Transport Phenomena
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Lecture 12
Equations of Change for Isothermal Systems (Cont.)

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We have so far covered the Navier–Stokes equation, the equation of motion, the special form of which is Euler's equation and we have seen how in conjunction with equation of continuity, the Navier–Stokes equation can be simplified for different situations and to obtain the governing equation for flow. Once the governing equation is obtained, it's easy to solve which boundary condition, which appropriate boundary condition which that defines the physics of the process. We, in the previous classes, we have seen the some of the simple problems which we have done using shell momentum balance. We have reworked those problems using Navier–Stokes equation. In this and two more classes we would look at different problems which are slightly more complicated wherein we would be able to appreciate the utility of Navier–Stokes equation to obtain the governing equation for slope in such complicated systems. So I would draw your attention to this

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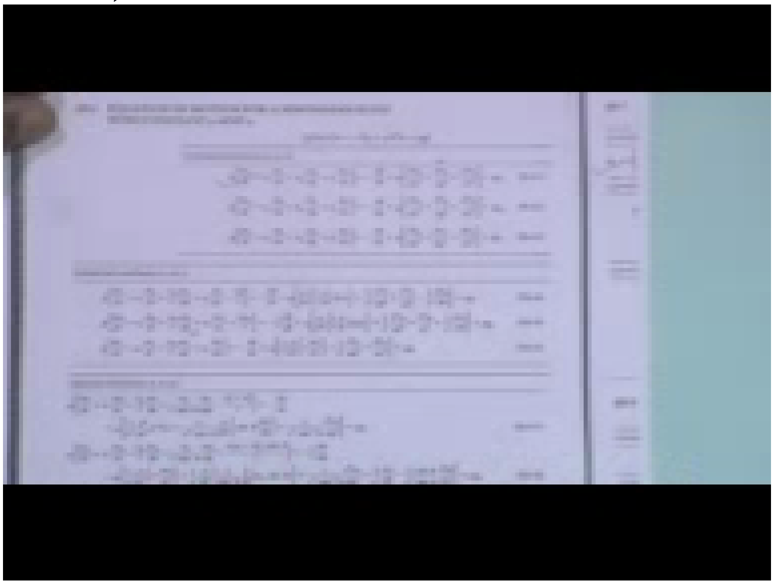

which is you, I think you should be able to see it more clearly in your textbook where the equation of motion for a Newtonian fluid with constant density and viscosity are provided for Cartesian coordinate systems,

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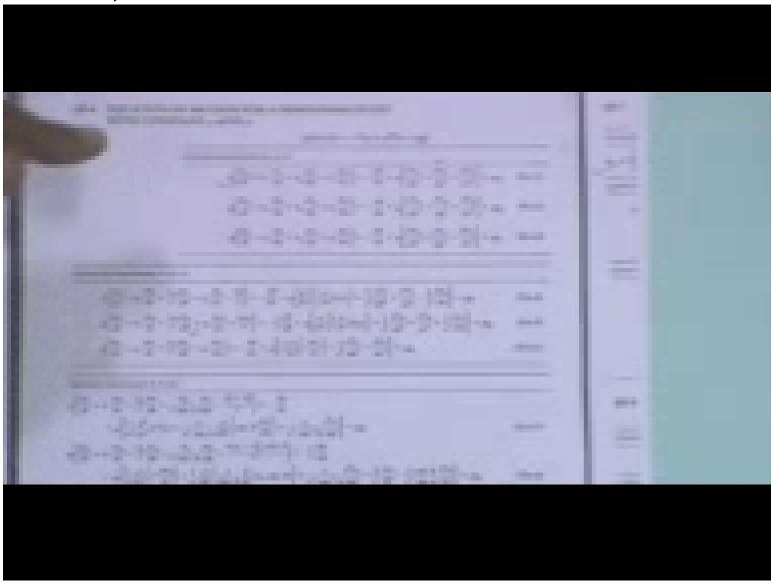

cylindrical coordinate systems and for spherical coordinate systems. In the previous table the same equations

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are provided in terms of

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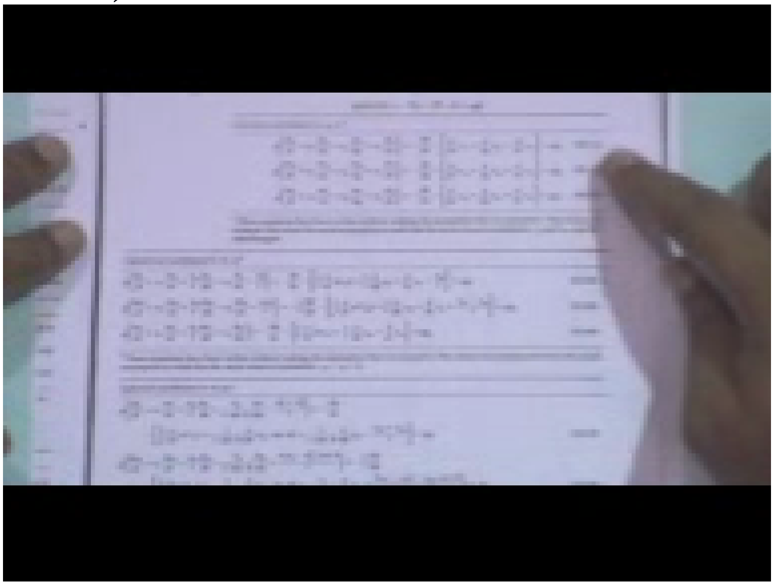

shear stress and the other equations is in terms of velocity gradient, again in Cartesian, cylindrical

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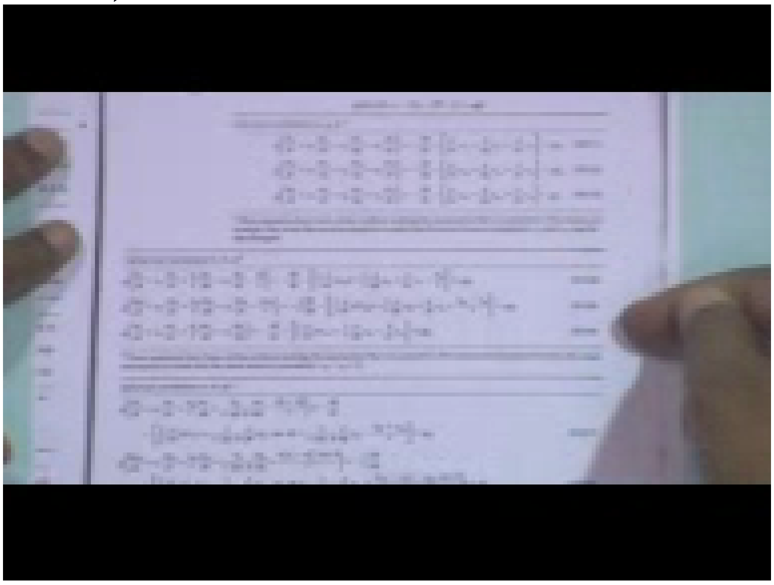

and spherical coordinates.

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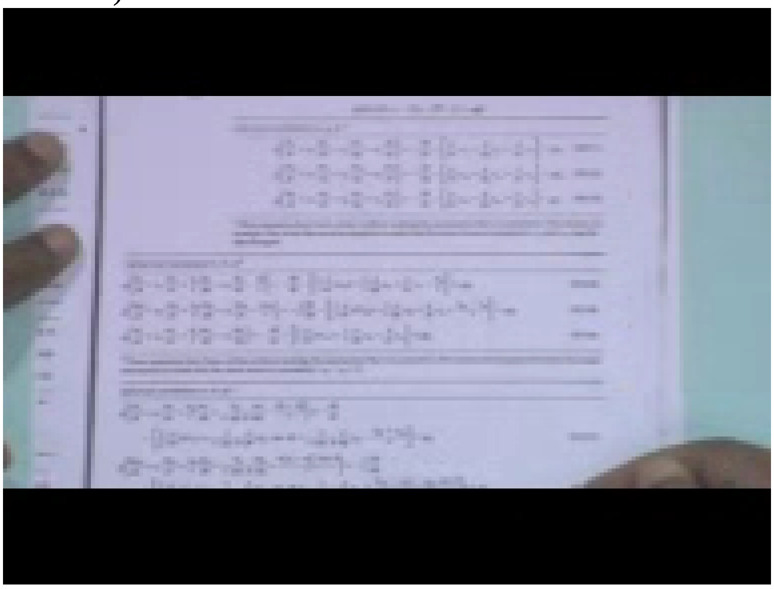

So depending on which kind of geometry you are handling, you would, should be able to choose which equation out of this

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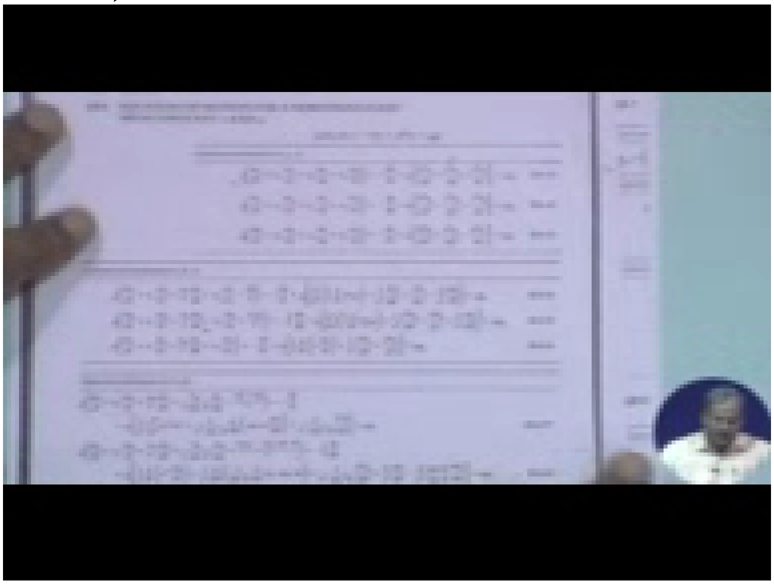

And in most of the cases what you do is

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you choose the equation which is in the direction of principal motion. There is flow in x direction, let's say on an inclined plane then you should choose the x component of the Navier–Stokes equation and cancel the terms which are not relevant to obtain the governing equation. What I would do in this class is I would give a problem for you to try on and I will also provide the answers. If you have any questions you can ask me and either me or the t a’s for this course would try to answer your query. So the first problem, I would only introduce the problem to you and give you the final solution, Ok. It would be your job to arrive at the solution based on whatever we have discussed so far.

The problem that we have is a flow between, a laminar flow between two

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infinite parallel plates. The upper plate moves to the right at, with a velocity u equals 3 meter per second.

