Transport Phenomena
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Lecture Number 13
Equations of Change for Isothermal Systems (Cont.)

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We will continue with our treatment of equation of motion, Navier–Stokes equation and apply it to different geometry the same way we have been doing before and before we end this part of the course I would like to show you two more problems slightly complicated but nothing that Navier–Stokes equation and common sense cannot handle. So the specific situation that we have is where we have a cylinder, two coaxial cylinders, one is going to remain stationary, the other will be rotated. So let us see, the outer cylinder is being rotated while the inner is kept stationary. So I have a cylinder which is stationary and another cylinder which is coaxial but it's being rotated with some velocity.

And the space in between two cylinders is filled with a liquid. Now you can clearly see that how much force, how much torque in this case would be required to move the outer cylinder at a constant speed, at a constant speed uh would depend on what kind of liquid we have in between. So if we have a highly viscous fluid, you would require more torque and if it's a very light fluid, very low viscosity then it would be easy to rotate the outer one with the same speed. Now this suggests that measuring the torque of such systems where one of the two coaxial cylinders is being rotated with some speed while the other is kept stationary, this torque can be calibrated with the viscosity of the liquid.

So in other words if you know the viscosity of the liquid, if you know the torque, if you can measure the torque you should be able to calculate what is the, what is the viscosity of the liquid. This is a very common; this is a quite common and accurate method to, to measure the viscosity of unknown fluid, unknown liquid. But before we can get up to that point, I need a compact expression that connects that torque with the viscosity of the liquid. So you as a transport phenomena expert, you are given the job to find out what is going to be the torque necessary to rotate one of these cylinders while the other is kept stationary and when we have the liquid in between, what is the torque required to rotate the outer cylinder and obviously the torque, the expression for torque will contain, from our common sense we can say it should contain geometric parameters and by geometric parameters I mean the, the radii of the inner and the outer cylinders, the length of each of these cylinders, what is the density of the fluid, what is the viscosity of the fluid and at what speed the outer cylinder is being rotated.

So my goal for this specific problem is to find an expression for torque, the expression of which would contain among other things the unknown parameter viscosity; so measuring the torque I should be able what is the viscosity of the liquid. So it’s very good model to measure the viscosity. Its, it’s a very good model for some of the viscometers which measure the viscosity of certain liquid. Now when we talk about the viscosity measurement, the first instrument that comes to our mind is the capillary viscometer where a liquid is allowed to fall through the, or fall through the very narrow capillary and you know from your high school physics how to connect the viscosity with the flow rate, nothing but the Huggin Poisson equation which you have derived.

But this becomes problematic if you have a highly viscous fluid, if you have a viscous fluid to deal with, then in order to collect sizeable quantity of the liquid to predict what is the flow rate it would take a very long time, Ok. So for those liquids, you must device other ways to measure the viscosity, not the capillary viscometer way. And the problem, the present scenario, the present situation that we are going to discuss and model is an ideal case to be used, ideal candidate to be used to measure the unknown viscosity of the liquid. So we are looking at the tangential annular flow of a Newtonian fluid and the system in which

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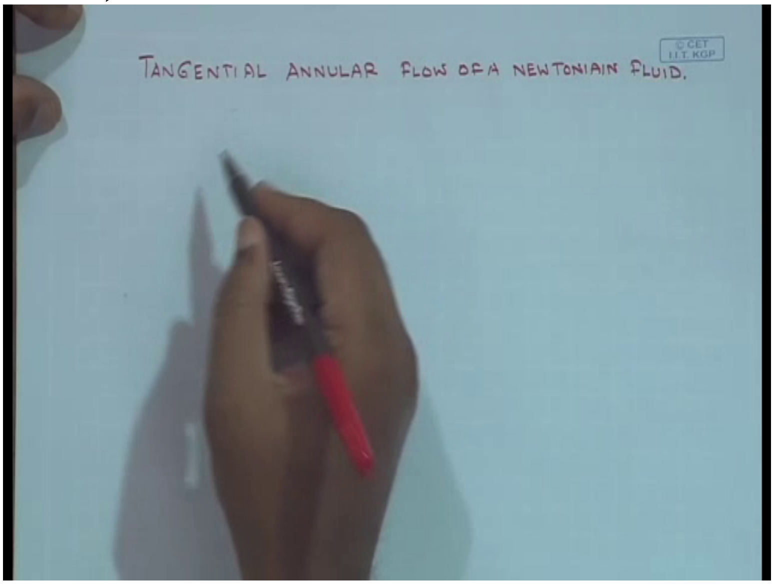

this annular flow takes place is in between two cylinders and I am looking at the top. In this one is moving with an angular velocity

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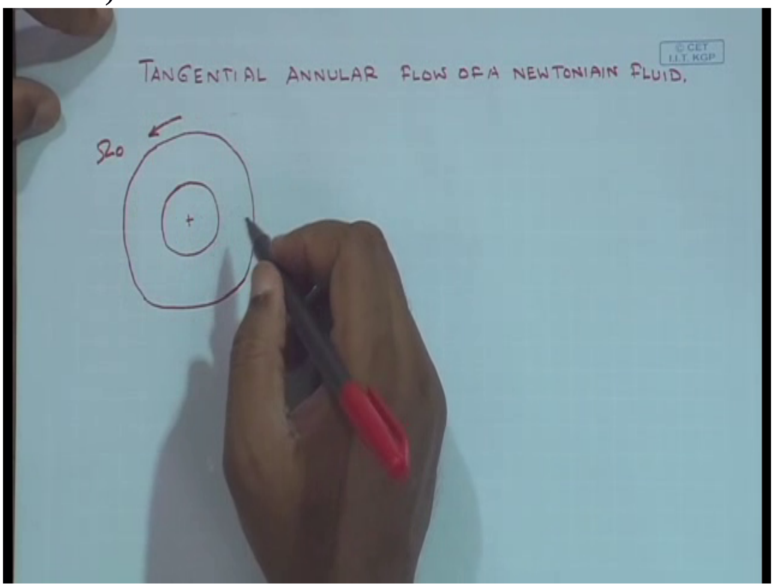

equal to omega naught. The inner radius is k r. The outer radius is equal to r

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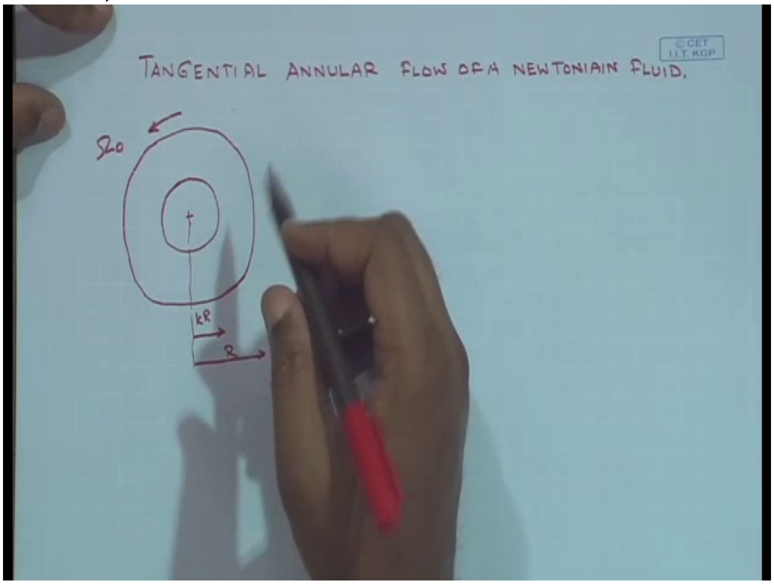

and the inner cylinder is stationary.

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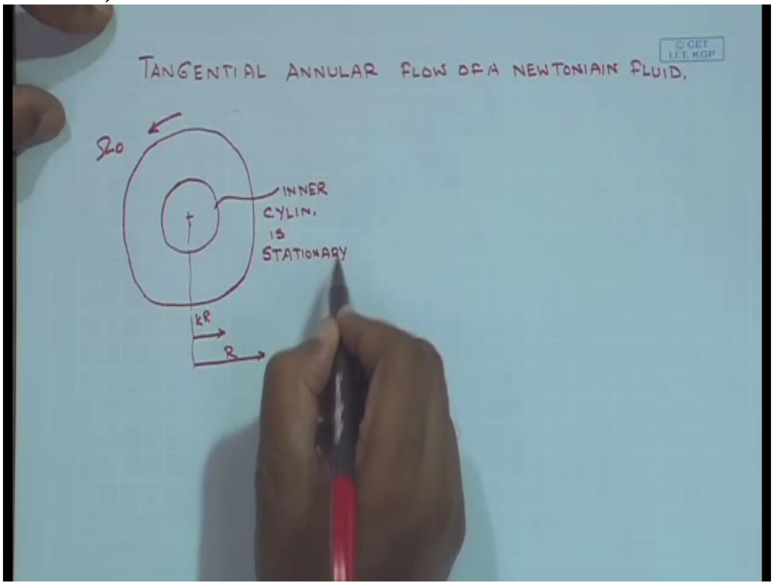

The outer is rotating. And of course you can

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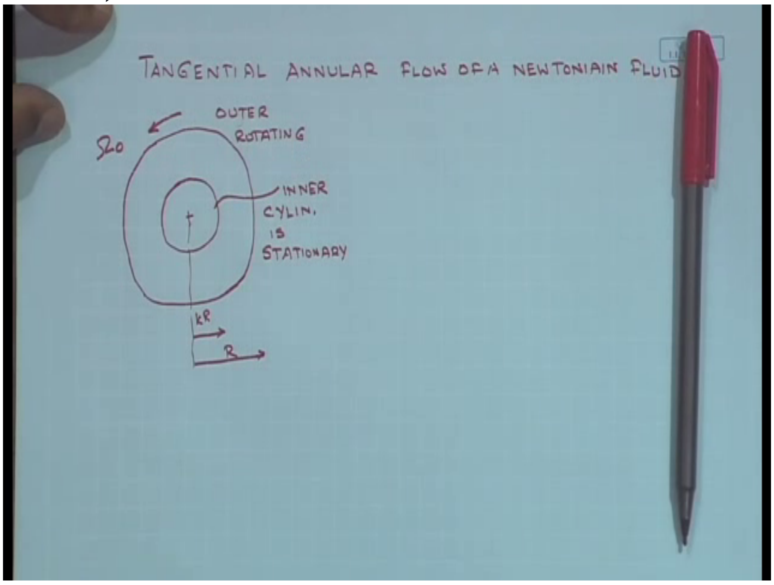

imagine what kind of a flow profile it's going to be; so here it will have a higher velocity and as progressively you come inside the velocity will start to decrease. So this is the kind of velocity profile which you would get and of course the velocity at this point must be equal to zero since

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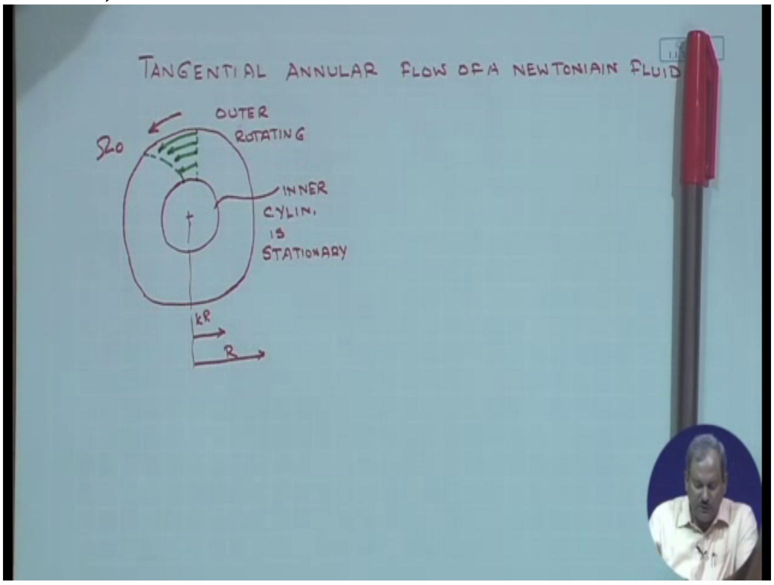

you have no-slip over here. So here you have no-slip and in this point also you have no-slip. So you, you can also write that v r and v z both are zeroes.

