equals nu and then what I have u and the value of y is going to be equal to delta; so the relation that you get over here, the approximate relation that you are going to get based on an approximation,

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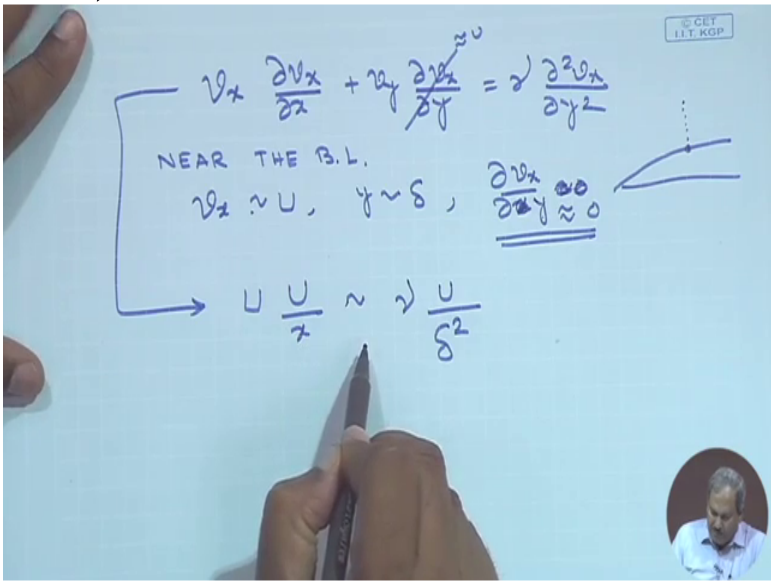

based on order of magnitude analysis, as scaling parameter what you would see is that delta square is going to be equal to nu times x by u. And your delta

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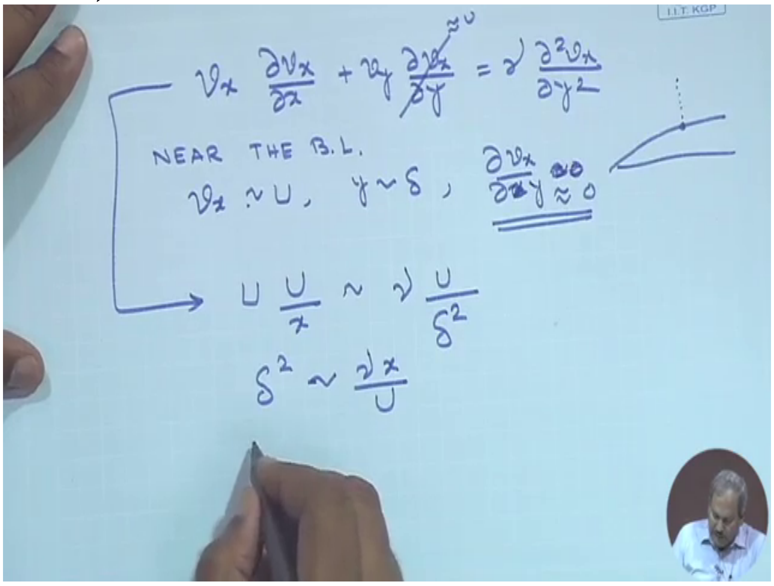

is simply going to be equal to or approximately equal to nu x by u; so if I define my new dimensionless variable eta to be y by delta as we have done before, this would simply be equal to y u by nu x. So this is the combination variable that

