Transport Phenomena
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Unsteady Flow
Lecture 15

So far we have seen and by now you must be comfortable with the concept of equation of motion. And how the right component of equation of motion can be used to derive the covering equation for flow in a channel or in a tube or in any other complicated geometry both in one dimension and in some special cases in two dimensions in which analytic solutions, solutions are possible and in all other cases one has to resolve to numerical solution techniques.

So before we find and leave this chapter behind I would quickly show you a result a specific keys which would have applications in a subsequent chapter that we are going to get into. This so far all the problems all the situations which we have dealt with are steady in their flow is steady in nature. That means the velocity at any given location does not change with time. But there will be many cases which in which the velocity can be a function of time. So if you fix the location the velocity at that location can also change. So I am going to give you just one example of unsteady state motion in which a compact closed form solution is possible. There are numerous other cases each involving more complicated mathematics or numerical techniques that would give you the values of velocity at a given location in unsteady flow.

So in order to analyze the simplest possible problem in unsteady one of the simple problem in a steady flow we think of the situation in which we have a plate and on top of the plate we have a layer a large layer of liquid, so initially the plate and the fluid above it are at rest, both are at rest so at t less than equal to 0, the velocity of the fluid is 0 everywhere, but at t equal to 0 suddenly the bottom plate adjoining the fluid is set into motion with a constant velocity v. So as time progresses the presence of the moving solid boundary will be felt at a greater depth of the liquid.

So initially when the plate starts to move only the layer very closed to that solid surface will sense that the solid plate has started to move. As time progresses the thickness of the effect of the motion of the solid plate will penetrate more and more into greater depths in this case greater heights of the liquid. So this would create a condition in which the flow is not only going to be a function of y which is distance from the plate but also it is going to be a function of time, how much time has elapsed before this the measurement of velocity at a given y has been done.

So it's the velocity in the x direction in the direction of motion of the plate is going to be function both of y and time. We realize that the plate velocity is not too large therefore the nominality of flow will still be maintained and once the plate starts to move it will try to drag the fluid with it because the viscosity to its viscous flow its a one dimensional flow in which the non vanishing component of velocity is V x in the x direction that is no vy and obviously there is no vz. And its just a full of liquid above a solid plate and all the motion is initiated because of the motion of the bottom plate therefore there is no pressure gradient and since the fluid and the solid plate they are horizontal so there is no effect of gravity as well.

So when you think of the process its a two dimensional its a problem its a unsteady state problem in which the velocity is the only the x component of velocity is present which is a function of y as well as a function of time. We are trying to see if what kind of a solution we can obtain for a situation like this.

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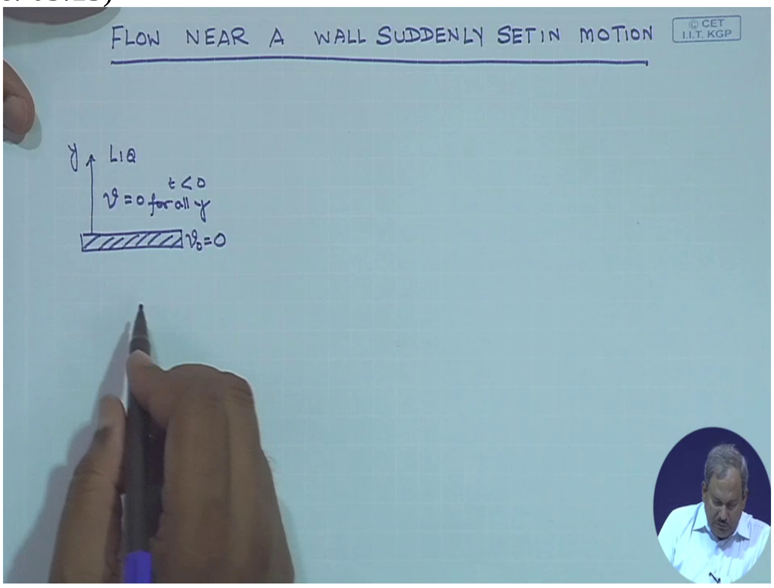

So I will draw the first scenario in which this being the solid plate and the time is less than 0 this is the y direction and I have a liquid in here. So till its in 0 there is no velocity that means velocity of the solid plate is equal to 0. So velocity of the liquid is 0 for all y, so at any point the velocity would be equal to 0.