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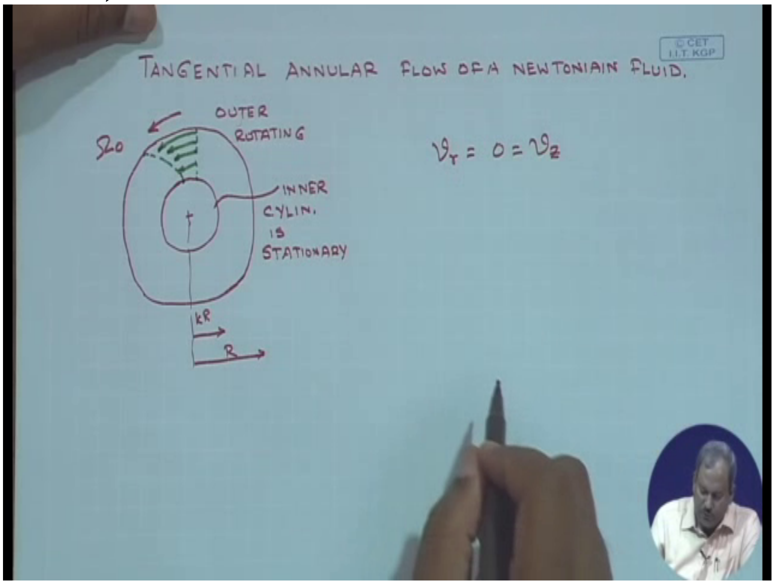

So it is 1 D flow that means liquid is only rotating, there is no velocity in the r direction and there is no velocity in the z direction so both in r and z direction the velocity would be zero. And since I don't have a pressure gradient, I also assume that there is no pressure gradient in the theta direction. So

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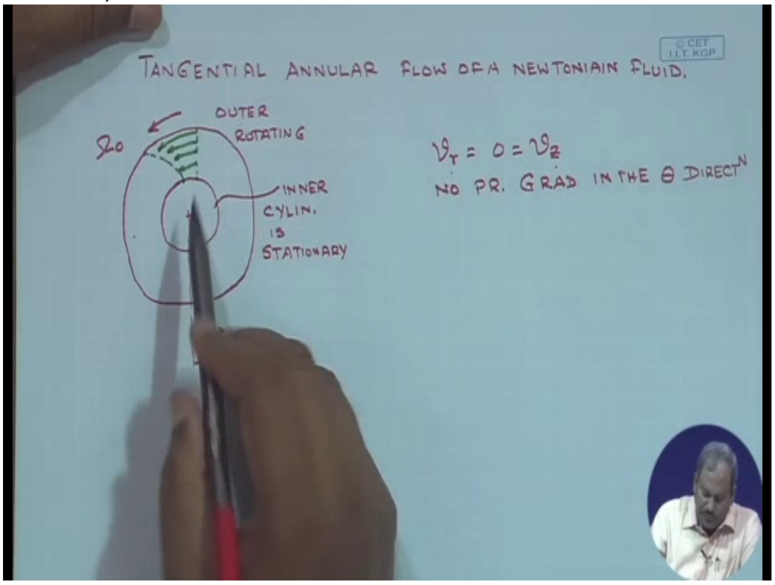

in the theta direction you can assume it is almost like a Couette flow because in this direction the liquid is dragged in the theta direction because of the motion of the outer cylinder. There exists no pressure gradient to drive the flow and since it's vertical therefore this plane is parallel to the ground, therefore there is no effect of gravity in the theta direction. So there is no pressure gradient and no body force in the theta direction. The only motion in the fluid in the theta direction is initiated by the motion of the outer cylinder which is rotating. So in that sense it's like a Couette flow, however there are

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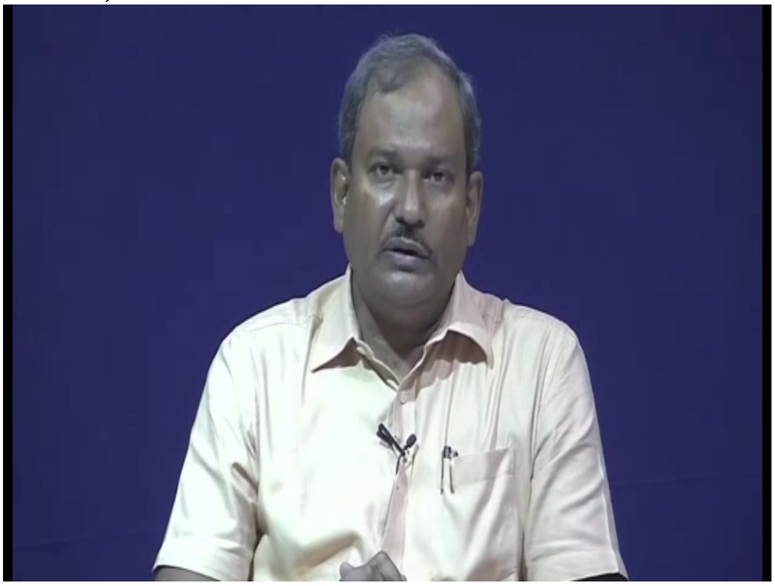

some dissimilarities. The Couette flow in this case is, is not in Cartesian coordinate system and the separation between the cylinders is significant, Ok such that the Couette flow approximation or the parallel plate approximation which we have done in the previous problems cannot be used in this case. So the two cylinders are sufficiently apart from each other which does not let you, does not allow you to use the approximation by which a cylindrical problem can be converted to a, to a Cartesian coordinate problem. So we have to deal with the equation, with the cylindrical Navier–Stokes equation, Navier–Stokes equation in cylindrical component.

So I have 3 choices. One is the component equation in r direction, the component equation for, the theta component motion equation and the z component motion. If you again think about this, since v r is zero there is no motion in the r direction, Ok. So the, since the two cylinders, the bottom of the cylinder is, is blocked therefore there is no motion in z direction as well. So if you solve the r component of Navier–Stokes equation and the theta component of Navier–Stokes equation you will simply get the expression for the pressure gradient. Because since there are no velocities the entire left, left hand side would be zero. Since there is no velocity gradient the viscous term would be zero so what you get is, for example for the z component minus d p d z is equal to rho g, Ok.

So rho, in the z, in the z direction since gravity is working so therefore we will simply have, if I write the z component equation, the variation in the, the variation of pressure in the z direction is to be related by a body force which in the case of the r comp, in the z component will simply be the gravity. So d p d z and d p d r, the corresponding body forces, in one case it will be the centrifugal force, and in other case it is going to be the gravity force.

So these can be used to obtain the pressure but these two expressions tell us nothing about the velocity expression. But my aim is to obtain the expression for velocity and somehow connect this velocity expression or the gradient of the velocity to the shear stress prevalent in the system. Once I have the shear stress evaluated, from the evaluated, from the known expression of velocity I should be able to relate this shear stress to force and force to torque. That is the approach that we should take. So I start with an equation that solution would give me the expression for velocity. Once I have the velocity I will find out what is the velocity gradient. If I know the velocity gradient, then I need to use the appropriate form for the stress of a cylindrical coordinate system which is slightly different from that, which is not as straightforward as that of a Cartesian coordinate system.

So from the velocity gradient and the expression for shear stress I should be able to get the complete expression for shear stress which definitely will contain the unknown parameter viscosity. Once I have the shear stress I then multiply it with the relevant area to obtain what is the force. Once I have the force I multiply that with the lever arm, in this case capital R to obtain what is the torque and again the, the viscosity that we had in the velocity expression, in the shear stress expression, in the force expression and in the torque expression will remain the only unknown in that expression.

So if I can measure what is the torque needed to rotate the outer cylinder with some velocity then I should be able to calculate using that expression what is the viscosity. Thus my aim is to obtain the velocity. In order to obtain the velocity I need to use that component of Navier–Stokes equation in which we can foresee some motion. And for that the only direction in which there is motion is the theta component. So we must write the theta component motion, theta component of the equation of motion in cylindrical coordinate system and try to see if we can obtain an expression for velocity. So that is what we are going to do next.

So if you write the theta component of the N S equation Navier–Stokes equation, the complete expression would be,

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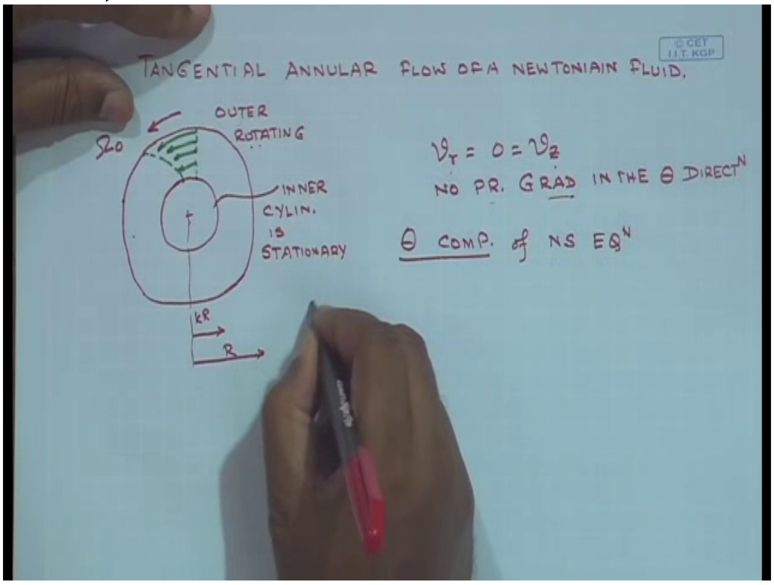

which is available in your textbook as well.