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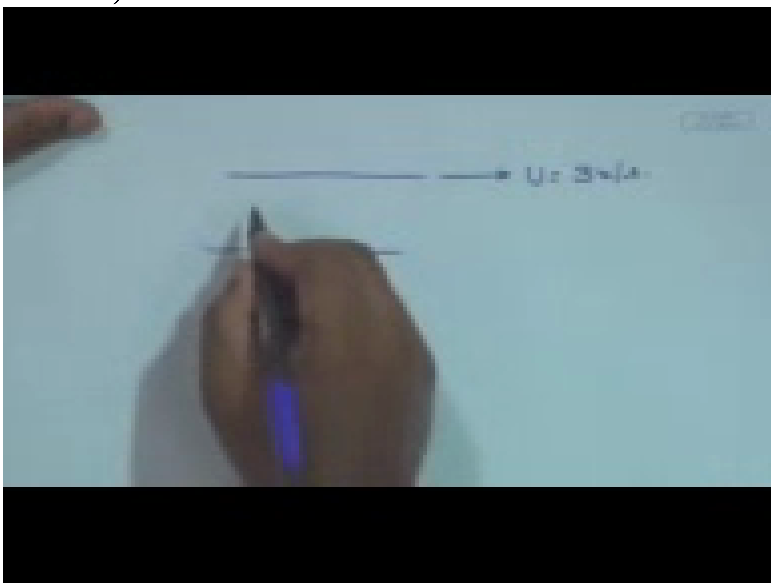

There is no pressure gradient, pressure variation in the, pressure variation in the x direction, so this is

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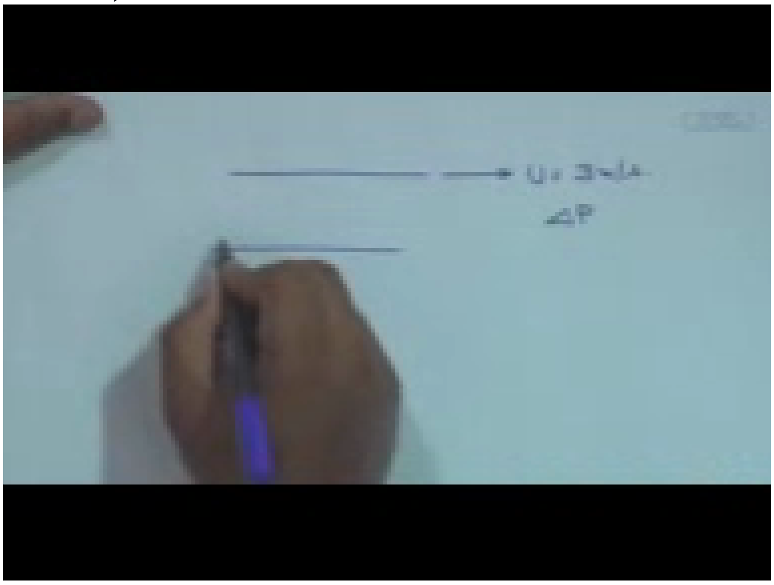

your y and this one is the x direction. So there is no pressure gradient in the x direction. However there is an electric field

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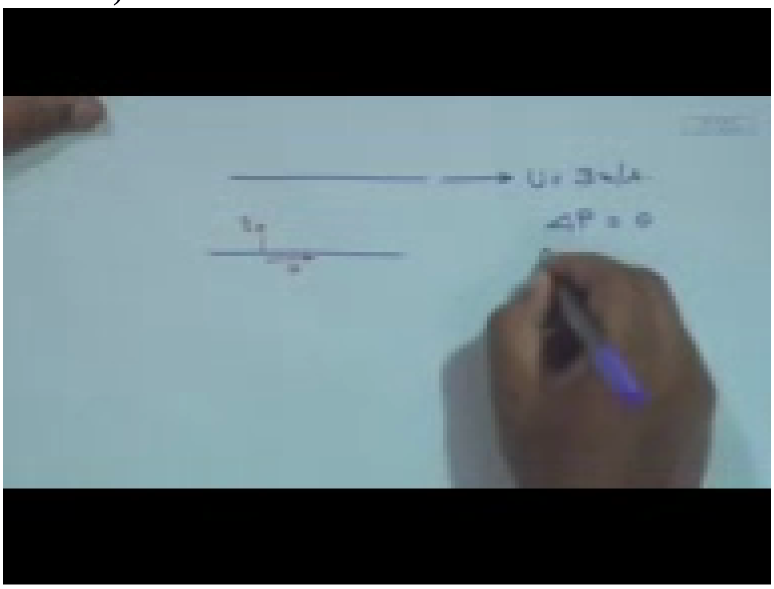

which is given by rho B x to be 800 Newton per meter square. So the body force provided, body force

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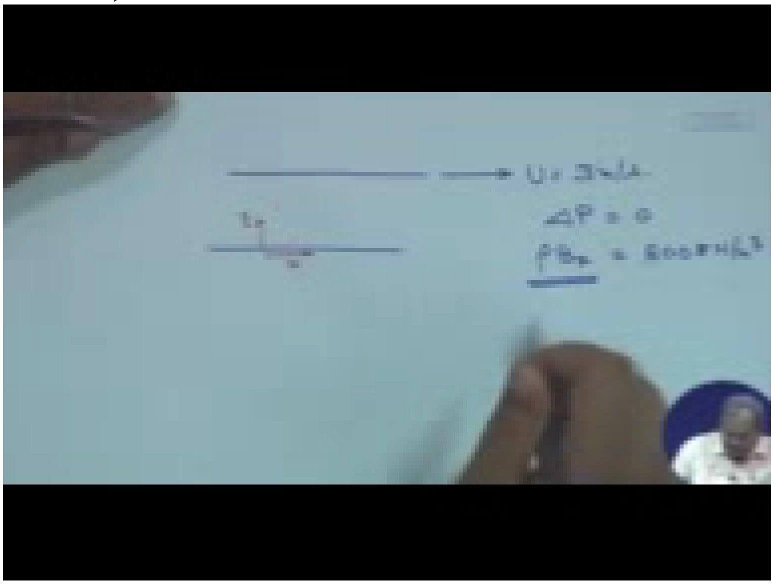

provided by an electric, electric field is 800 Newton per meter cube. The gap between these two plates which I call as h is point 1 millimeter and the viscosity of

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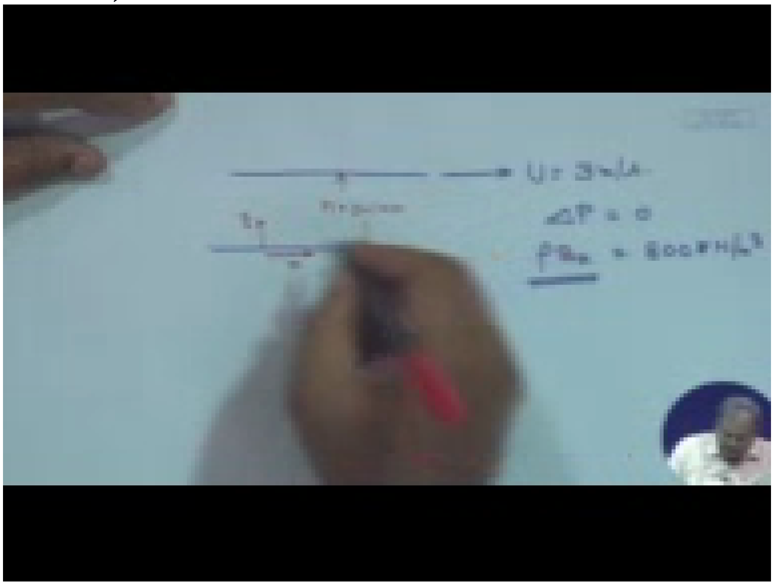

the liquid, mu of liquid is 0 point 0 2 kg meters second,

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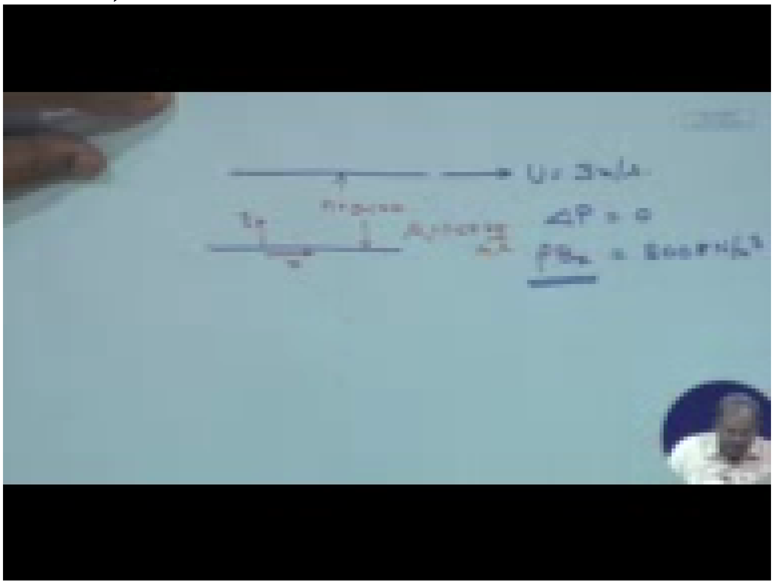

Ok. What you need to do is you have to find out what is the velocity profile, what is u as function of y, the expression of u as a function of y and

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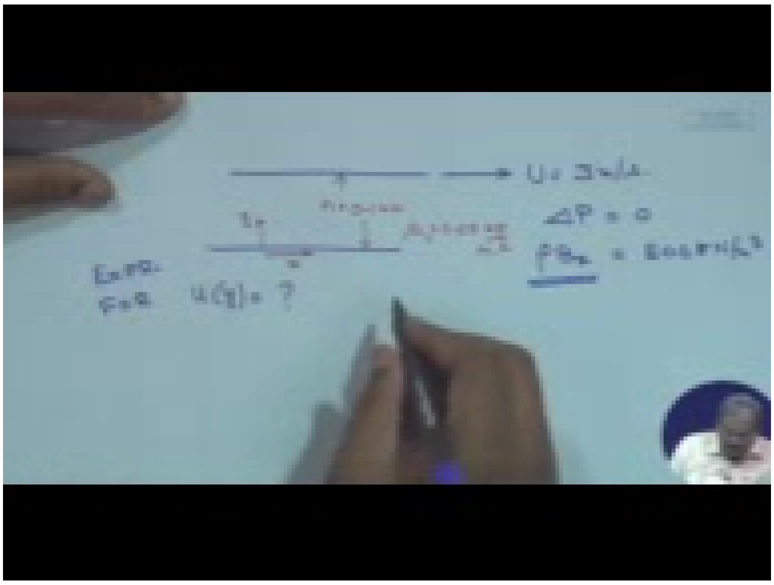

the coordinate system is given like this, so that y equal to zero is at the lower plate. So part 1 of the

