Transport Phenomena
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Lecture 11
Equations of Change for Isothermal Systems (Cont.)

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This is a going to be a tutorial class on the concept that we have, we have covered in for equations of motion, mostly Navier–Stokes equation. What we have seen in this, in the previous two classes is the, from the fundamental concepts of derivatives, there are different types, the equations of continuity and the equation of motion, we have a comprehensive system right now which could address any of the fluid mechanics problems at least up to the point of governing equation. The concepts behind choosing the boundary condition will remain unchanged from whatever we have discussed previously. So just a quick recapitulation of what we have done over here,

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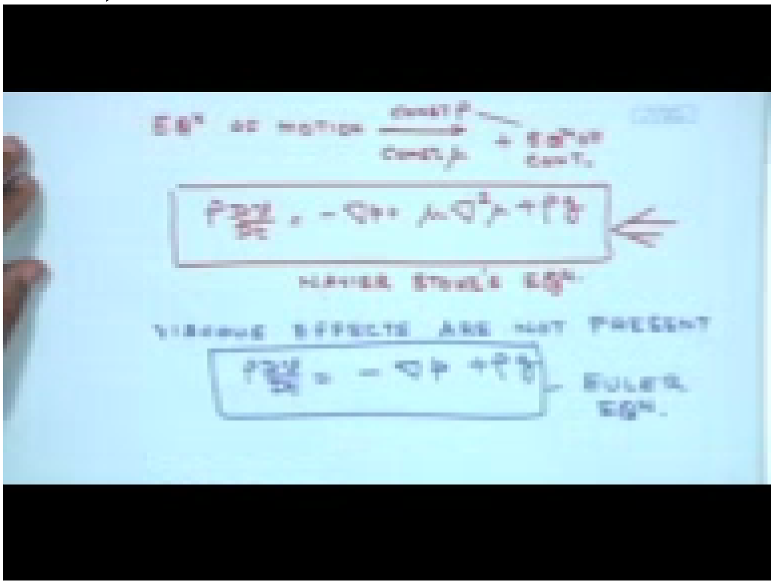

What you see this is the equation of motion. When in the equation of motion you add the constant rho and constant mu and equation of continuity what you get is Navier–Stokes equation. So this is mass per unit volume multiplied by expression, so this becomes force per unit volume. Similarly del of p is also force due to pressure due to unit volume. This is viscous force per unit volume and rho g is the gravitational force or the body force per unit volume, so all terms in Navier–Stokes equation are nothing but force per unit volume. In the special case would emerge if you assume that it is an inviscid fluid or the viscosity of fluid is insignificant. Since the viscosity of the fluid is insignificant what you get from Navier–Stokes equation is something which is known as Euler's equation for inviscid flow of liquids where this mu term would simply be dropped and what you get is this Euler's equation where rho times turbidity is the sum of surface force mainly pressure and body force which is gravity. Now the expressions for different components x, y, z components of Navier–Stokes equation in Cartesian, cylindrical and

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Spherical components are given in any textbook. You can refer either to Bird, Stewart and Lightfoot or you can also, any of the or Fox McDonald, any of the textbooks will contain the expression for Navier–Stokes equation in different coordinate systems, the x, y, z components in Cartesian, cylindrical as well as spherical coordinate system. So what you need to do in that is first see what is the, what kind of a geometry you have in hand. Then accordingly choose whether it is Cartesian, spherical or cylindrical geometry has to be chosen. Then find out what is the principal direction of motion.

If it is one-dimensional flow, let's say flow in the z direction then you choose the z component of equation of motion and simplify the terms which are not relevant for these in that context. If it is, if it is flowing two dimension, then you have to, let's assume you have to, you have to analyze both the x component of equation of motion as well as y component of equation of motion and then see what kind of simplifications you can suggest in order to make the problem, make the set of equations solvable hopefully by the analytic method. If not we have to think of other methods including numerical techniques to solve such problems. So the table for these equations would look something like this. so

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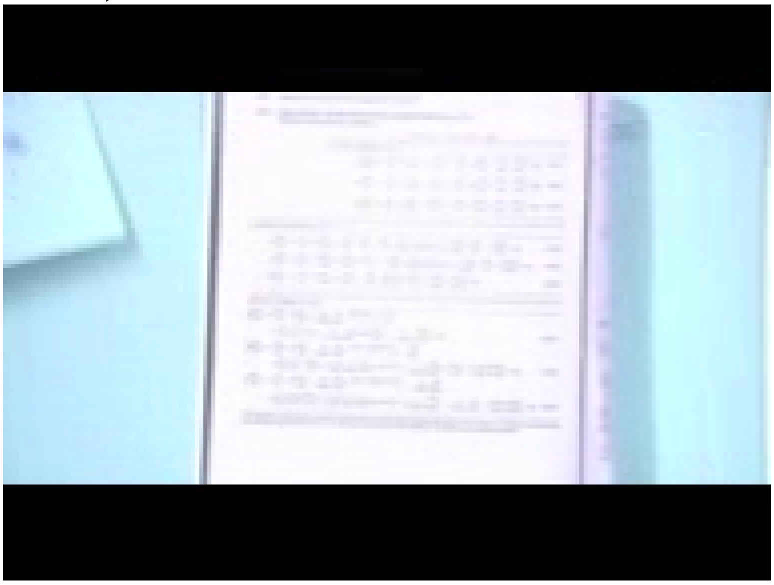

this is the, this is from Bird, Stewart and Lightfoot and what you see here are the Cartesian coordinates

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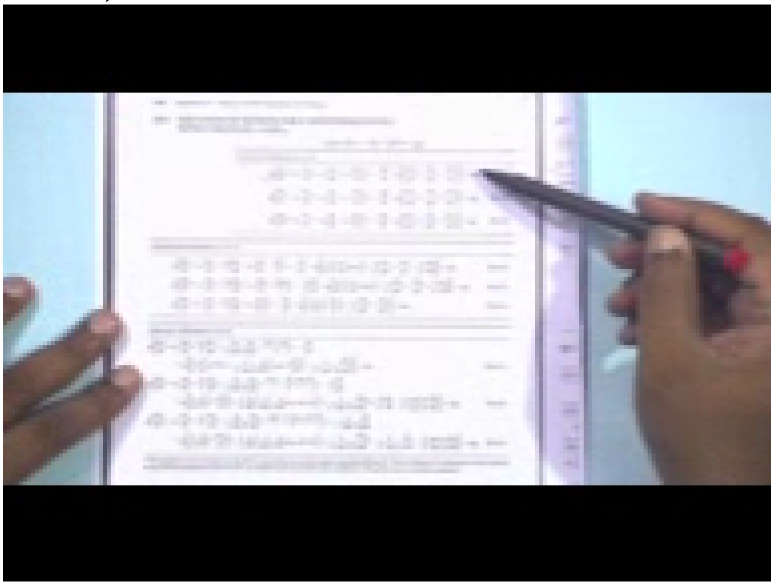

x, y, z; what is the, what is the equation of motion,

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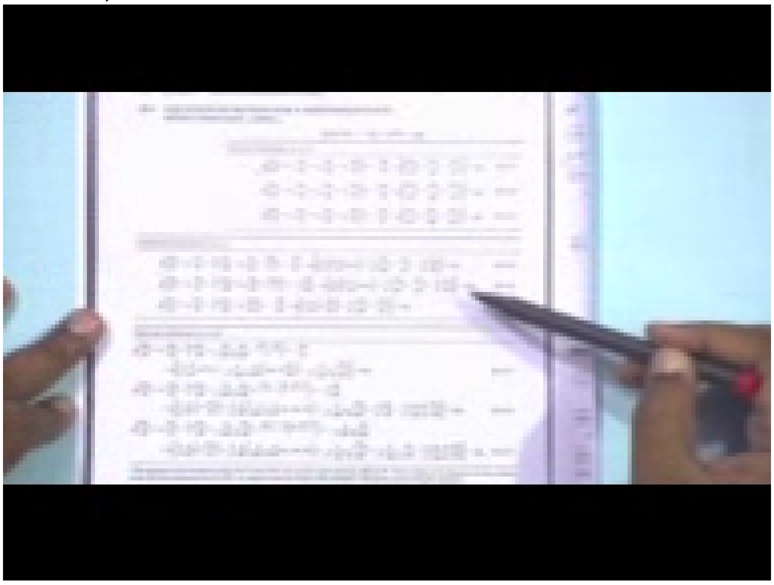

cylindrical coordinates r theta and z, what is the r component of equation of motion,

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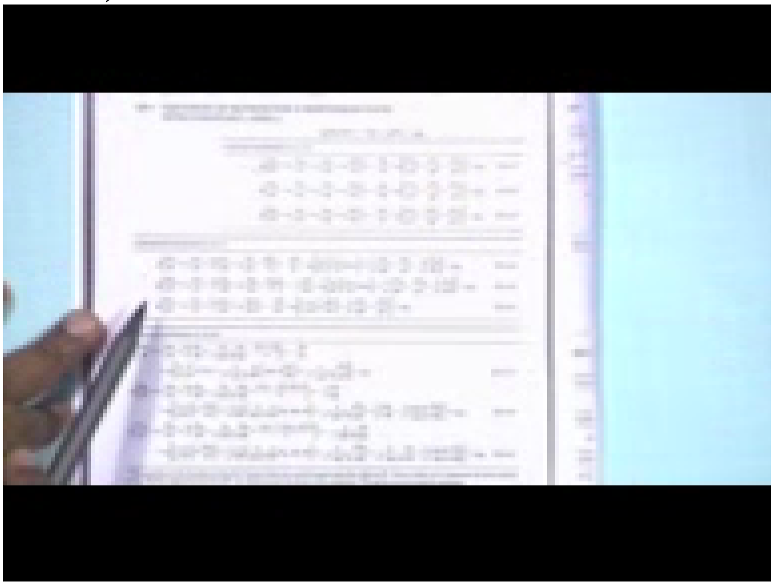

theta component of equation of motion, and z component of equation of motion and in spherical components r, theta and phi be different components of motion in three different directions. So these are available in any standard textbook. This is from Bird, Stewart and Lightfoot. So what you need to do is from these nine equations, first you have to see which equation is going

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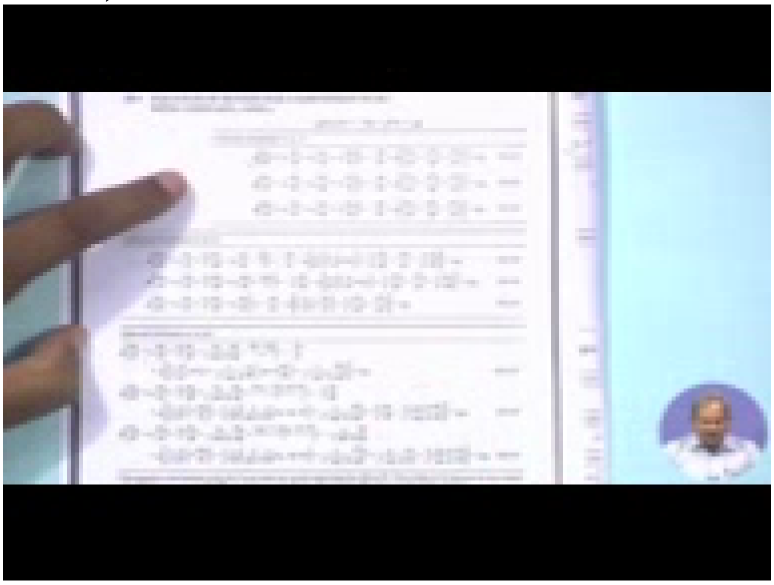

to be relevant in your case. Is it Cartesian, is it cylindrical or spherical? Let's say

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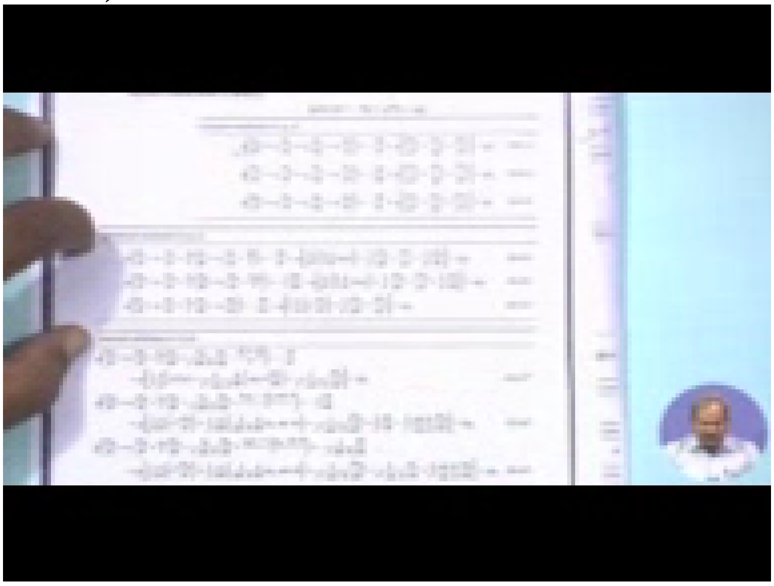

for the case of flow through the tube. For flow through the tube that we have analyzed so far the direction of the motion of the fluid was in the z direction.

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So first of all in a tube we must choose the Cartesian coordinate system. Once I choose the Cartesian coordinate system I will also have to choose what is the principal direction of motion; that is the z component. So in that table that lists the different, the different components of Navier–Stokes equation, I must choose the z component of the cylindrical, cylindrical version of Navier–Stokes equation because my principal direction is in the z direction and then cancel the terms which are not relevant. So I will, in this tutorial part of the, of the course I will pick three problems that we have done in our, using shell momentum balance, one where a fluid flows along an incline. It is a freely falling liquid film. That means there is no imposed pressure gradient in the direction of flow.

The flow takes place only because of gravity. In the second problem, the one that we have done is where the flow is taking place in a tube and there is a pressure difference as well as the gravity is acting downwards. And the third problem that we will look at is where we have a tube, the flow is from below, the liquid which is the top, the top of the tube spills over and starts flowing along the sides of the tube. So these three problems we have put considerable effort in obtaining the difference equation and from the difference equation, the differential equation. You would see and I am sure all of you would agree with me towards with the end of this class is that the Navier–Stokes equation, the use of Navier–Stokes equation is the way to go for solving the problems of fluid mechanics, solving the problems of fluid flow, the differential fluid analysis of fluid motion.

So we start with the first case where there was flow along an inclined plane, no pressure variant only gravity. So this was, this was the case which we have drawn.