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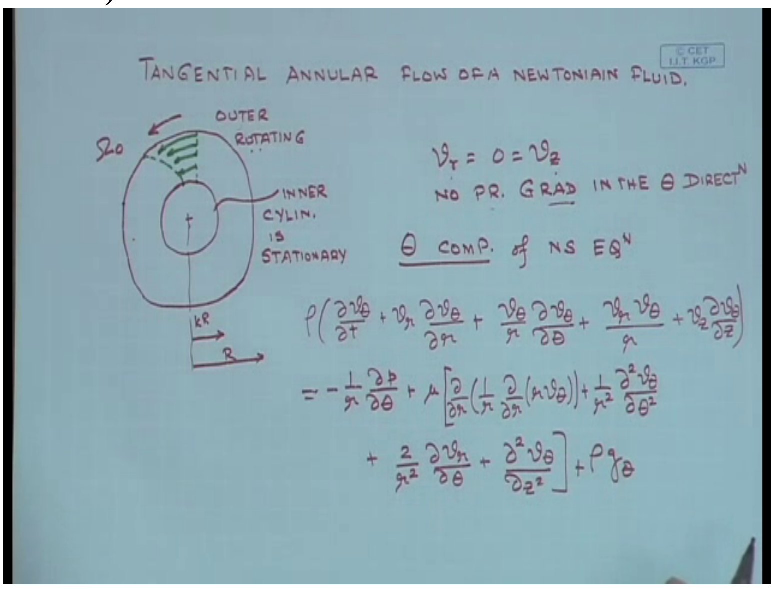

So this is the final form of the equation and even though it looks ominous to start with, but you would see that after you cancel the terms it is going to be very compact. So let's start with the first one which is the temporal term. So this would obviously be zero since we are dealing with steady state.

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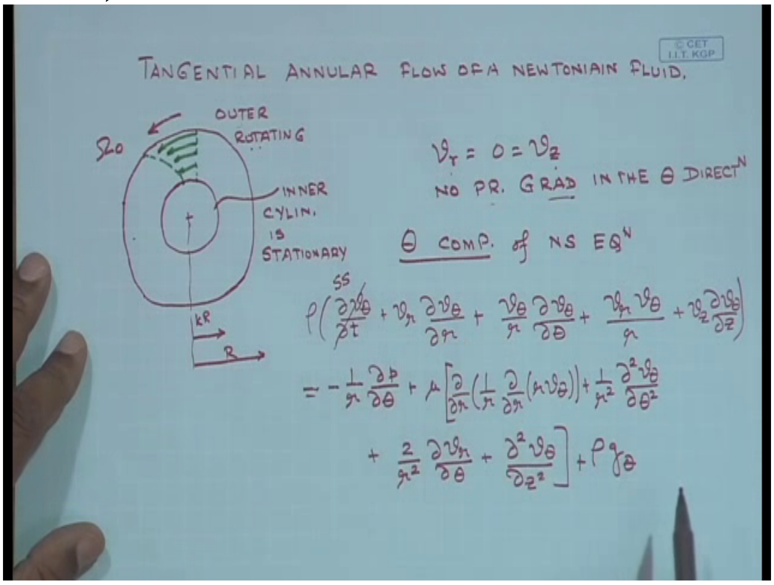

So since it's a steady state case then del theta del t would be equal to zero. Come to the next term. v theta, variation of v theta with r and I have a v r in here but there is no velocity in

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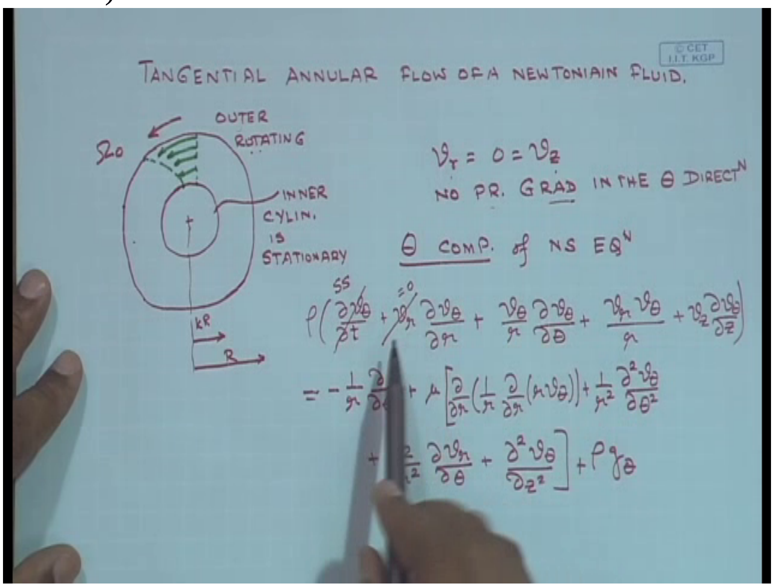

the r direction so v r would be zero. Here v theta is non zero but the del v theta del theta, this is equal to zero

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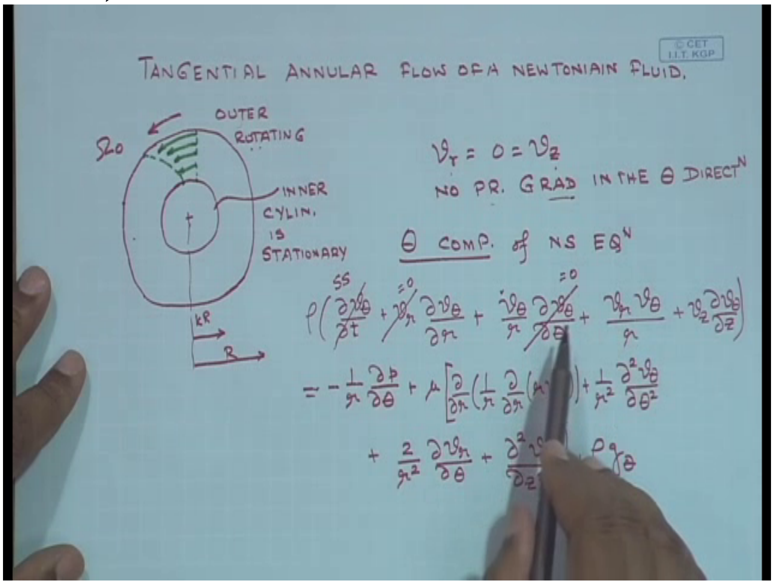

which would simply says which is the statement of the fact that theta, the velocity is not a function of theta. So if you fix the r the velocity is not a function of theta. Velocity is a function of r but velocity is not, if you fix the r the velocity is not a function of theta. And here v r is simply going to be zero

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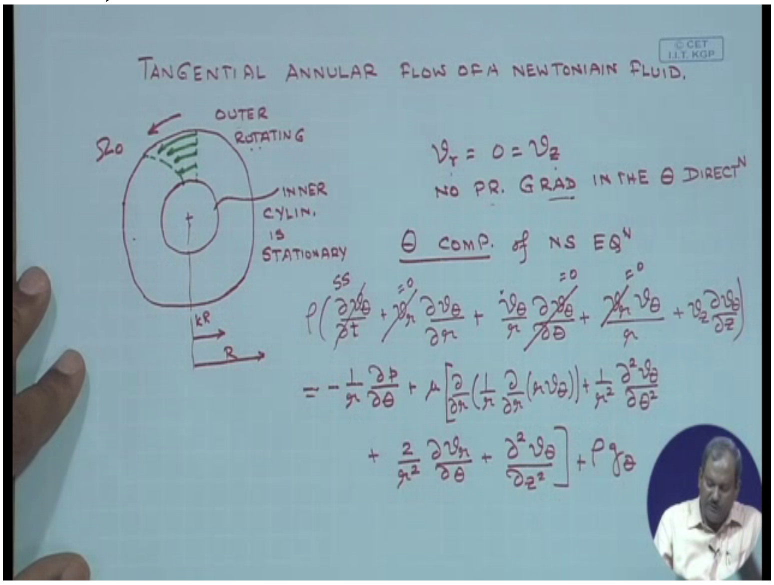

and in this case v z is going to be equal to zero.

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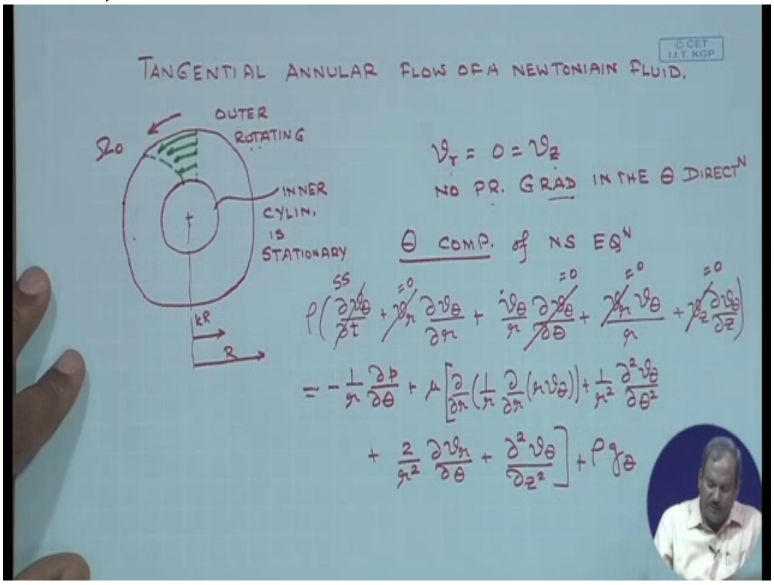

We have mentioned, we have assumed that there is no pressure gradient in the theta direction so it is simply the outer one; the outer one is simply rotating. There is nothing to force the liquid, no pressure gradient to force the liquid to move in the theta direction, so therefore this is going