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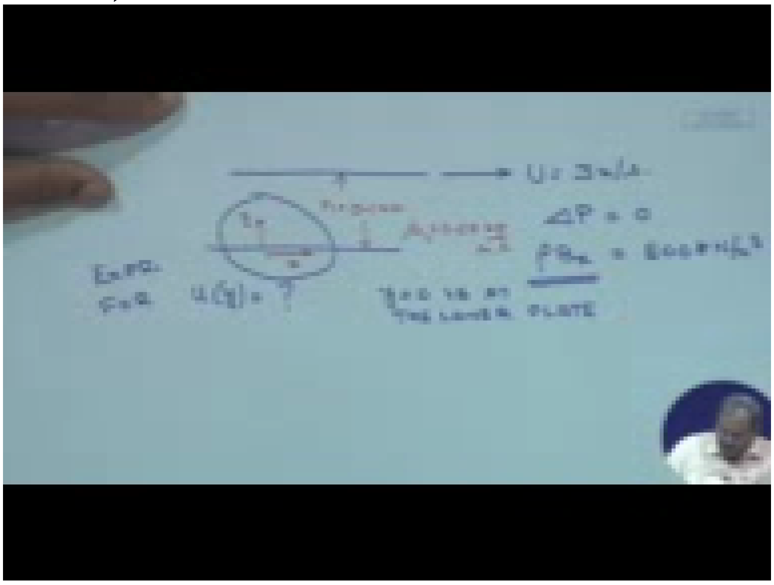

problem is find an expression for u y and the second part of the same problem is compute the volumetric flow rate, volumetric flow rate past a vertical section. And here you can assume the width of these plates to be 1 meter, to be unity, Ok.

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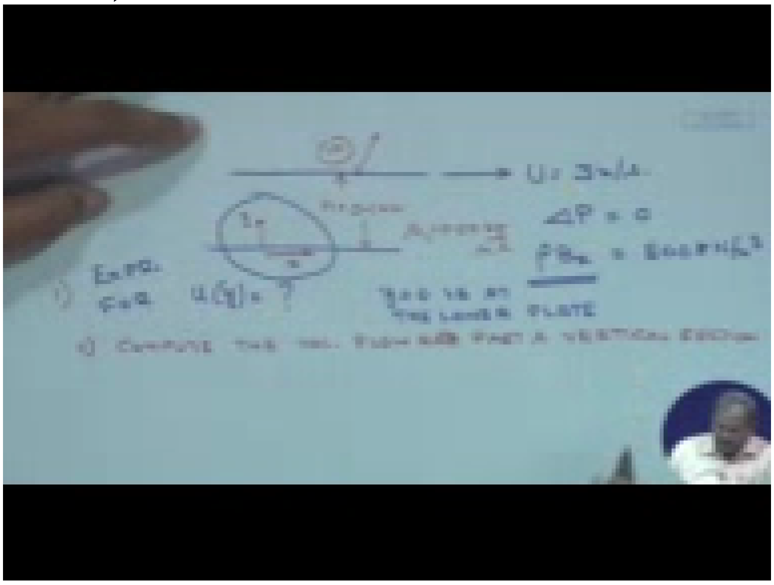

So there are two parts to the problem. The first thing is to obtain what is a velocity profile, u as a function of y, its one dimensional flow so you have to write those assumptions. It's a one dimensional flow therefore you are not going to get any, one dimensional steady incompressible flow, so there is no variation of velocity u which is x component of the velocity. So there is no variation of u with x but you can clearly see that u is going to vary with y. It’s going to be no-slip condition at this point and a no-slip condition at the top plate. So you should use the Navier–Stokes equation for Newtonian fluid and since your velocity, the boundary conditions are in terms of velocity, So known velocity at this point and known velocity at this point due to no-slip condition so probably it would be better if you use the velocity gradient form of the Navier–Stokes equation.

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Since your boundary conditions are in terms of velocity so therefore the, the Navier–Stokes equation you should choose it, choose the velocity gradient form of it and not the shear stress. That would probably have been useful or appropriate if you have, instead of a liquid solid interface at both ends, if at one end you have a liquid vapor interface. If you have a liquid vapor interface then the prevalent boundary condition at that point would be in terms of shear stress, would be in terms of tau. So tau would be zero at the liquid vapor interface. So if that is the condition that you have in your system then it is probably better to start with the tau form, the shear stress form of the equation of, equation of motion rather than the equation of motion expressed in terms of the velocity gradient. So look at the problem, see what are the boundary conditions you can use and then choose your governing, the relevant governing equation.

So what we would, what we should do here is we choose the boundary, choose the governing equations and since the velocity, since the motion is in the x direction, we must choose the x component of motion for Navier–Stokes equation for a fluid which is Newtonian and we understand the, the other parameters which are relevant here is that the velocity, the x component of velocity denoted by u is a function of y only. It does not vary with x, it does not vary with z and there is no body forces, it is a horizontal system so there is no body force present in the system. However, no gravity force present, no gravity present in the system, however there is an electric body force denoted by rho B x, the value of which is provided.

So the additional body force term in this case should be rho B x and we also note that there is no pressure gradient present in the system so the d p d part in the Navier–Stokes equation would be zero. And if you work out this problem then you would see as before the entire left hand side of the Navier–Stokes equation which has one temporal term and the other term, the other three terms denote the convective transport of momentum, they will be zero and you would be left with the right hand side, the first term on the right hand side which is d p d x, that would be zero because that is provided, that is what is stated in the problem, andthen you have the viscosity terms, the terms which denote shear stress and a body force term.

So you would also, you would clearly see that in this specific problem, the principle motion, the principle momentum is in the x direction so x momentum is getting transported because of a variation in the velocity in the y direction; so x component of momentum gets transported in the y direction. So that's the only shear stress term that would be left in the governing equation. So you will have one shear stress, rather the variation of shear stress and the body force term. So these two terms would remain in your governing equation which you are going to solve and I will provide; I will just simply give you the final expression for the velocity and the expression for the volumetric flow rate. So the expression for velocity which would work, you would find that, so this would be the expression for velocity,

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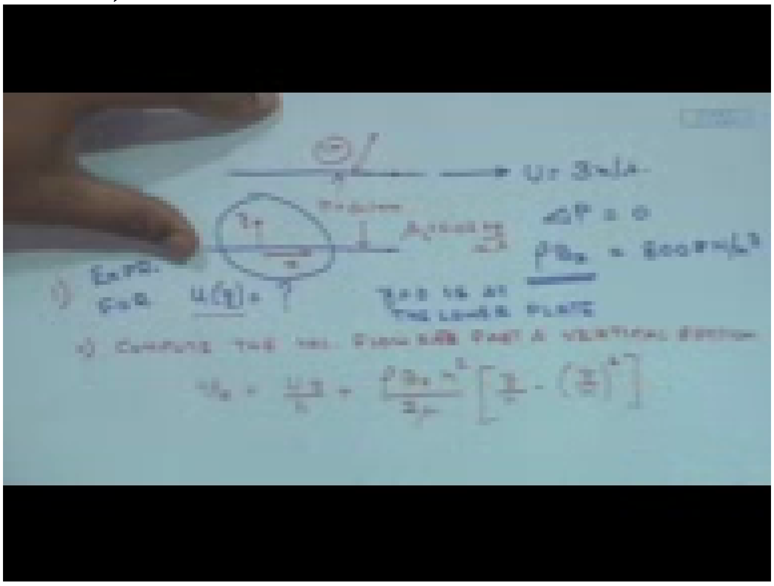

x component of velocity. If you did not have the body force then this entire term would be zero and what you end up with is the Couette flow expression. The Couette flow expression

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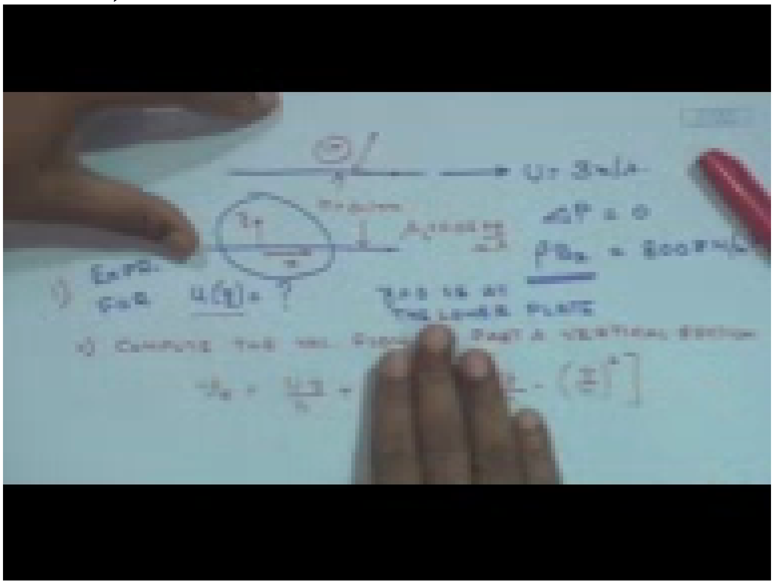

which simply says that the velocity varies linearly with y and its maximum value of velocity would be at the top plate which is equal to u. So this is the Couette flow part and because of the body force you have these additional terms present in the expression. So once you have the velocity you should be also able to obtain what is the area average velocity and you should get it to be this form,

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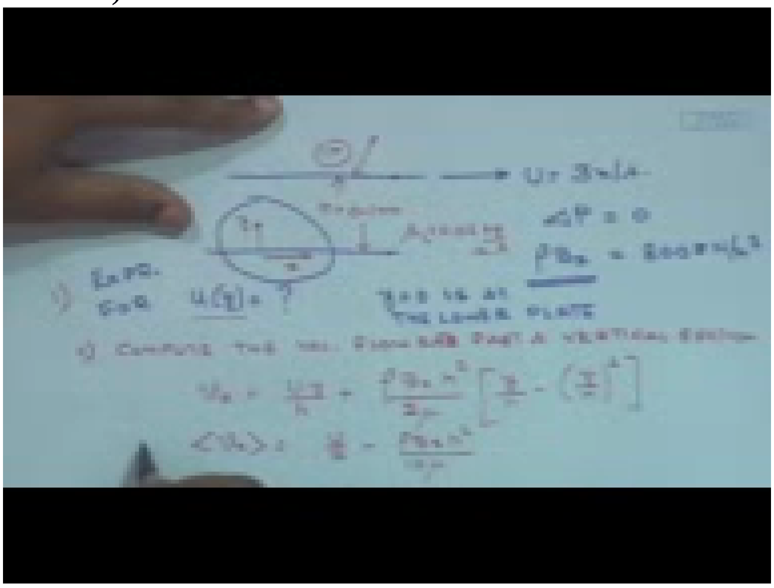

this is the area average velocity that means if you put a plate like this and area which is perpendicular to the direction of flow, which is perpendicular to the direction of flow and average the velocity