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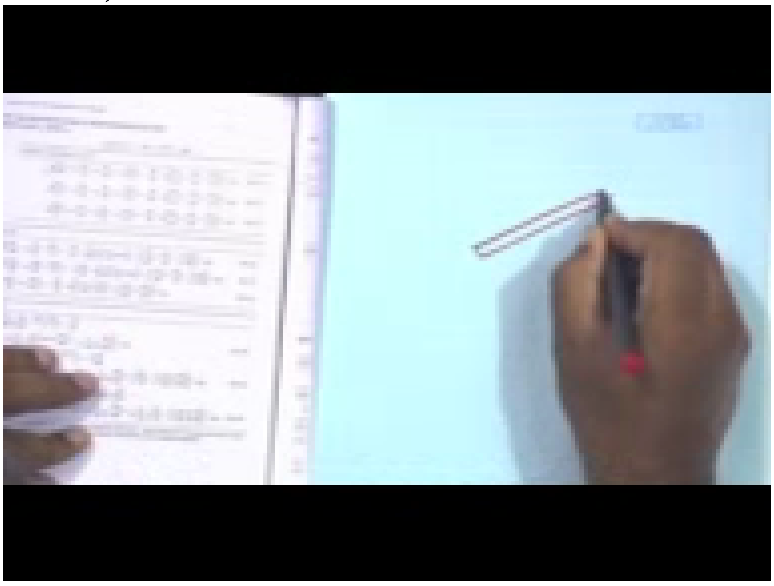

This is the plate and I have a flow of liquid, the dimensions, this is x direction, this is z direction and in the y direction the film is assumed to be the, the plate is assumed to be very wide. This is the length and the angle is beta. The thickness of the falling film is equal to delta.

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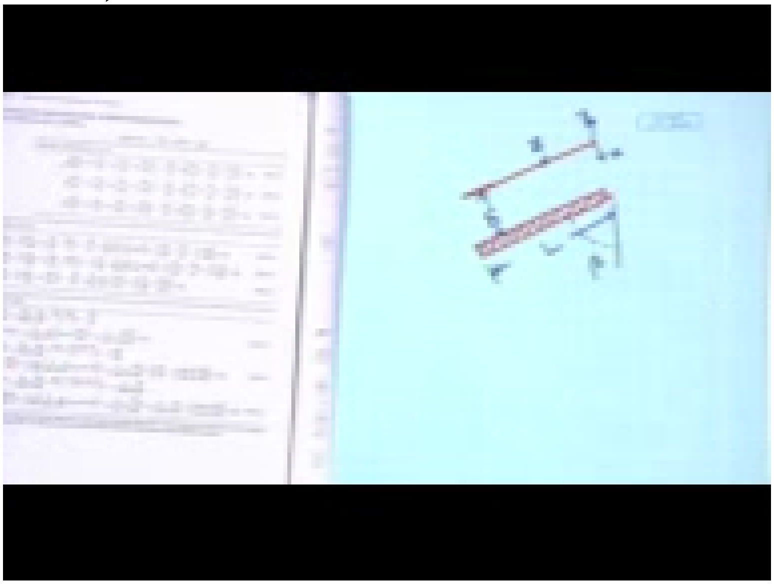

So let's see if we can simultaneously find out which equation that we need to choose from this table. First of all, one must see that I have to choose, if this is a Cartesian coordinate, cylindrical or spherical coordinate, so obviously it is going to be a Cartesian coordinate system. So I am going to restrict myself to

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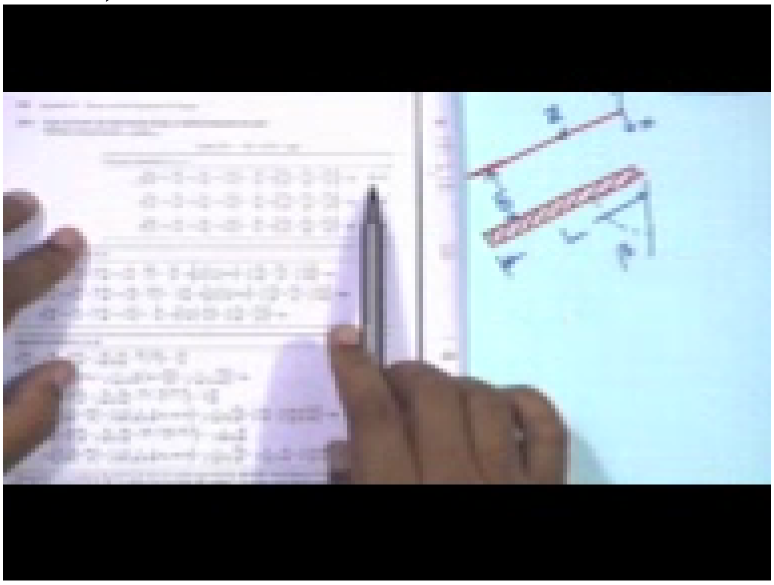

B 6 dash 1, B 6 dash 2 or B 6 dash 3. The principal direction is in the z, principal motion is in the z direction. So I must choose the z component of equation of motion,

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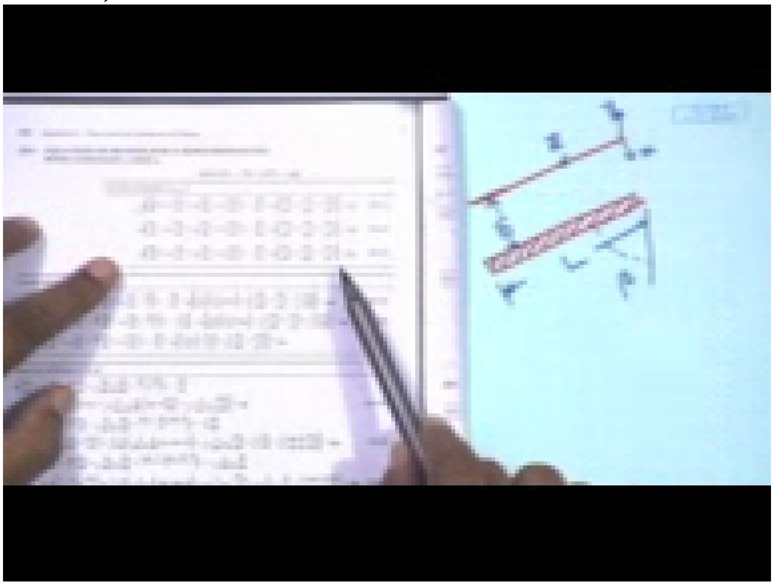

Ok. So if I choose the z component of equation of motion, then I am going to write the z component,

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either the expression that you see here, everything

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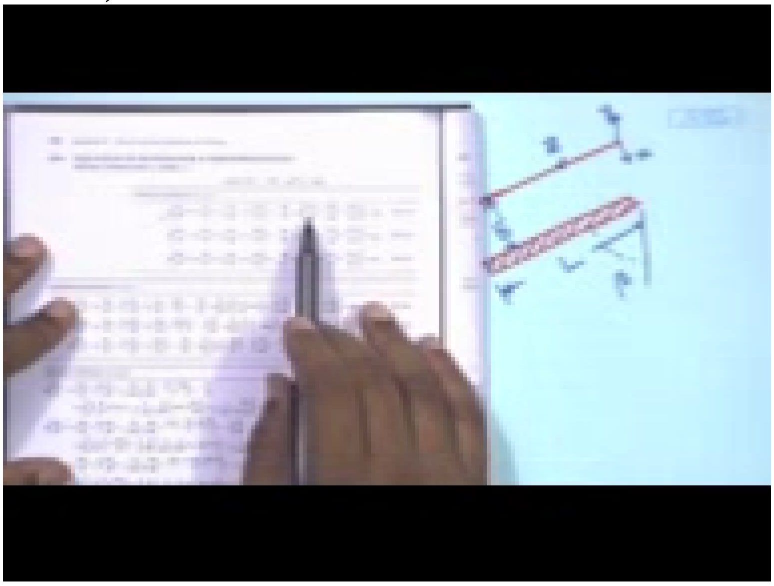

is expressed in terms of velocity, in terms of velocity. But you can also, also use the component, the form of the equation where instead of everything in terms of velocity; it is expressed in terms of shear stress. That is the same thing except mu times del v del v x del x is simply going to be del square v x by del x, we can express it in terms of shear stress. So that's the same thing. This is what

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I am talking about.

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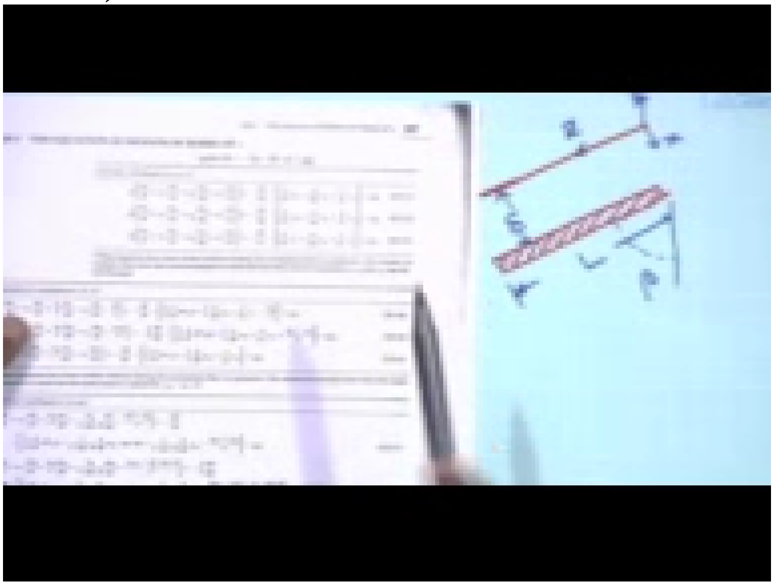

In here the same equation that is seen in the next page

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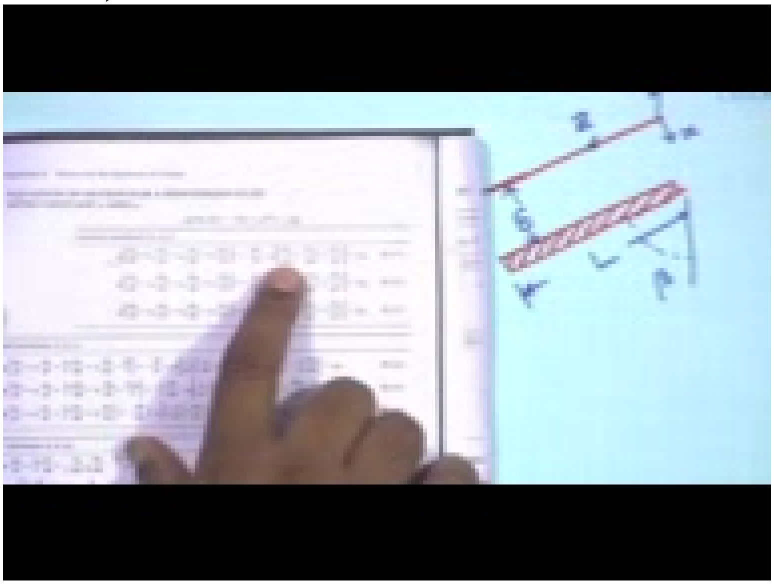

where everything is in terms of velocity, here

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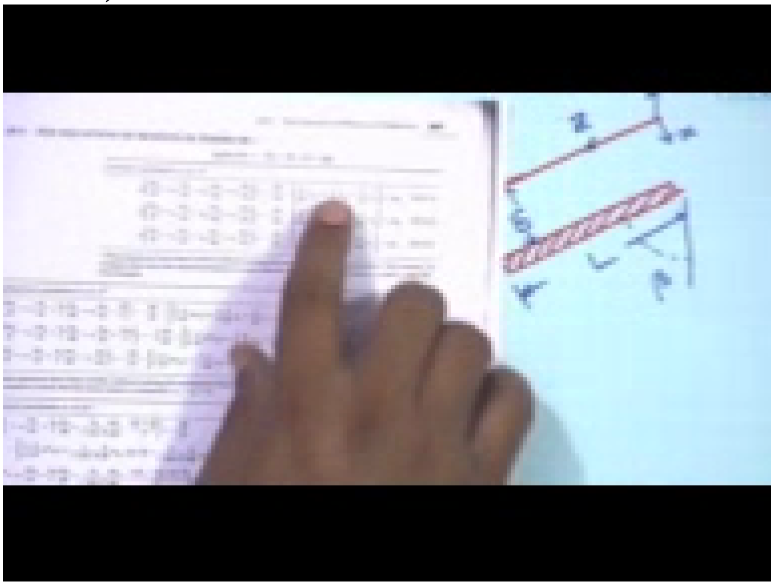

everything is in terms of shear. So

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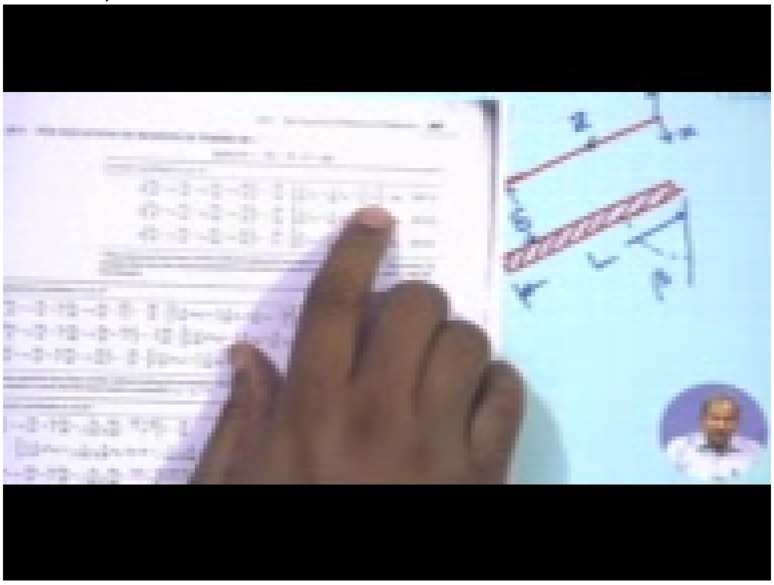

in some cases it would be, it would be beneficial to work with the shear stress form, in some cases it would be more convenient to use with

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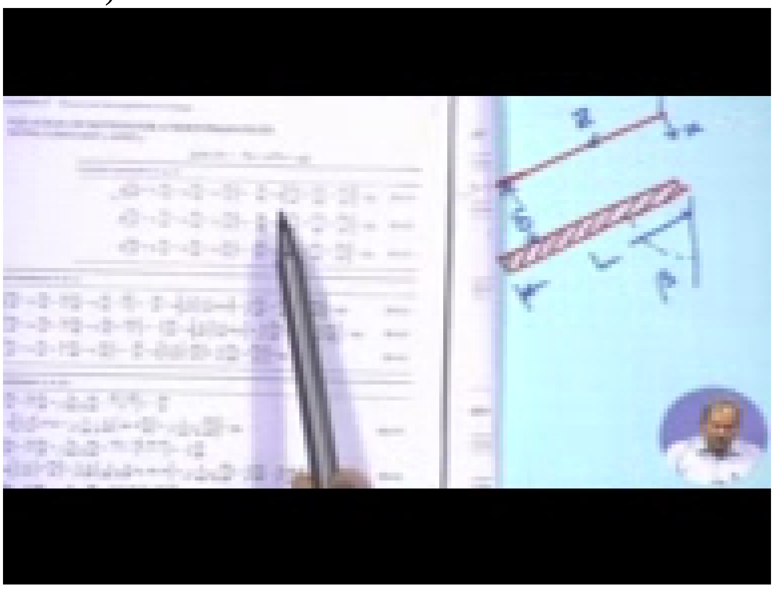

the velocity gradient form, Ok. So you can choose which one is, which one you are comfortable with. But the conceptually they are the same. So I am going to start with,

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in order to bring parity to what we have done in our previous class while solving this problem, I am going to choose the z component of Navier–Stokes equation in Cartesian coordinate system and we are, I will use the shear stress form of the equation. Because you would remember that the governing equation that we have obtained for that case was in terms of shear stress. So I will choose the shear stress. The same problem can be done using the velocity form of the Navier–Stokes equation. That is fundamentally, conceptually nothing, no difference between the two.