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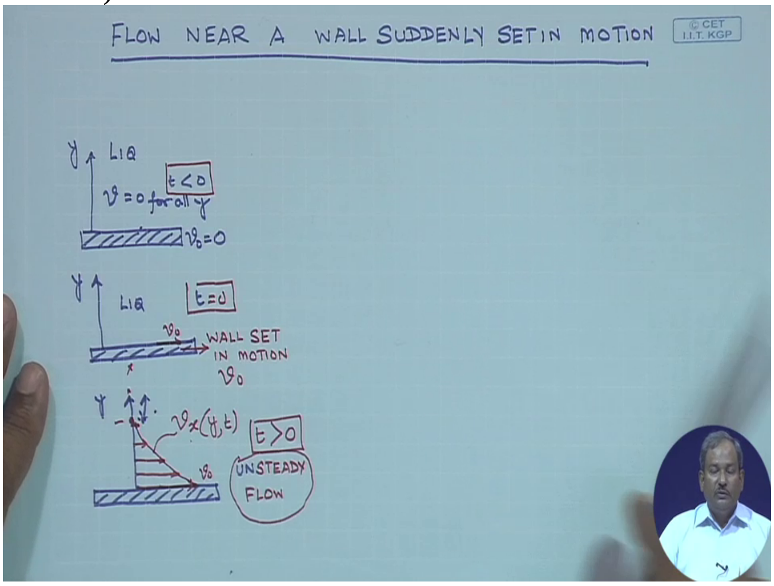

The second scenario is at t equal to 0 so this is t less than 0 at t equal to 0 b wall is set in motion with a velocity v 0 so I have now this is moving with some velocity and this is still my y and the velocity over here is V 0. Then comes the third part where again the same wall has some time has passed it is moving with velocity equal to V 0 in I have the same y in here and the layer so this is a case of a unsteady flow. The velocity of the plate is V 0 in what you are going to see in here is the velocity of a layer just above it will be slight less than V 0 over here is going to be lesser until the point has come, so this is the point where the velocity is still 0.

So I have the velocity profile something like this, so this is my velocity profile which is a function of y as well as its a function of t. So this is t greater than 0. So if you look at the three situations that I have drawn over here this is the initial condition where the velocity is 0 and therefore the velocity of the adjoins liquid is also 0. And t equal to 0 the wall is set in motion with a constant velocity v 0 the liquid is still stationary. So we have the liquid which is still stationary but as time progresses t greater than 0 and unsteady flow pattern sits in to the adjoining liquid with the velocity being maximum due to no stick condition on the velocity at this point at this solid liquid interface is going to be equal to V 0 but as we move deeper and deeper into the fluid the velocity decreases and at some distance in some point the velocity will come to 0 so that means from this point onwards the velocity of the fluid is 0 and the fluid here does not know that a moving plate exists at some distance from it.

Now you can clearly see that as time progresses when t becomes quite large the penetration depth of the effect of the plate will keep on increasing and the velocity profile will also be different. So the velocity profile at the penetration depth of the effect of the solid plate of the motion of the solid plate will be felt at a higher distance as time progresses. So this is definitely an unsteady flow problem and you would like to see if we can get an analytic solution for other close form solution for this specific case.

So what we are going to do first is we are going first going to write the governing equation for this specific case. Now when we write the governing equation we keep in mind that its only one component of velocity V x, V y is 0 V z is equal to 0, and v x does not vary in the x direction so if you think about the (())(10:03)equation the Cartesian coordinate system in the x direction because x is the direction in which the flow takes place the first term is del V x del t. Now previously in all our previous cases since we were dealing with steady state del V x delt was set equal to 0.