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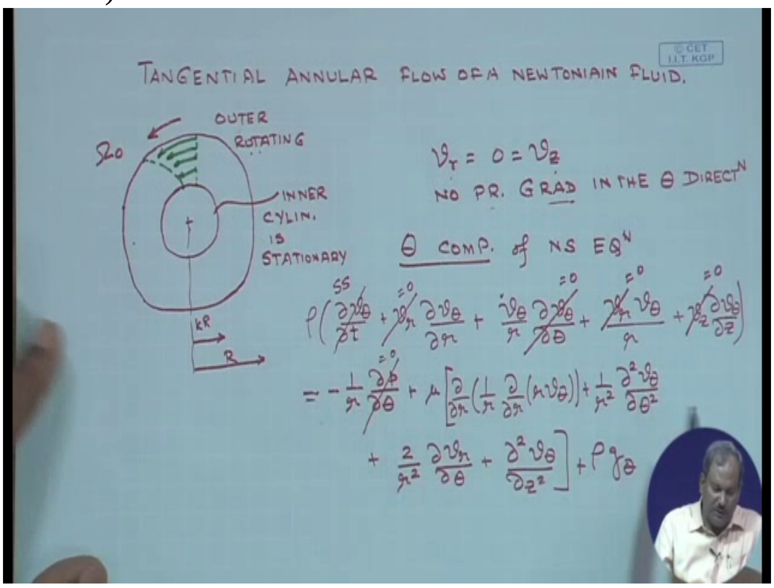

to be equal to zero and v theta is a function of r, v theta here is a function of r so I cannot, I cannot neglect, I cannot make a statement, I cannot neglect this term. So this term will remain in the governing

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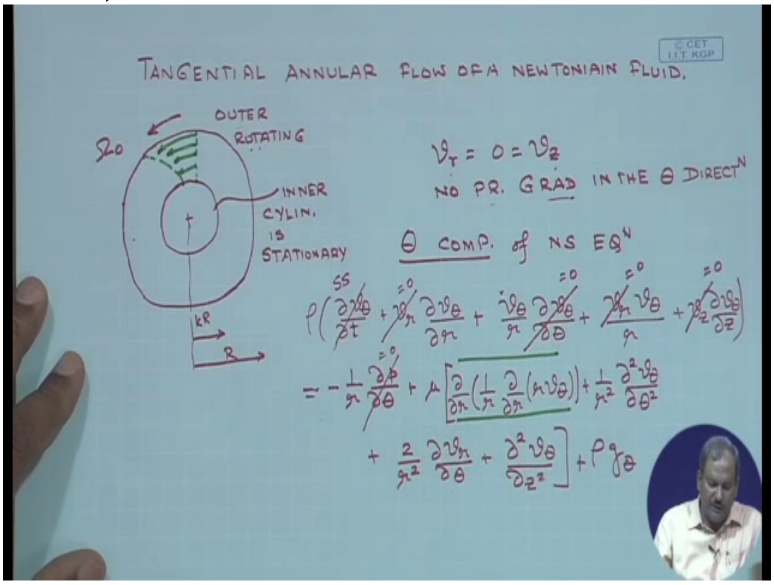

equation. If I move to the next term, v theta is not the function of theta so this will be equal to zero.

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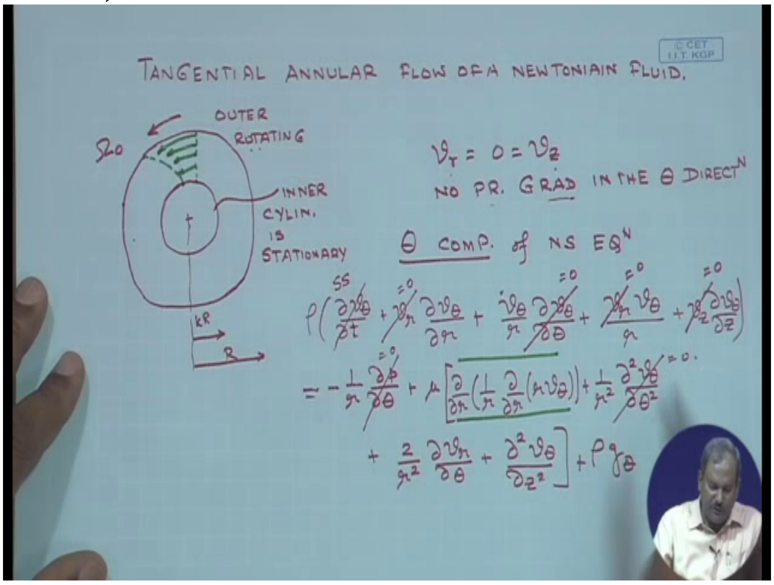

Then there is no question of v r so this part is going to be equal to zero.

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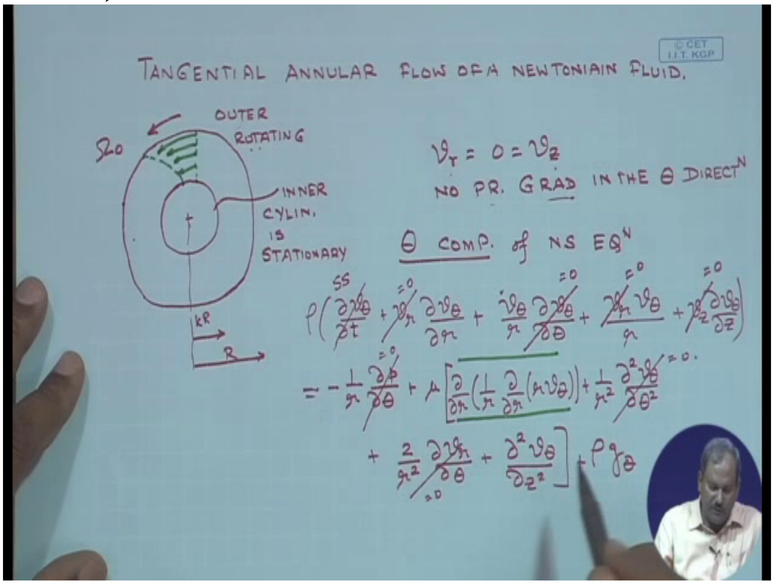

And v theta is not a function of z so this is going to be equal

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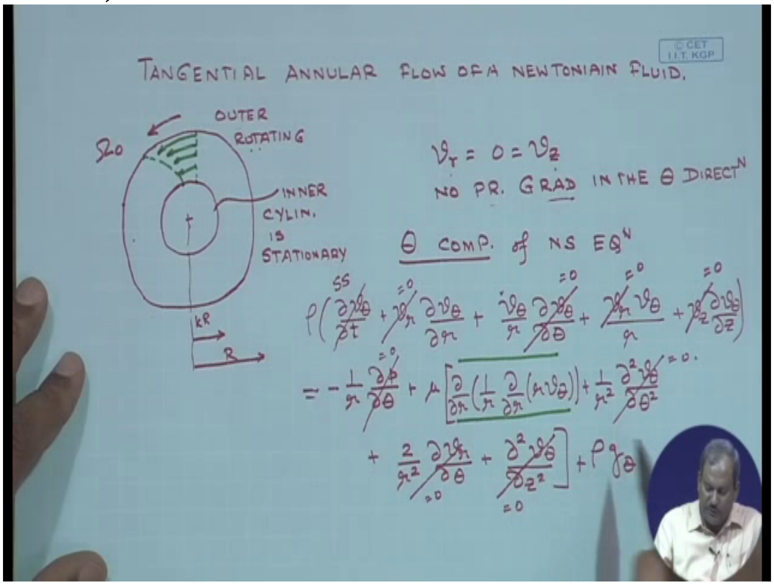

to zero as well as there is no body force, no component of g in the theta direction, in a vertical cylinder therefore this will be zero

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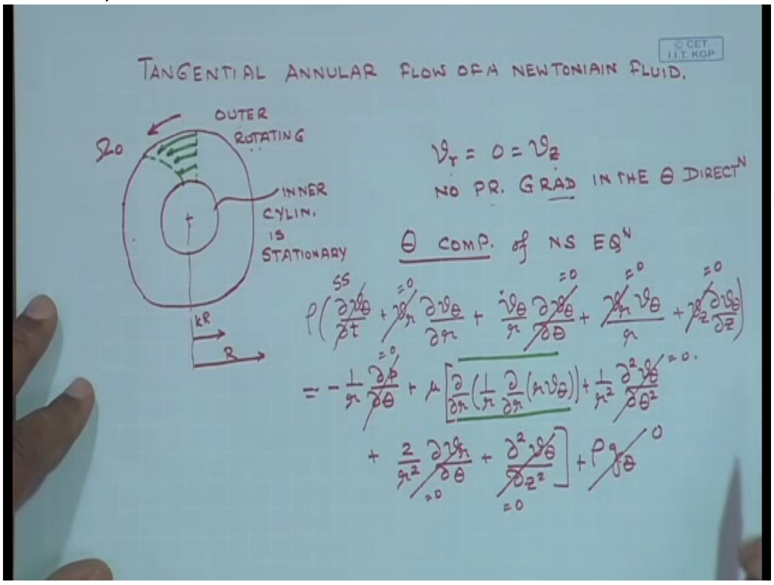

as well. So what you see from here is that the, from the entire equation, the entire Navier–Stokes equation, theta component of Navier–Stokes equation all terms are zero and you get the governing equation out of this to be zero equals d d r of 1 by r d d r of r v theta is equal to zero. So this is your governing equation. And look at it; look at how easily you could

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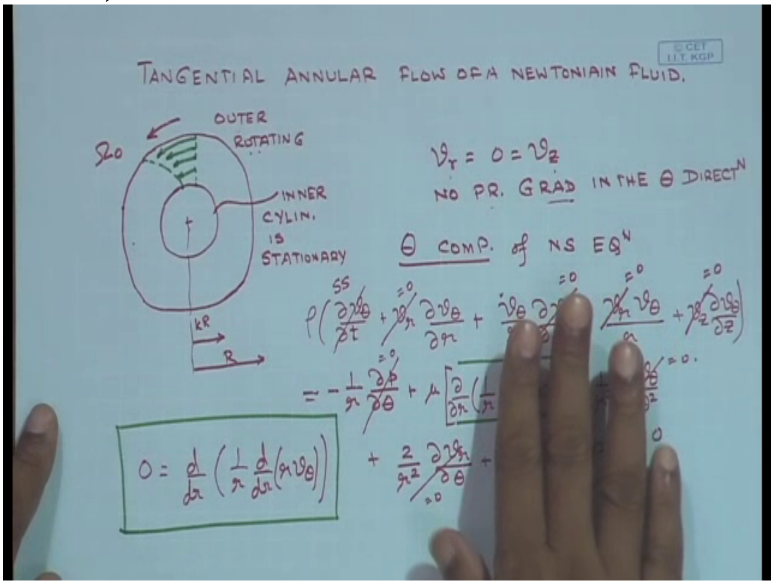

obtain this governing equation starting with a general equation that takes into account all possible variations. But you do not have these many complications so you can directly get from the theta component of Navier–Stokes equation what is your governing equation. Now think of the difficulty or the potential for errors if you are going to imagine a shell around this and try to figure out what are the momentum in term, out term,