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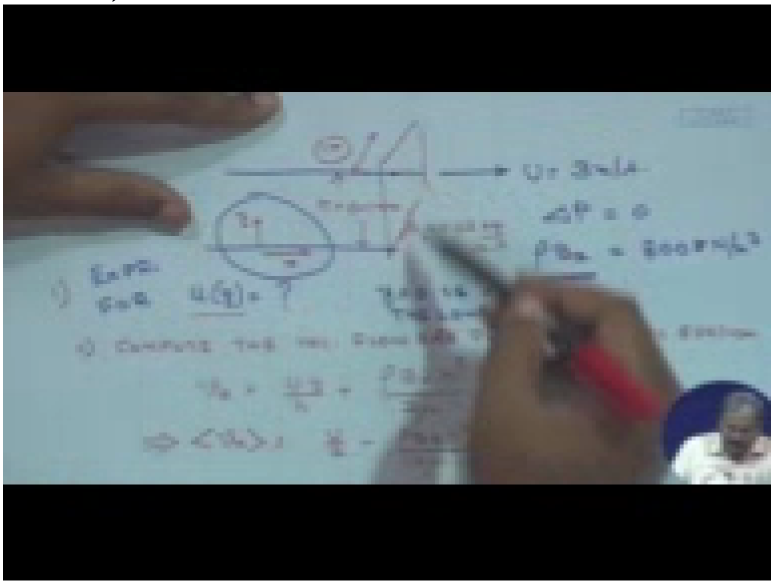

out as whatever we have done before, this is what you are going to get as your average velocity and the volumetric flow rate would be the average velocity multiplied by the area which in this case is h times 1 since we have assumed that

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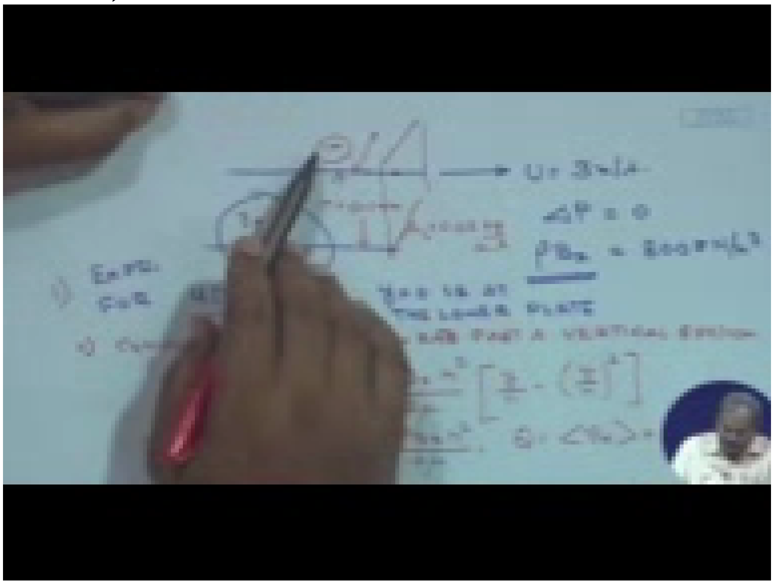

the plates are one meter in width, so this together is the area and this is the velocity and when you plug in the numbers

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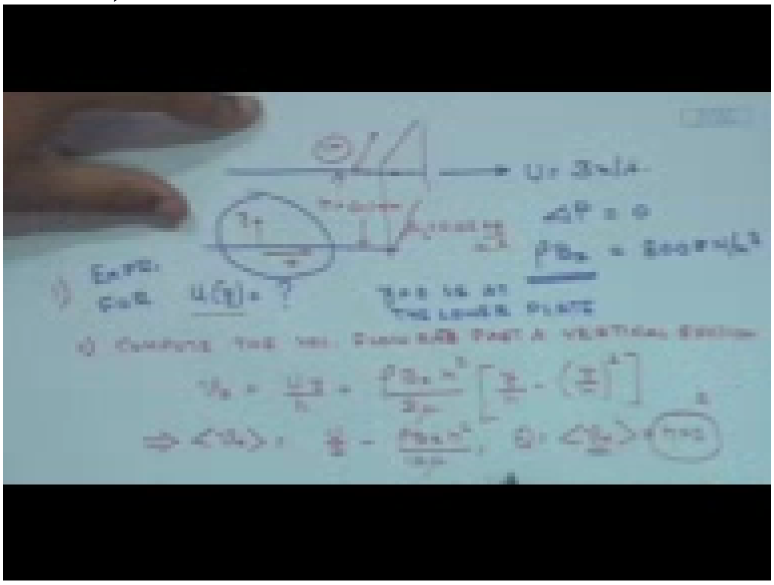

the value of Q would be equal to 1 point 5 into 10 to the power minus 4 meter cube per second. So this is the quick problem which would give you some idea about what is the, what is the, how to use

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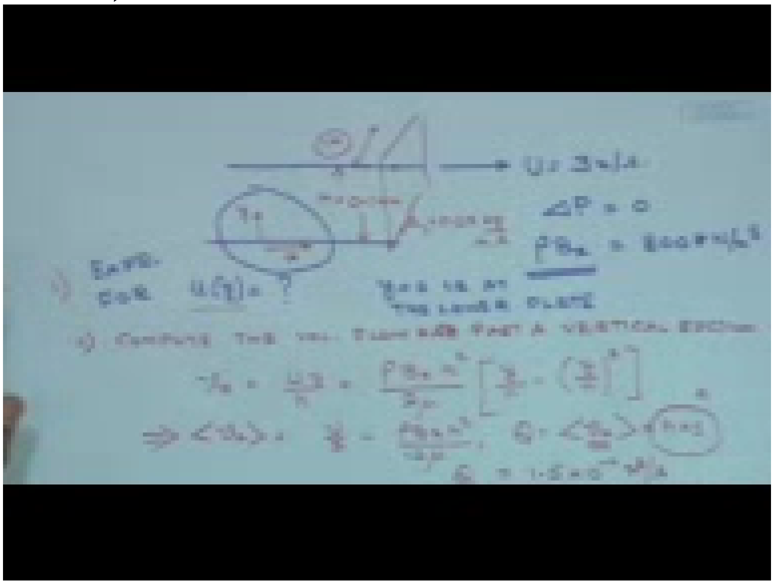

Navier–Stokes equation, how to get a form, how to get the velocity, how to get the average velocity and so on. So this problem is for you to work and to see if you are getting the right expression, right expression.

Next problem that we are going to deal with

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is slightly more complicated. Herein we have a piston and a cylinder. So it is a piston cylinder apparatus assembly where there is sufficient pressure which is generated or which is provided on the piston. As a result of which the piston slowly starts to come into the cylinder. The cylinder initially contains a liquid, an oil, viscous oil which is used as a lubricant. So as the piston starts to come inside the cylinder the oil which is contained in the cylinder must come out in between the thin gap between the piston and the cylinder. So as the piston, if this is the piston and if this is the cylinder, as the piston starts to come inside,

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since the, since the liquid present inside is incompressible it must leak through the very small gap that exists between the cylinder and the piston. So it's a piston cylinder assembly, very close fitting that means the outer diameter of the piston is slightly smaller than the inside diameter of the piston. Or in other words, the gap in between the piston and the cylinder is extremely small. We have to, we have to make , make a statement here, make a , we have to make an assumption in this case which is very common for systems in which the curvature is small compared to the radius of the system.

Now if the piston and the cylinder are very close fitting and if the piston has a larger, large radius then what would happen is that the flow, for a very small section in the piston cylinder assembly, it is as if the flow is going to be in between parallel plates. Or in other words, for cases where the gap between the two surfaces are very small as compared to the radius, as compared to the curvature of the system which is the case in the piston assembly then a cylindrical coordinate problem can be transformed in a Cartesian coordinate problem. So when the piston starts to get inside the cylinder,

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as the gap is small compared to the curvature of the system, I can visualize the system as if it's between two parallel plates. So I open both the piston and the cylinder. They are very close to each other and the piston going down would simply be this plate which was the piston going down with a thin gap in between that is filled with the, that is filled with the viscous liquid. So the piston cylinder assembly or in, in any, or in any such situation where the gap is very small compared to the curvature of the system you can assume those two surfaces, those two surfaces which are essentially cylindrical in nature but would behave as if there is, they are flat plates.

So a piston cylinder assembly like this can now be opened so that they become plates. And once they become plates, then we will be able to use the Cartesian component, Cartesian component of the Navier–Stokes equation. It is a very common practice in many cases to resolve the cylindrical problems into Cartesian coordinate system, Cartesian coordinate system problems. That one has to be very, very careful. It must be explicitly written or you must understand; you must evaluate whether the gap is really small as compared to the curvature of the system.

So what we have in this case is the piston going down in the cylinder with a very thin gap in between so we can very safely assume that it is going to be the flow between two parallel plates where the plate which is representing the piston is going down and the plate which is representing the cylinder remains static. So it is as if the piston is going down as a plate with a very thin gap in between. So whenever you come across these problems, first try to see, can it be resolved in a Cartesian coordinate system which would, if you use the cylindrical system it is not going to be, it is fine, you can still do that but the problem that, the advantage of using the Cartesian coordinate system is the terms are most simple, Ok. Since the terms are simpler, then it is fairly easy to handle a problem in Cartesian coordinate system as opposed to that in a cylindrical coordinate system. So we will always try to use Cartesian coordinate system as far as possible.

So herein is a case which is ideal for transforming from cylindrical to Cartesian coordinate system. So I will draw the system and tell you the parameters in the, and the problem. So the problem that we have in here is a piston cylinder assembly. This is the cylinder, Ok and herein we have a piston. There is a very thin gap between the piston and the cylinder, Ok.