There is a Rho outside so Rho times del V x del t, that term cannot be neglected in here. All the other terms either contain V y or V z or it contains del V x del x so the second term on the left hand side of (())(10:47) equation which is V x del V x del x is 0 since del V x del x is 0 the third and the fourth term which contain vy and v z since both are equal to 0 the entire left hand side only term which will remain is the unsteady state term which is Rho times del V x del t. Now we come to the right hand side of (())(11:14) equation. The right hand side of the (())(11:17) equation the first term is the gradient in the safest force and principally we are talking about the applied pressure. And since its a case of applied which is moving in a static liquid with no applied pressure in the direction of flow.

So del p del x the first term on the right hand side of (())(11:40) equation would be 0. Then comes the viscous terms the viscous terms and viscosity Mu times del square V x by del x square which is 0 since V x does not vary with x del square V x by del y square which denotes the transport of molecular momentum viscous momentum in a direction perpendicular to the motion in the x direction that means in the y direction del square V x by del y square that term must be present in the governing equation. Because that is the term which acconts for the motion of the plates above motion of the layers of the fluid above the plate so entire flow is caused by the viscous interaction of the layers one top of the other. So Mu times del square V x by del y square should remain in the governing equation.

The third term would be Mu times del square V x by del z square and since V x does not very with z that term can also be dropped. The last remaining term of (())(12:52) equation in this case will be Rho times gx where g suffix x denotes the gravity component in the x direction which this system being a horizontal system would be equal to 0. So we enter right hand side of (())(13:12) the pressure term is 0 the body force term is 0 one term only one term of the viscous transport will survive which is Mu times del square V x by del y square.

So your enter governing equation now becomes Rho times del V x del t is equal to Mu times del square V x by del y square. So that is our governing equation and now since you know how to use (())(13:40) its quite easy to do that simply by looking and cancelling the terms that are not relevant in the present scenario.

So I am going to write the governing equation for this case.

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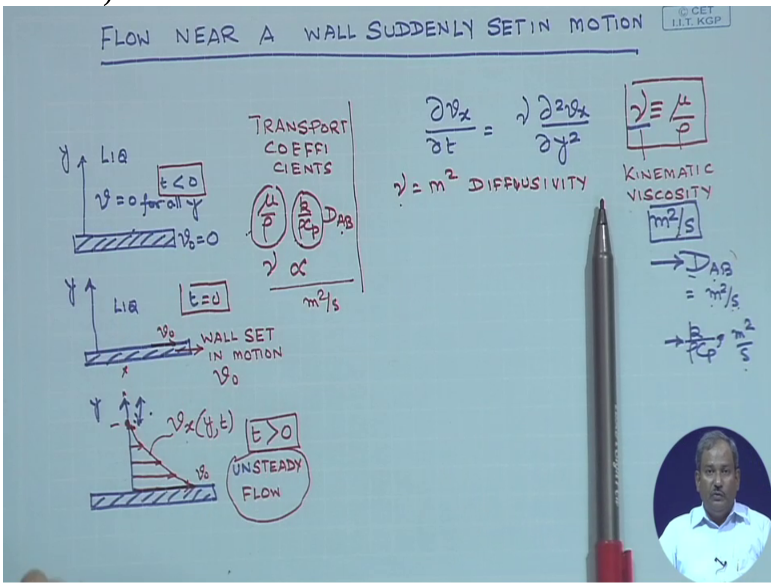

So the governing equation for this case would simply be del V x by del t is equal to Mu times del square V x by del y square where this Mu is defined as Mu by Rho. This is called the kinematic viscosity. Now if you look at the limits of the kinematic viscosity Mu you would see that it is going to be in meter square per second. This meter square per second has some significance because you would see later on that all the transport coefficients for example when we talk about fix law the transport coefficient there was DAB which is a deficient coefficient which has also meter square units of meter square per second.

