Transport Phenomena
Prof. Sunando Dasgupta
Department of Chemical Engineering
Indian Institute of Technology, Kharagpur
Lecture Number 14
Equations of Change for Isothermal Systems (Cont.)

(Refer Slide Time: 00:16)

We would look at one more problem involving Navier-Stokes Equation and this would probably be the last problem to deal with in this part, in this specific part of the course and we would move on to something different. However I would supply you with list of, number of problems with answers which you can try on your own and if there are, there are any questions I would be glad to answer those queries. So far what we have done, what we have seen is that the velocity we were dealing with 1 D velocity that is the velocity is the function it is a one dimensional velocity case. But there can be situations, many of the situations that we, that have come into our every day experience, the velocity can vary with more than one dimension.

It could be a function of x and y, r and theta and so on. So in, in those cases you would see that it is still better to start with Navier-Stokes Equation, it's impossible to use the shell momentum balance in such cases so we start with Navier-Stokes Equation and you get the governing equation and identify what would be the boundary conditions. In some of the cases it is possible to use certain simplifying assumptions. So those assumptions must be mentioned clearly in order to obtain a closed form solution for, for velocity. In these assumptions sometimes can be used as the asymptotic solutions of the governing equations under certain special conditions which would tell us something about the physics of the process and it could be extremely helpful in many such situations.

So what we are going to do in the last problem in this series, we are going to look at the flow between two parallel plates where the fluid enters through one of the plates at the center and then distributes, distributes itself at the, in the intervening space between the two, two disks. So we have two disks, circular disks one on top of other with a certain separation in between and through the top disk at the center, through a hole liquid enters the space in between the two disks and then they start radially flow, then they start flowing, radially flowing in the direction the way my fingers are pointing. So it's a radially outward flow in between two disks once they enter from the top at the center.

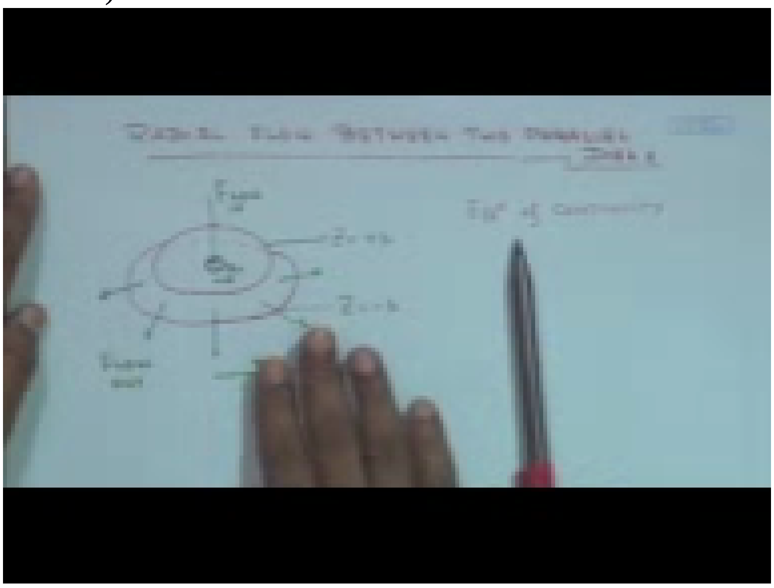
And you would, there is a, there is a pressure gradient and since these two disks are horizontal there is no effective body force to speak about, its only the pressure gradient which drives the fluid radially outward. And you, it is also clear that the no-slip conditions on the top plate, and on the bottom plate must be, must be uh adhered to so therefore the velocity is going to be zero at the, at this point and zero on the bottom plate as well. So the principle motion as it's clear from the figure that I have drawn over here, so we are looking at the radial flow between parallel disks and two disks that I have drawn, they have the, the top one has a hole at the, at the center and then in between, in between the two disks the fluid starts to move outside. So it's definitely a cylindrical coordinate system. But in this cylindrical coordinate system there can be three components of velocity, v r which is in the radial direction, v theta which is in the theta direction or v z.

So if you look carefully for the, at the, at the space in between the two, two disks it is only v r, the velocity in the radial direction which exist. All other components of velocity namely v z and v theta are zero. However there is an assumption involved in this. Think about the hole at the top plate. The liquid enters through that hole, comes the space in between the two, changes its direction and then starts to flow radially outward. So there is a region very close to the inlet where the flow is, where the flow changes its direction, where the flow can be a function of, function of z as well where it could be a function of r and so on. So in our analysis we are not taking into account the region which is very close to the inlet.

So all our analysis is valid from this point, that means from the hole to the outside, outside of the disk and not right under the hole where the flow, situation is extremely complex. So now you, you see that in, in the, in the space between two disks you only have velocity in the r direction. Now this velocity in the r direction is the function of z where you are with respect to each of these plates and as the fluid moves outward the area available for flow keeps on increasing because the area available for flow is simply twice pi small r times h if h is the distance between the two disks.

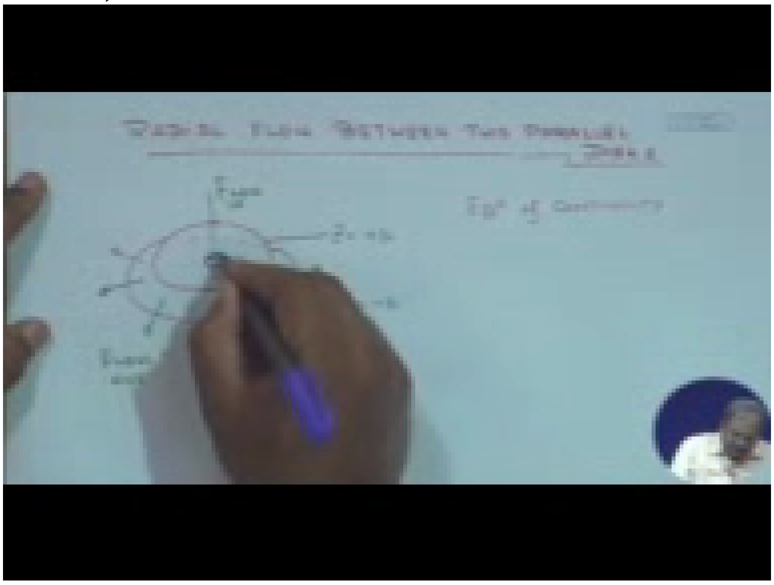
So as the flow goes to higher and higher radius the flow area keeps on, the flow area which is perpendicular to the flow direction which is twice pi r times h that also keeps on increasing. So as the flow area increases the velocity must decrease in order to satisfy equation of continuity. So v r is not only a function of z, it's also a function of r. This has to be kept in mind. So we are dealing with a velocity which is a function of r and as well which is a function of z; whenever we come across such a problem we first try to see if use of equation of continuity can somehow simplify the situation. So that's what we are going to do first. We are going to use the equation of continuity

(Refer Slide Time: 07:19)



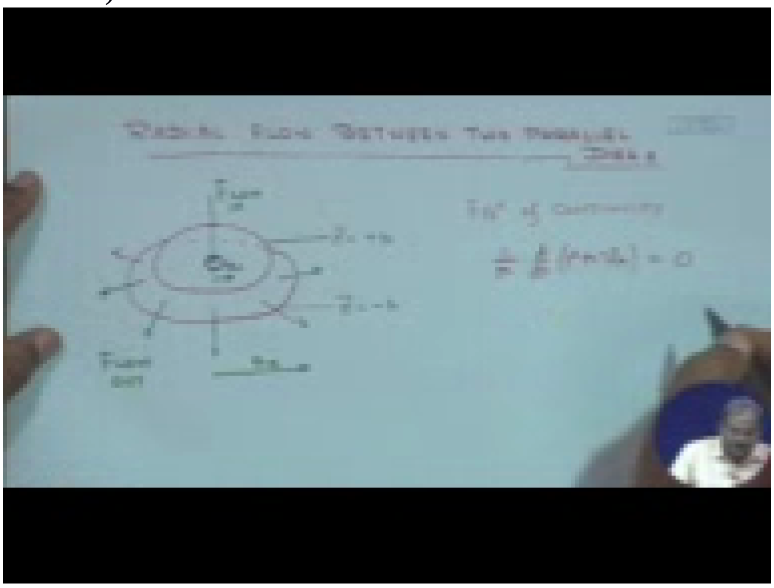
and this; we are trying to see if uh, a simplification is possible. So as I was saying, the top disk, this and bottom disk which have separation of twice D between them and the flow comes in through a hole whose radius is equal to r 1, the radii of both the disks, they are equal to r 2 and the flow is moving radially outward. As the flow comes towards the outer edge the cross-sectional area increases so the velocity reduces and we are not going to deal with the region below

(Refer Slide Time: 08:00)



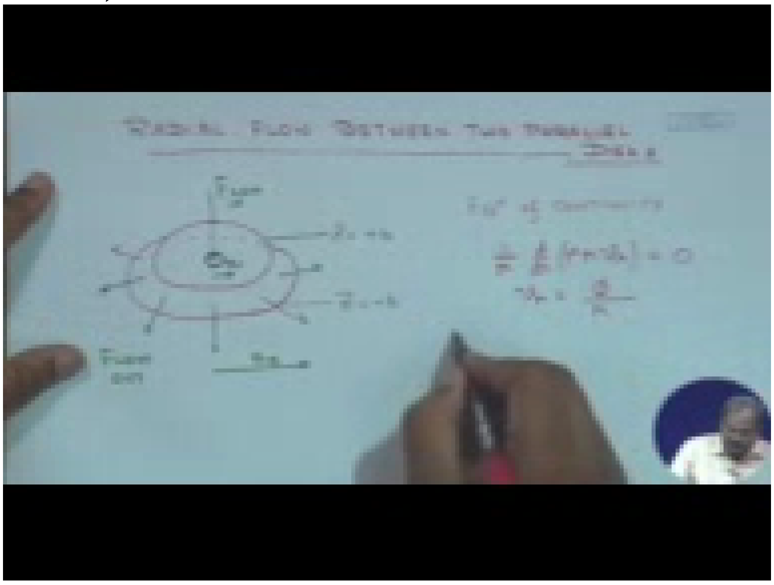
r 1 because we understand that the flow situation over there is extremely complicated. So as I said the first thing to do is to use the equation of continuity and from the equation of continuity, try to see if any compact formation is possible. So the r component of equation of continuity in cylindrical coordinate system would simply be rho r v r where v r is the r component of velocity.