So let me write this first.

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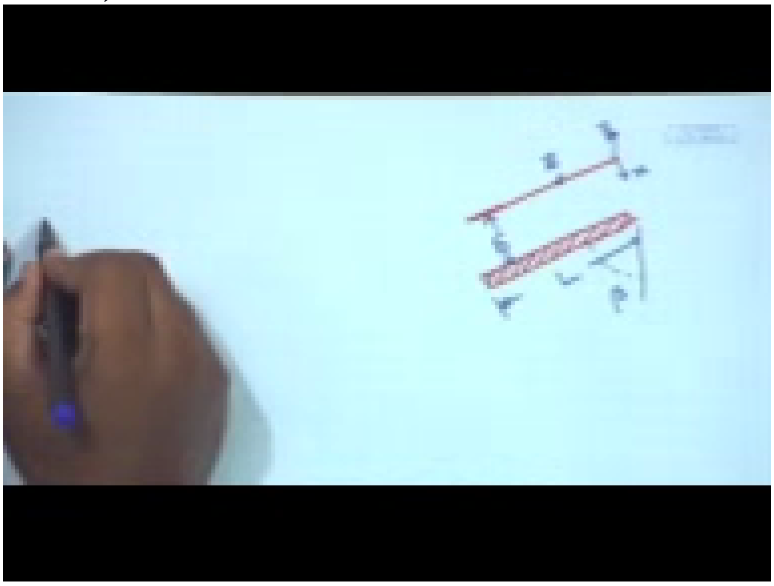

This is the z component of this equation, Navier–Stokes equation; first one is the temporal term or the time varying term. These are the convective part of the momentum transport. This is the conductive or viscous transport of momentum. Let's see what these terms are, once again. This is the temporal term. Any term that contains the velocity component separately, these are convective transports of momentum. So the entire left hand side of Navier–Stokes equation is the, is the convective transport of momentum; the convective transport of momentum and temporal transport. When you come to the right hand side, this is the surface force part, this is the shear and this is the body force. So let's see what are the, how did we, how can we solve this problem. What were the assumptions that we have made while solving these problems? The first one, it was a steady state, the second was one D flow where you only have v z which is not equal to zero but all other components are going to be equal to zero. No pressure, no pressure acting on the system and you only have gravity which is the component of the gravity in this direction. So

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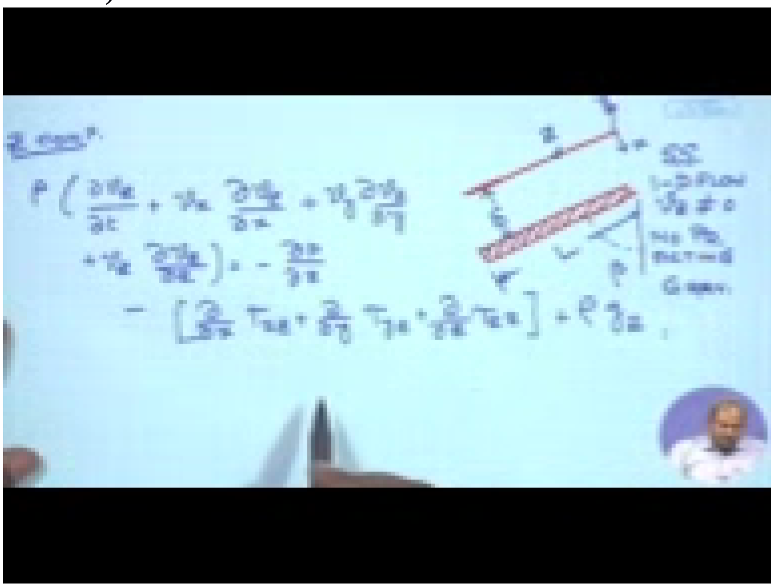

the first term would obviously be zero since the velocity in the z direction is not a function of time, so

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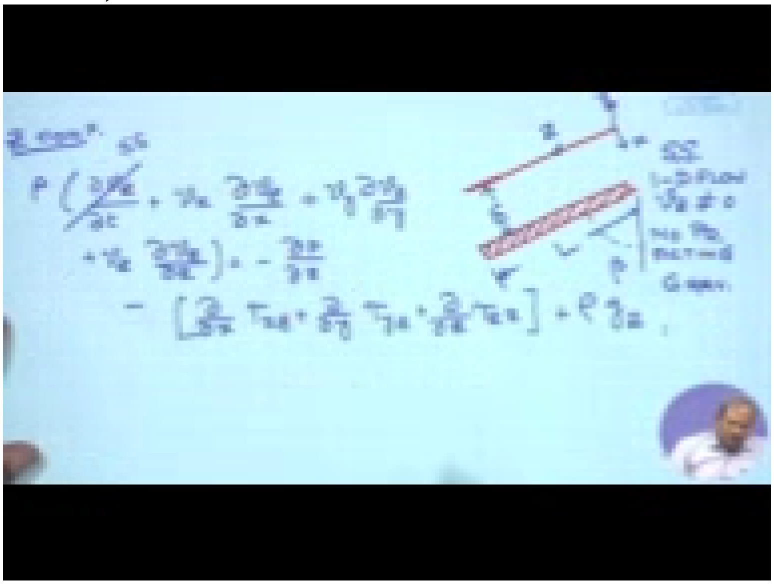

the reason for that I am, since we are writing s s or steady state. Now if you look at the second term, the v x, the x component of velocity, x component of velocity, it is not there. Only v z is non zero

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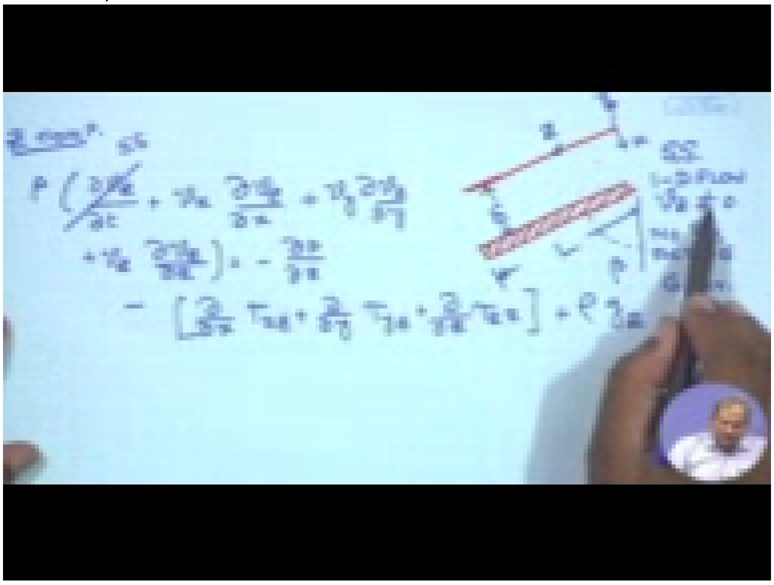

in this case but both v x and v y, v x and v y both are equal to zero. So this part is zero since v x is zero.

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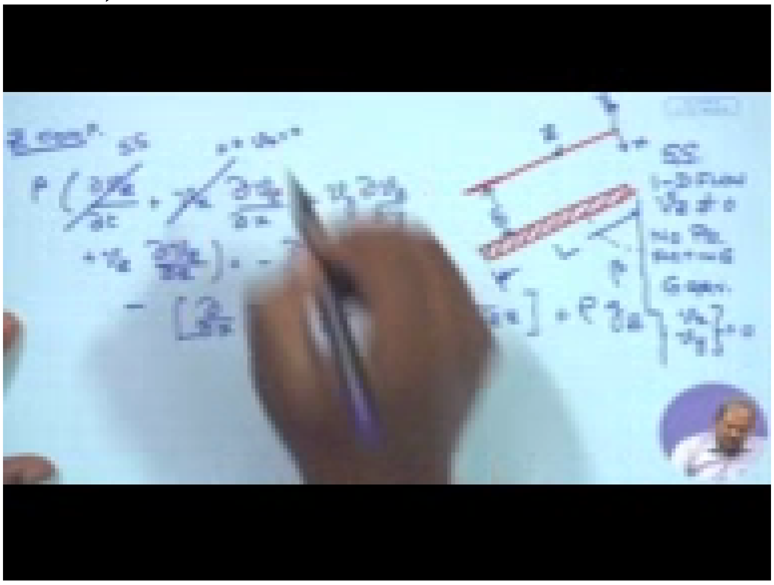

Similarly this is zero since v y is zero.

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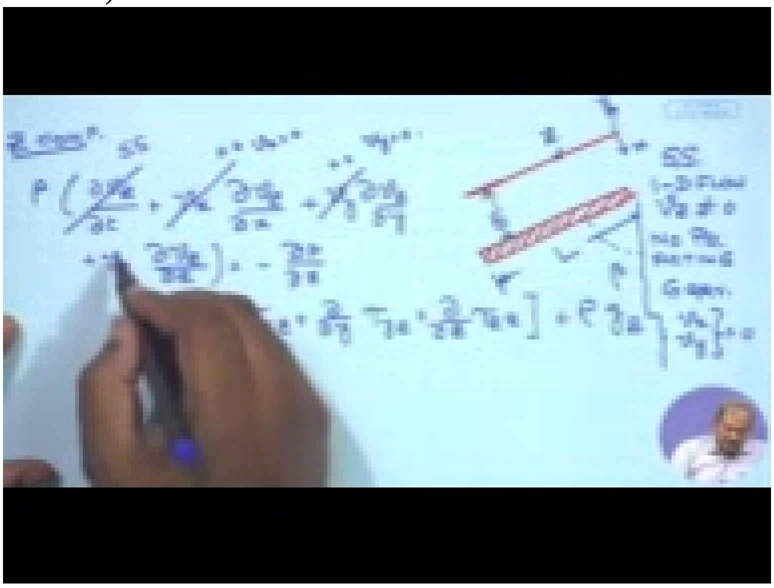

However in the fourth term v z is not equal to zero

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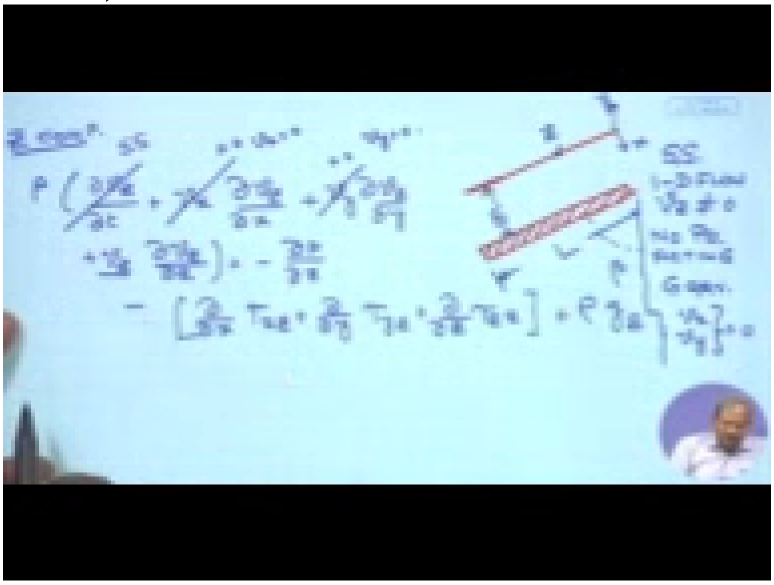

but equation of continuity since we are also going to assume that this is an incompressible fluid. If this is an incompressible fluid

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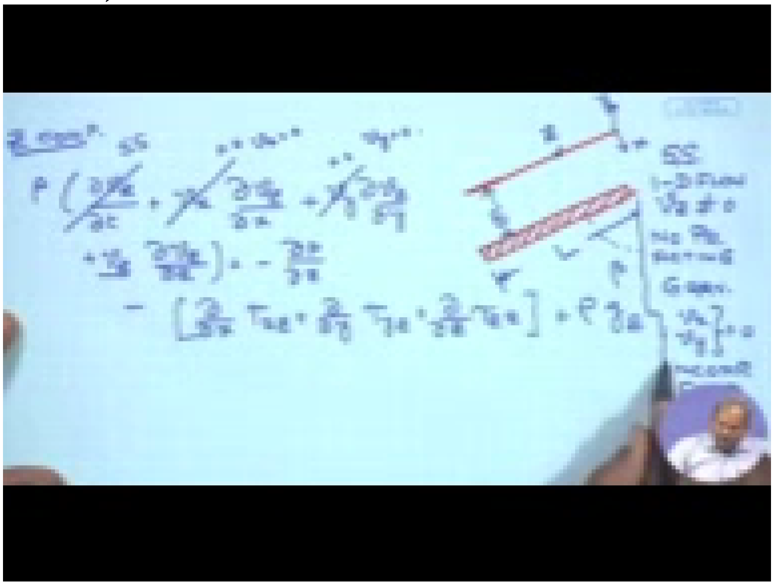

then what we have is that v z is function of z, is not a function of z only. So v z if you look at the figure is a function of x, v z is not a function of either y or z. Depending on where the film is located, with respect to it,