So in mass transfer the transport coefficient has units of meter per square per second and when you look into the heat transfer I will introduce that later but there is a term k by Rho cp which also has units of meter square per second. So this is called the mass diffusivity this is called the momentum diffusivity and this is called the thermal diffusivity.

So this is this Mu is essentially the momentum diffusivity. So the momentum diffusivity Mu it is units of meter square per second similar exactly similar to the mass diffusivity which is denoted by DAB as units of meter square per second and thermal diffusivity which has units of meter square per second.

So when we talk about transport coefficients when we discuss about the term transport coefficients we do not talk only about viscosity but it is rather viscosity by Rho also denoted by Mu or k by Rho cp the thermal diffusivity generally denoted by alpha and DAB which is all of them has units meter per second. So this is kinematic viscosity this is thermal diffusivity, kinematic viscosity or momentum diffusivity, thermal diffusivity and mass diffusivity. So when we talk about transport coefficients which are important in disturbing the flow of momentum the flow of conductive flow of heat or the conductive flow of species from one point to another due to the presence of a concentration gradient we talk in terms of diffusivity the momentum diffusivity and the thermal diffusivity not just Mu it is Mu by Rho which is important not just k the thermal conductivity its k by Rho cp which is important and which dictates what is going to be the rate at which momentum gets transported from one point to the other the heat gets transported or the species mass gets transported from one to the other.

So when you talk about transport coefficients transport parameters defining the flow of momentum heat or mass, we generally talk in terms of diffusivity. All of which will have units of meter per second.

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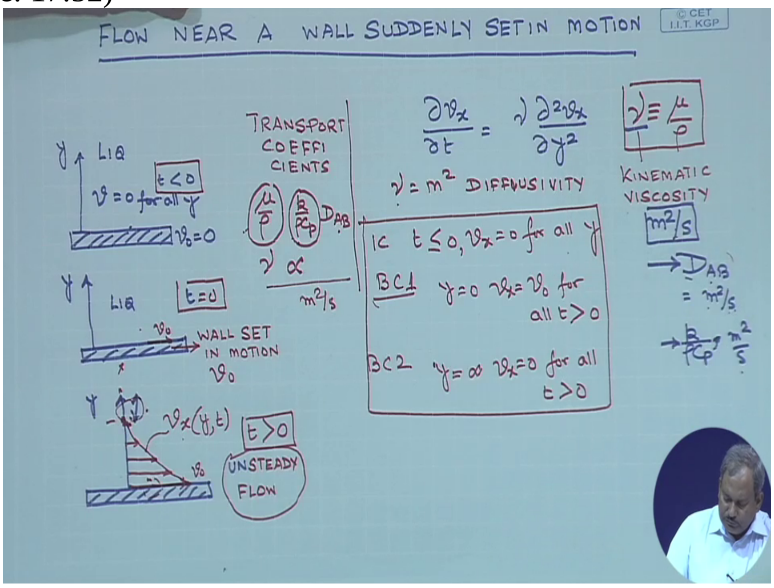

But any way coming back to the problem this specific equation has an initial condition which says that t less than equal to 0 V x is 0 for all y, so this is essentially the first figure which I have drawn that is t less than equal to first and second figure so this t less than equal to 0 V x is 0 for all y. And the boundary condition 1 is at y equal to 0 V x is equal to V 0 for all t greater than 0. That means if y equal to 0 the V x will always be equal to V 0 or the velocity of the top plate due to no strip condition for any t greater than 0. And the second boundary condition, so the boundary condition 1 refers to this region and the boundary condition 2 refers to this region which says that as mathematically speaking as y tends to infinity then V x is equal to 0 for all t greater than 0, so these three are the initial and the boundary conditions which we must use to solve this PDE which are a statement of the our physical understanding of the system.