(Refer Slide Time: 08:33)



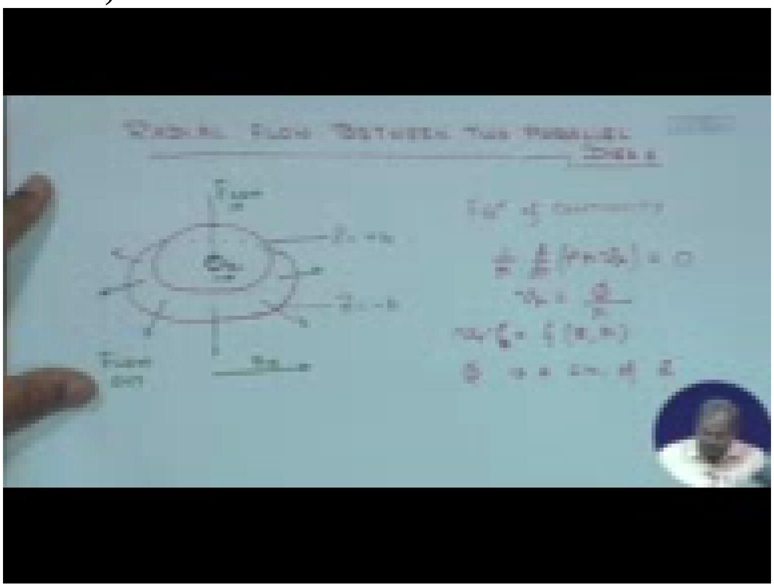
And since v theta and v z are zero this is the only non-zero term in the equation. And therefore v r would be some phi by r. Now we realize that

(Refer Slide Time: 08:48)



velocity in the r direction as we have, as I have mentioned before is going to be, is a function of z and r. Now since d d r of r v r is a constant, or d d r of phi is equal to zero, d d r of phi is equal to zero then from this relation it is clear that phi is a function of z.

(Refer Slide Time: 09:17)

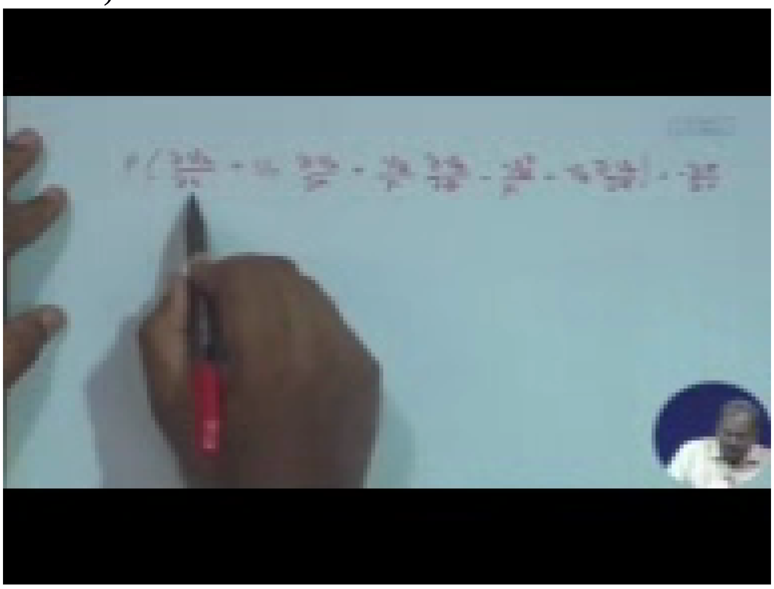


Once again, d d r of r v r is zero so if I bring r on this side, d d r of r v r is equal to d d r of phi which is equal to zero. So phi is not a function of r. However since v r is a function of z and r, phi has to be a function of z. So from this relation, from this relation we understand that phi cannot be a function of r but since v r i is a function of z and r so phi has to be a function of z. So that's the first thing that we, we can obtain and we have an approximate, we have an expression for v r just by looking at the equation of continuity where the only thing we need to do is try to see what, how can we evaluate phi? Now which component of Navier-Stokes Equation that we are going to find, that we are going to do next.

(Refer Slide Time: 10:19)

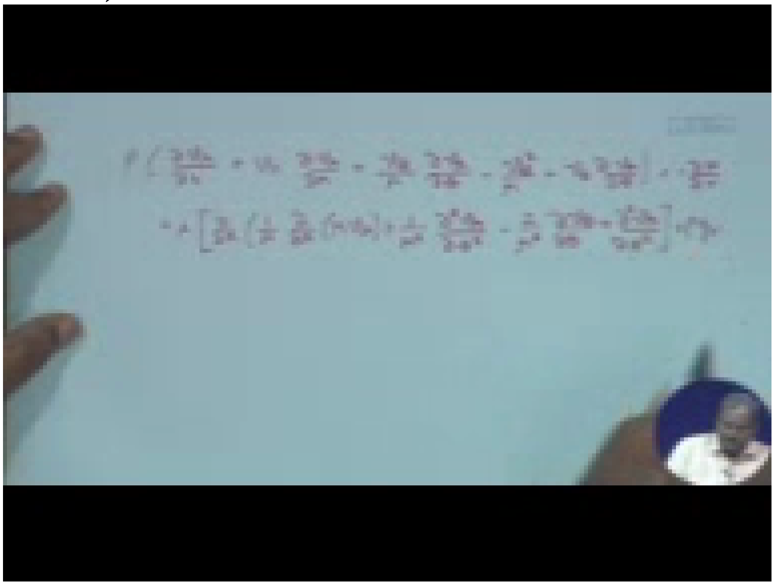
Now it's a cylindrical coordinate system v theta is zero v z is zero so we must start with the r component of Navier-Stokes Equation in cylindrical coordinate system that will describe the flow in two parallel disks when they are separated by a distance twice b and when there is a pressure gradient forcing the liquid to move radially outward; so we start with the r component of the Navier-Stokes Equation. So I will write the r component which would be, so this is the entire left hand side, so you can again uh

(Refer Slide Time: 11:36)



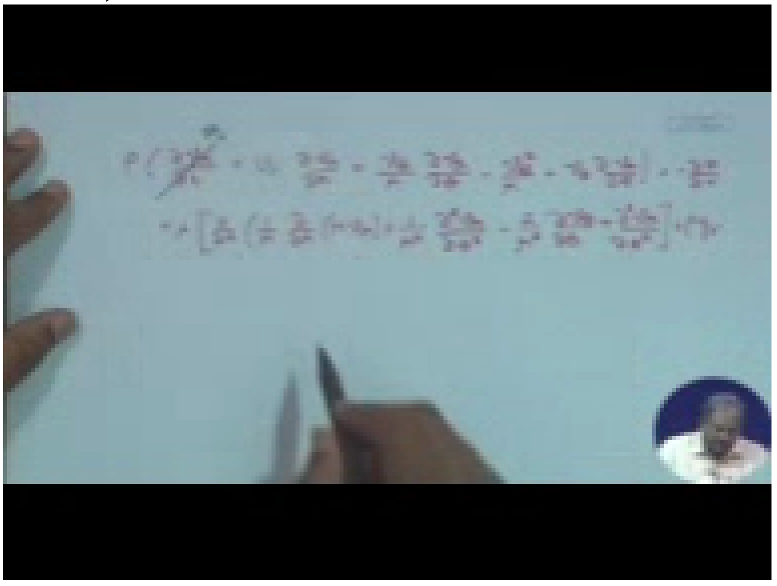
identify that this is the special term and all these four terms, since they have velocity explicitly present in them, they refer to the convective transport of momentum.

(Refer Slide Time: 12:25)



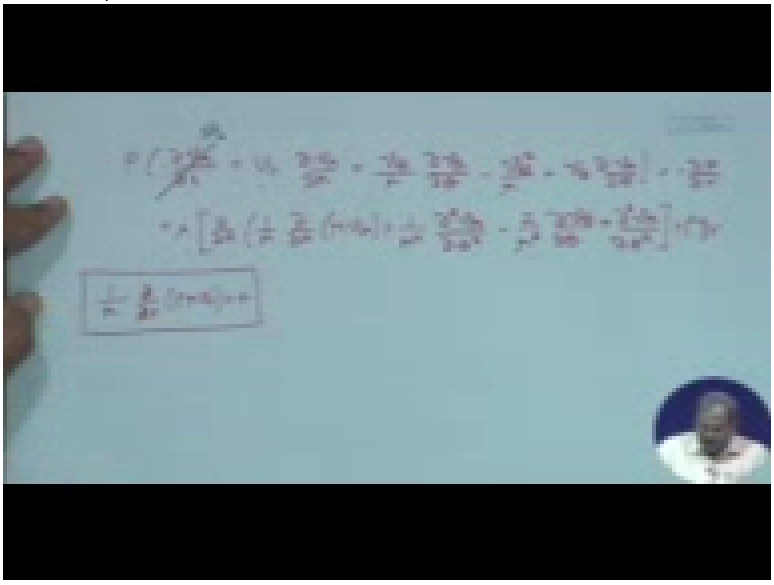
So these are the convective transport of momentum. This is the pressure gradient in the r direction. What you have are the viscous transport of momentum; 1, 2, 3, 4 terms and then you have a body force term. So as before we are going to see which of these terms can be neglected. First of all it's a steady state. So this is equal to zero.

(Refer Slide Time: 12:52)



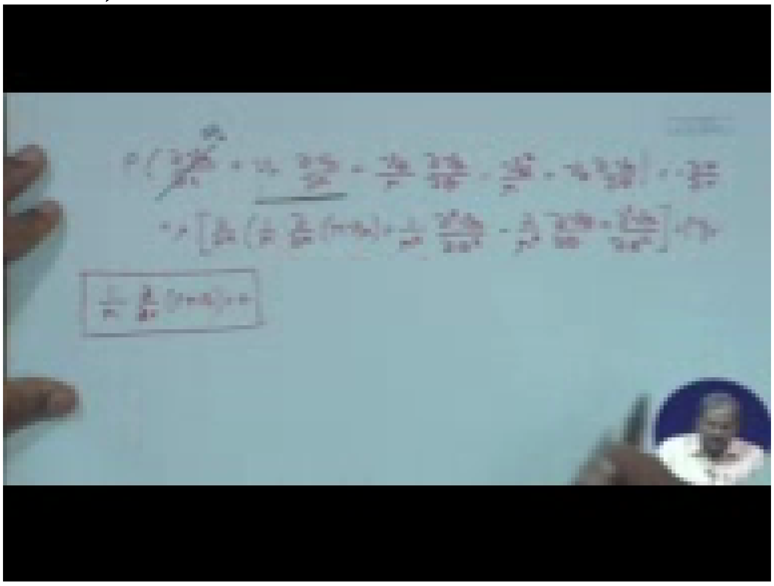
v r does not vary with t. Coming to the end I would also write this, the continuity equation, this is for reference which was 1 by r. This was the equation of continuity which we have seen

(Refer Slide Time: 13:18)



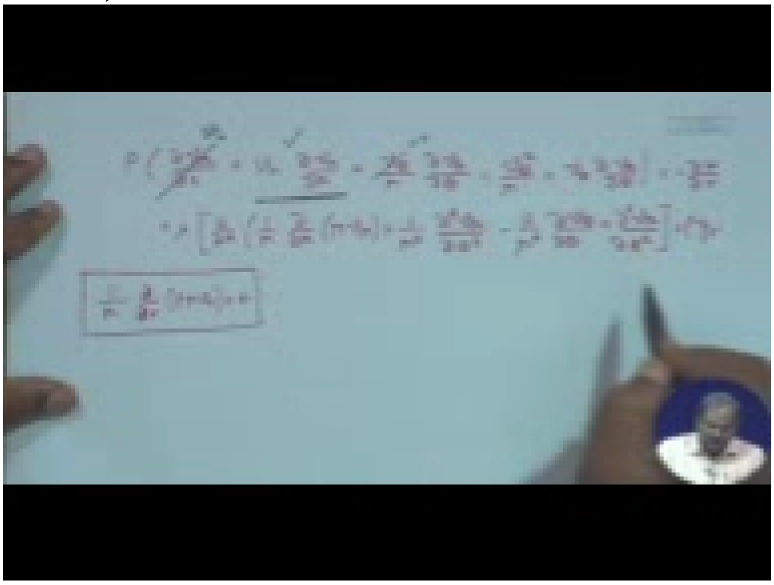
in the, in the past this thing. So here v r is not zero and del v or del r is also not zero so as the, as the, as the liquid moves towards larger and larger values of r the v r has to reduce so v r slows down with r therefore we cannot say anything about, we cannot equate this term to be zero.