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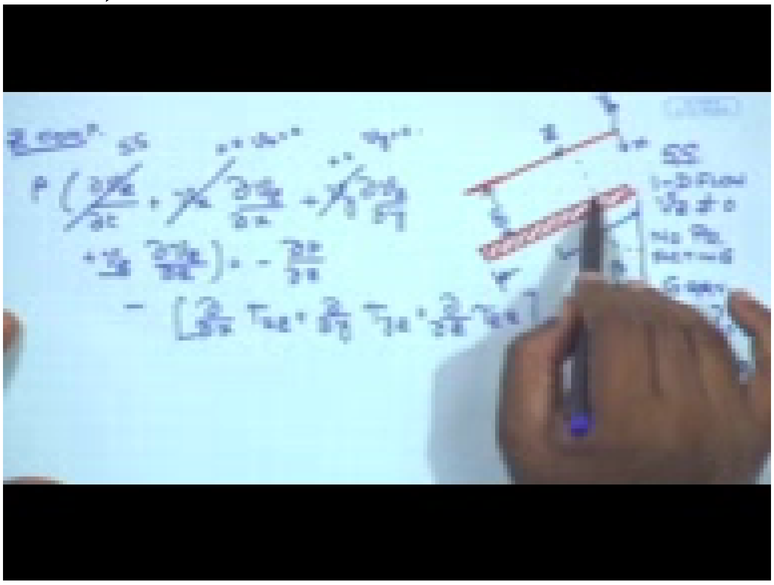

distance from the solid wall the velocity varies. So the velocity is zero

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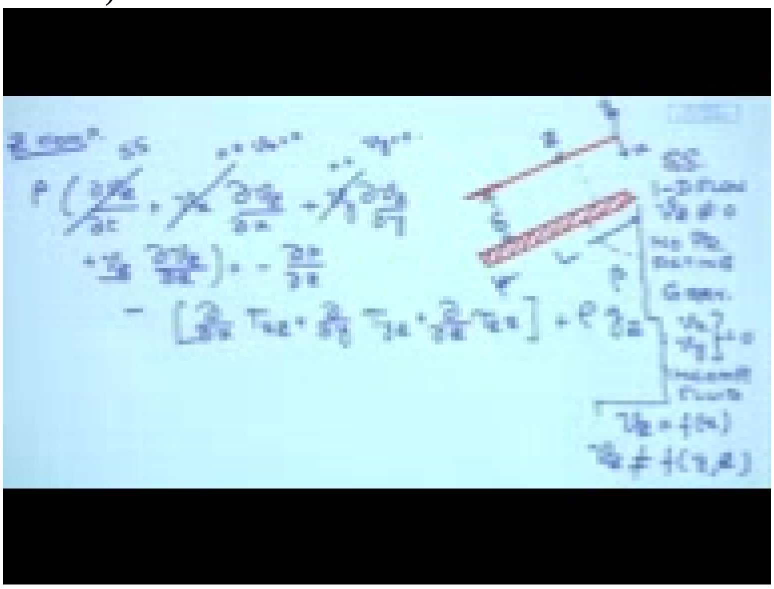

on the solid liquid interface, that is at that this point the velocity would

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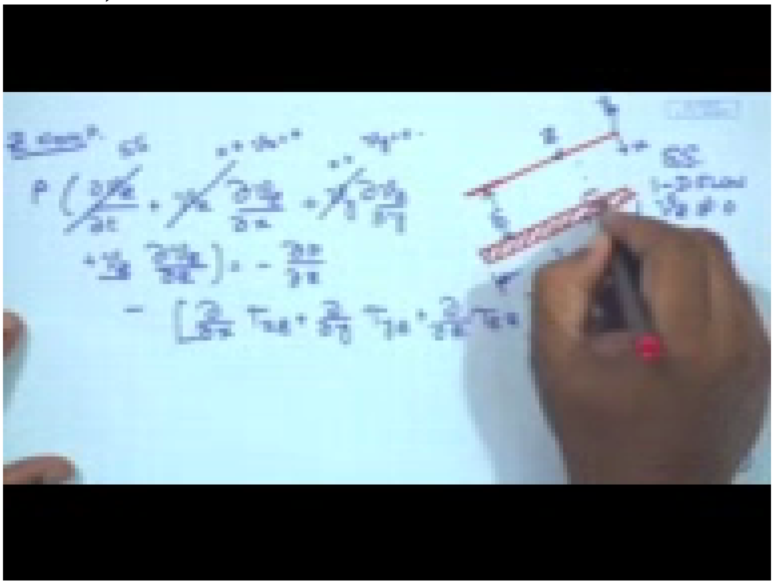

be zero. And the velocity would progressively rise as we move away from the solid plate. Therefore v z is the function of x.

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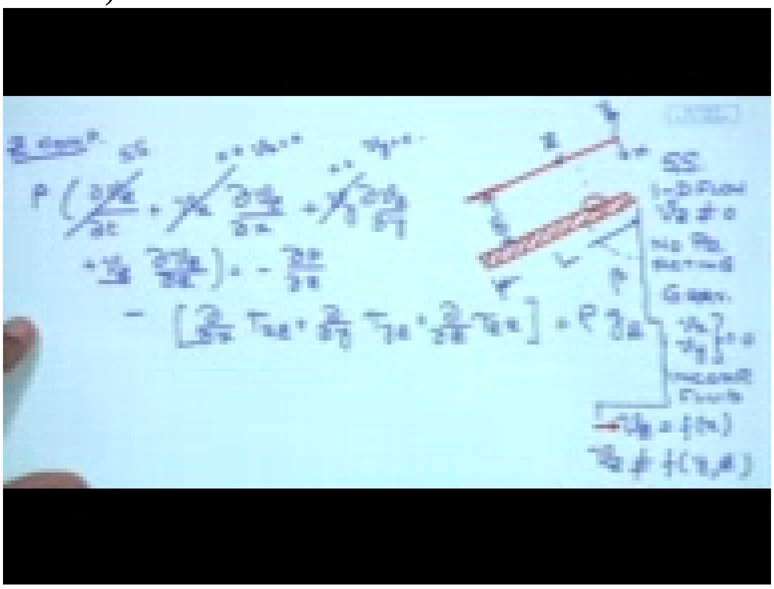

But v z is not a function of y, v z is not a function of z or it's a function of y. So therefore del v z del z, since v z is not function of z, this is going to be zero since v z is not

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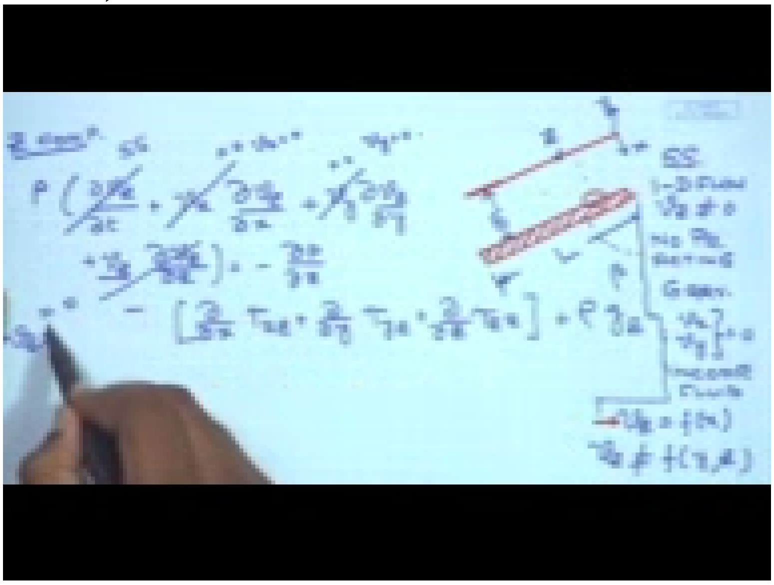

a function of z. So what I have in the entire left hand side is going

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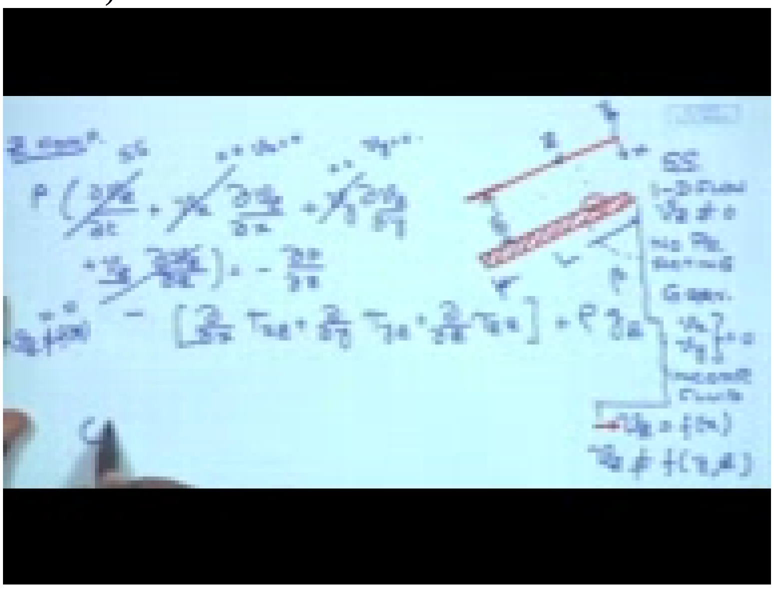

to be equal to zero, Ok

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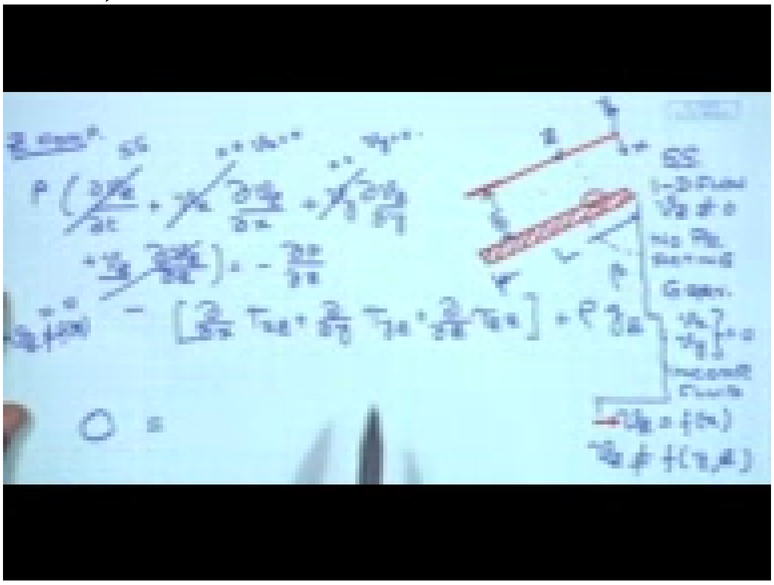

and then comes this del p del z. This is a case where no pressure is acting,

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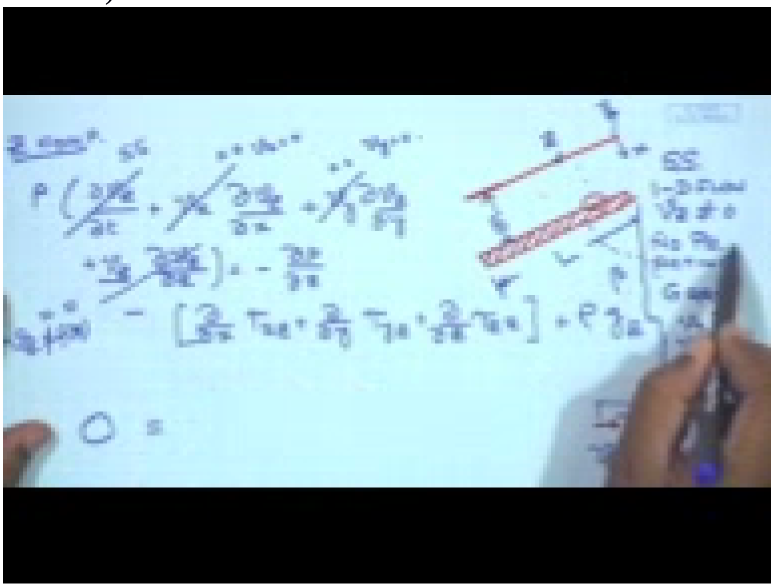

no applied pressure gradient on the system so this could be equal to zero.

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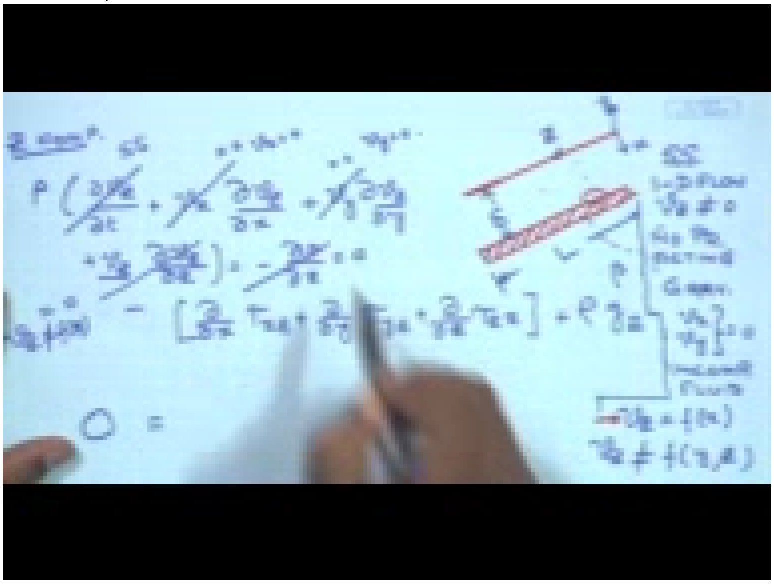

In case if this is going to be be equal to zero then what you have in here is then the z component, the now we are going to add, look at the shear stresses. The z, for the z component to be transported in the y direction there must exist a velocity gradient in the y direction. So for z component of momentum to get transported in the y direction there must be a y variation of velocity or there is no y variation of velocity. So v z is not a function of y and therefore

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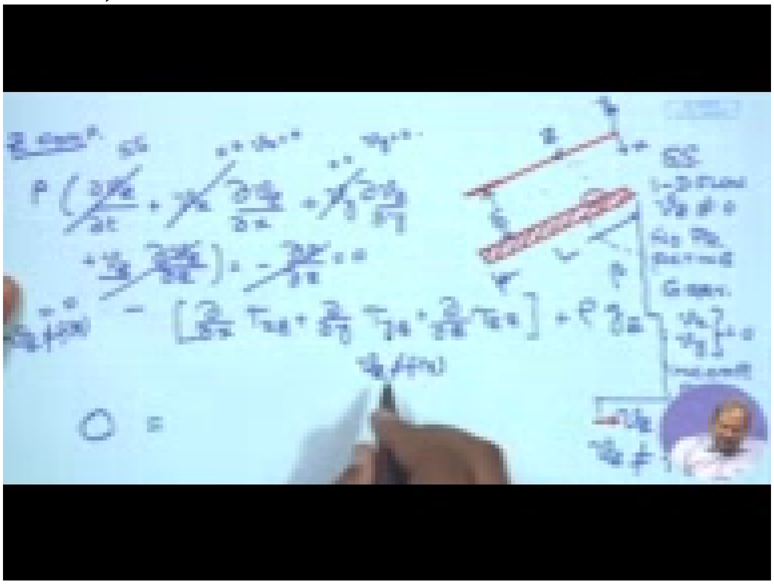

tau y z is equal to zero. Similarly

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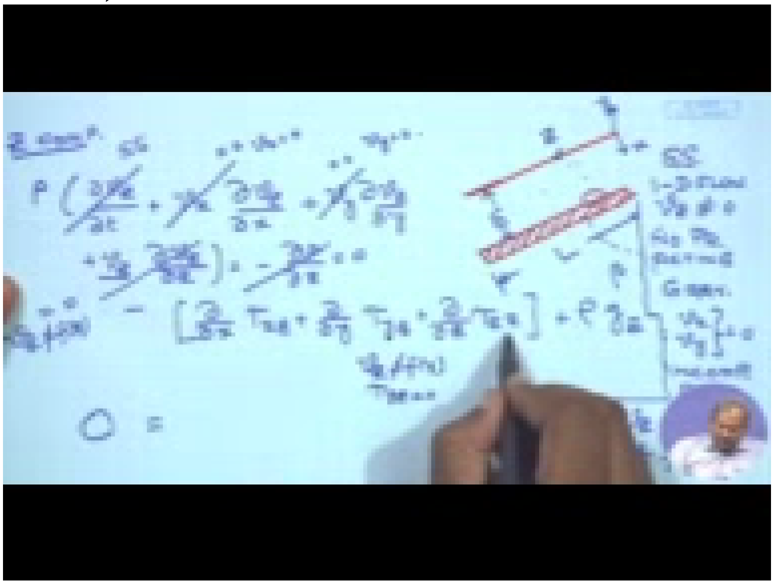

for z component of momentum to get transported in the z direction, v z must be a function of z