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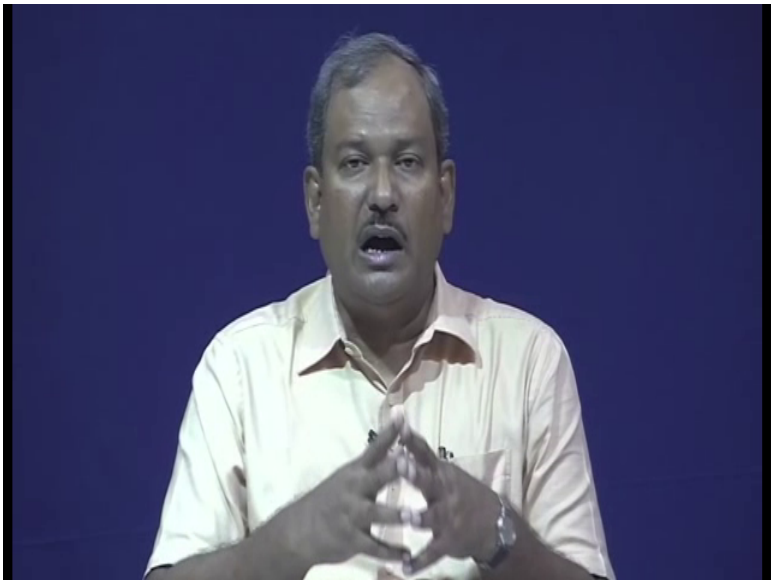

shear stress, pressure, gravity and so on. So here you don't have to do anything. That's the beauty of Navier–Stokes equation. You start out with the equation, cancel out the terms that are not relevant and you get a very compact expression in, in a matter of minutes, Ok and you can never be wrong so if you use Navier–Stokes equation. So with this equation we are now going to work on, on this equation to obtain what is the velocity expression. So we start with, we started with the expression is zero is d d r of 1 by r d d r of r v theta is equal to zero which,

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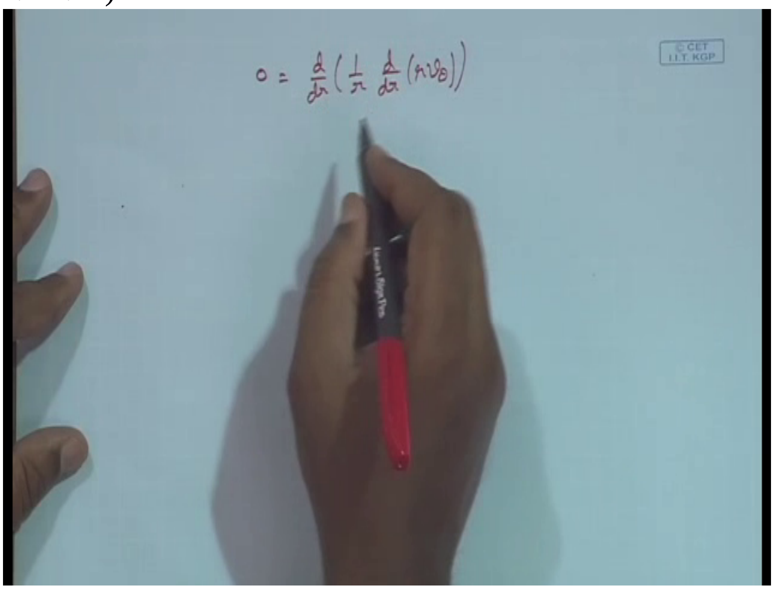

after integration would give you v theta is c 1 by 2 r plus c 2 by r and the two boundary

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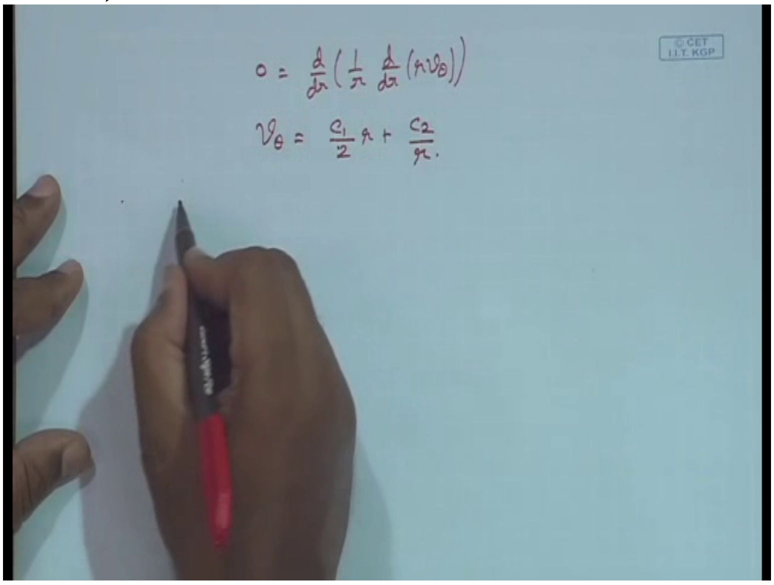

conditions are the no-slip conditions; that is at r equals kappa r, v theta is zero. So r equals k r, this is essentially the inner cylinder where as at r equals capital R

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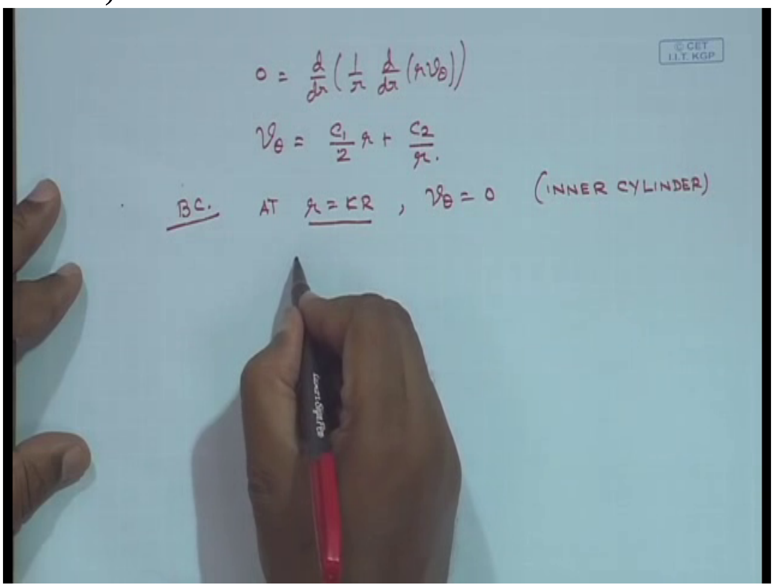

theta would be equal the angular velocity times r. This is what you have on the outer cylinder.

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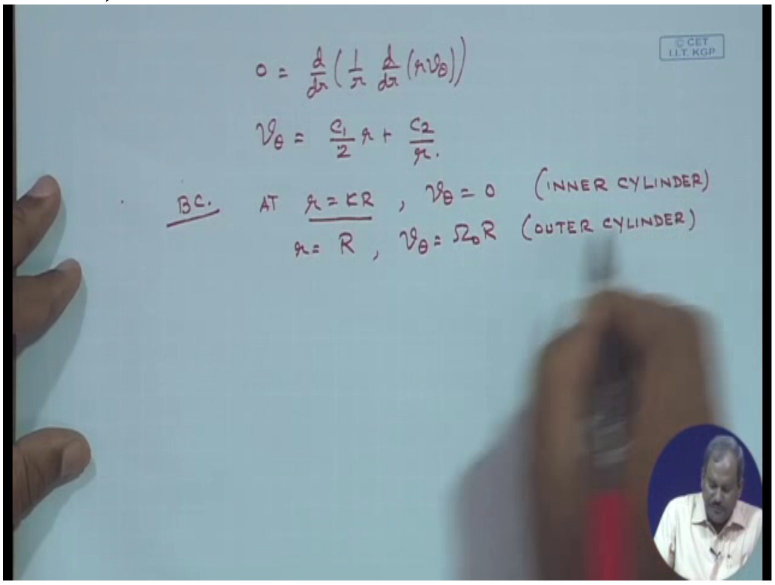

So the expression for velocity and the boundary conditions

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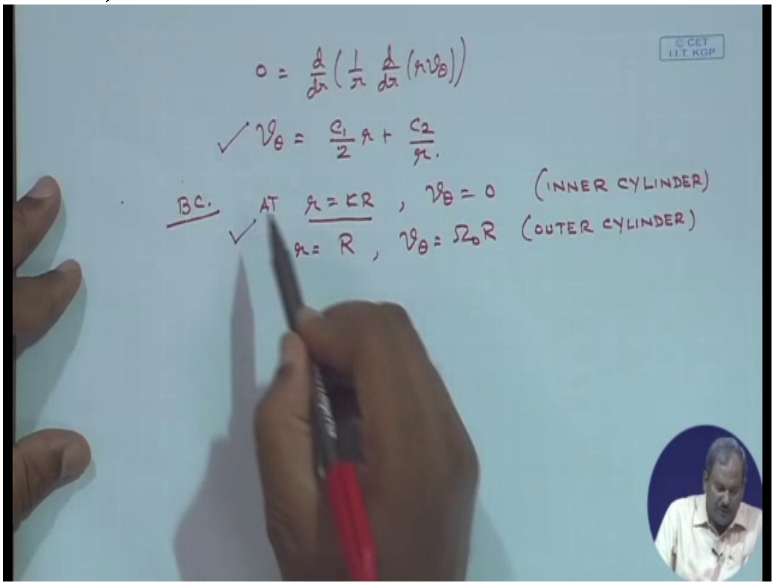

would, when you put these two together you obtain the expression of velo, v theta to be omega naught r k r by small r minus r by k r divided by k minus 1 by k. The velocity is a function of the imposed

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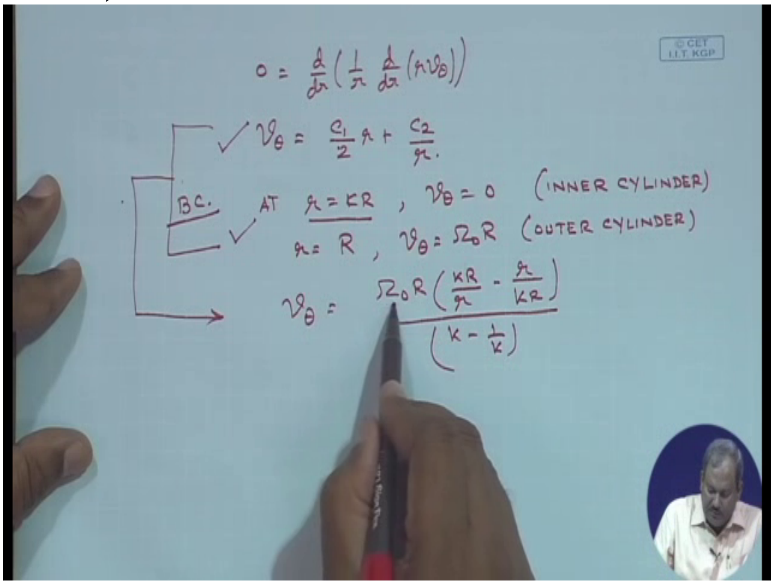

condition which is the velocity with which the outer one is being rotated and all the rest are geometric parameters and therefore v theta is a function, function of r only, function of r.