(Refer Slide Time: 13:46)



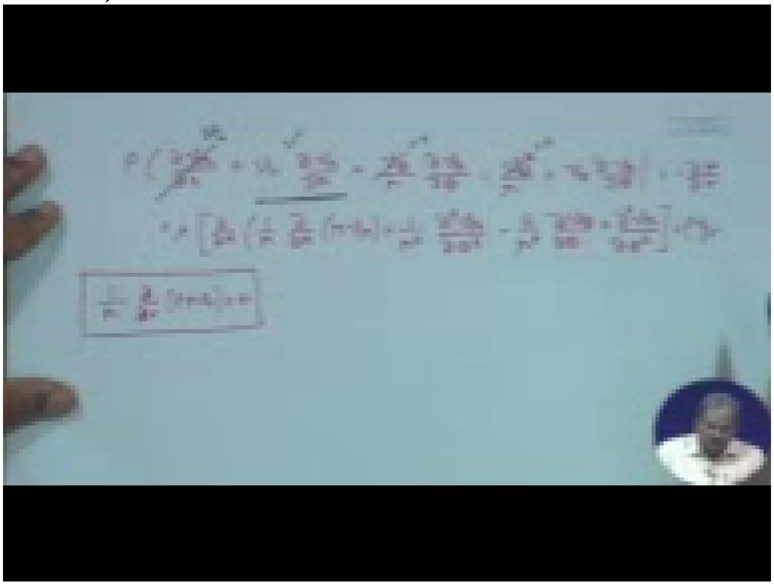
So this will remain in the governing equation. Here we have a this term, v theta would be zero

(Refer Slide Time: 13:56)



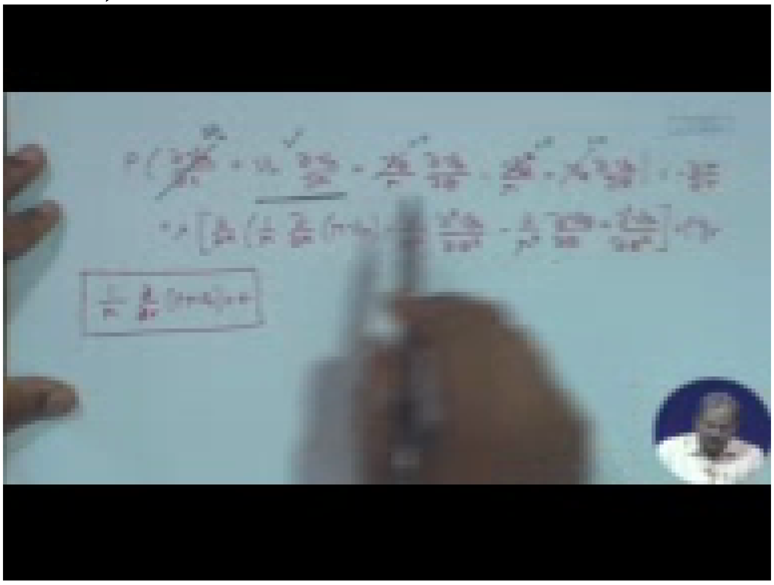
and in this case, v theta would be zero once again

(Refer Slide Time: 14:00)



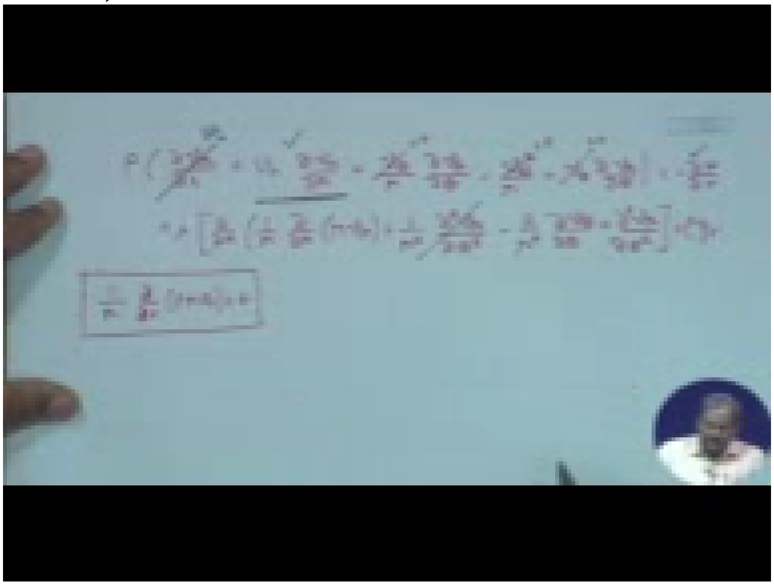
and even the v r, v r varies with z however v z is zero so

(Refer Slide Time: 14:09)



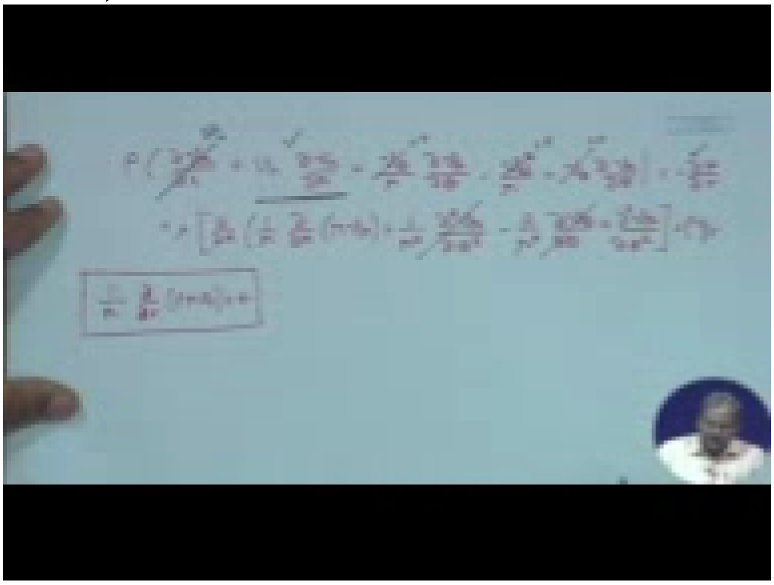
therefore of the entire left hand side the only term which remains is rho times v r del v r del r. There is a fixed pressure gradient present in the system so I must keep del p del r in, in here. I will come to these at a later point. If I think of this term, v r is not a function of theta so therefore

(Refer Slide Time: 14:35)



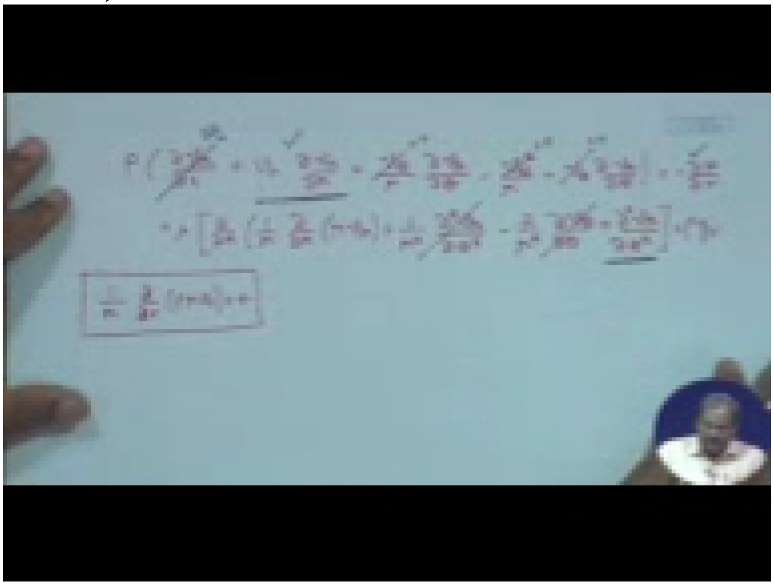
therefore this is going to be zero. There is no v theta present in the system so this term

(Refer Slide Time: 14:40)



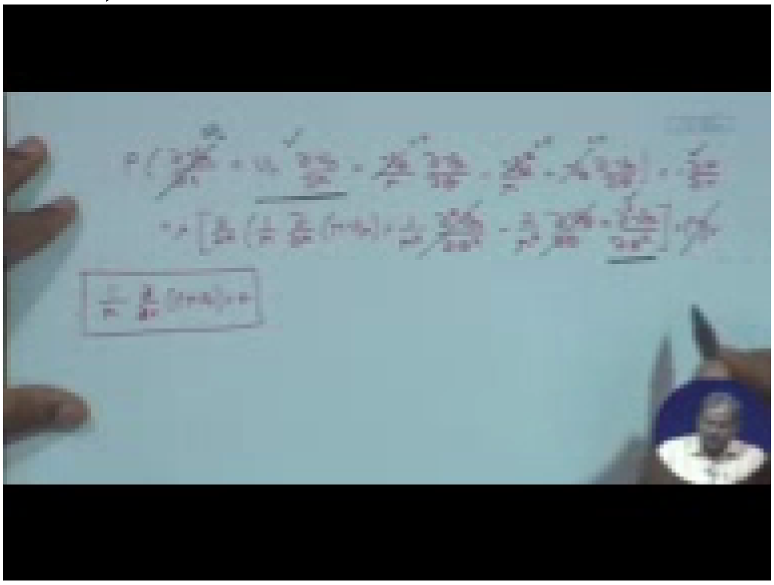
would also be zero. However del square v r is a function of z so I cannot neglect, I cannot make this term, equate this term

(Refer Slide Time: 14:51)



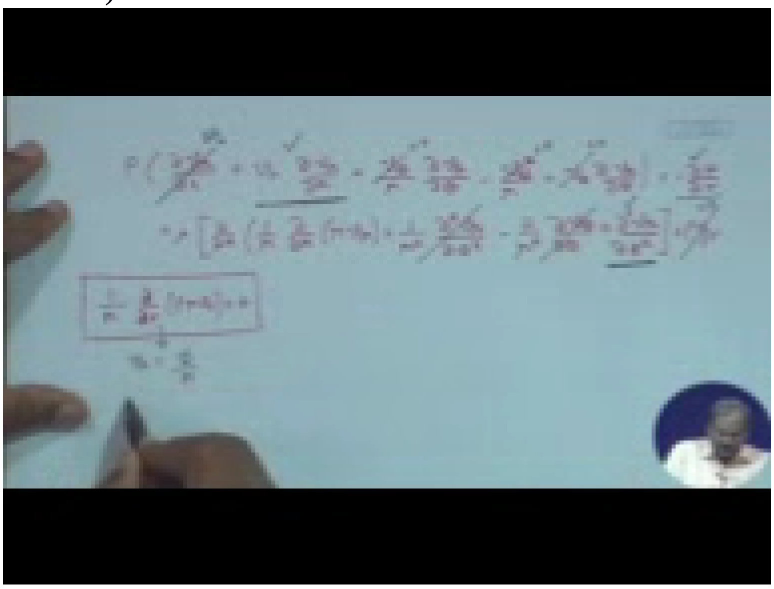
to zero. So this will also remain in the governing equation. And since the disks are horizontal so there is no body force