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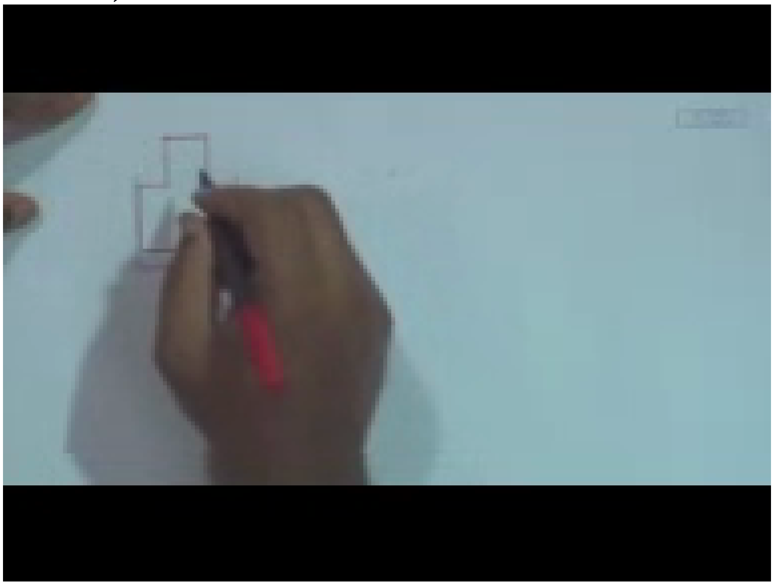

So this, on top of this piston let's put a mark of M

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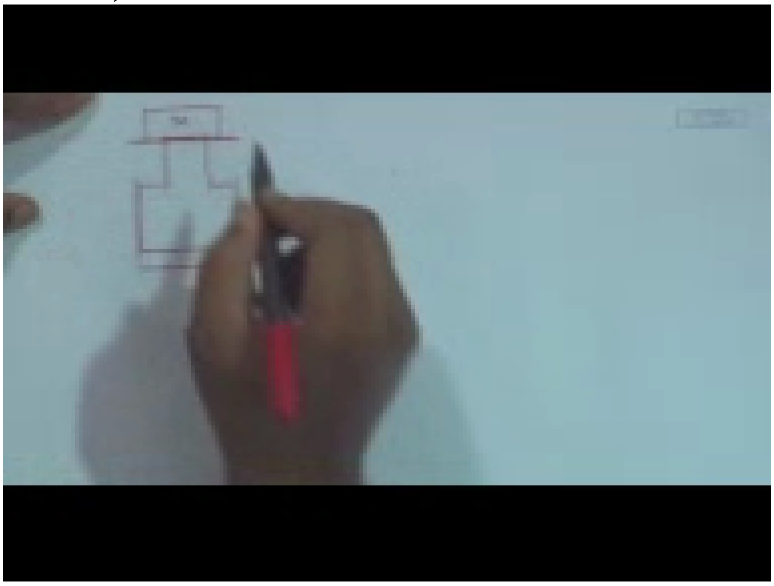

which is going to create a, and herein we have the oil. As the piston starts to come, starts to come down, the oil has to leak in between the thin intervening space between the piston and the cylinder. So herein we have the piston, and this one is the cylinder. The diameter of the piston

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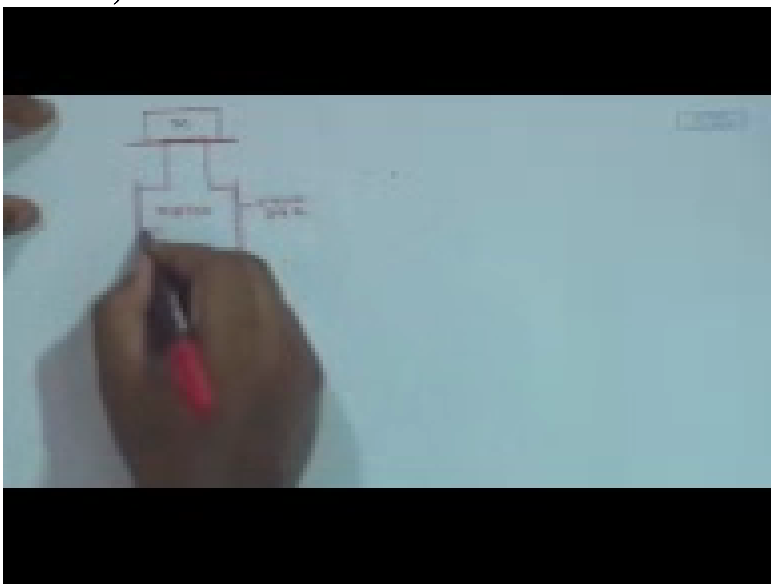

D is 6 millimeter

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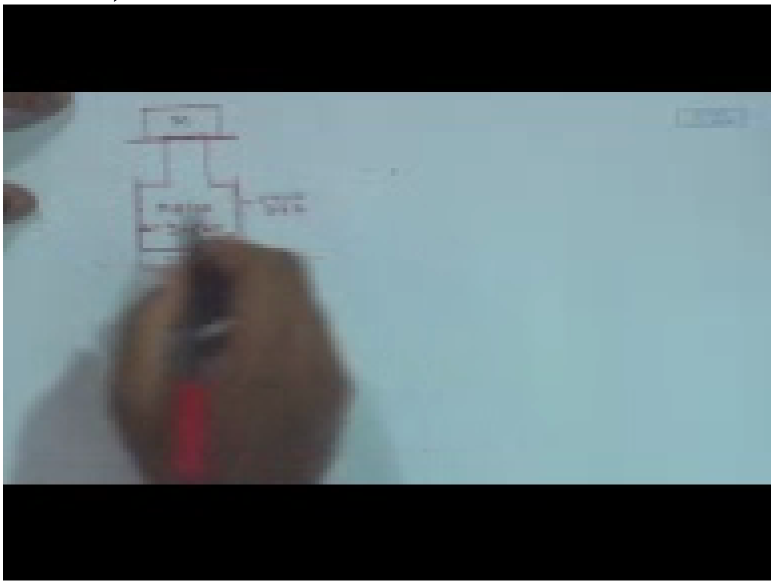

and the length of the piston is L equals twenty five millimeter

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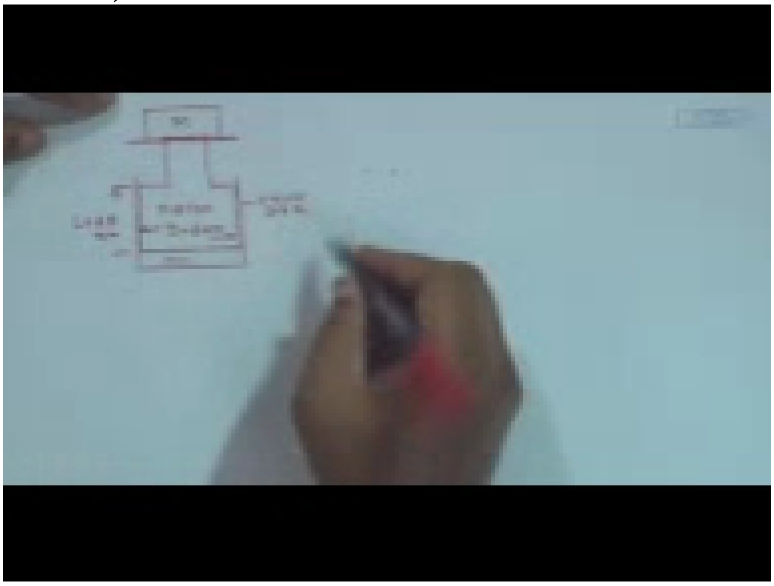

and the piston is coming down with a velocity, the velocity of the piston is equal to 1 millimeter per minute. So you can see the piston is coming down at a very slow

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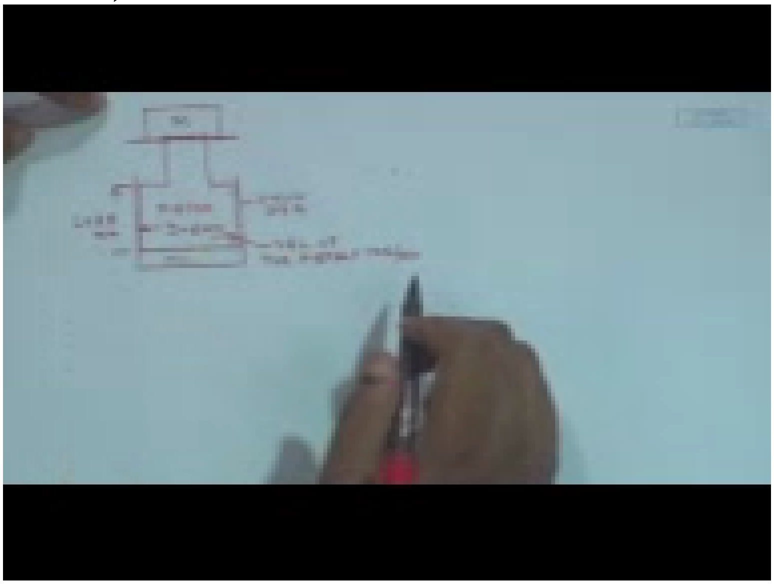

velocity and the, as a result of which the oil is leaking from here. Now the first part of the problem is find the mass m that needs to be placed on the piston to generate a pressure equal to 1 point 5 mega Pascal inside the cylinder. That is the first part of the problem,

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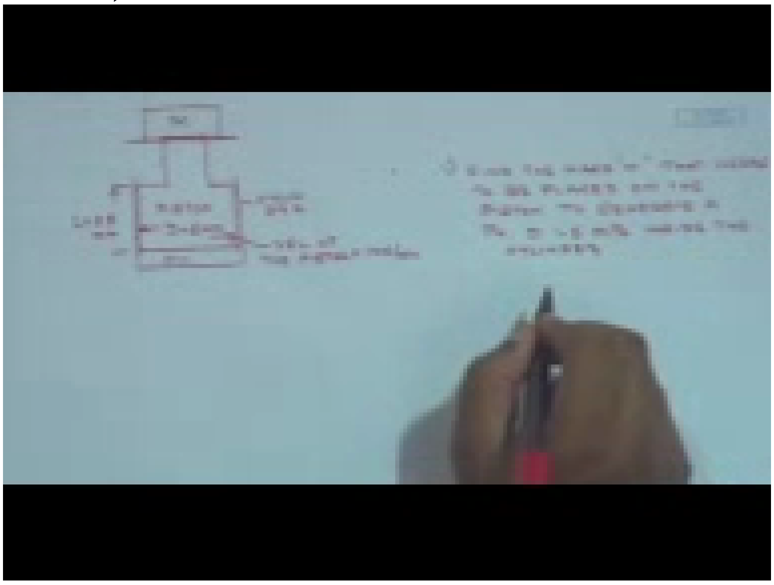

and the second, the second part of the problem is; let's assume the gap between these two is equal to a. Find the gap a between the cylinder and piston such that the downward motion of the piston is one millimeter per second. So this is the entire problem.