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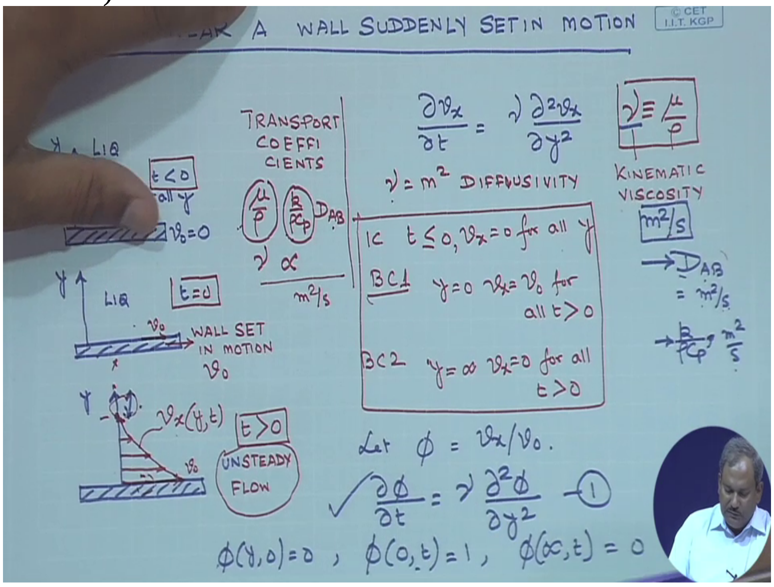

So first thing we will do is we will denote a dimensionous velocity as V x by V 0 therefore the governing equation would simply be Del Shi by Del t is equal to Mu times Del square shi by Del y square and we call this as our equation 1. And the reduced the changed boundary conditions in this case will be Phi(y,0) time would be 0 since V x is 0 that is one condition Phi (y) equal to 0 for any value of t is equal to 1 due to no slip condition on the solid liquid and that Phi for y equals infinity at any time greater than 0 would be 0 that means the effects of the motion of the solid plate has not penetrated beyond this distance and since it is infinite distance no matter whatever be the time the fluid over here at an infinite very large distance from the plate will never know that the plate has started to move. So these are the three conditions that we need to use to solve the governing equation.

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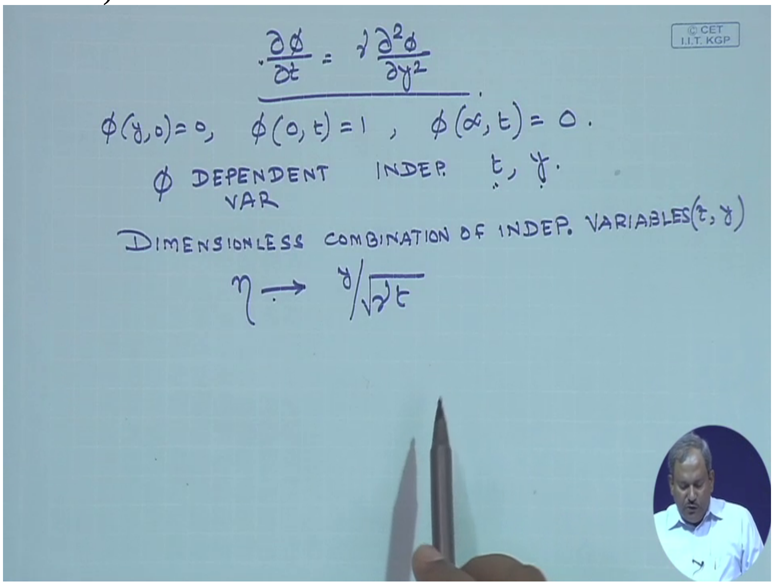

Now how can we solve that, that is the next question. I will quickly write the equation once again just for our reference which is del shi by del t is equal to Mu times del square shi by del y square that is the governing equation and the boundary conditions are y at times 0 is 0 Phi at y equal to 0 at any time greater than 0 is equal to 1 and Phi at infinite time infinite distance any time greater than 0 is going to be equal to 0.Here we see that the independent variable is Phi sorry dependent variable is Phi. And the corresponding independent variables are t and y so if we could express some function of t and y in such a way that that new defined variable the combination variable of t and y can express can be substituted in here obtain an expression only in terms of Phi and the new variable which is the some sort of a combination between t and y.