(Refer Slide Time: 15:02)



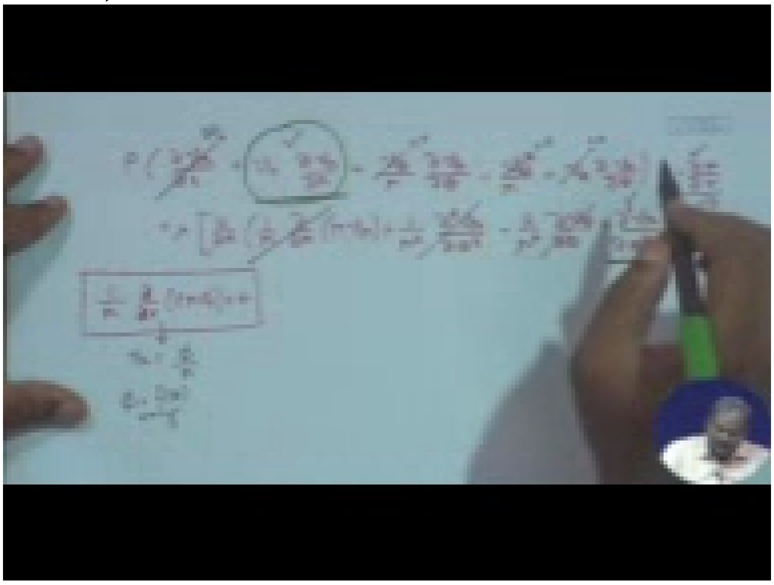
acting on it therefore this will also be zero. So of these, of all the terms in the right hand side, this is non-zero which will remain in the governing equation. This is non-zero, everything else is zero and now we come back to this term. Now if you, if you remember from what we have obtained from the, from the continuity equation is that v r is equal to phi by r

(Refer Slide Time: 15:32)



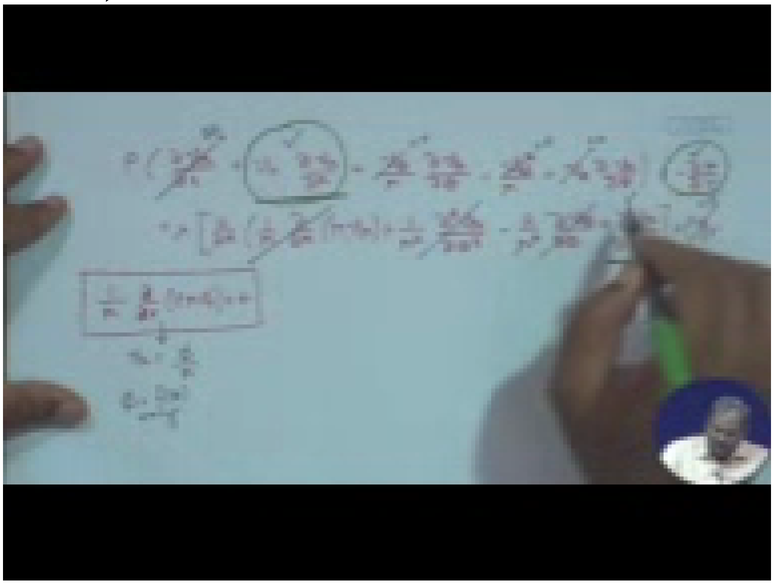
and we also realized that it's a function of z only. So if it is a function of z only so this is simply del del r of r v r and del del r of r v r is simply phi. So I have phi, I can replace this with phi and since phi is a function of z only the differentiation of phi with respect to r must therefore be equal to zero which would allow me to cancel this term as well. Once again, this r v r, r v r is equal to phi and we have realized from our previous discussion that phi is a function of z only. phi is not a function of r. So if phi is not a function of r, therefore del del r of phi would be zero, therefore this term is going to be equal to zero. So my governing equation would consist on the left hand side as

(Refer Slide Time: 16:30)



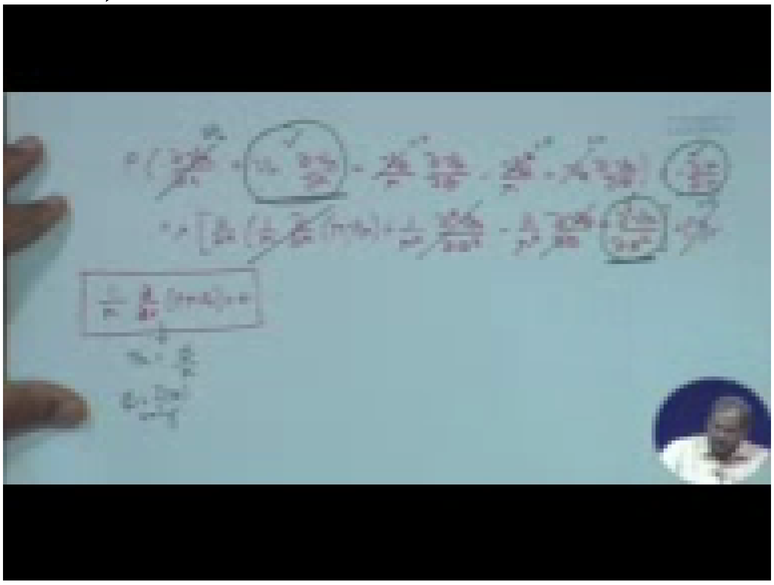
this term, on the right hand side, this one

(Refer Slide Time: 16:33)



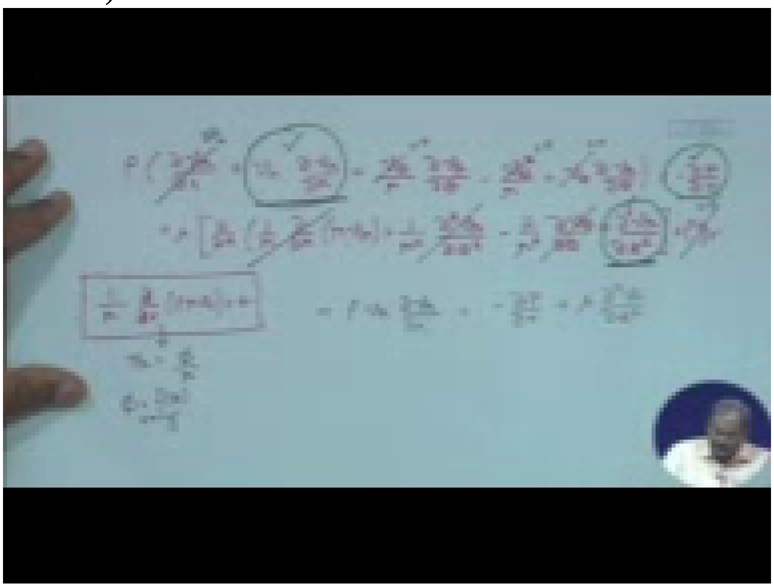
and this term.

(Refer Slide Time: 16:36)



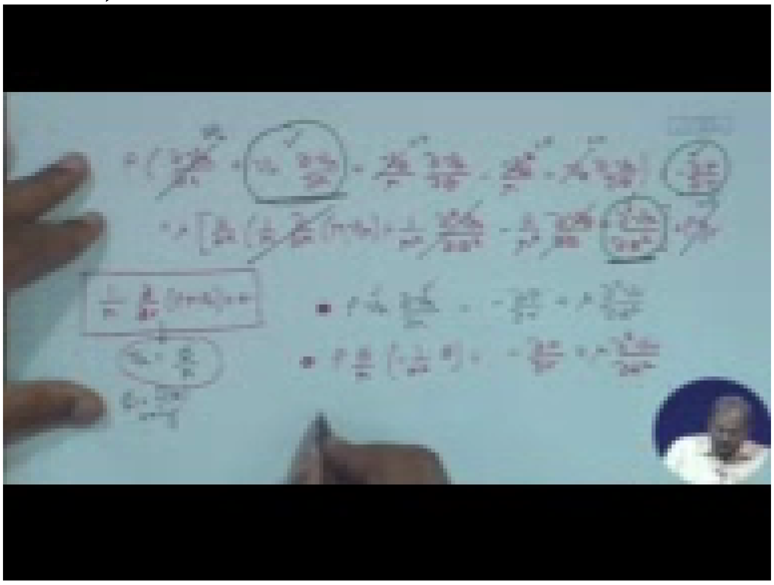
So the governing equation would simply be equal to minus rho v r del v r by del r equals minus del p by del r plus mu times del square v r by del z square. So this is the governing equation

(Refer Slide Time: 17:03)



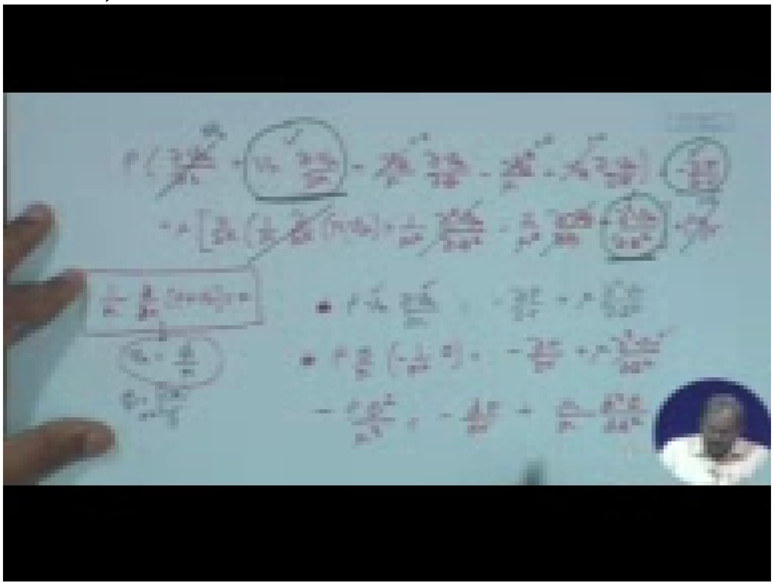
for flow in such a system. And I am going to substitute v r and del v r del r from this relation so what I would get is minus rho phi by r into del v r by del r which would be minus 1 by, sorry this is plus, this is not minus; this is not minus, this is plus, 1 by r square in 1 by r square; so del v del r if you see, del v del r it would be 1 v by r square into phi is equal to minus del p del r plus mu times del square v r by del s square so final form of this

(Refer Slide Time: 18:00)



would be minus rho phi square by r cube equals minus d p d r, I am consciously using d p instead of del p because p is the function of only r. It does not vary significantly with z or with theta so your, your this thing won't be, and if I substitute this in here, my governing equation would simply be;

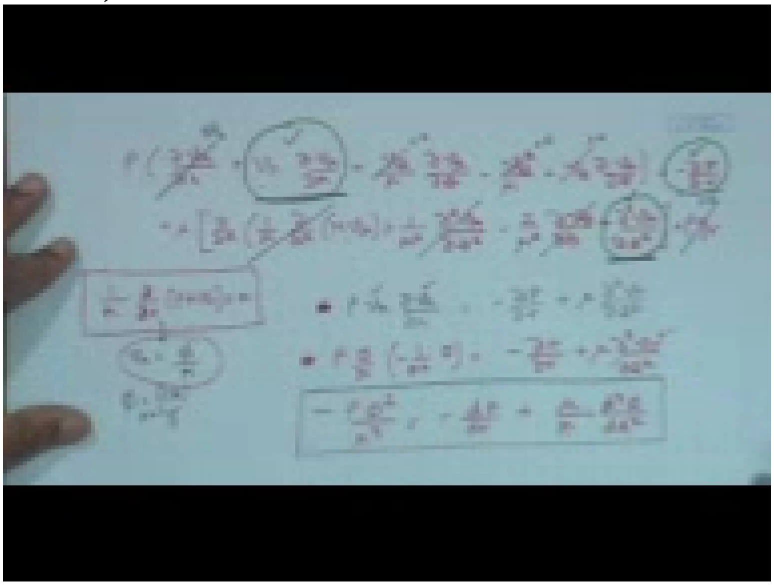
(Refer Slide Time: 18:41)



so substituting v r in here you would, you would obtain d 2 phi by d z square and since phi is a function of z only so I can drop the partial sign, it would simply be d 2 phi by d z square and r, the r over here would simply come at this point.