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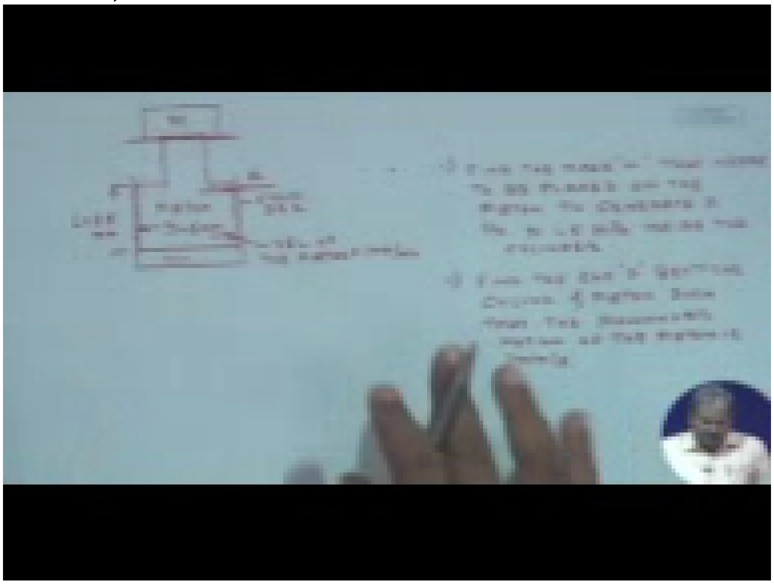

The piston is loaded with the mass that creates a pressure of 1 point 5 mega Pascal in

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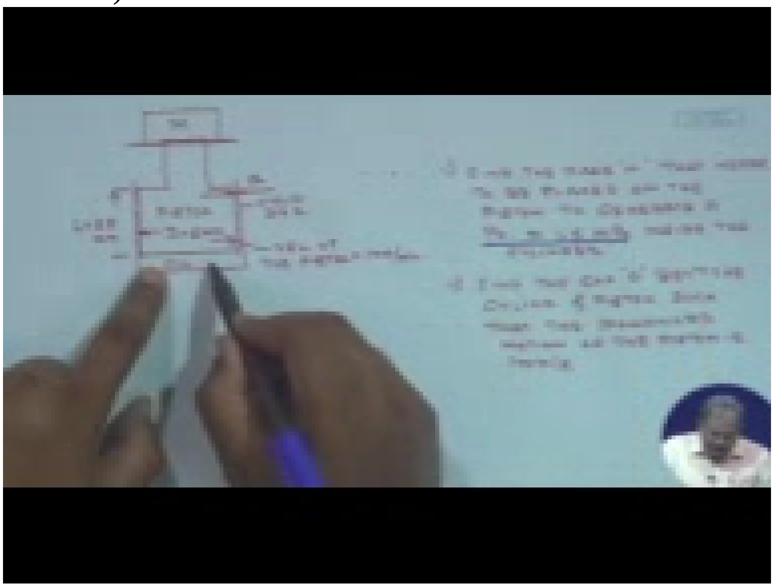

here and the piston is slowly coming down. There is a huge pressure difference between this point and the point outside which is open to atmosphere. So you have a huge pressure gradient that is acting from this point, between this point and this point. At the same time you have the cylinder coming, coming down but the cylinder velocity is fixed at 1 millimeter per minute. So you must, you must find out what is the,

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what is the space in between the cylinder and the piston that would allow the system to work in the specified form. So if your a is large then the liquid, the oil will come out with a very high velocity and as a result of which the piston will fall with a, with a higher velocity. On the other hand if a is very, very small then the leakage rate of oil would be such that it would not allow the piston to come down with a velocity of 1 millimeter per second. So there is only one value of a which would give you the correct leakage, oil leakage rate such that the piston is going to come down with the velocity, with the specified velocity while the pressure difference, the pressure is maintained to be 1 point 5 mega Pascal inside the cylinder. So let's first evaluate how do we get the value of m in this case.

The value of m is going to provide a force equal to m v on the platform on which it rests on the piston. So this m v, that is the force, the force exerted on the piston by the weight must be, must be supported by the pressure gradient which exists inside the cylinder and outside of the cylinder. So whatever be the area of the piston which has the diameter of 6 millimeters, that area of the piston multiplied by the pressure difference between the cylinder and the atmosphere must balance each other. In other words m g must be equal to pi by, pi r square times p inside minus p outside, Ok. So, so this formula should be used to obtain what is the value of the unknown mass in, that must be placed on the piston to generate the pressure of 1 point 5 mega Pascal inside the cylinder. So if I write it in this form, then it is simply going to be pi d square by four where d is the diameter of the piston multiplied by pressure inside, minus pressure outside which is atmospheric pressure must be equal to m g. So this equation, the balance equation where

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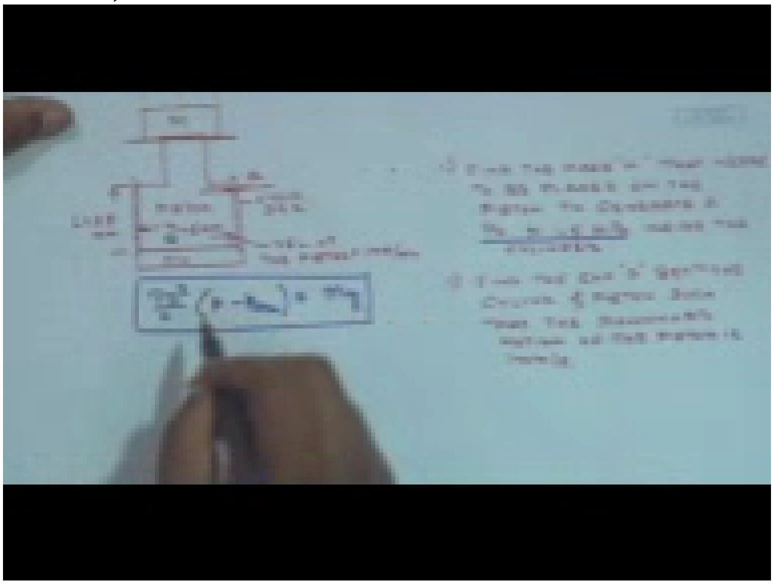

D is 6 millimeter, p is one point 5 into 10 raised to power 6 Pascals, p atmosphere is known to us, g is known, so the only unknown is m and the value of m can be obtained as 4 point 3 2 kg, so the first part is done.

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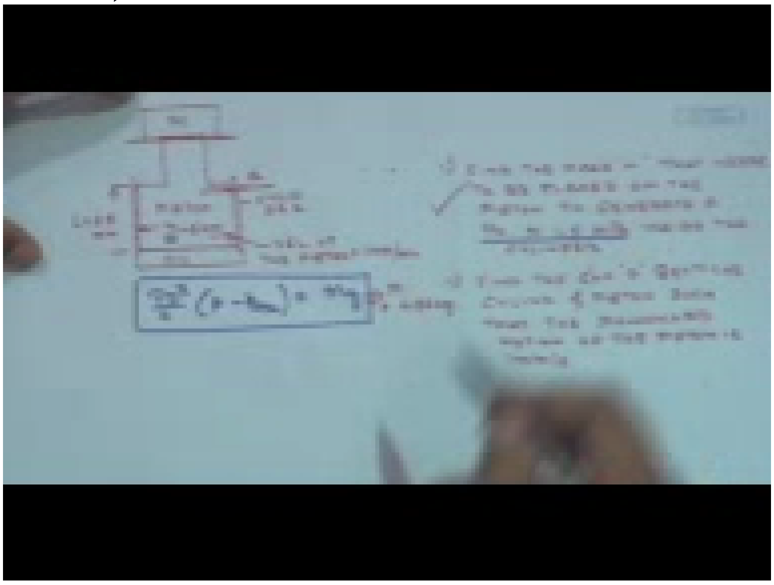

About the second part, as I said since the gap is very small in comparison with the, with the curvature then we are going to think of it as 2 parallel plates, this is the x direction and here we have the y direction. This wall represents the piston and this wall is the cylinder. So in between I have oil present inside.

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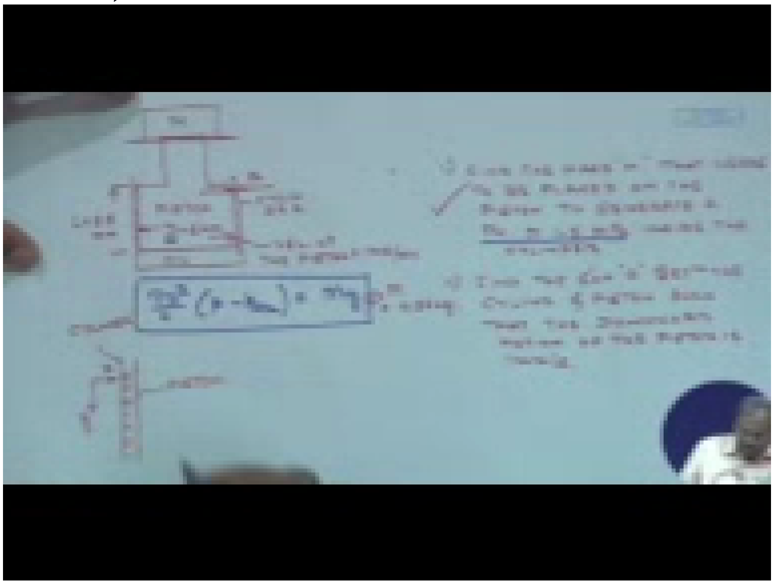

The piston is slowly coming down with the velocity which is provided 1 millimeter per second, the gap in between the cylinder and the piston is a

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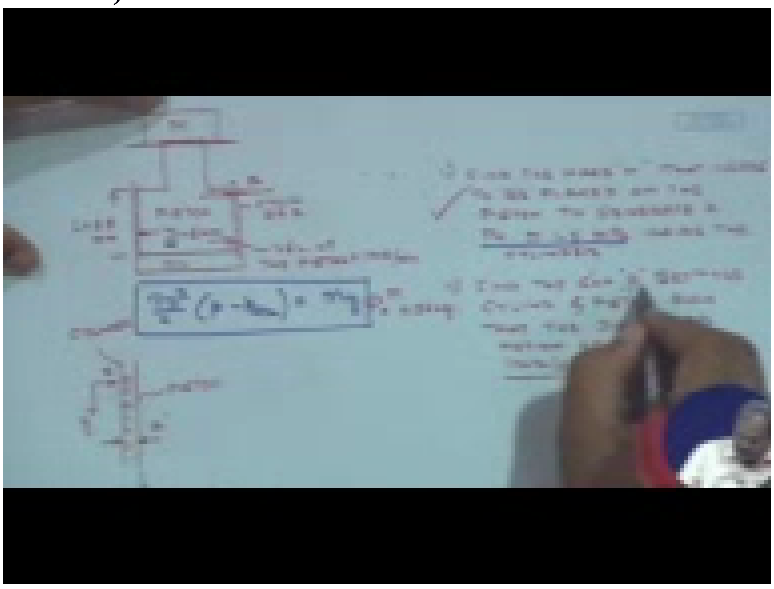

that is over here and we simply have to write the balance equation to obtain what is the expression for a. So of course now you are experts of Navier–Stokes equation, you know which component of Navier–Stokes equation to use in this case and

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you can clearly see that you have, this is the motion in the y direction, you have to choose the y component of Navier–Stokes equation and as before it is a steady state one dimensional flow problem and therefore the entire left hand of Navier–Stokes equation would be equal to zero, Ok. There would be no temporal term and no convective momentum flow term. So what you have on the right hand side is the d p d x. This d p d x must be taken into account because we have a huge pressure gradient in between, in between the two ends of the piston. At one end, its 1 point 5 into 10 to power 6 Pascals; at the other end you have the atmospheric pressure.