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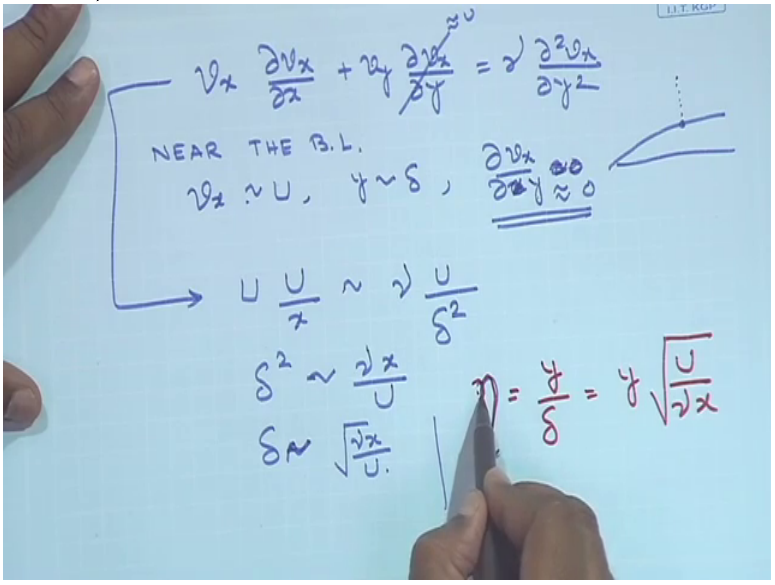

contains both y and x in a specific functional form and the functional form is essentially y by root over x and we have these two terms which come from an order of magnitude analysis of the equation near the boundary layer at the edge of the boundary layer. So approximately it gives me a form of delta in terms of x,

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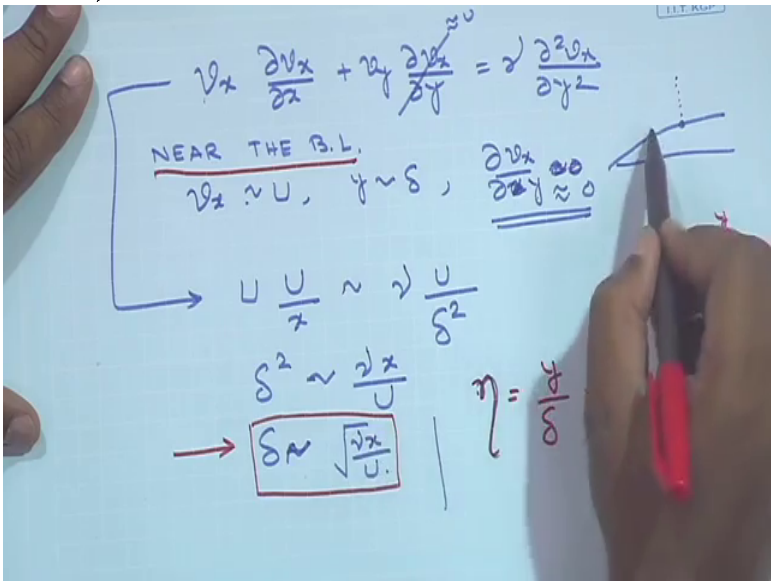

delta is the film thickness which is only a function of x; delta is the boundary layer thickness which is only a function of x. So it gives me delta as a function of x and since the dimensionless distance is defined as y by delta, so it is of the form of this only. So therefore my equation, the equation that I have over here, the equation of motion

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which is v x del v x del x plus v y del v x del y is equal to nu times del square v x by del y square. This can

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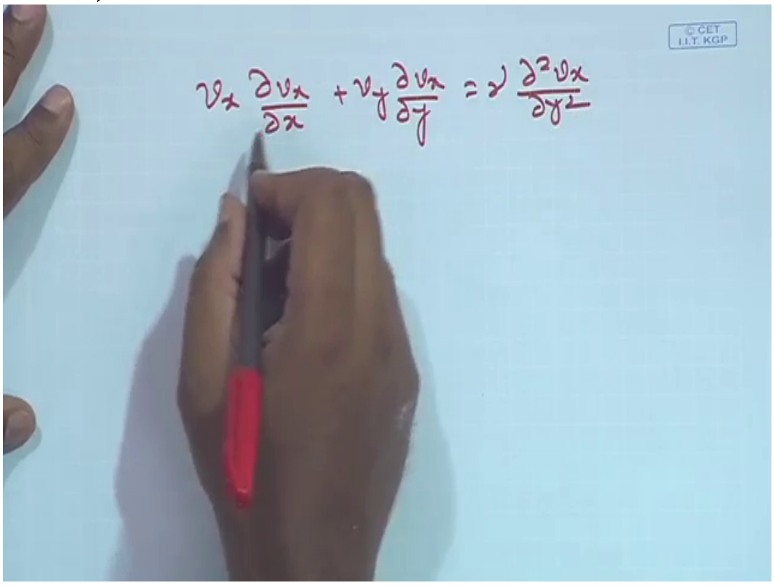