So my reduced form of the governing equation for flow between two circular disks as a function

(Refer Slide Time: 19:11)

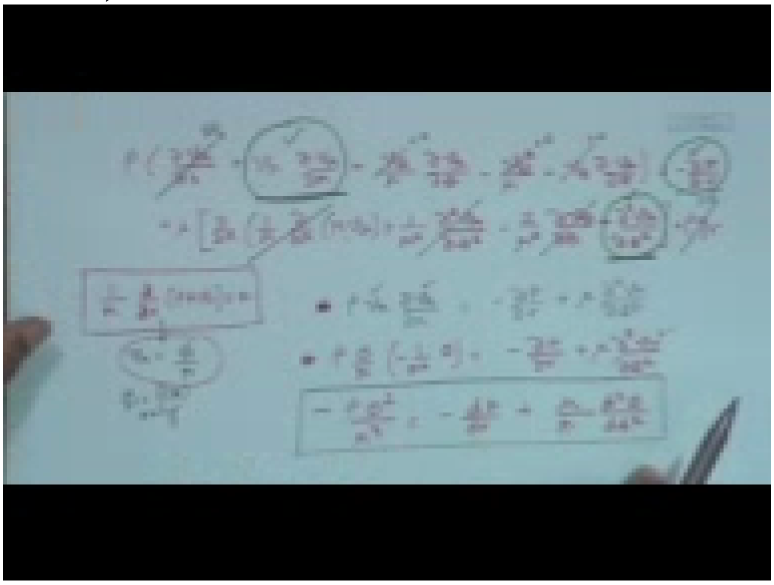


of applied pressure gradient would simply take this form. So again you see the utility of Navier-Stokes Equation, how easy it is to at least arrive at the governing equation. There is no need to think of any complicated shell

(Refer Slide Time: 19:29)

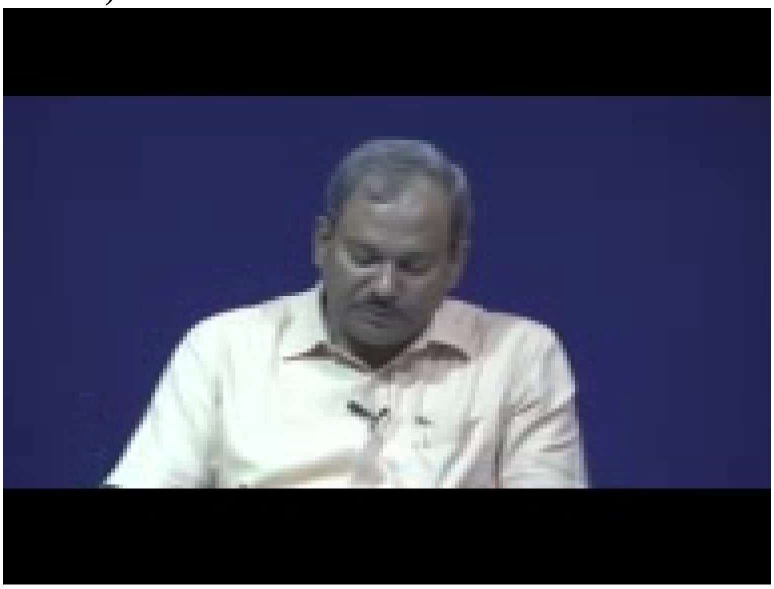
which would be very, complicated in this case. We would simply pick the form of the, pick the direction in which velocity varies and try to solve it. So this is your governing equation.

(Refer Slide Time: 19:41)



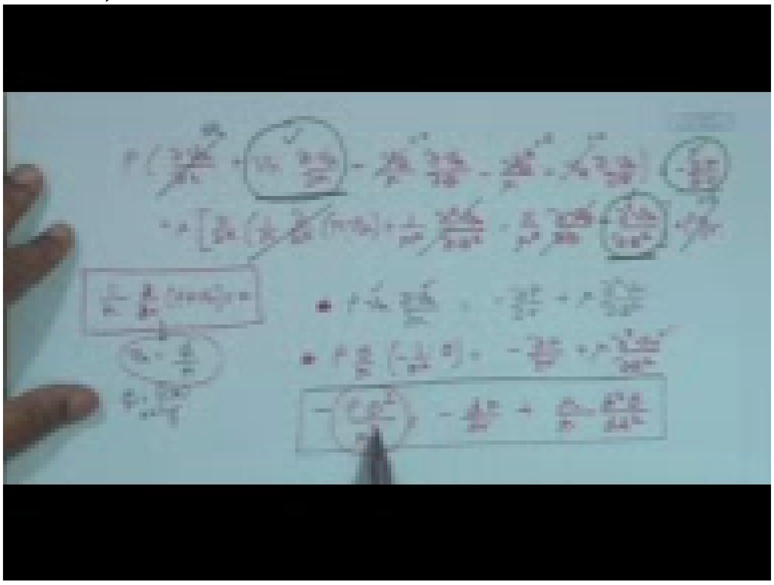
The problem with this, the problem with this specific equation that is is non linear, it's a non-linear equation. The non-linearity comes because of the presence

(Refer Slide Time: 19:51)



of this term on the left hand side. Since

(Refer Slide Time: 19:58)



the presence of this term makes it non-linear, the solution there is no way to obtain an analytic solution for this case but if we try to think about the genesis of this

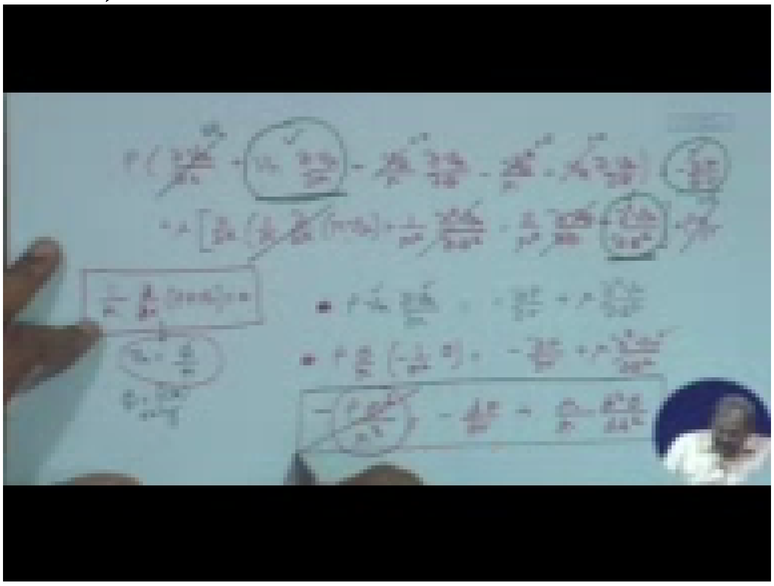
(Refer Slide Time: 20:12)

non-linearity, then probably it would give us some idea of some asymptotic conditions in which this non-linear term, the effect of this non-linear term would not be significant. So if you again look at the derivation, you would see that the non-linear term arises from the left hand side of the Navier-Stokes Equation and the left hand side of the Navier-Stokes Equation as I said many times before is due to the convective transport of momentum. Now what is convective transport, what is the root cause of convective transport of momentum? It's because you have a flow and with the flow, the flow carries some momentum along with it which is, which is nothing but the convective momentum.

Now if there are situations in which you can say that the effect of this convective momentum is rather small then you would be able to drop the non-linear term on the left hand side. Now since it is related to velocity, since this non-linear term is related to velocity, the only way when you can drop this term or when you can disregard the contribution of this term into the overall scheme of things is only when the flow is very slow. So if the flow is slow the convective transport of momentum can be neglected but not the convective, not the conductive transport or the molecular transport or the viscous transport of momentum.

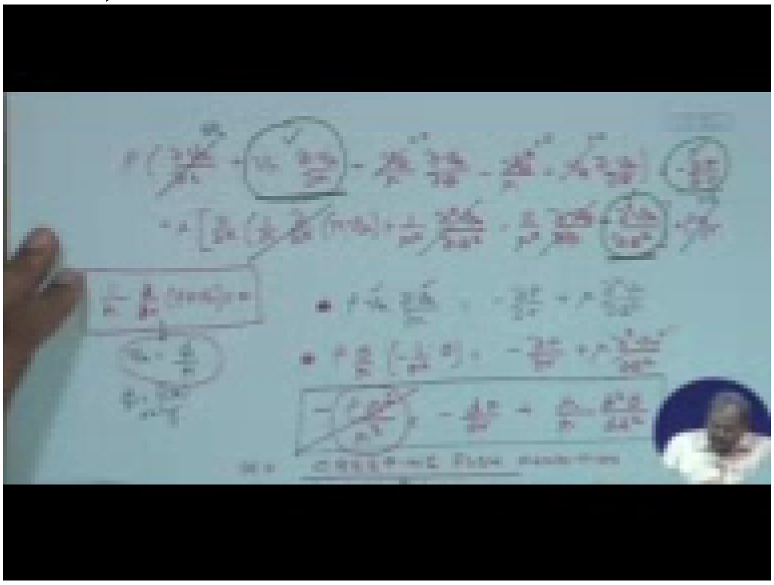
Because unlike convective transport of momentum, the viscous transport of momentum does not depend on velocity. Rather it depends on velocity gradient. So reduce, so keeping the velocity low, assuming it is a low velocity situation will let you drop the convective term but will not necessarily, you are in a position to cancel the viscous terms ok. So if this, the contribution of the terms mathematically speaking, is truly zero then those special flow conditions in which the convective term, the effect of the convective term can be completely neglected are known as the creeping flow situation. So the flow, the case where this entire term

(Refer Slide Time: 22:37)