So I would show you what it is? How it can be done? This Phi is dependent variable and independent two variables t and y So I am trying to find out a dimension less combination of independent variables and the independent variables are t and y I use eta which is the new dimensionless variable this is y by root over Mu times t. So what we see here is that my partial differential equation I need to resolve the partial differential equation by an ordinary differential equation. What I see is that my dependent variable is a dimensionless quantity which is Phi which is simply the velocity at any y and at any t divided by V 0 where V 0 is a constant is a velocity constant velocity of the moving plate.

The dependent variable Phi there are two independent variables one is what is the distance of the point of concentration from the solid plate that is y and the second independent variable is how much time has elapsed after the plate has started to move. So y and t is these two independent variables can be combined in a specific way yet to be determined way so that two independent variables are merged or combined into one variable and that independent variable the new independent variable in itself is dimensionless in my Phi is not a function of y and t, Phi is a function only of that combination variable, so instead of two independent variables instead of the dependence of Phi on two independent variables which makes the equation a partial differential equation what I am proposing is I am merging these two combining these two into one variable.

So therefore Phi is now a function of only one variable. So when Phi is a function of only one variable the equation no longer remains a partial differential equation it will become an ordinary differential equation. So that is what is our goal to define a new independent variable in such a way that the combination variable becomes dimensionless and the combination variable will be it is a substituted in the main governing equation all those y and t they will disappear they will cancel from each sides what that equation ideally should have finally is Phi and the new combined variables. If that happens then I do not have a partial differential equation I just have an ordinary differential equation so this method of resolving partial differential equations into ordinary differential equations is known as the combination of variables.

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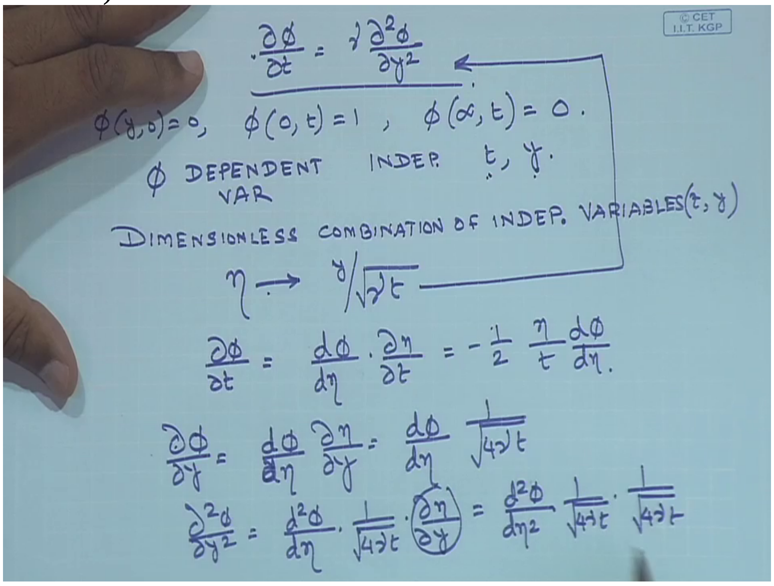

So the new combination variable is defined as Eta equals y by root over Mu times t in what you get then is I am going to substitute this in here so my del Shi by del t is I can express it as d Shi by d Eta times del Eta by del t which del Eta by del t would be simply be equal to minus half Eta by t dPhi by d Eta.