Now as I mentioned

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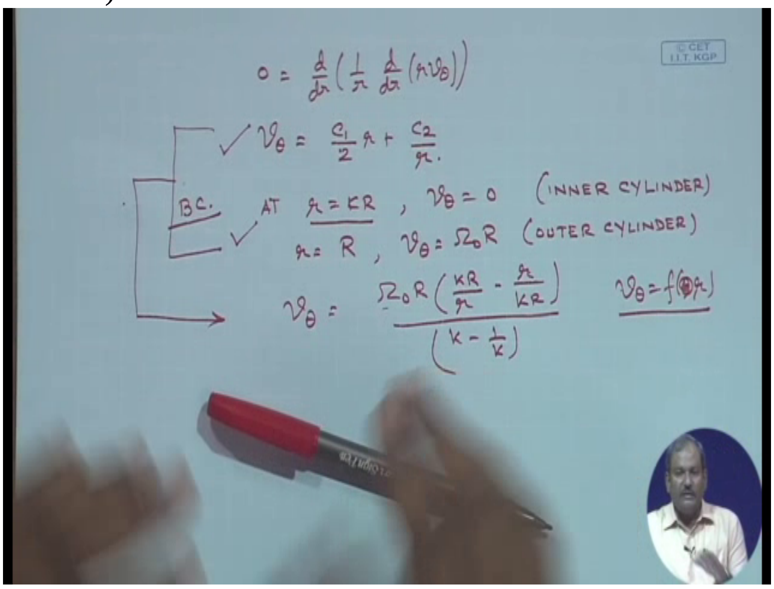

to you before

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is what we know as the shear stress for a simple Cartesian coordinate system will have slightly different form because of the transformation from a Cartesian coordinate system to a cylindrical system. So the expression for a shear stress in a cylindrical or a spherical coordinate system would simply not be equal to mu times the velocity gradient. There would be other components present in it and in any text you would see what is the corresponding form of the equation, corresponding form of the shear stress for cylindrical coordinate systems and for spherical coordinate systems.

Here you would see that the non-zero component of the shear stress, the velocity is in the theta direction and the theta component of the momentum, because of the variation of velocity in the r direction would get, would get transported in the rth direction. So tau has the first subscript to be equal to theta because theta denotes the direction, the momentum the theta component of momentum which is non zero in this specific case. So theta is the direction in which you have velocity, theta is the direction in which you have momentum. Due to viscosity, this theta momentum gets transported in the rth direction as the velocity varies in the rth direction.

Velocity does not vary in the theta direction so therefore tau theta theta is zero, velocity does not vary in the z direction. Therefore tau z theta is zero. However since velocity varies in the r direction, tau r theta will be non zero. So the only shear stress expression that you should read, you should find out from your text is the expression of tau r theta from your text book. So once you have the expression of tau r theta from your textbook for a cylindrical system then your job is done and that tau r theta, expression for that tau r theta will contain the velocity gradient. The theta, the theta component, velocity, theta component velocity gradient of that. So the job is to identify which theta, which tau would be non-zero which is simple in this case and find the expression of that tau from your textbook and plug in the expression of velocity which you have obtained by the solution of Navier–Stokes equation. So let's see how that's done.

So here what you have is the expression for velocity and you realize that tau r theta is the only non-zero tau

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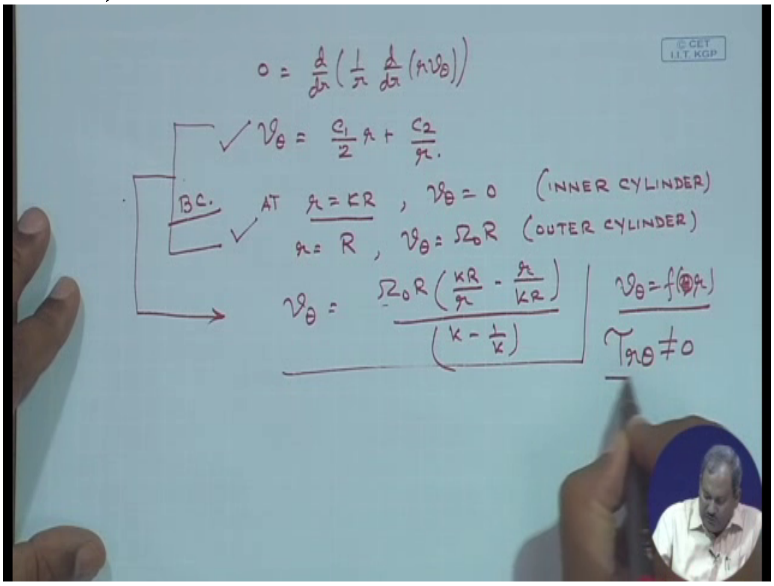

you have in the system and from the text you know that, you would be able to see that tau r theta is mu times r del del r of v theta by r plus 1 by r del v r by del theta. So this is the expression for

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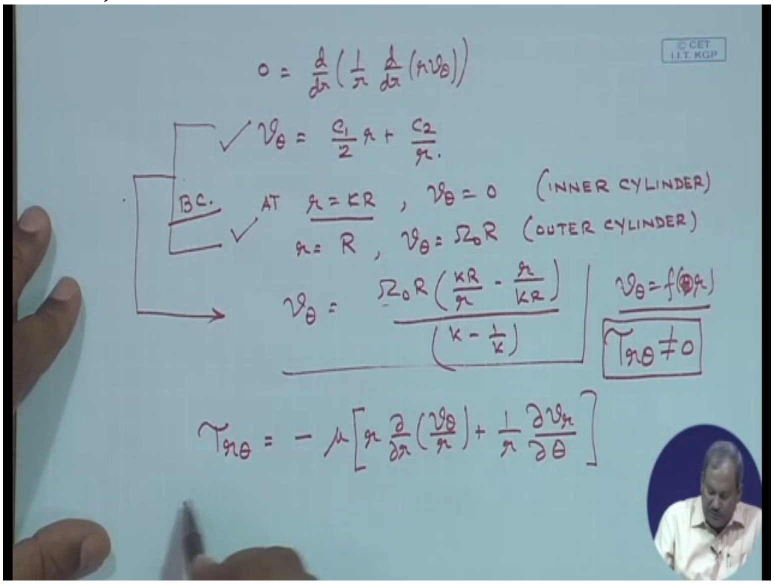

tau r theta for a cylindrical coordinate system and here you know that v r is zero so therefore this is going to be equal to zero so

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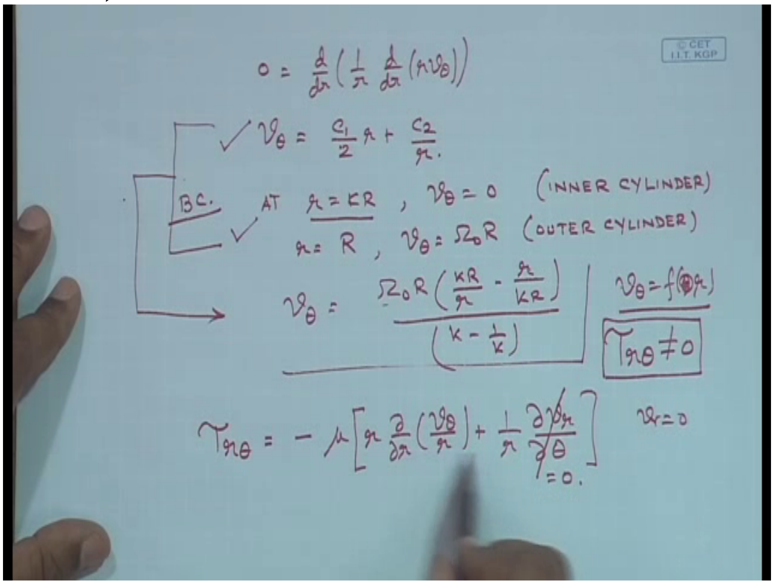

your tau r theta would simply be equal to the first term. So this would be the expression for tau r theta.

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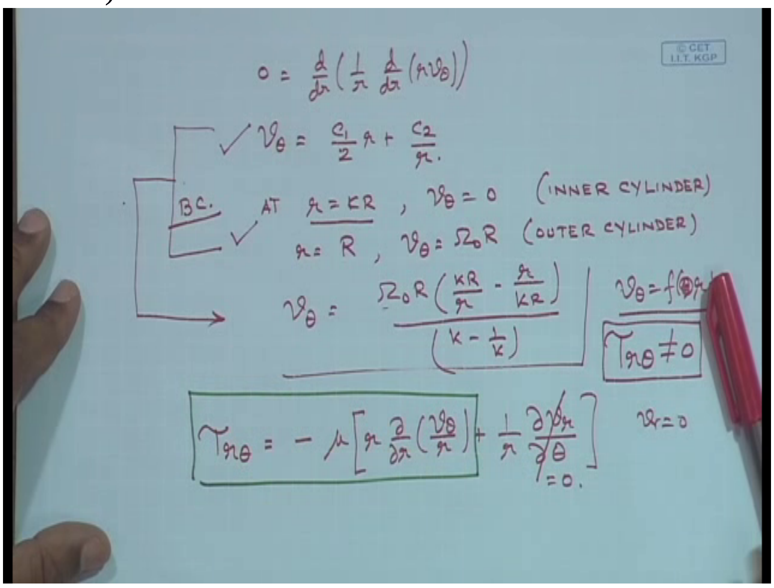

And in this expression we have to put in the expression for v theta in

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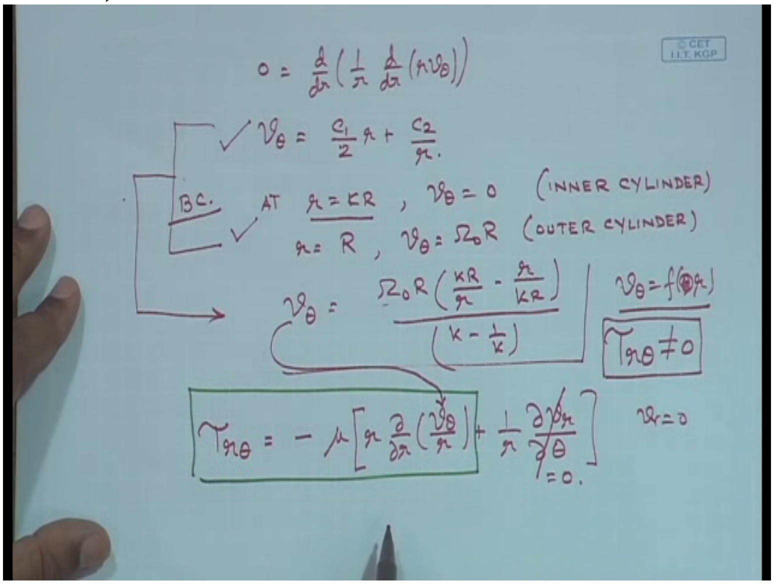

here and do the del del r of that and this would look like as mu times r d d r no need to use the partial signs in v theta is the function of r only it is not a function of theta or of z and then what you get is, so this is the expression that you would get