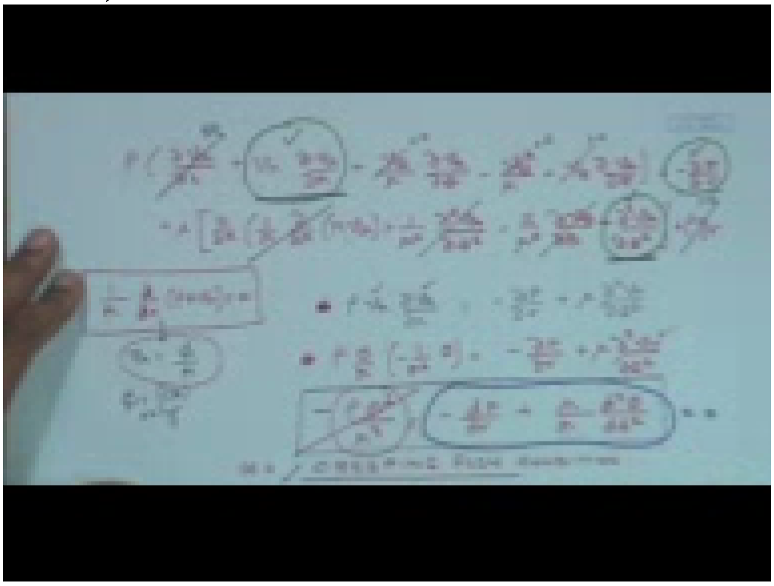
is equal to zero, they are called the creeping flow, Ok. So this creeping flow condition, which the name suggests, it's a very slow flow

(Refer Slide Time: 22:58)



so the effect, the convective transport is small but the viscous transport may not be small. So when this term is equal to zero, that is known as the creeping flow condition. So the analysis that we are going to do from this point onwards is only valid for creeping flow or close to creeping flow solution where the non-linear term can be dropped. So my governing equation then becomes only this is equal to zero

(Refer Slide Time: 23:32)

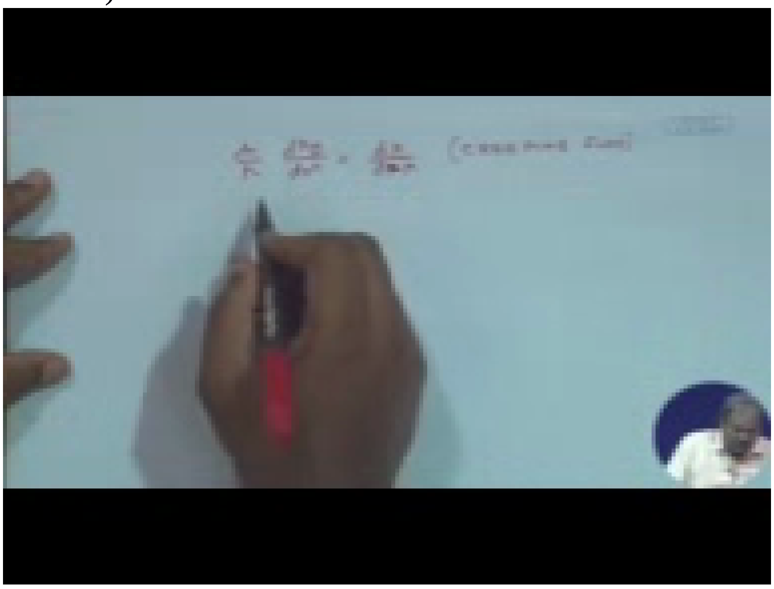


since the left-hand side, I have, I have made it equal to zero by assuming it's close to creeping flow situation.

(Refer Slide Time: 23:40)

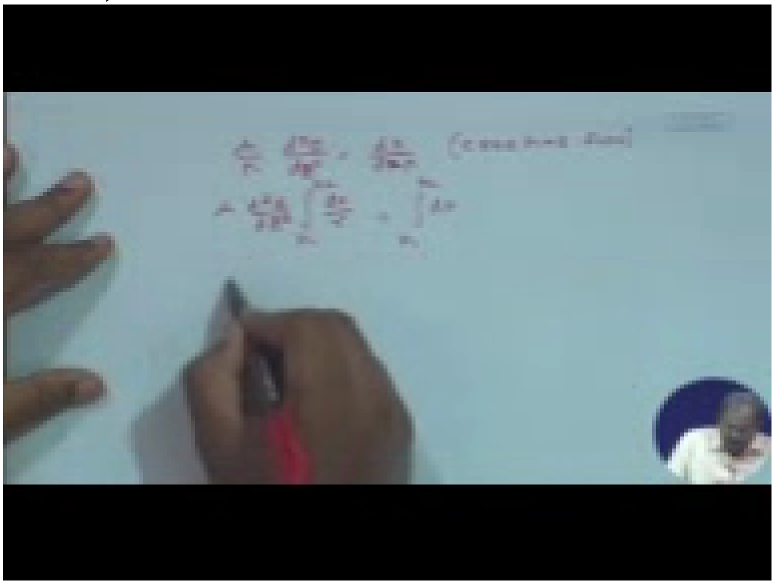
So what then I have, what I have then is mu by r times d 2 phi by d r square is equal to d p by d r. So this is creeping flow. This can now be integrated.

(Refer Slide Time: 24:08)



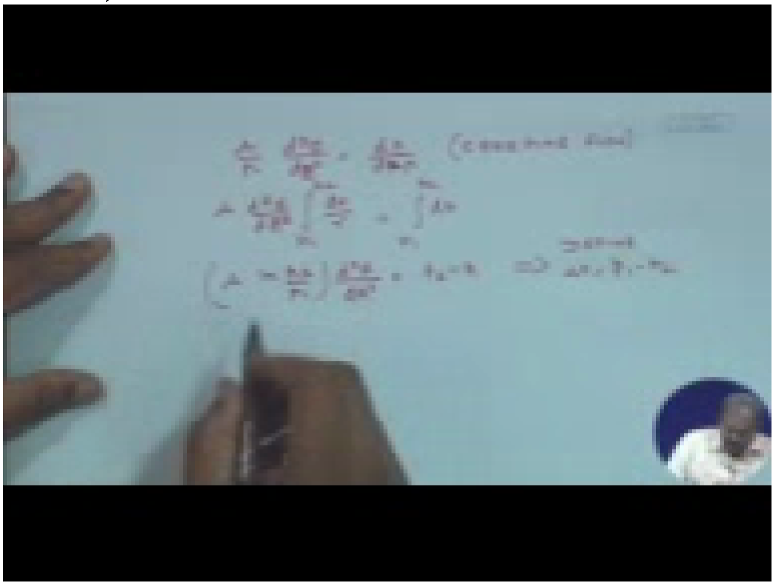
phi is not a, sorry d z square; phi is not a function of r. So I can keep it outside the integration sign, this is d r by r from r 1 to r 2 and this is d p, let's say the pressure at these two locations are p 2 and p 1 and therefore

(Refer Slide Time: 24:38)



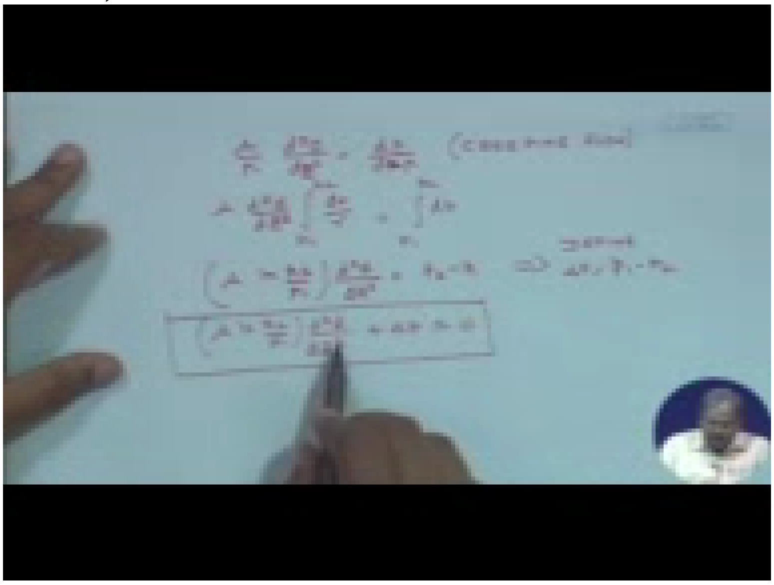
you can write l n r 2 by r 1 upon integration times d 2 phi by d z square would be equal to p 2 minus p 1. So if we define delta p is equal to p 1 minus p 2 then the equation

(Refer Slide Time: 25:03)



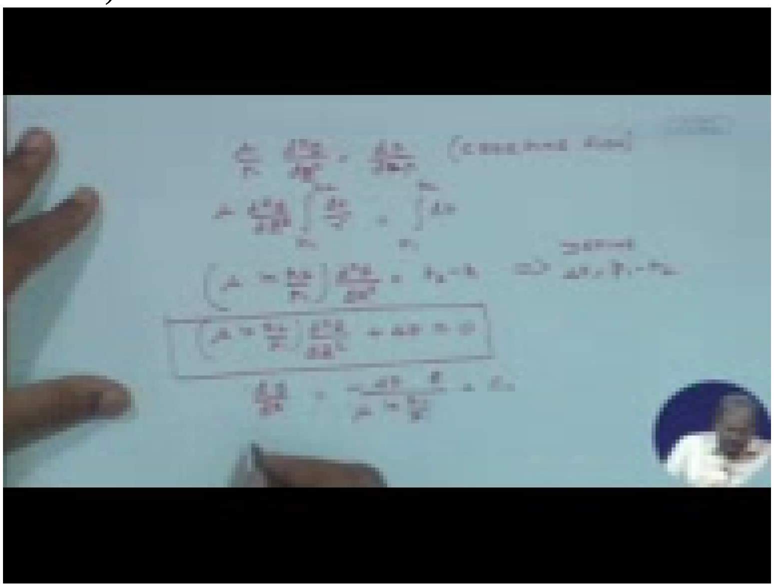
would transform to r 2 by r 1, d 2 phi by d z square plus delta p is equal to zero. So this is now

(Refer Slide Time: 25:20)



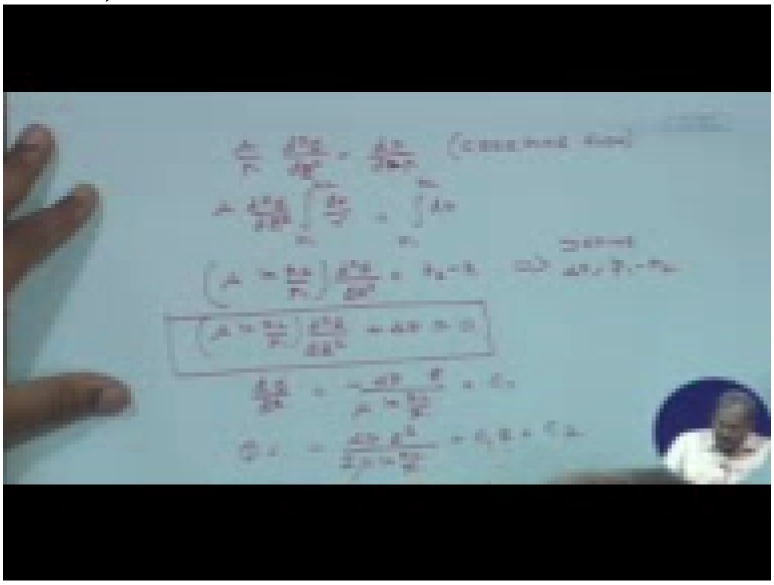
a straight forward equation, second order equation in terms of phi. So creeping flow lets me simplify the equation, governing equation considerably to obtain a compact equation where phi is a function of z, phi is not a function of r. Pressure is a function of r, so therefore I perform the integration and I obtain this expression for the, for phi. So what I do next is I find out, I find out what is d phi d z which would be minus delta p z by mu times l n r 2 by r 1 plus c 1 and