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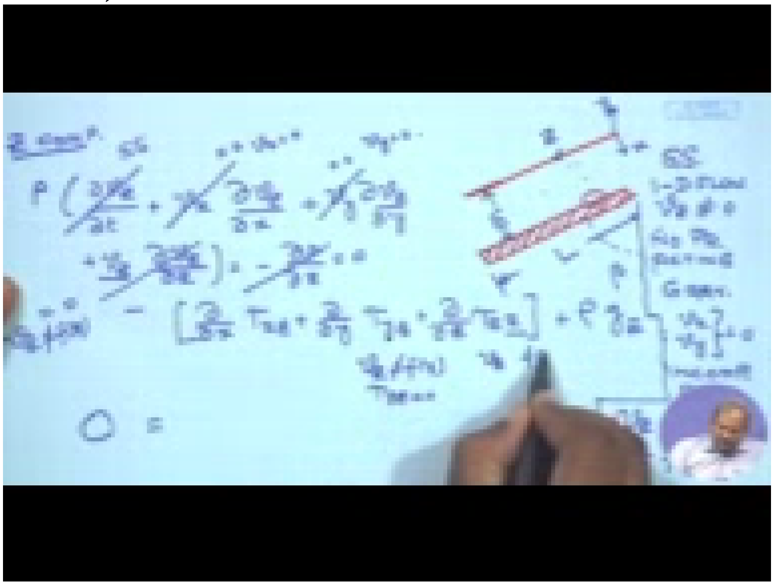

but we know that v z is not a function of z therefore tau z z would be equal to zero. I will repeat it once again. The tau y z represents the transport of z component of momentum in the y direction through viscous means. Now transport of z momentum in the y direction by viscous means will only happen when there is a variation in v z with y. So without the velocity gradient velocity variation, there cannot be any transport of viscous momentum. But we understand that v z is not a function of y. v z

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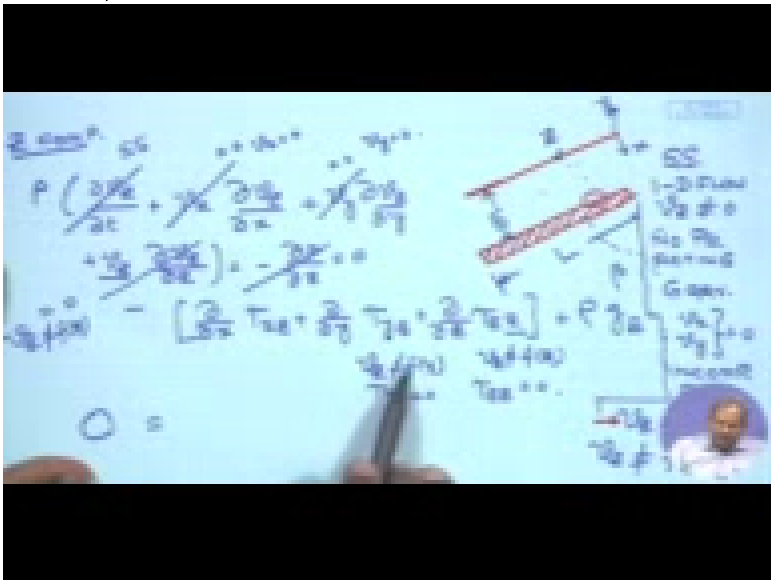

is not a function of y, there is no gradient in the y direction, there is no gradient of v z in the y direction. If there is no gradient then tau y z must be equal to zero. Similarly when we come about tau x, x tau z then in order for

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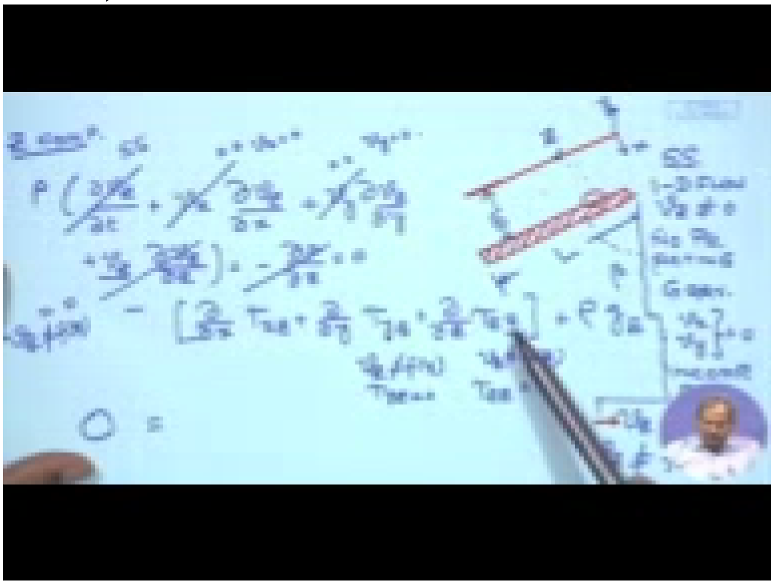

z component to have a viscous transport in the z direction the velocity must vary. But we understand that the velocity in the z direction is not the function of z. So therefore tau z z is also equal to zero. Now we come to this part. For z component of momentum to get transported by viscous means in the x direction v z must be a function of x. So need to see,

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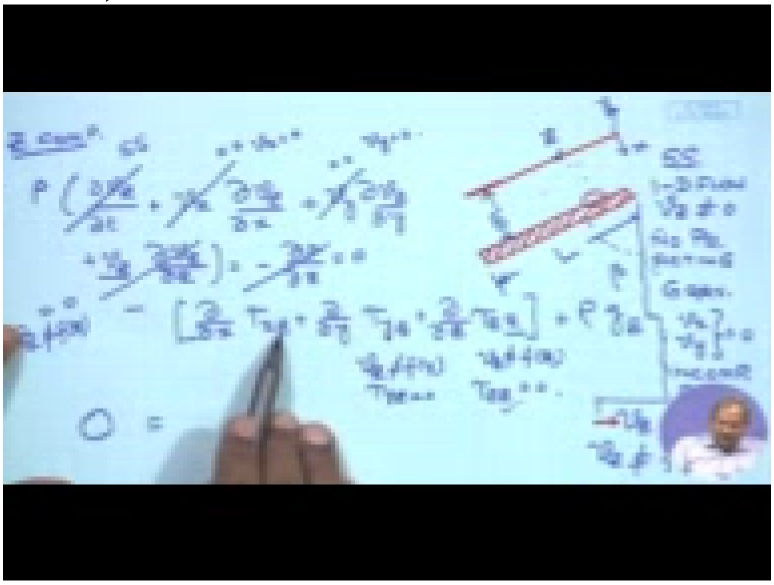

is really v z a function of x? If we look at the picture over here, v z is definitely the function of x, Ok. v z varies with x

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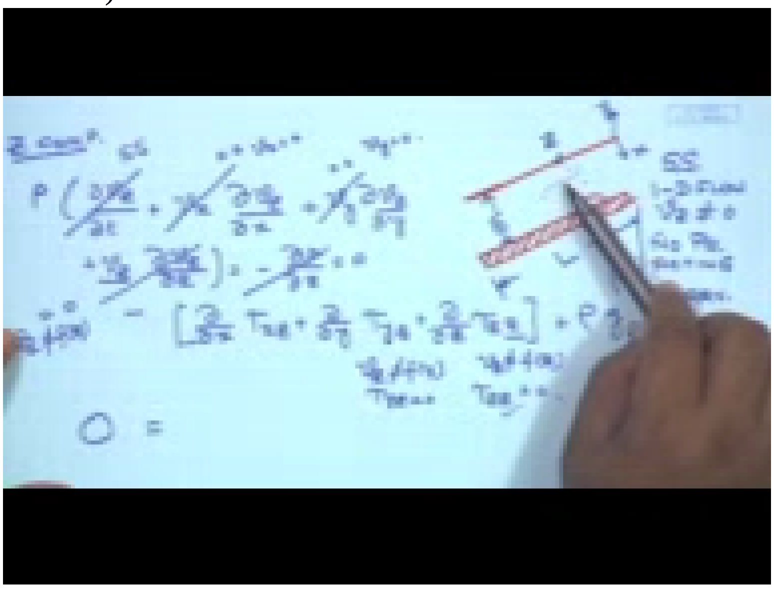

being zero over here and maximum over here. So there will be a transport of z momentum in the x direction since v z is a function of x.

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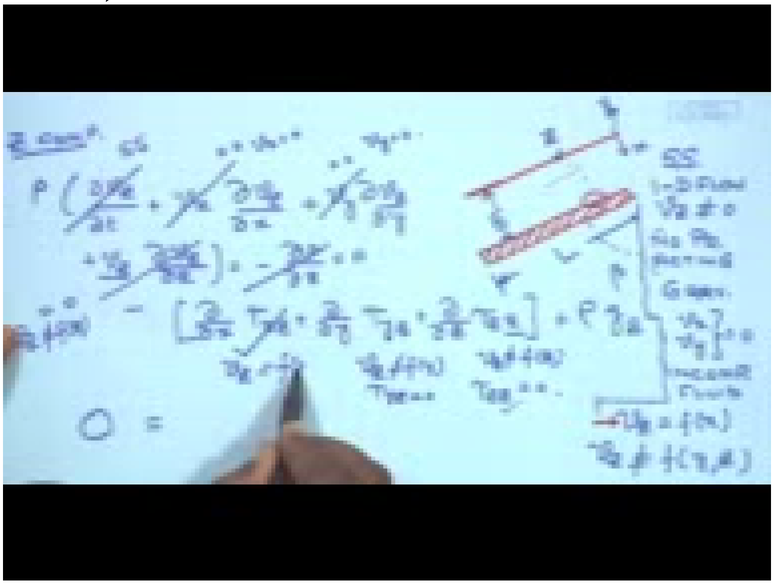

So what I have then d d x of tau x z in, I have dropped the partial sign because it is only a function of x, tau x z or any quantity is only a function of x. They are not functions of; they are not functions of y or z. So of the three terms of the viscous transport of momentum, only one will remain. The rest are zero because of our conditions, because of our understanding. Now come, we come

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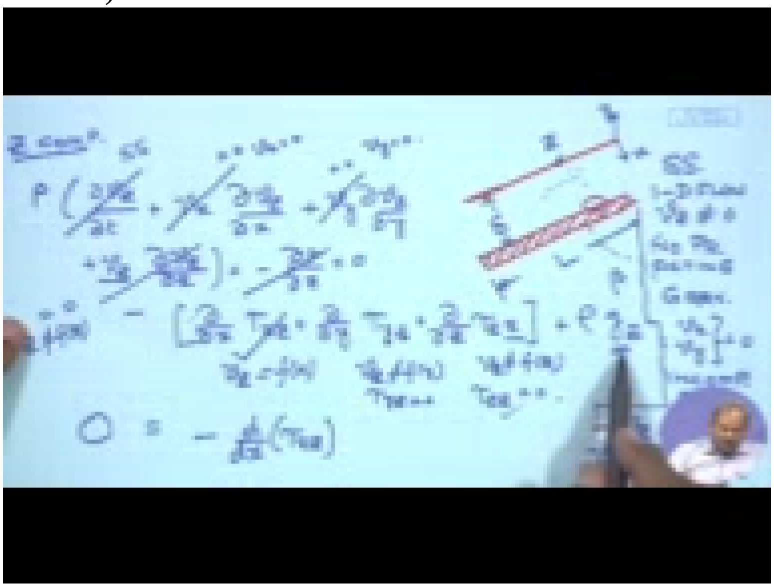

over here. g z is the component of the body force in the, in the z direction and from the figure you can clearly see that the component of gravity in the z direction is simply g cos beta. So your, this would simply be equal to g cos beta.

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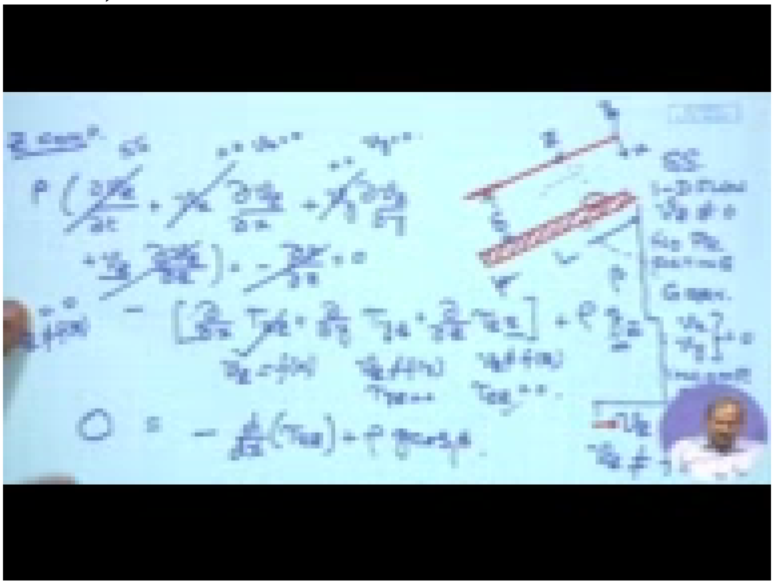

And if you look at the previous, the one that we had done

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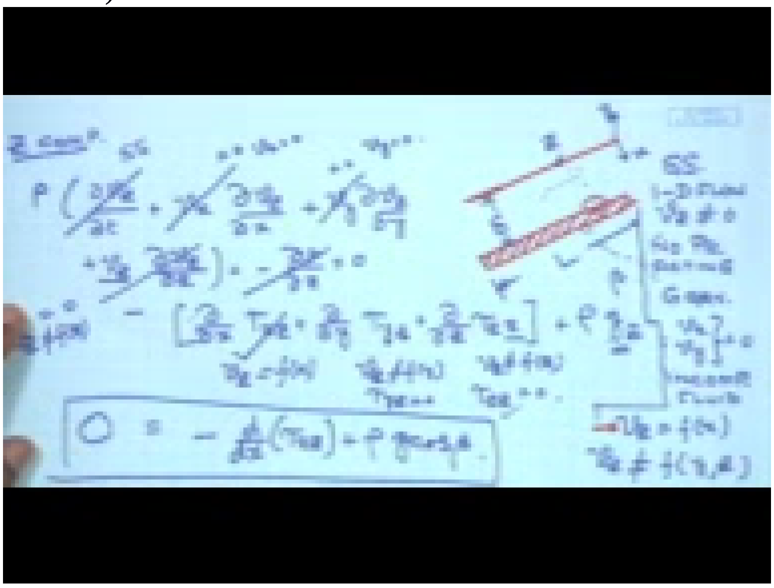

in the last week, this is exactly the same

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governing equation that we have obtained. So there is no need at all to think about a shell, make balances along, along and across the surfaces, find out what are the pressure forces and so on. The only job that you need to do is simply you choose the relevant, the right equation component for Navier–Stokes equation in the appropriate coordinate system. After that use your understanding, the description, the physics of the problem, cancel the terms which are not relevant. What you will be left with is the governing equation. So it's a very simple way to arrive at the governing equation and once you arrive at the governing equation the rest will be identical. That means we are going to integrate in the same way, we are going to use the same sort of, same boundary conditions and you will end up with the same solution but in a much more structured and easy way.