Similarly Del Shi by del y would simply be equals d shi by d Eta times del Eta by Del y which would be equal to d Shi by d eta times 1 by root over 4 Mu t and del 2 Shi by Del y square would be equal to d 2 shi by d eta times 1 by root over 4 Mu t times del Shi by del y so this would be d 2 shi by d Eta square times 1 by root over 4 t and again another 1 by root over 4 Mu t and again another 1 by root over 4 Mu t because of this.

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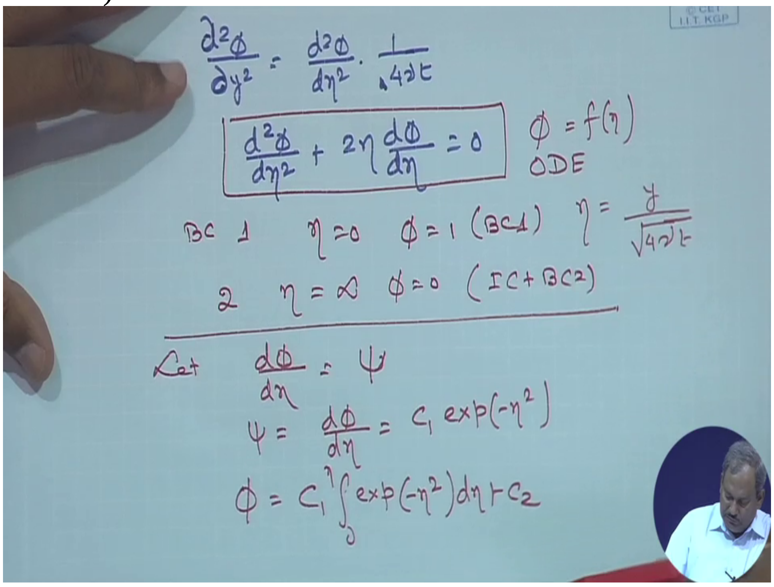

So ultimately what you get is d 2 shi by del 2 shi by del y square would be equal to d 2 shi by d Eta square times 1 by 4 Mu t. When all of these substituted into my previous equation, equation 1; the equation 1 becomes this. And the important point to note here is that my Phi is a function of only Eta so its an ordinary differential equation its no longer a partial differential equation. And the two boundary conditions are at Eta equals 0remember Eta is defined as y by root over 4 Mu t so Eta equals 0 Phi is equal to 1 and the boundary condition 2 is at Eta equals infinity Phi is equal to 0.

So this is the initial condition plus the boundary condition two of my previous discussion and this is simply boundary condition 1. So this now becomes easy to integrate first you have to just define that d shi by d eta is equal to some other d phi by d eta is some other function which is going to be the integration factor and when you do this integration factor then this is simply going to be d shi by d eta which is equal to C 1 exponential (minus eta square) which is essentially plus 12 calculus and once you substitute once you integrate it once you get from 0 to infinity exponential minus eta square d eta plus c 2. 4So C 1 and C 2 are the constants of integration.

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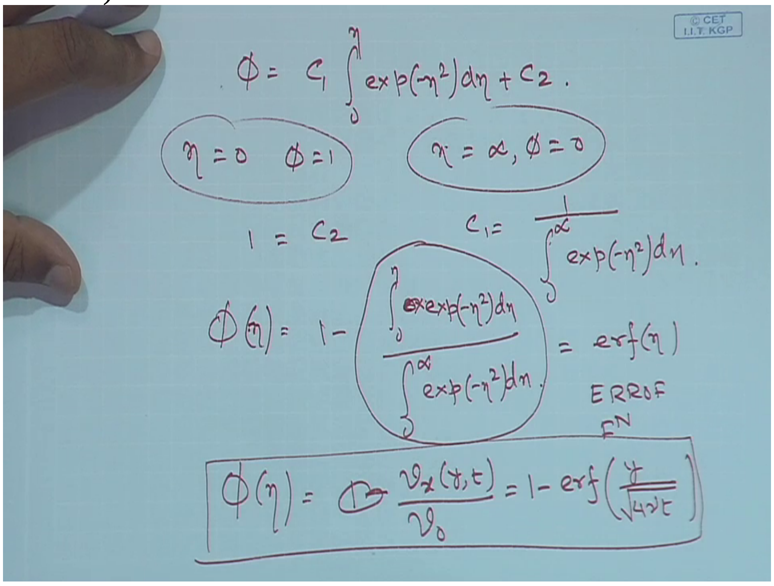