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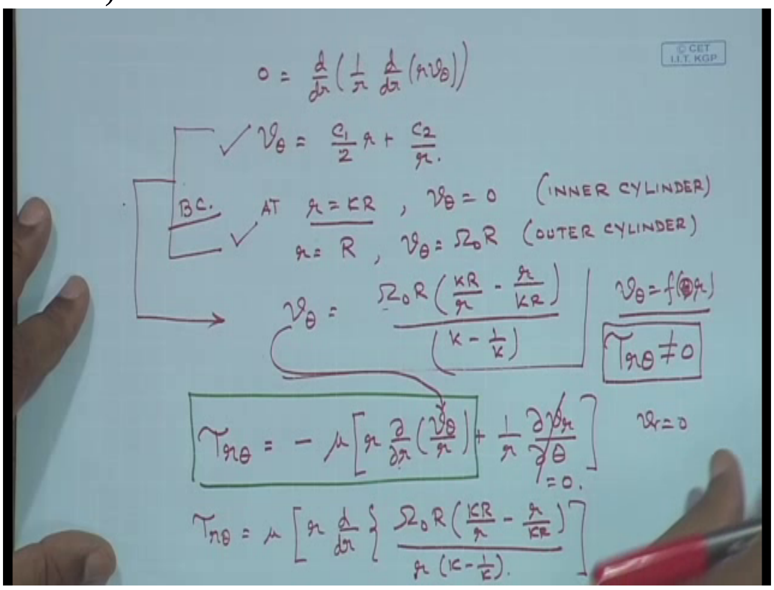

and once you do this I will skip the intermediate steps as they do not require any more understanding part tau r theta final expression would be minus 2 mu omega naught r square 1 by r square times kappa square minus 1 minus kappa square. So this is the expression for tau r theta.

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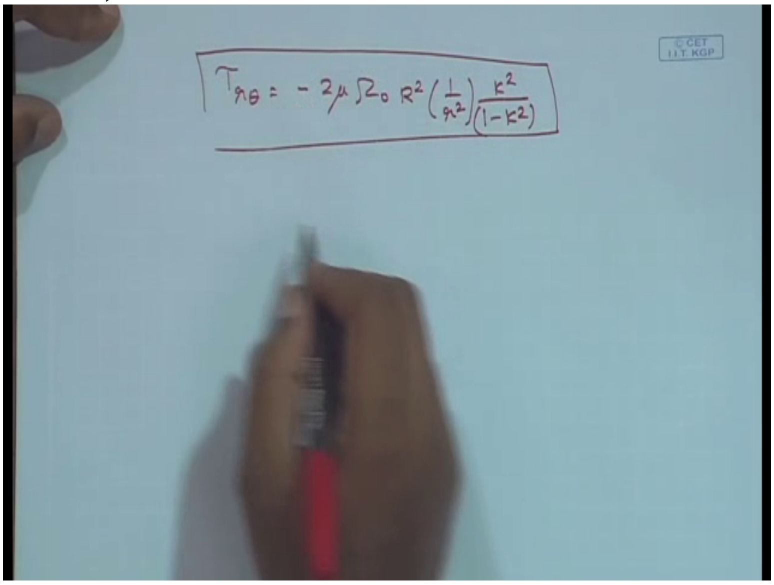

The torque t required to turn the outer shaft is equal to the force. The force would be tau r theta but it is evaluated at r equals capital R since it is the torque needed to rotate the outer shaft, so this is tau r theta and since this is on the fluid, on the shaft would be minus tau r theta multiplied by the area which would be equal to twice pi r L so this is area, this is stress and so together

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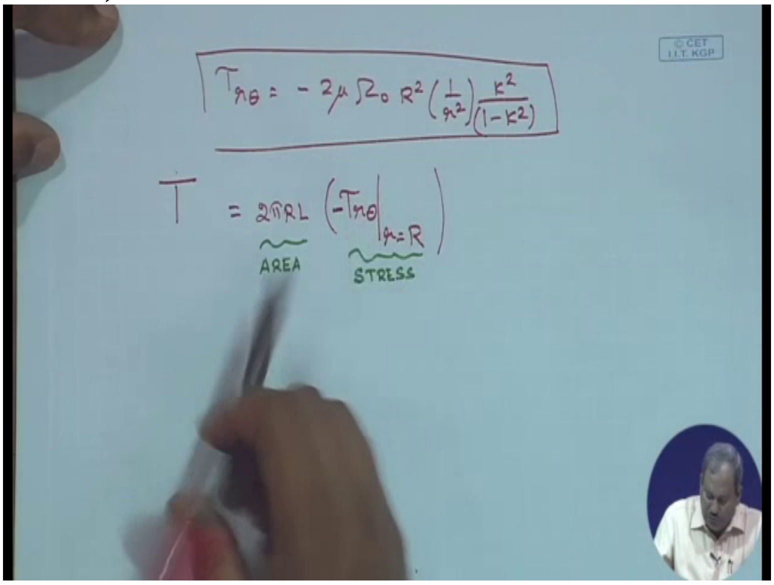

it's force and it should be multiplied with the lever arm which

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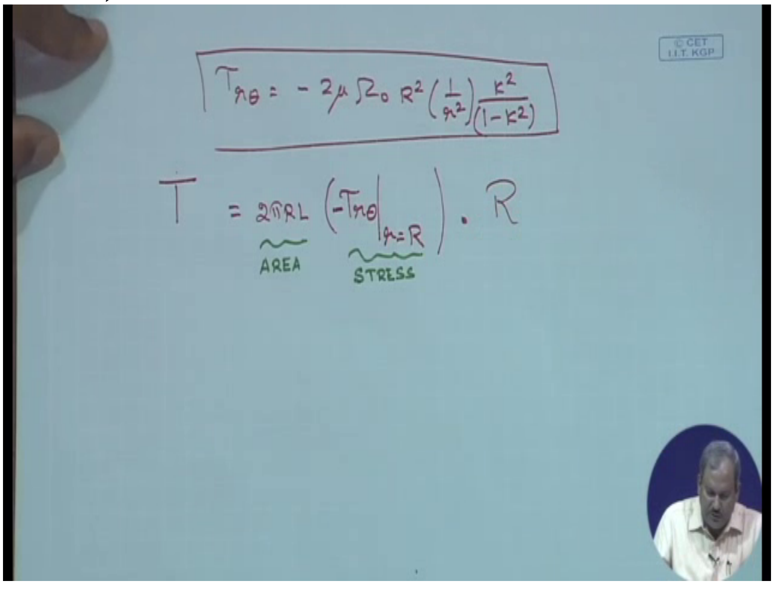

in this case is r. So this expression for tau r theta is then plugged in here and you get a

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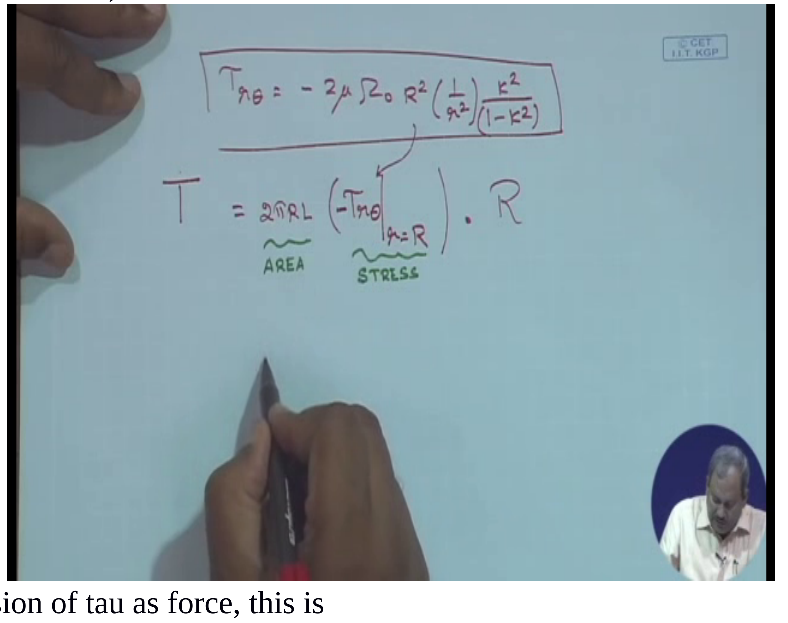

compact expression of tau as force, this is

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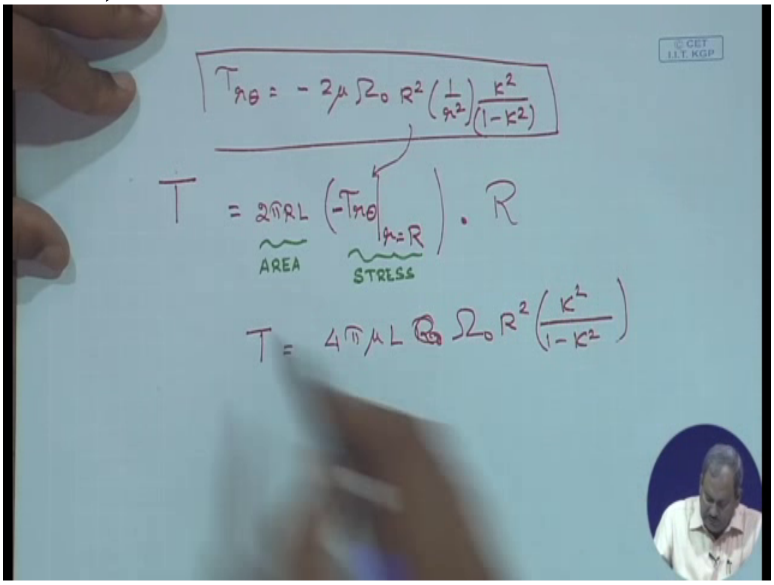

the final expression for shear stress, for torque that you would obtain. So this is a very good model for friction bearing and the viscometers which are based on this expression or this concept are known as Couette Hatschek viscometer.