So in the next problem what we are going to see is the same problem where we have a flow through a tube in which there is going to a pressure gradient and there's going to be, there's going to be the action of gravity which has led to the Huggins Poisson equation in the problems that we have dealt with before. So our next problem is analysis of flow through a vertical tube when there is a pressure gradient active in the system. So what we have then is

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the, a tube of, this is the radial direction,

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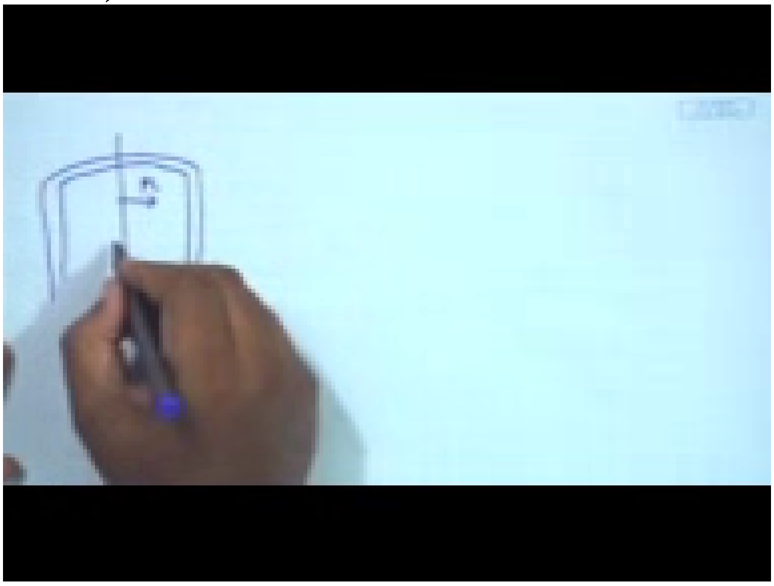

this is the z, actual

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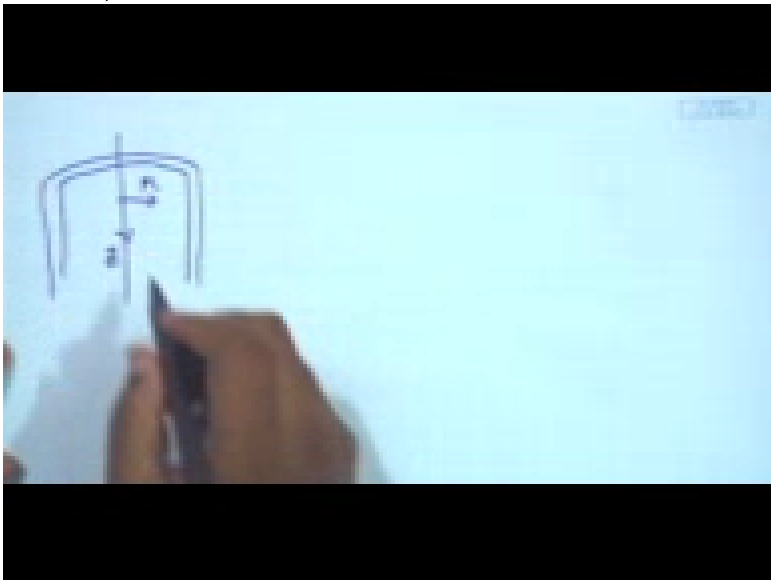

direction. You have some pressure p naught over here and you have some

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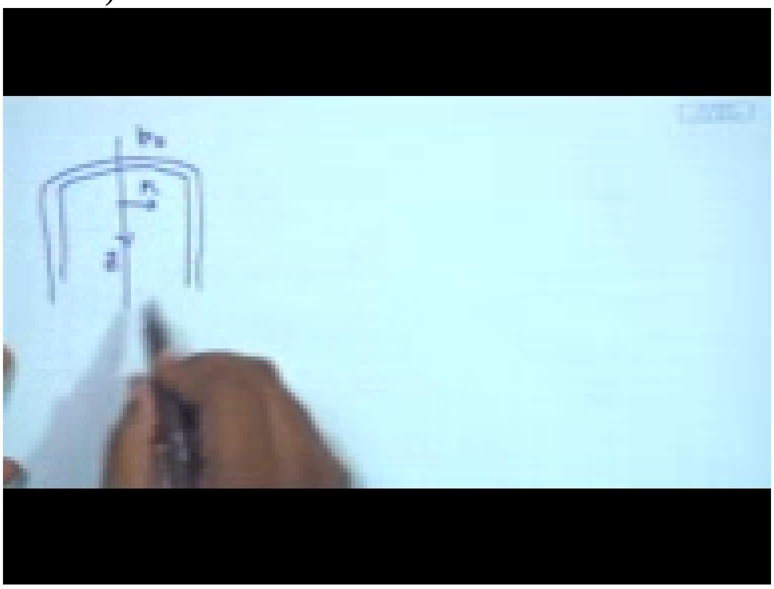

pressure p L

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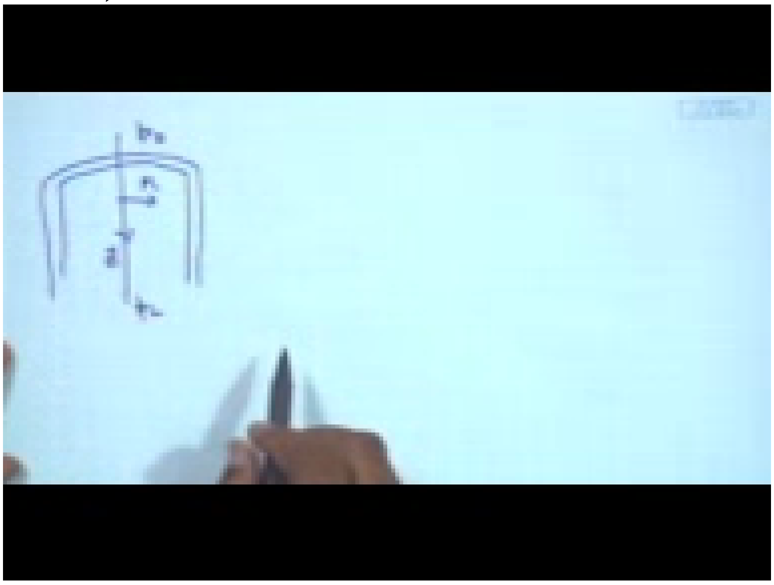

over here and as a result of which you are going to have flow in and flow out of the system.

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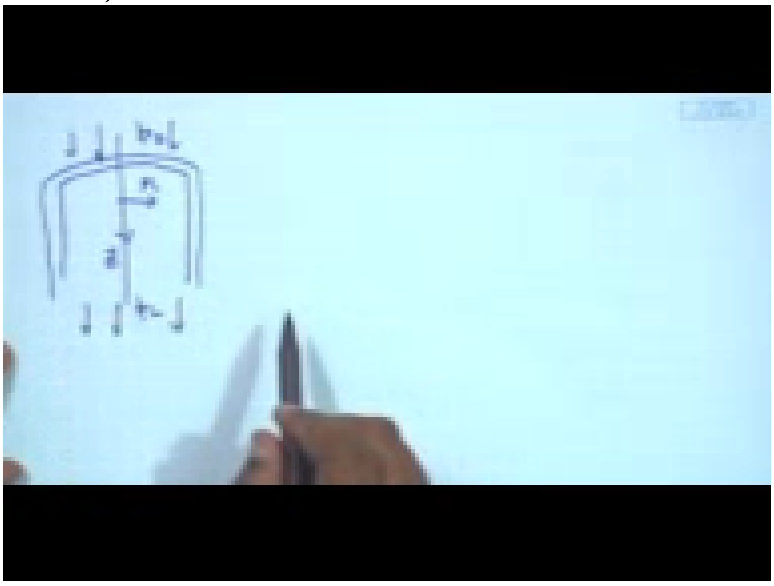

If you remember previously we had to think of a shell like this, and in that shell we have found out what is the amount of liquid coming through the annular top surface,

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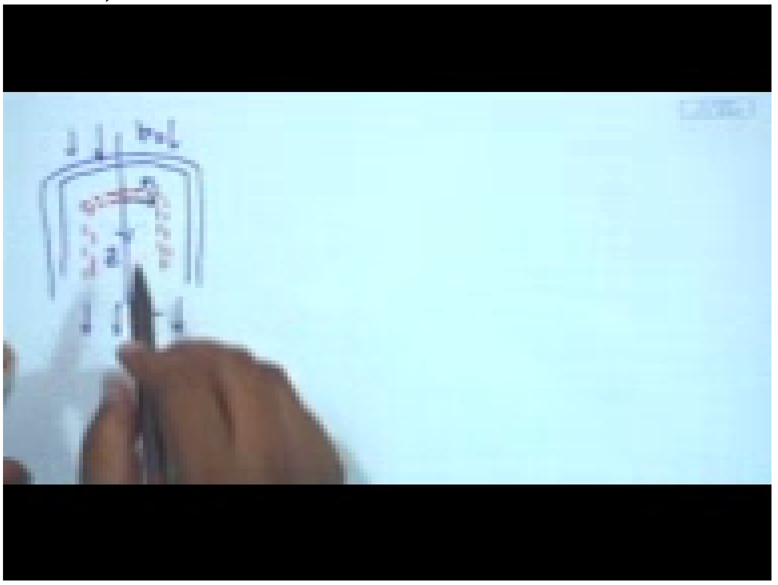

what is the shear stress that is acting inside and so on. We need not do anything, anything of that sort right now, Ok. What I need to do only is

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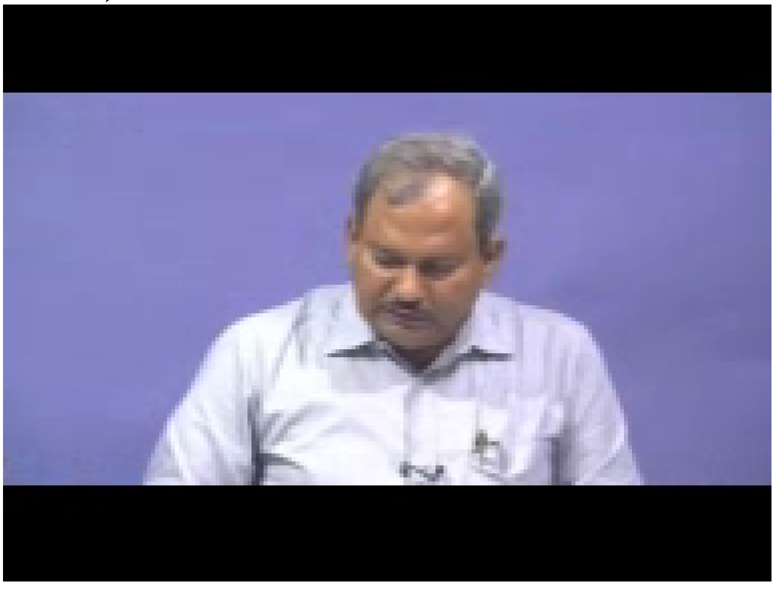

find out, use the, again

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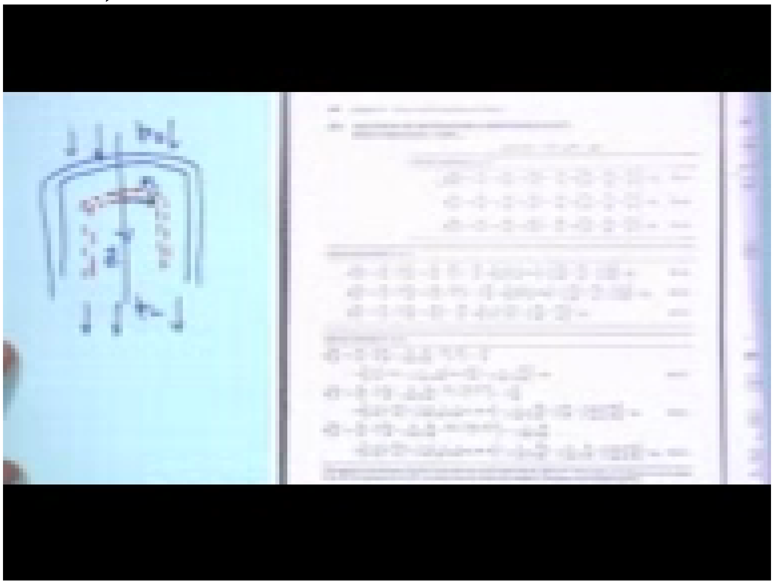

the right component the right component

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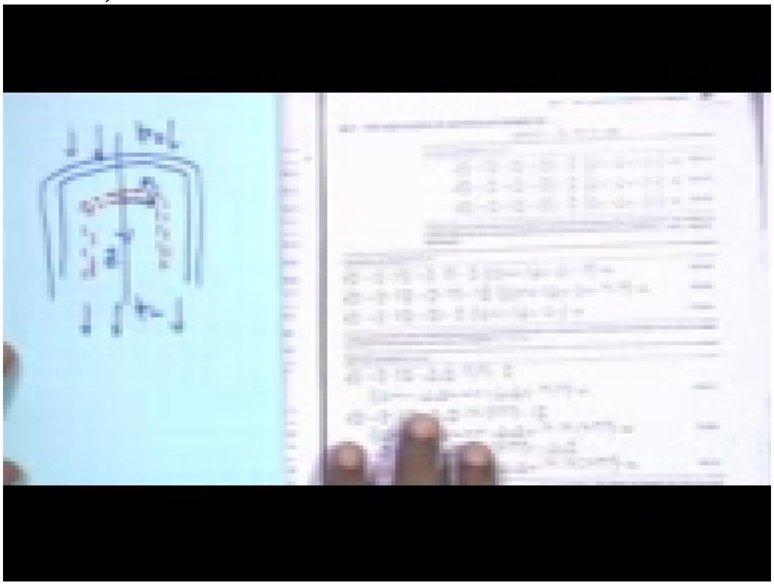

of the equation over here. This is a cylindrical coordinate problem, r, theta, z. I am going to choose the cylindrical coordinates.