So once again Phi is going to be C 1 from 0 to Eta exponential (minus eta square) d eta plus C 2 and the boundary conditions at Eta equal to 0 Phi equals to 1 and Eta equals infinity and Phi equals to 0. These are the two conditions. So let us first use this condition and then what you get is 1 is equal to c 2 and then when you use this equation you are simply going to get c 1 as 1 by from to infinity exponential (minus eta square) d Eta.

So your final equation which is a function of Eta is simply going to be 0 to Eta exponential (minus eta square) d Eta by 0 to infinity exponential (minus eta square) d Eta. This part has a special name this is called the error function, so this is called the error function and your Phi Eta is simply going to be 1 minus which is V x(y and t) divided by V 0 is 1 minus error function Eta where Eta is y by root over 4 Mu t. So this is a complete expression in terms of a known mathematical function which is error function and this one more property of error function which I must say before I conclude this.

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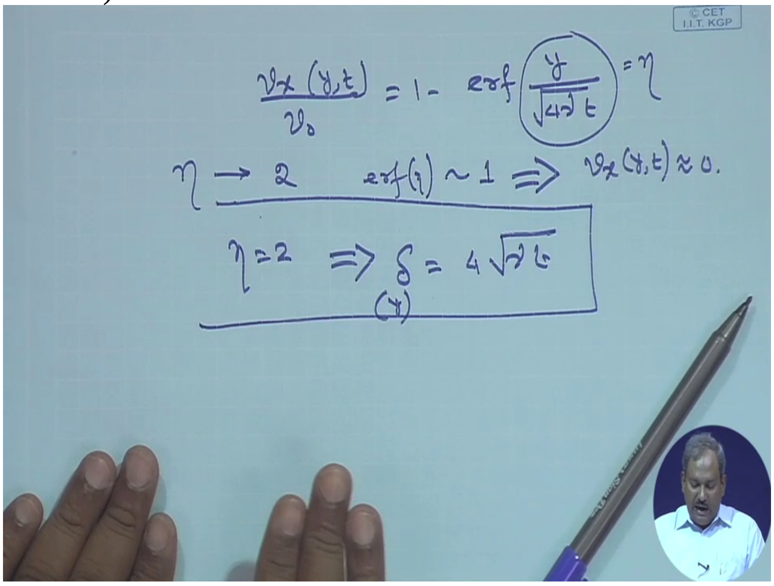

Again this is V x(y,t) by V 0 is 1 minus error function y by root over 4 Mu t. The property of error function is when Eta tends to 2 this entire thing is Eta. The error function Eta tends to a value equal to 1. So you would see that this would give rise to a value of V x (y,t) to be approximately equal to 0. So this value of Eta equals to which would give you delta equals 4 Mu t, delta is essentially y this would be a natural length scale to define what is going to be the distance at which the effect of the plate can be felt by the fluid. So we would get an error function solution in this specific case and the behaviour of error function is when Eta is close to 2 then the value of error function is equal to 1 and the velocity becomes equal to 0. So its a combination of time and distance which would give you an idea of the penetration depth of the effect of the motion of the plate below it and beyond that point there is no effect of the motion of the bottom plate. So this is essentially giving you an idea of the effect at a specific location and at a specific point of time.

So this is just one example of use the solution or the treatment of an unsteady state problem when the velocity is an function of one variable one spaced variable and time as well. But it is also giving us some idea that there exists a large region of the fluid where the motion is not felt at all. So this is what is giving us the concept of something which is known as boundary layer which I will introduce in the next class.