(Refer Slide Time: 26:09)



finally phi to be equal to minus delta p z square by 2 mu l n r 2 by r 1 plus c 1 z plus c 2

(Refer Slide Time: 26:24)



where c 1 and c 2 are constants of integration. If c 1 and c 2 are constants of integration in then they should be evaluated using the appropriate boundary conditions. See what the way we are, we are trying to handle this

(Refer Slide Time: 26:43)

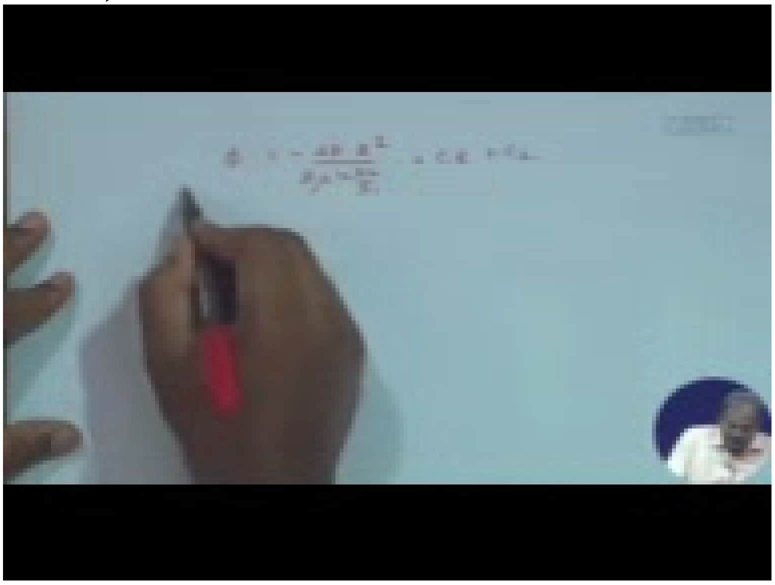
problem is we understand that v r is simply going to be equal to phi by r; phi is a function of z only. It's not a function of r. So my, our aim is to obtain an expression for phi. The expression for phi can be obtained if we write the r component of Navier-Stokes Equation in cylindrical coordinate system. That's what we have written but we have seen that unlike in previous cases, there is a contribution from convective momentum. There is a contribution from the left hand side of Navier-Stokes Equation. On the right hand side, the terms which would remain are the pressure gradient and the variation of velocity v r with z Ok. The other terms can be, other terms can be cancelled based on our understanding and based on the, on the use of the continuity equation.

The first term inside the third bracket, the first viscous term can be cancelled because of our, because of the use, through the use of the continuity equation so there will be 3 terms in the equation. One, the convective transport of momentum, one is a pressure gradient and the other is a viscous transport, mu times by del square v r by del z square. The presence of the convective term makes the equation nonlinear. We have to get rid of it if we would like to find analytic solution. So fundamentally we understand that this contribution, the convective contribution comes from velocity, the velocity in the r direction and the change in the surface area in the r direction, so we have the convective contribution.

When this convective contribution can be neglected, will be small when the velocity itself is small; so velocity is small, will let me allow as a special case in the limit when the convective term can be completely ignored. Mathematically that is known as the creeping flow case. So the convective term, the non linear term is neglected since we are dealing with a very low values of v r but that does not mean that the viscous transport can be neglected because it depends on the gradient and not on the value of the velocity itself. So if we use the creeping flow then we have only 2 terms present in it. One is the pressure gradient term and one is the variation of velocity with respect to z, that term.

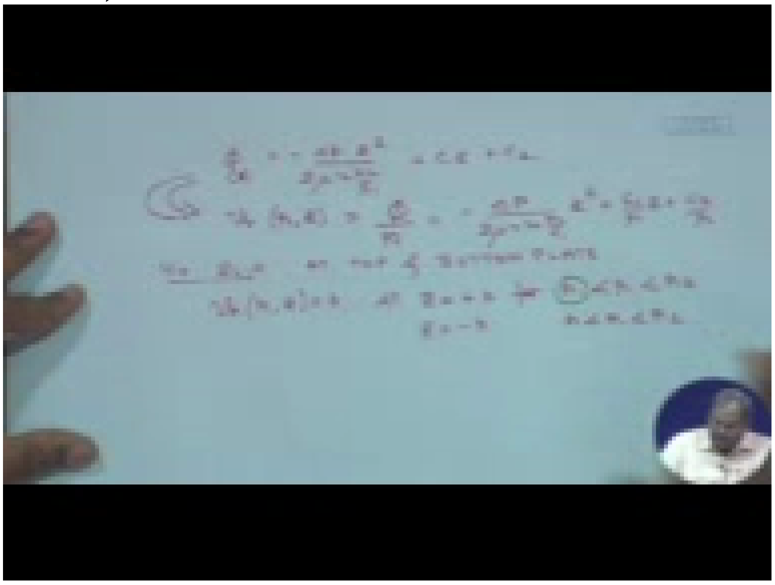
We already have an expression for v r which is, if you look at any, which is phi by r. So somehow I have to obtain an expression for phi. So I put that expression into the governing equation and what I obtain is an expression of phi only. This equation can now be, so this equation can, this is the equation which can now be integrated and this is the final form of phi where C 1 and C 2 are two constants of integration which will have to be evaluated. phi which is minus del p z square by two mu l n r 2 by r 1 plus C 1 z plus C 2, this would give me

(Refer Slide Time: 30:08)



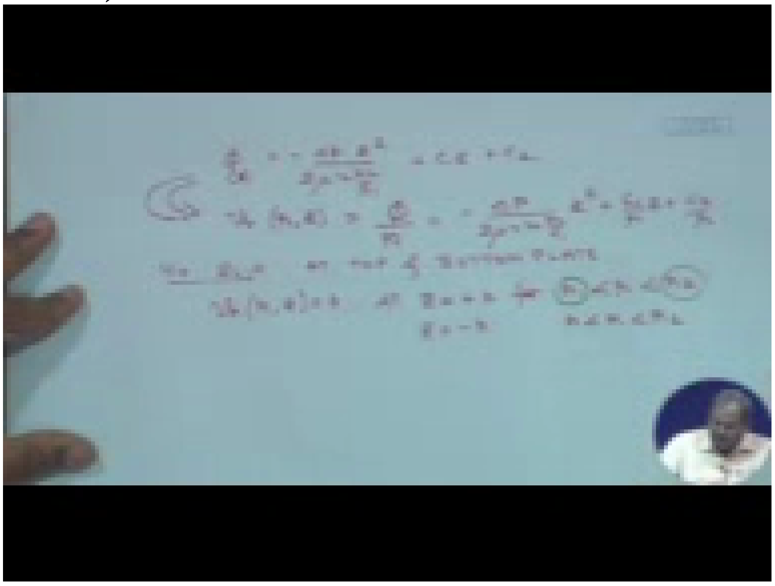
v r which I realize is the function of r and z. phi is a function of z only but v r is a function of r and z as we have seen from our equation of continuity is phi by r. So this would be minus delta p by 2 mu r l n r 2 by r 1 times z square plus C 1 by r z plus C 2 by r. So see how, how we have got the expressions for v r from the expressions for phi identifying it's a functional z and this is from equation of continuity, the specific form of v r. What are, what are the boundary conditions? The boundary conditions are essentially no-slip at the top plate, at top and bottom plate. In other words, mathematically v r which is a function of r and z would be zero at z equals plus v and at z equals minus v at the top plate and at the bottom plate. However we must realize that it's valid in a region where r is greater than r 1 but less than r 2, where r 1 is the radius of the hole at the top through which the liquid comes in to the space between the two discs

(Refer Slide Time: 32:00)



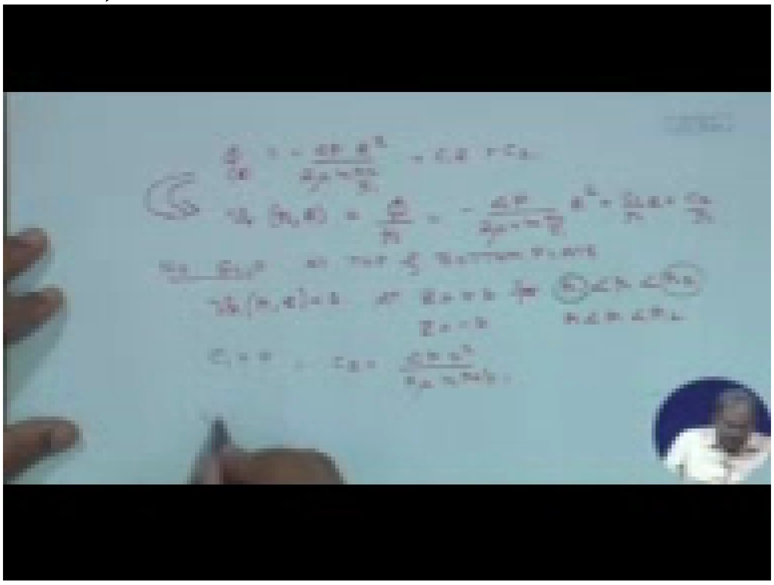
and r 2 is the radius of the two discs, outer radius

(Refer Slide Time: 32:05)



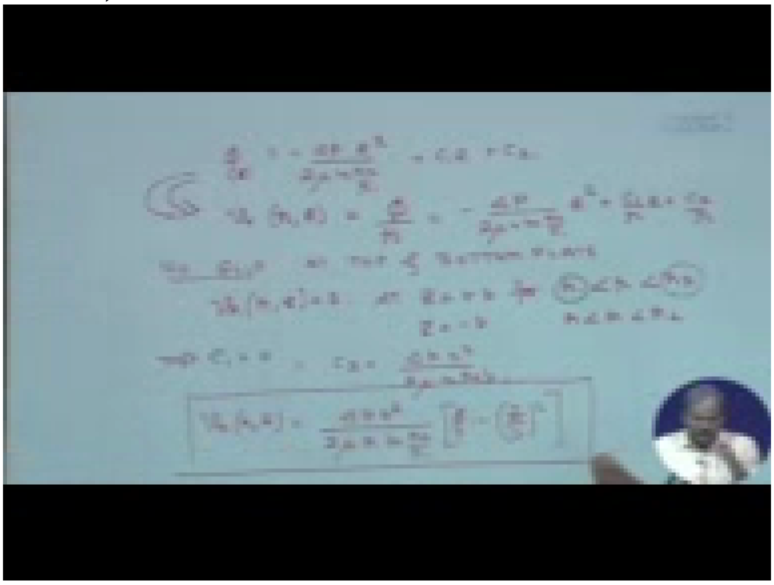
of the two discs. So the domain of, domain of applicability of these two equations, these equations, this equation is only between r 1 to r 2. So that's what it simply means, that it is going to be between r 1 and r 2. So with these two boundary conditions one should be able to evaluate what is C 1 which would turn out to be equal to zero and the expression for C 2 would be equal to delta p d square by twice mu l n r 2 by r 1. So

(Refer Slide Time: 32:47)



your v r which is a function of r and z, with the, with this expressions for C 1 and C 2 would finally come to be delta p v square 2 mu r l n r 2 by r 1, 1 minus z by D whole square. So this is the final form of the, this is the final form of