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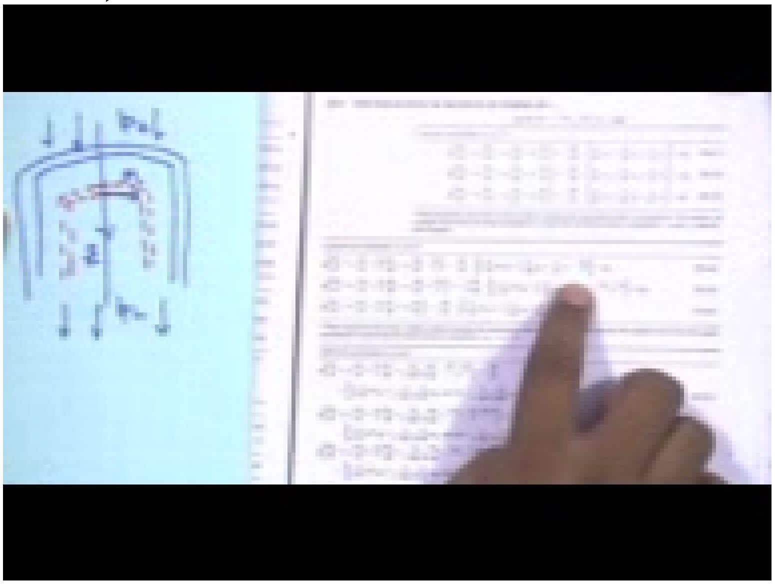

The principle direction of the motion

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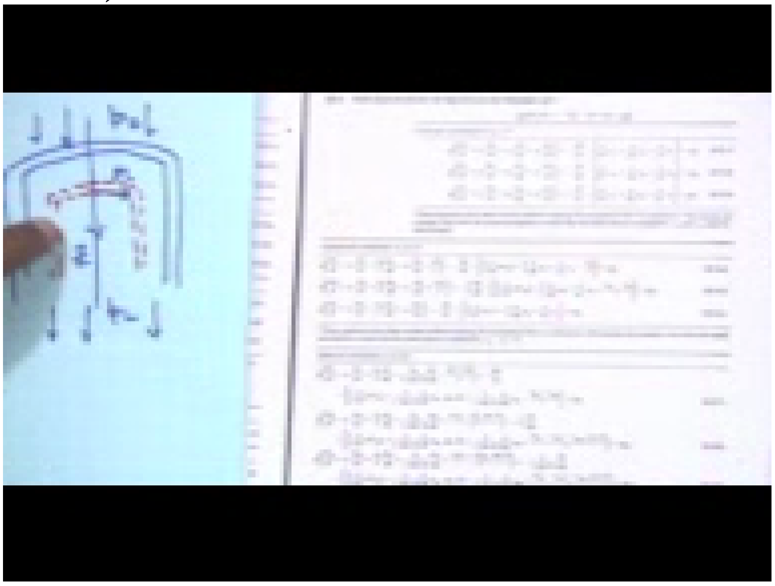

is in the z direction. So I would choose the equation B dot 5-6

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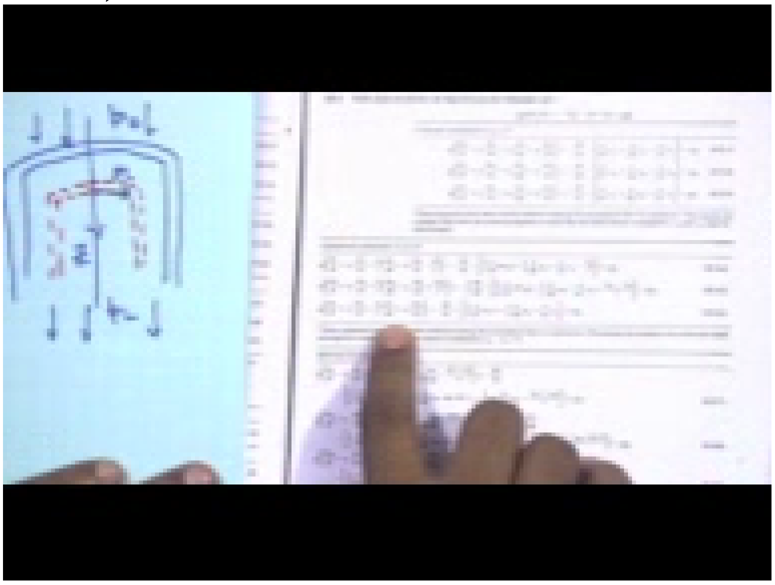

which is nothing but the z component of equation of motion in cylindrical coordinate. So this is the last, this last equation I am going to choose as the starting point to derive the governing equation for this specific problem, Ok. So I am going to write this equation in first. The equation

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is nothing but this z component of equation of motion which from the text would simply

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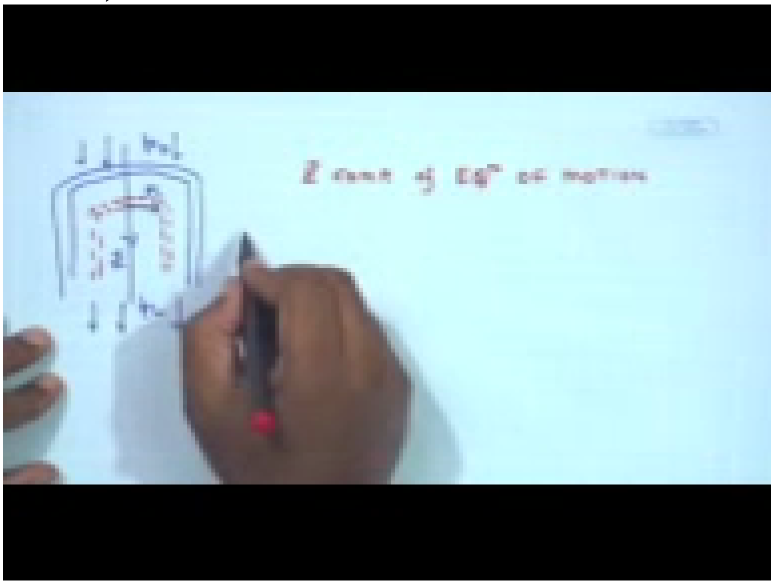

tell you row del v z del t, the temporal term v r del v z del r plus v theta by r del v theta by del theta plus v z del v z by del z since it's a, since it's a

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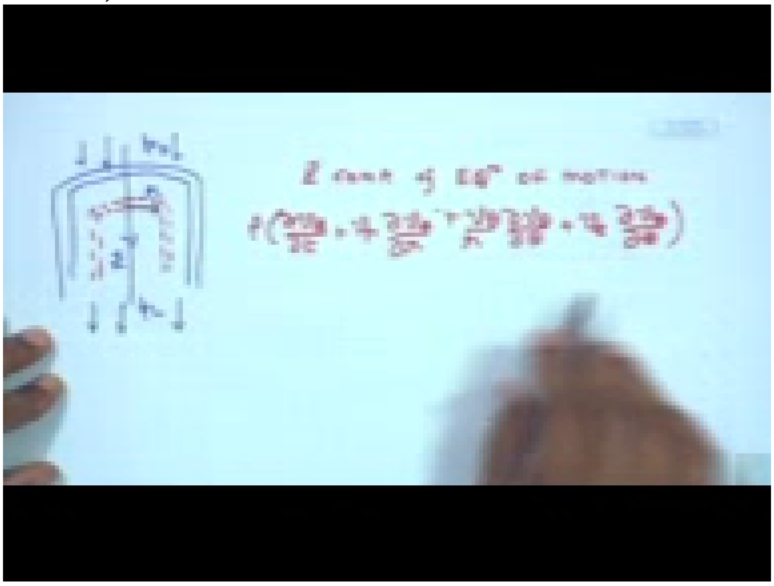

cylindrical coordinate system the terms are slightly more involved and complicated but ultimately it would not matter because we would be able to simplify it to a large extent. This is a pressure term and then what I have over here is the viscous transport of momentum and also you have the body force term.

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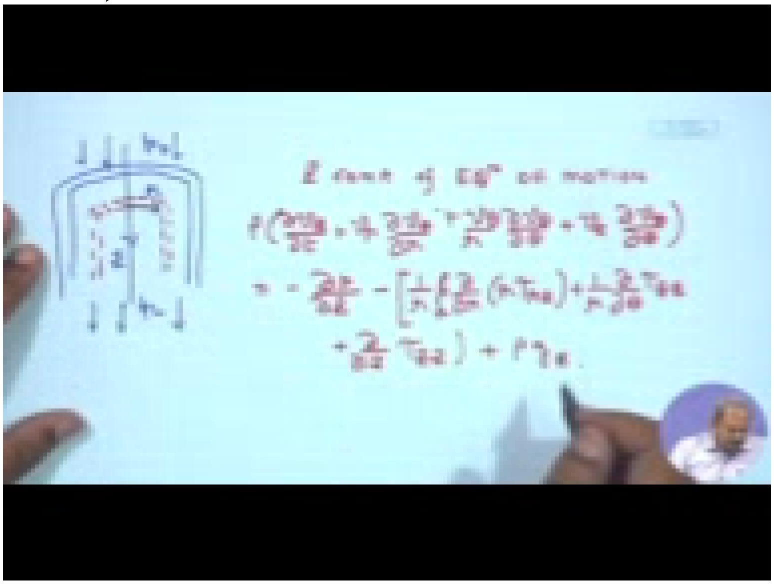

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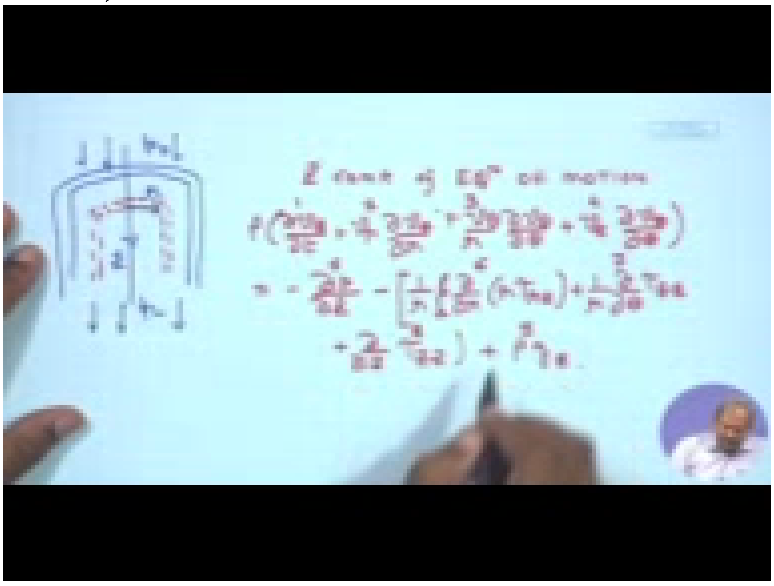

Now I will write this as term 1, 2, 3, 4, 5, 6, 7, 8 and 9.

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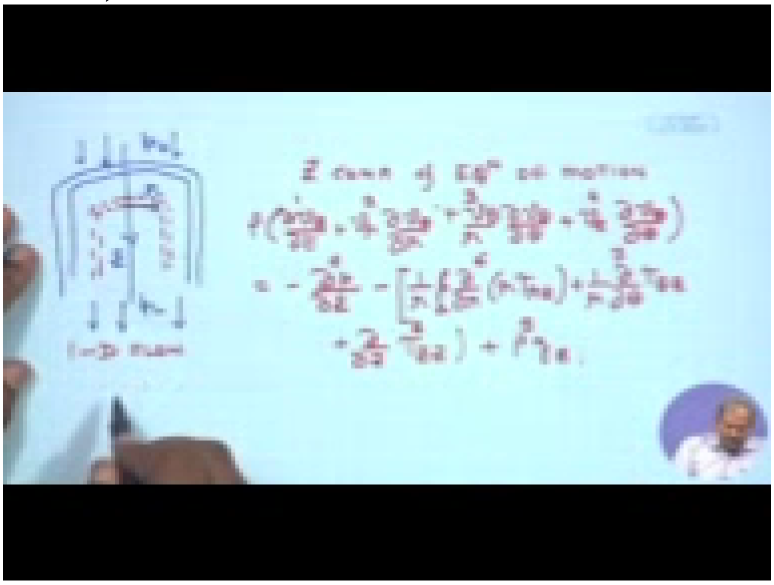

So what were the descriptions of the problem that we have, we have seen over, we have done over here? That is, that is one D flow, that was the same assumption that we have used, I am just repeating the assumptions that we had made over there; flow only in z direction, it's incompressible flow,

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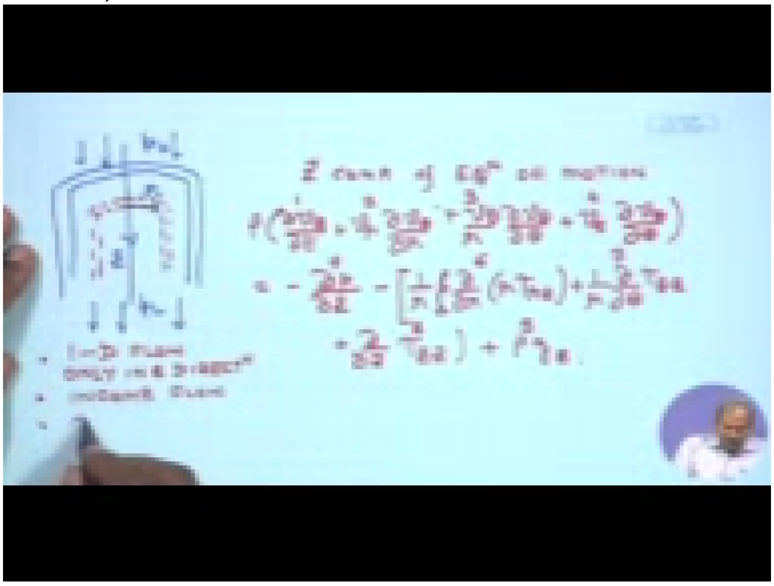

v z is not equal to zero and v z is a function of r, v z is not a function of theta or of z

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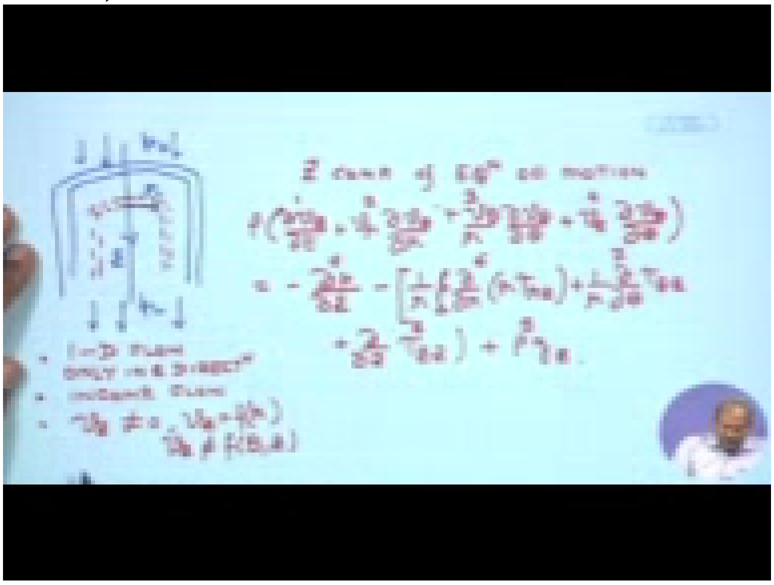

and all the other components v r and v theta would equal to zero.