now be, now be handled first by introducing a stream function and second is by

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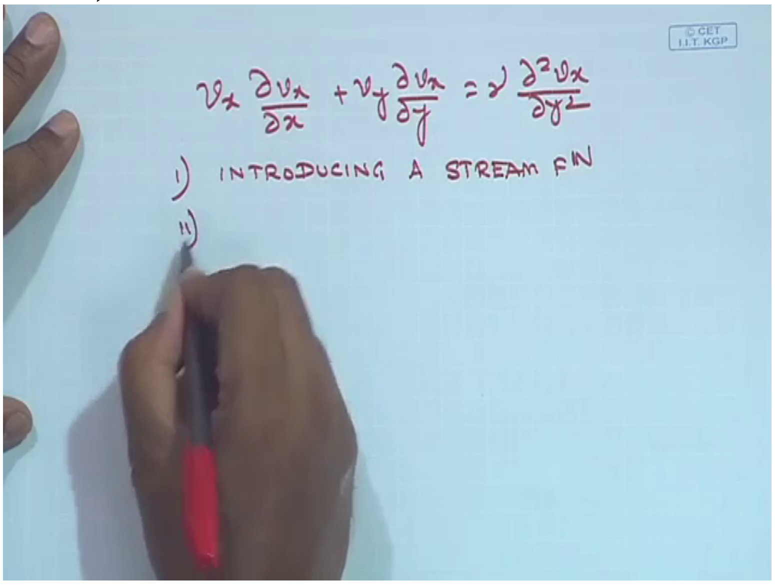

invoking the method of combination of variables And the stream function that we have defined is chi and the, combination variable that we have defined is eta equals y root over u by nu x. And since stream function, since I am going to make everything dimensionless,

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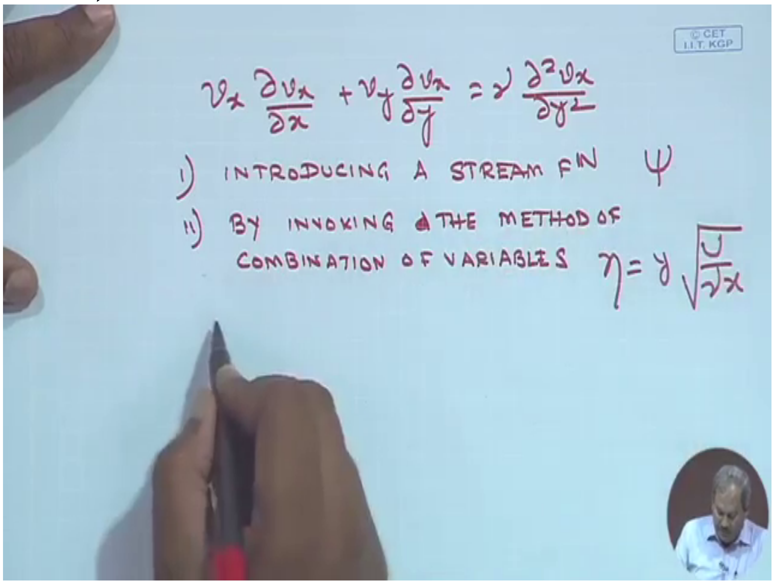

I am defining a dimensionless stream function which is f equals chi by root over nu x u. This dimensionless stream function is a function only of eta;