So d p d x and d x is crucially the length of the, of the piston which has also been provided. So the d p d x term cannot be neglected. It must be there in the governing equation. Then you have this, the viscous term which you have y velocity, velocity y which is varying in the x direction. So the velocity, mu times del square v y by del x square, that term will be present in the Navier–Stokes equation, in the y component of the Navier–Stokes equation and of course since it is vertical, you are going to have the effect of gravity as well. So the governing equation for such system is going to be, it can be written as mu times d v by d x square is equal to d p d x and minus d p d x plus rho g is equal to zero.

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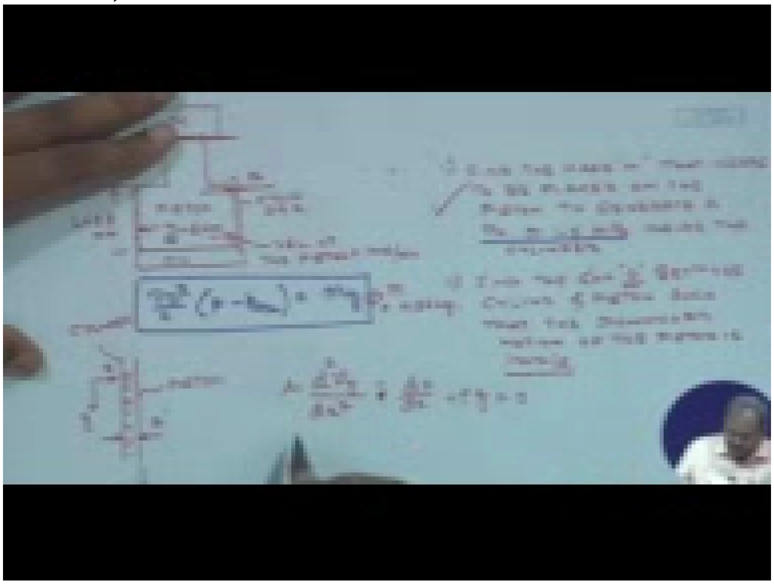

There is one more thing which you have keep in mind is that at times the problem becomes simpler if you would be able to cancel some terms based on their, based on their magnitude. So this order of magnitude analysis, whether a specific term is going to be large, very large as compared to the other term which would, which would

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tell you something about which of the terms in the Navier–Stokes equation, even though it is present, even though it is non-zero but it is so small in comparison, in comparison to other terms that it can be neglected. There are certain cases which would be apparent specially if you look into this problem. So I have two terms, one is the pressure gradient term; other is the rho g term. So what's the rough approximation value of rho g? rho is, of the, generally of the order of 10 to the power of 3, and g is of the order of 10. So the term rho g is going to be of the order of 10 to the power 4.

Well what is d p d x? We understand that d p d x; delta p is of the order of 10 to the power 6 that is the mega Pascal. So delta p is of the order of mega Pascal, so it is 10 to the power 6 and d x which is the length of the piston is 25 millimeters. So it is 2 point 5 into 10 to the power minus 3, so roughly this d p d x term is going to be of the order of 10 to the power 6 by 10 to the power minus 3 which would be roughly about, because I have 2 point 5 in here, it would be of the order of 10 to the power 8 where as this rho g is of the order of 10 to the power 3 for rho and 10 for g so it is going to be of the order of 10 to the power 4. So

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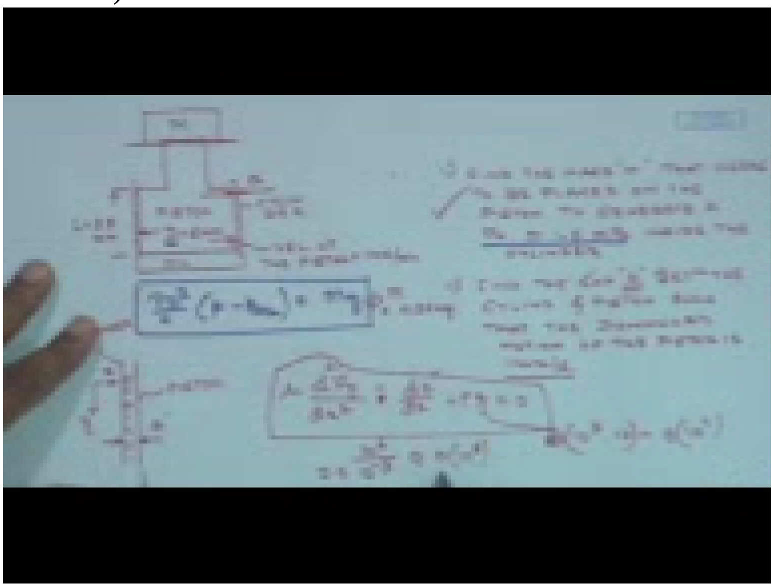

you could see the difference in order between these two is, it is safe to drop this term from

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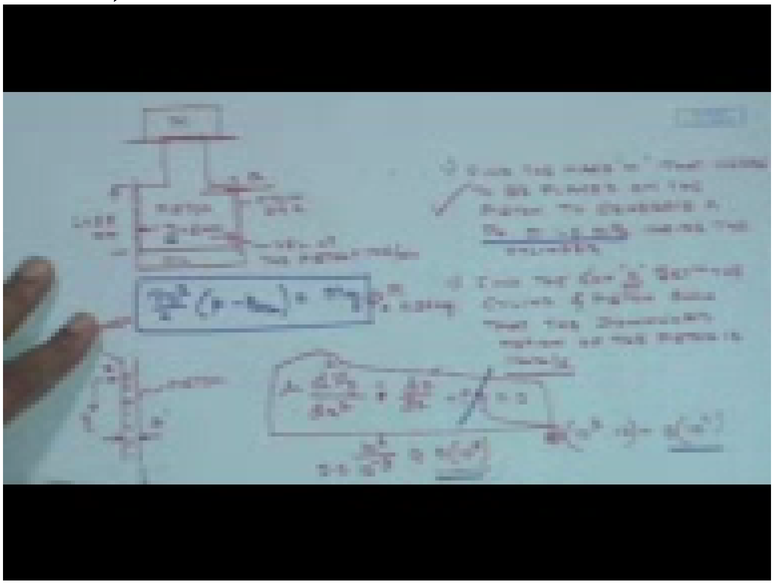

your governing equation as well and you will simply have this as your governing equation. So sometimes order of magnitude analysis of the different terms present in the governing equation lets you further simplify the Navier–Stokes equation, so that is something which you should always look for. So your

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governing equation now becomes mu times d 2 v y is equal to delta p by L and therefore your v y would simply be 1 by 2 mu delta p by L x square plus C 1 x plus C 2 and the two boundary conditions are

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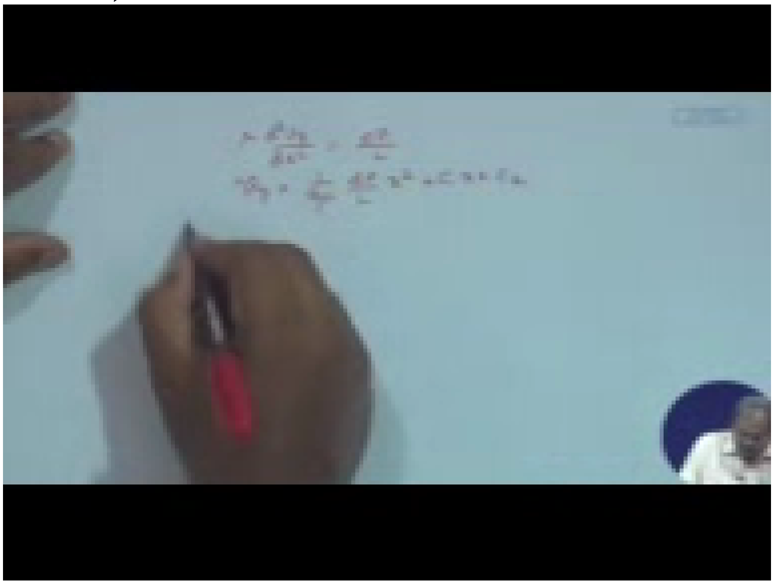

at x equal to 0, your velocity is equal to zero, no-slip condition which would give C 2 equal to zero

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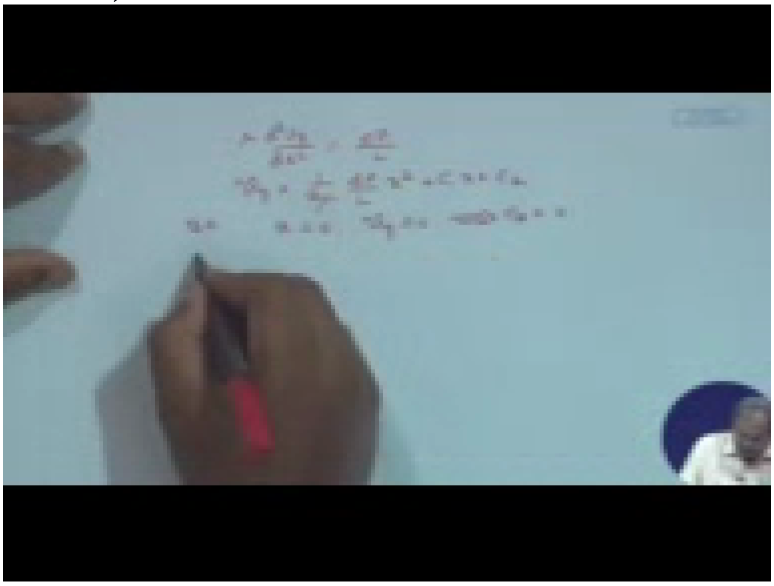

and the boundary condition too would be at x equals a that is on the piston, the v y is going to be equal to v and this would give you some value of C 1.