(Refer Slide Time: 33:20)

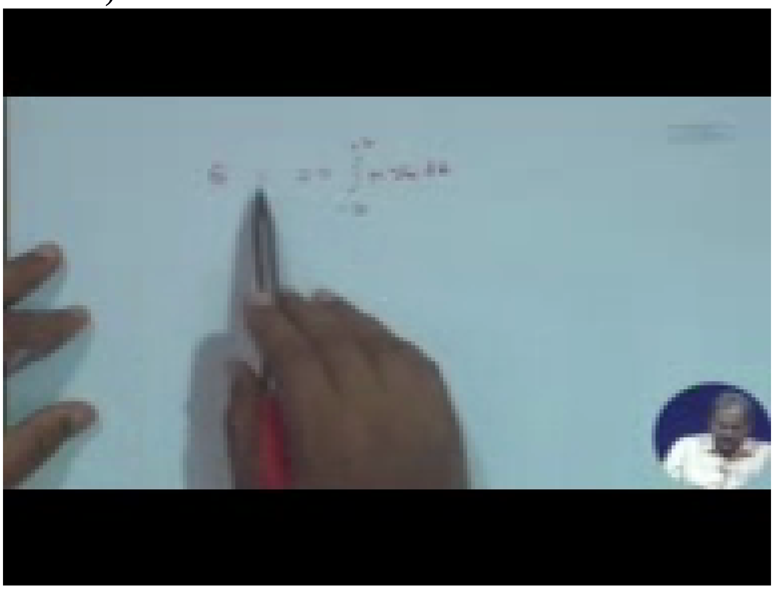


the velocity expression for the case of flow between, for the case of pressure driven flow, pressure gradient driven flow between 2 parallel plates which are separated by the distance twice b where the geometric parameters are r 1 is the radius of the hole through which the liquid comes in, r 2 is the, is the, is the outer radius of the both the plates, delta p is the applied pressure gradient and mu is the viscosity of the liquid in between. If we have, since we have the value of v r, it is then easy to calculate what would be the value of

(Refer Slide Time: 34:01)

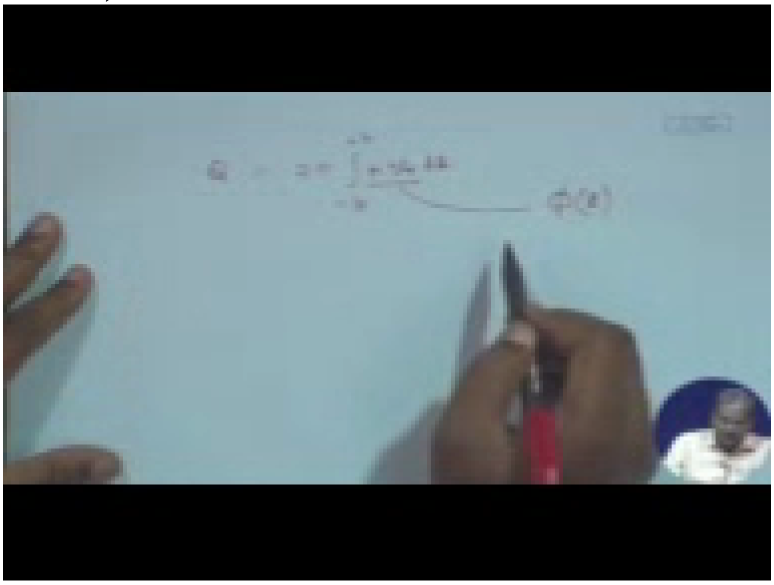
Q. Because here, most of the times we are not interested in finding out what is the velocity at every point in the flow field. We would rather, we are more comfortable in dealing with the, the average values, for example the average velocity, or from the average velocity one can obtain what is the flow rate. So the flow rate would simply be twice pi minus b to plus b r v r d z. So this is

(Refer Slide Time: 34:36)



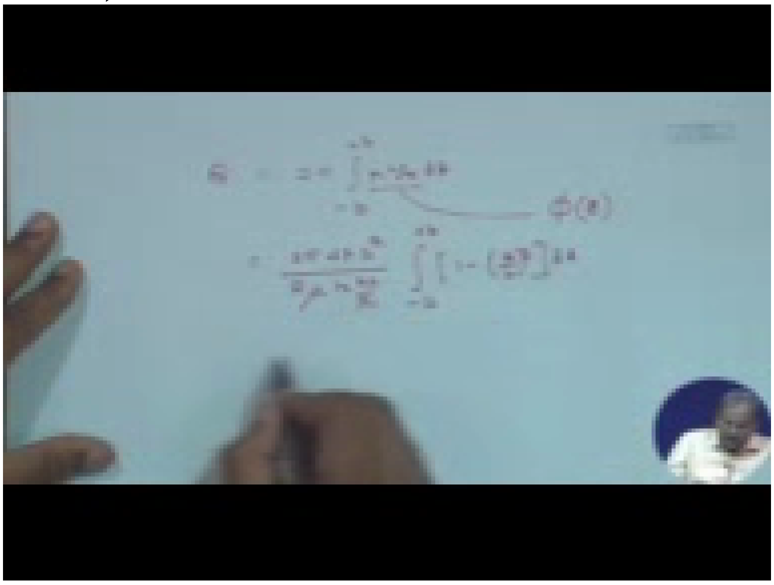
area averaged volumetric flow rate where the d z can vary from minus b to plus b and what you have and r v r is simply equal to phi from our equation of continuity and we understand that phi is a function only of z. So this is consistent, this expression of Q is consistent with

(Refer Slide Time: 35:03)



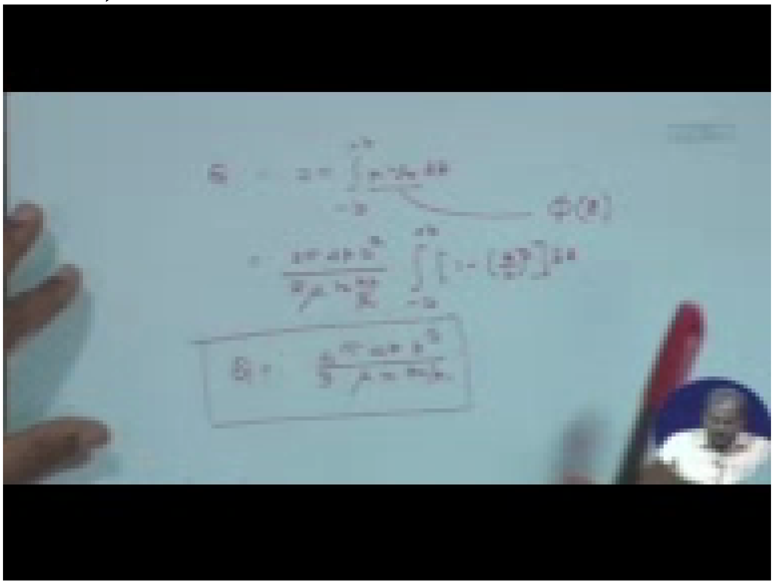
our understanding and it; it follows the physics of the problem. So Q would simply be equal to twice pi delta p, after you put the expression of phi in there, v square by 2 mu l n r 2 by r 1 from minus b to plus b 1 minus z by b whole square d z. And

(Refer Slide Time: 35:32)



after one or two steps, r 2 by r 1;

(Refer Slide Time: 35:40)



so here you see how it is, how we get

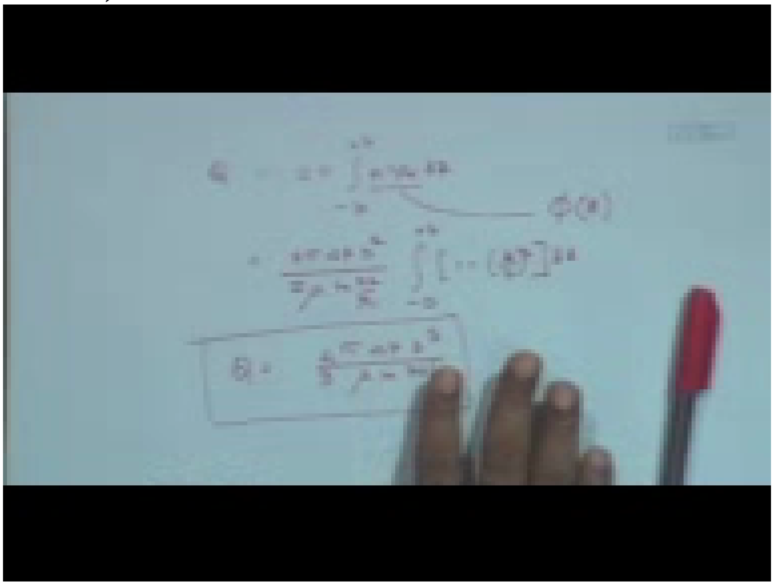
(Refer Slide Time: 35:44)

compact expression for the flow of liquid, Newtonian liquid in between 2 plates in laminar flow when there is a pressure gradient and when there is no effect of gravity. But the major assumption that we have incorporated, we have put in here is that the, the flow is very slow. That means the flow is called and can be termed the creeping flow. What is the specialty of creeping flow? There is no contribution from the convective transport. The only contribution to momentum transfer is due to the viscous transport or the pressure gradient and body force if it is present but momentum transport due to convection is absent in the case of creeping flow.

So a non-linear equation which we have quickly obtained from the, from our analysis of Navier-Stokes Equation has led to the complicated expression which with the use of the right, uh right approximation, assumption has given rise to an expression, governing equation that can be integrated, that to obtain the value of phi where phi is the z component dependent, the z direction dependence of velocity and we can obtain the point, the expression for velocity, the expression for average velocity and expression for volumetric flow rate as we have done here. So this is another nice example of the use of Navier-Stokes Equation to solve problems like this.

There can be many other problems, more complicated problems of Navier-Stokes Equation, some of which I would try to give as an exercise, as exercise problems to you and there would be more complicated problems which are beyond the scope of this specific course. So, but whatever it is, I would like to summarize so far what you have understood, what I tried to convey to you is that shell momentum balance is for beginners, it's a good thing because it gives you some hands-on on the concepts involved, but the moment you deal with slightly complicated problems you feel the need for a more generalized approach which is provided by Navier-Stokes Equation. And I think I have solved 5 or 6 problems in this, in this part of the course to give you some idea of how to handle such situations. And we would, we would solve similar few more courses, few more problems in,

(Refer Slide Time: 38:35)



in our tutorial part. So I think I will end

(Refer Slide Time: 38:38)

here and this would be the, this would be the like conclusion of our part of Navier-Stokes Equation and what I am going to do from next class onwards is try to introduce the concept of boundary layers, which this boundary layer, the concept of boundary layer is extremely important in transport phenomena and I would give you an overview of the boundary layer, how it can be solved, how they are connected with many of the problems that we encounter in everyday lives and I would try to give you examples to which you can relate to, and this would give us some idea about what is the effect of boundary layers, the utility of boundary layers, how you can modulate the boundary layers to get more transport and you would see most of the transport is confined near the interface, near the solid liquid, solid fluid interface. So if we can manage that, understand the physics of that then we are in a very good position to alter the boundary layer and to get the desired transport from the specific system. So that's what we are going to start in the next class. Thank you.