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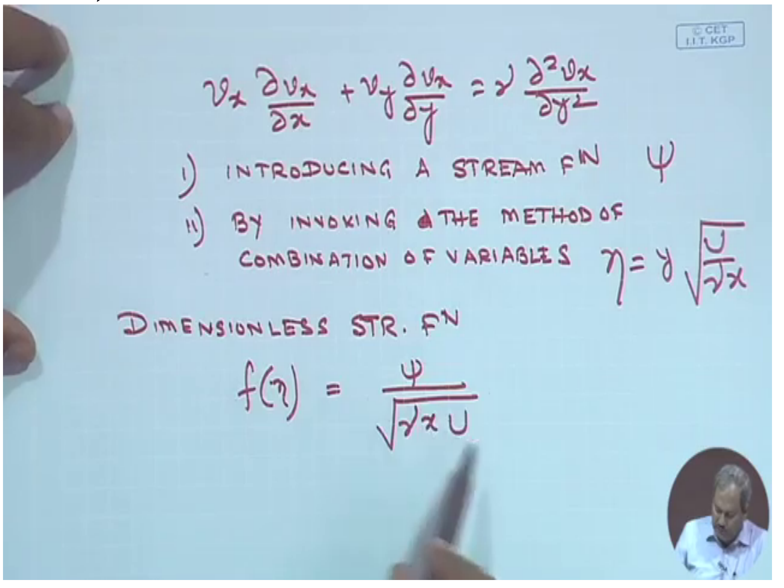

so next what is remaining is to convert the equation, the governing equation in terms of the stream function and in terms of the new independent variable eta; so instead of

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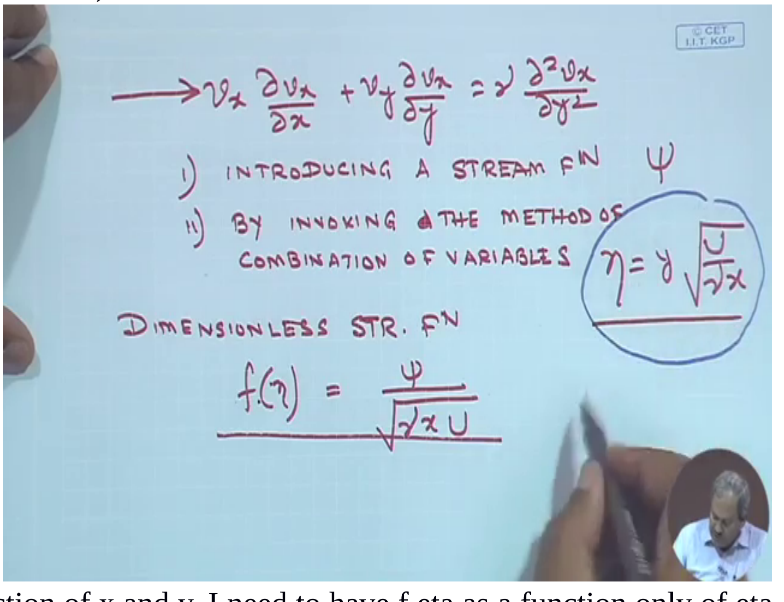

v x, v y as a function of x and y, I need to have f eta as a function only of eta. If I can do that, this gives rise to a p d e but since f eta is a function

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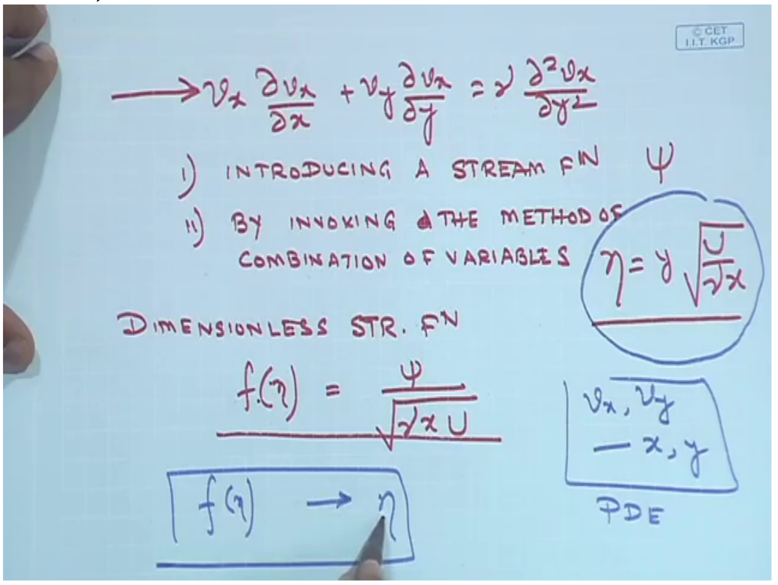