So what we have done so far is we have obtained an expression for torque when we have two cylinders, one is kept stationary, the other one, outer one is being rotated and the torque expression contains the geometric parameters which are L, r and kappa, the operational parameters which are omega naught and the property of the fluid, property of liquid that is of interest as tau.

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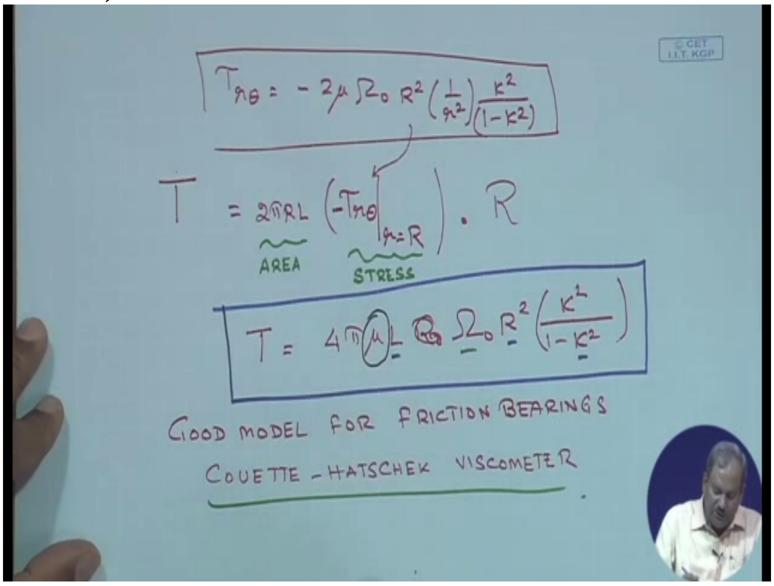

So once T is measured, the torque is measured, the velocity, the geometry, the length et cetera are known then the only unknown mu in this case can be accurately calculated. The viscometers which are based on this principle are known as Couette Hatschek viscometer. There is only one more thing to add before we close

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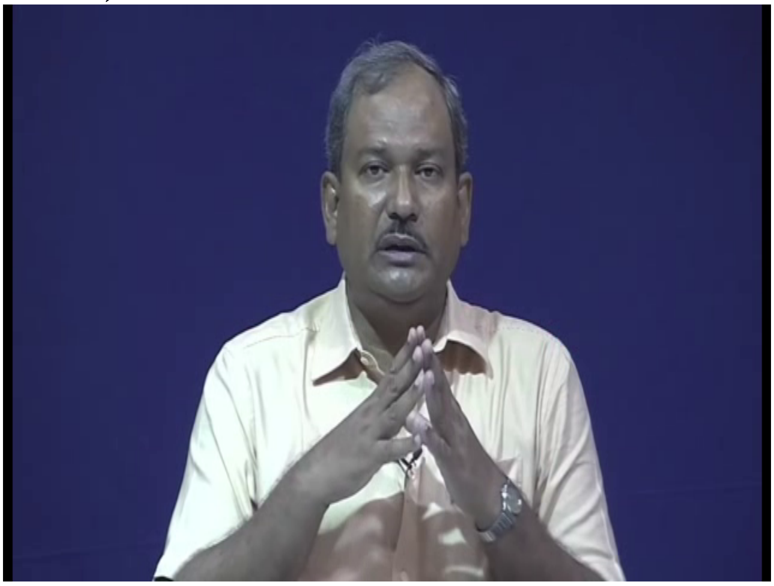

the discussion on rotating viscometers, viscometer where one is rotated. Here you have seen that inner cylinder is kept stationary, the outer one is being rotated. What happens if we do the reverse way? That is the outer is kept stationary, the inner is being rotated. If that's the case, then also this formula, slight variation of this formula will still hold but

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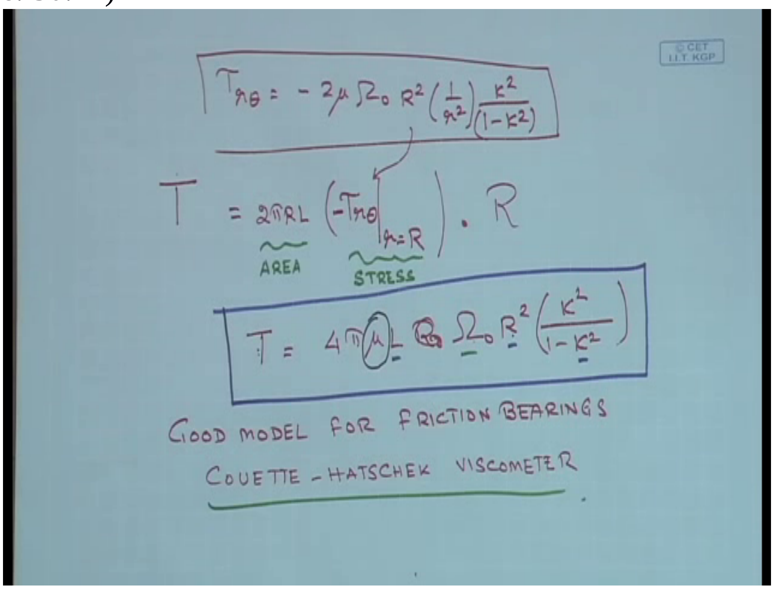

the

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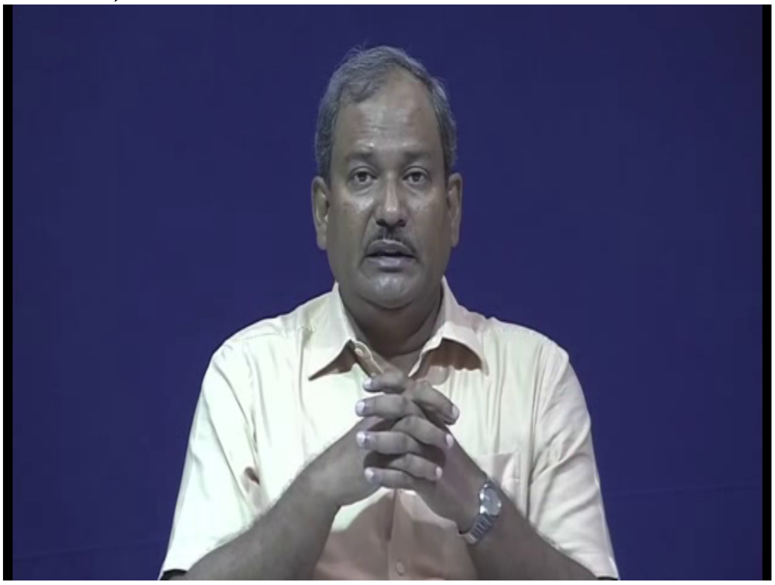

the , the expression that we have obtained for this specific case, let's review what are the assumptions that we have made in this. The first and prominent assumption that we have made, that it is one dimensional steady state laminar flow. The moment you start to rotate the inner cylinder keeping the outer cylinder as fixed, the packets of fluid will have a tendency to move towards the outer edge across the flow because the packets of fluid which are close to the inner rotating inner cylinder, they have a high velocity. What they see next to them is a region of low velocity.

So the tendency of that fluid packet, faster moving fluid packet would be due to centrifugal action to move towards the outer edge across the flow. When that happens the straight streamline nature of the flow is going to get disturbed and our assumption that we, our assumption of laminar flow will be put into severe test. So in order to maintain the assumptions in the system it is always customary to rotate the outer one and not the inner one such that the laminar flow can be maintained for a longer duration for a higher value of omega naught, higher value of the rotational speed and you can still use the expression to obtain what is the unknown mu.

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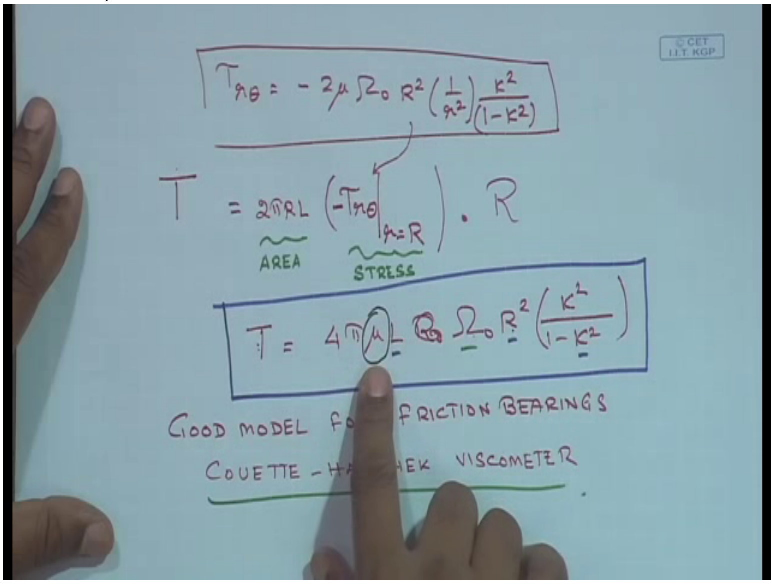

So the applicability of this relation in terms of laminarity would be sustained to a higher value of omega naught if the inner cylinder is kept stationary and the outer one is rotated. So that's something one has to keep in mind while working with viscometers where one cylinder is kept stationary, the outer, the other one is rotating. It's always the outer one which rotates and not the inner one,

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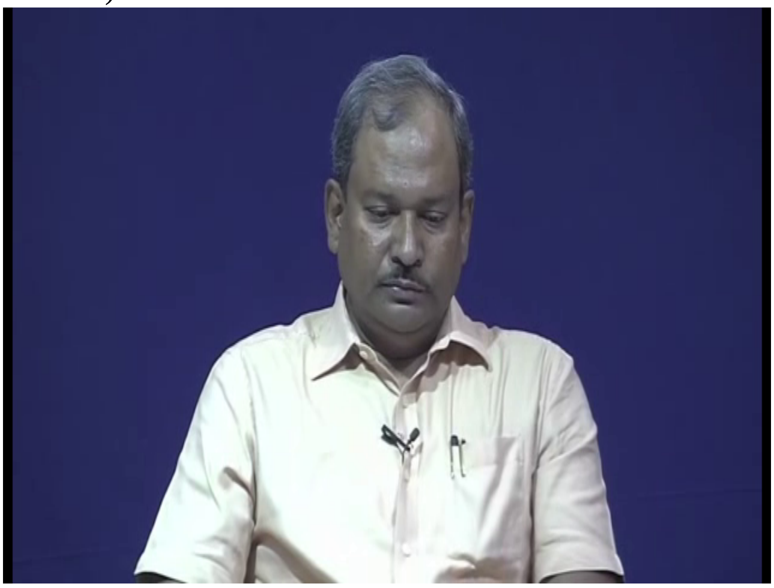

thank you.