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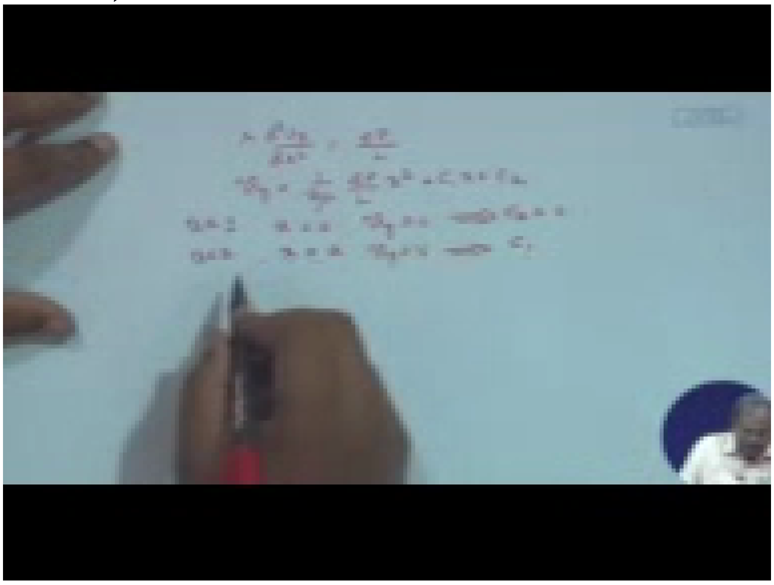

So I will not do everything in this part. You should do, you should check on your own that the expression for velocity would turn out to be delta t by l, x square minus x a plus v x and

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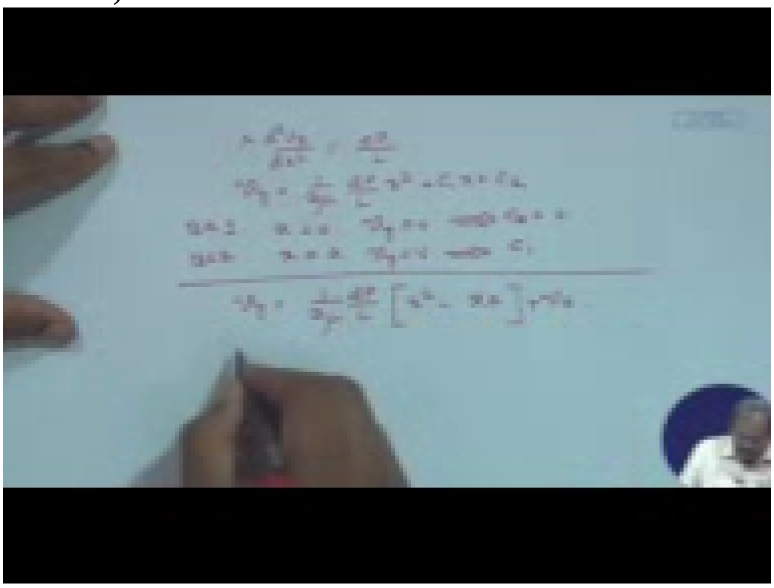

the average value of the velocity which is nothing but 1 by a from zero to a v y d x this would turn out to be minus 12 mu delta p by L a square plus v a by 2. So

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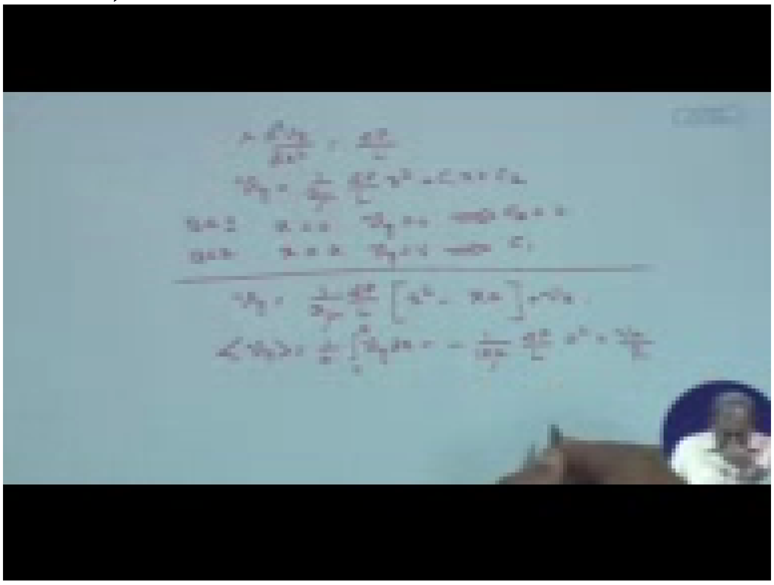

the governing equation, boundary condition should give you the velocity, local velocity and the average velocity. Once you have the average velocity, then you should be able to obtain what is the flow rate which is a times pi d, velocity times the area and you know that the expression for which you should be able to obtain from here, Ok. So for downward movement,

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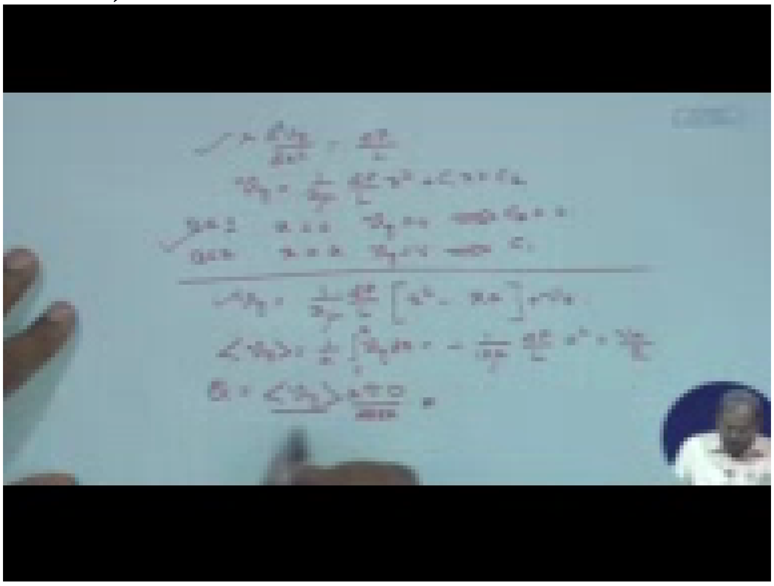

the volume displaced would be this Q, would be equal to pi d square by 4, which is the area of the piston, the cross-sectional area of the piston multiplied by v, this is the area, this the volume displaced per unit time and which would come to be 4 point 7 1 into 10 to the power minus 10 meter cube per second. So since I know the velocity is equal to 1 into 10 to the power minus 3 meter per second, the value of d is known to me so I know that when the v in 1 second, in 1 second

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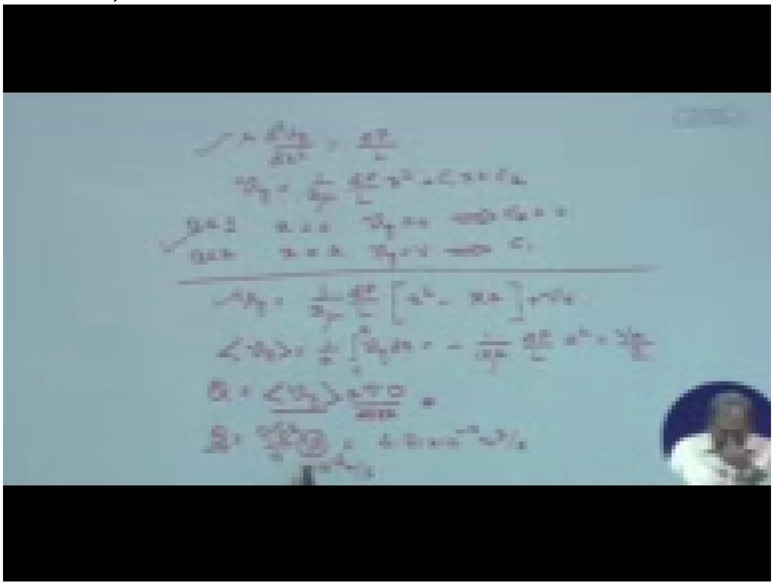

the v comes down by a distance of 1 into 10 to the power minus 3 meters so the total amount of volume displaced would be equal to 4 point 7 1 into 10 to the power minus 10 meter cube per second. And this must be equal to v y times a times pi d from here. So in this, in this expression everything is known except a which you should be able to evaluate as 1 point 2 8 into 10 to power minus 5 meters. So that is the answer, that is

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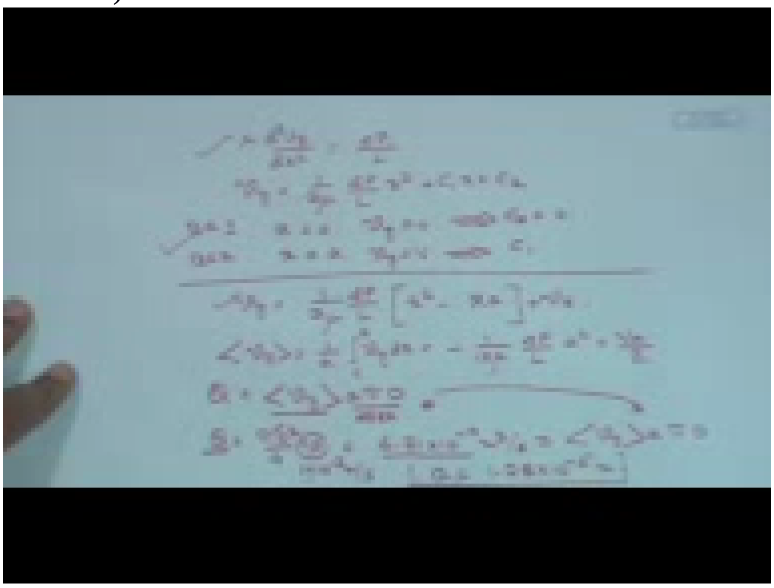

the separation between the piston and the cylinder that should exist which allow, which allows the piston to come down with a velocity of 1 millimeter per second when the pressure gradient generated, the pressure generated inside the

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piston cylinder assembly is 1 point 5 mega Pascal; so this problem is important because it shows you how and when to convert a cylindrical coordinate system into a Cartesian coordinate system. Whether it is possible to further simplify Navier–Stokes equation by looking at the possible magnitudes of the different terms present in the final form of the equation and if so a seemingly complicated problem can be resolved into a simple problem of flow between parallel plates with a pressure gradient present in the system which would allow you to, to find out what is, from, from the average velocity and the volumetric flow rate what should be the separation between the solid, between the piston and the, the cylinder that allows the piston to come down with a certain velocity where the pressure generated inside is known to us. So it is nice example of the use of Navier–Stokes equation and some common sense to solve a problem which is fairly common in many of the mechanical engineering situations.