only of eta this will give rise to an o d e. So in the next class I am going to just see how this transformation

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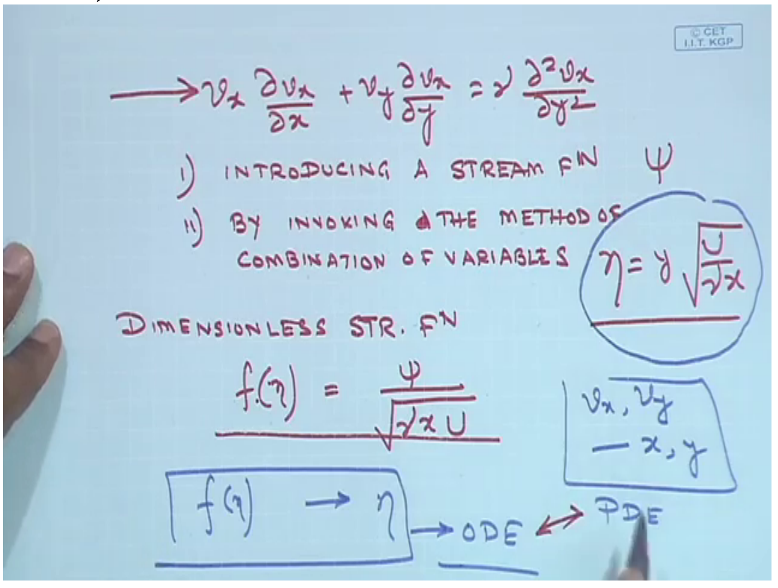

from p d e to o d e takes place and that would allow us to solve this

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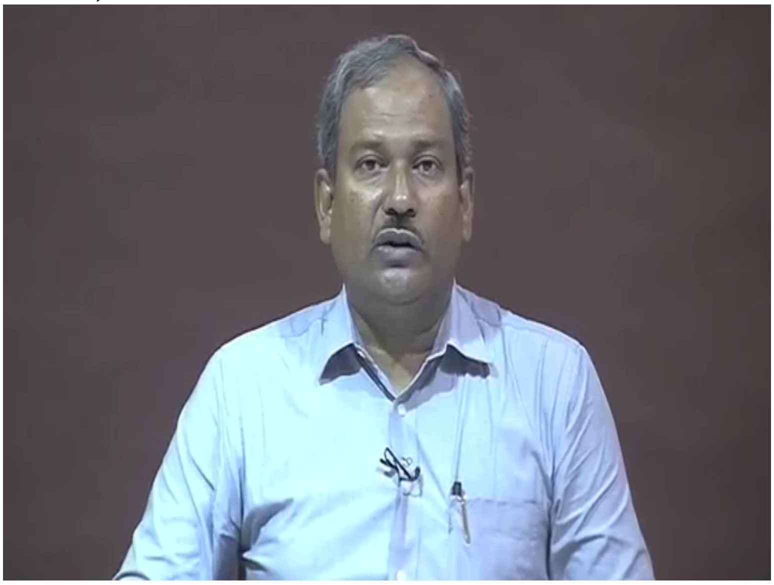

equation in a more meaningful way, in a more easy manner also eliciting fundamental information about what happens at the edge and on the solid surface for flow inside a boundary layer