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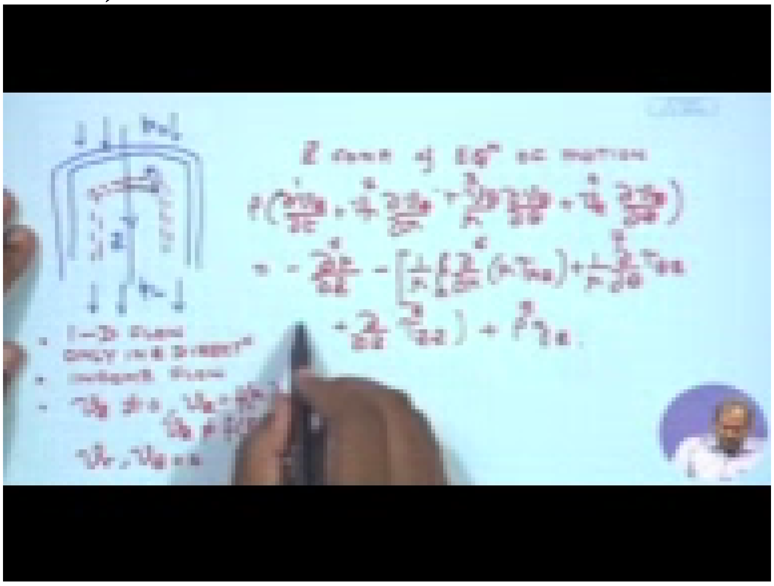

So these were the basic assumptions that we have, we have made in solving the previous problems, and the another one which I missed is it's a steady state condition,

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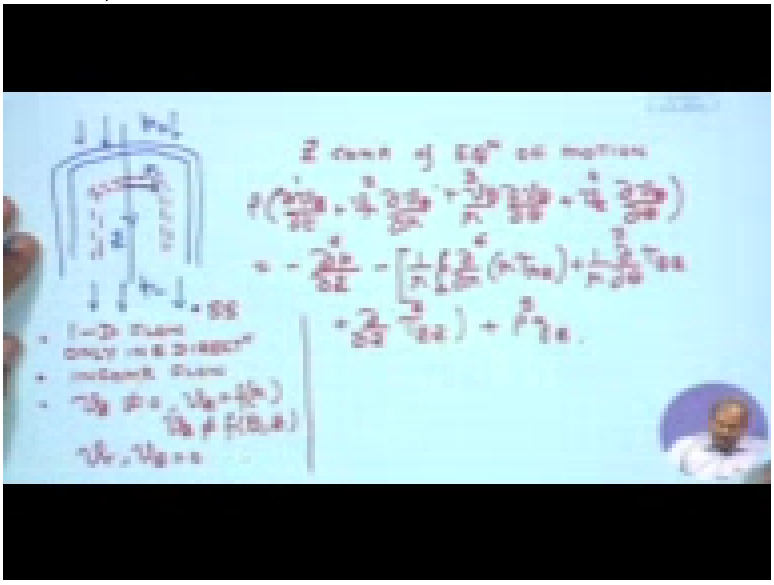

Ok. So we would like to see whether or not it's, what happens to 1, term 1 over here. Will it remain in the governing equation or I can cancel it? If you look at term 1, and see steady state so term 1 has to be equal to be zero since it's a steady state problem.

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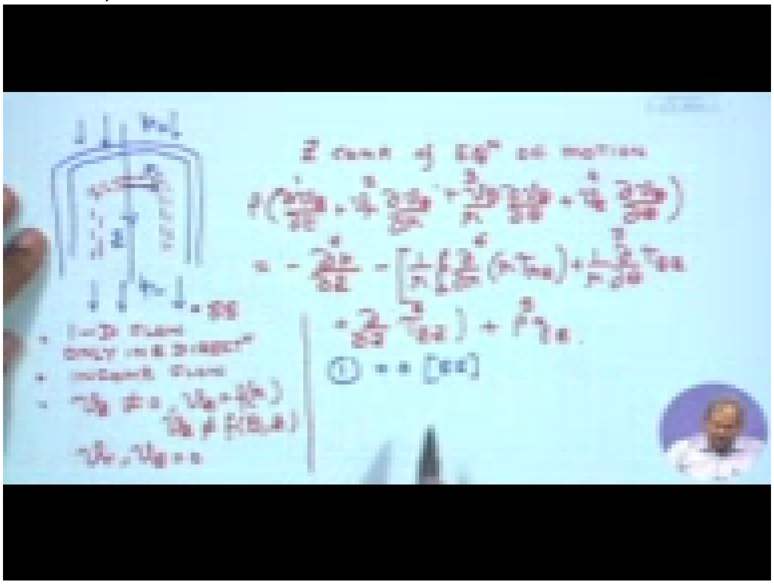

What about term 2? Term 2 is this term and if you look here it would simply tell you the term 2 would also be equal to zero since your v r is equal to zero.

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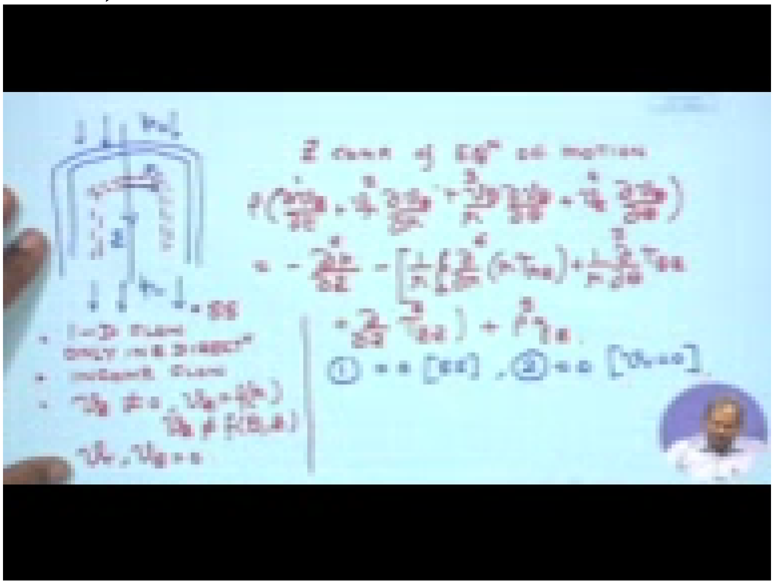

About term 3, the same thing term 3 would also be equal to zero; this is my term 3 since v theta is equal to zero.

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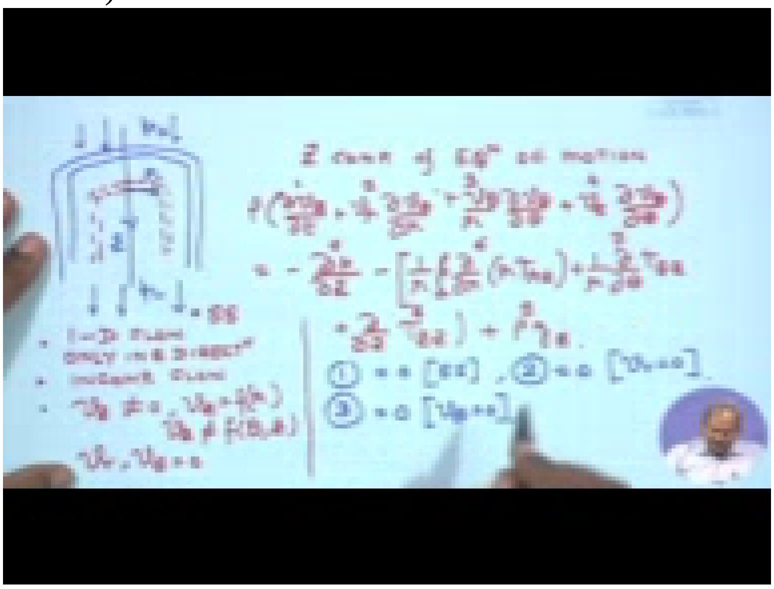

Term 4, term 4 v z is not zero, but del v z del z is zero since v z is not a function of z, Ok. So this is zero since v z is not a function of z. So the entire

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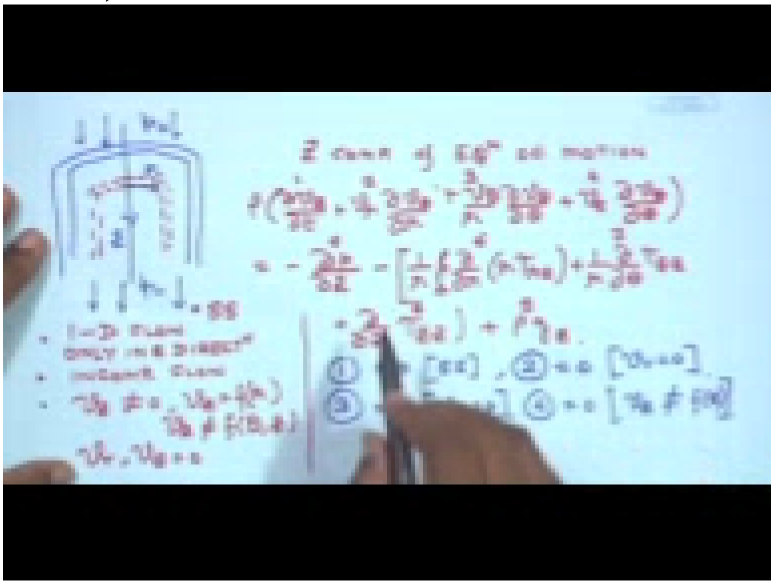

left hand side of the expression is

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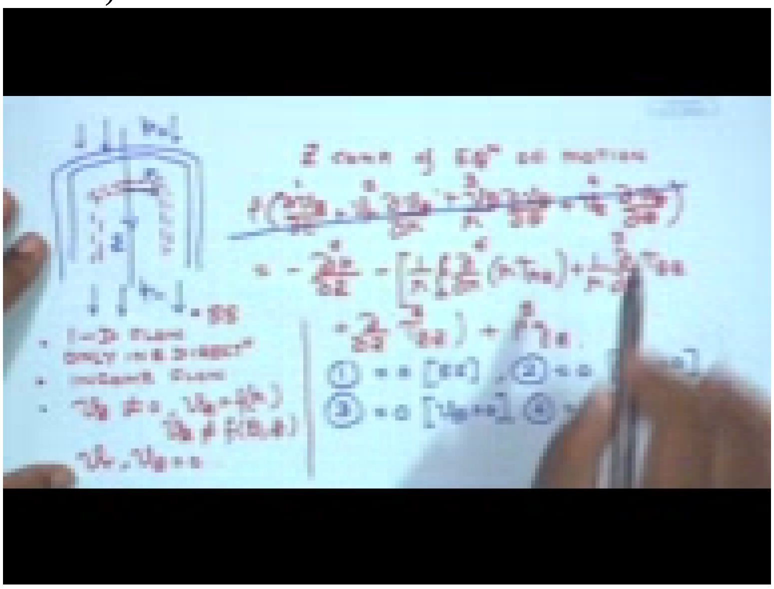

zero. The temporal and the convective transport of momentum in, for this specific problem is going to be equal to zero. Now what about term 5? The variation of pressure with z, I know that the pressure over here is p naught, the pressure over here is p L, so 5 is not equal to zero. There is a pressure variant that acts on, that acts on the system. Now remaining is the z component gets

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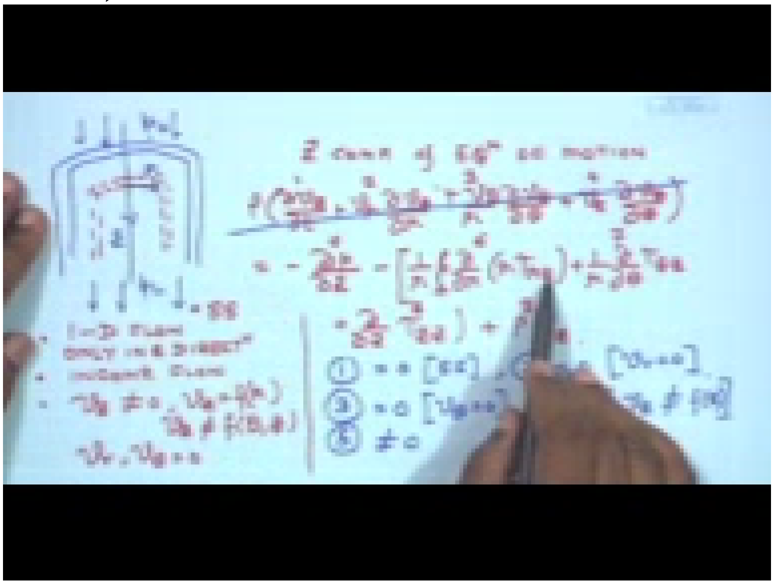

transported in r direction is the significance of tau r z. So in order for z component to get transported in the r direction there must be variation in v z with r. Do we have variation of z in r? Let's see over here. The velocity v z is the function of r so obviously the velocity varies with r and therefore tau r z is non-zero and cannot be neglected from this equation. So this is not going to be equal to zero

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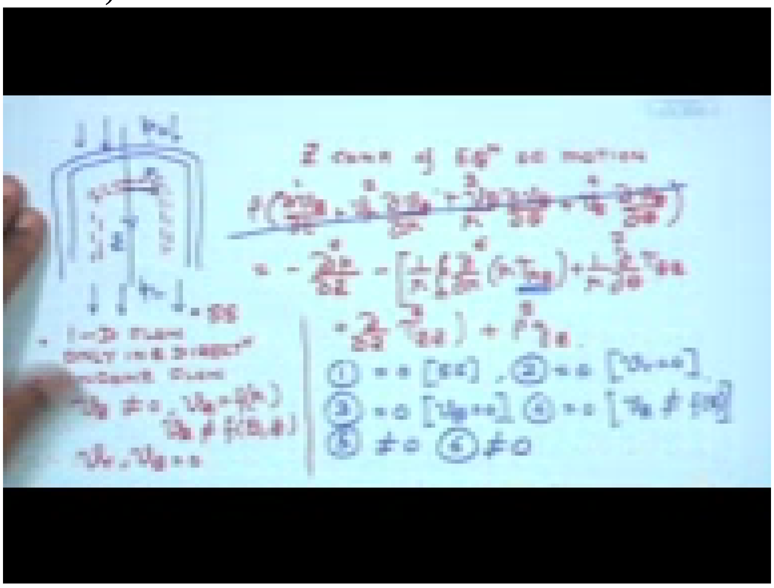

but how about z component getting transported in the theta direction? In order for that to happen the velocity must change with the theta direction which it does not because velocity is not a function of theta so your seventh term is equal to zero since v z is not a function of theta therefore it leads to tau theta z to be equal to zero.

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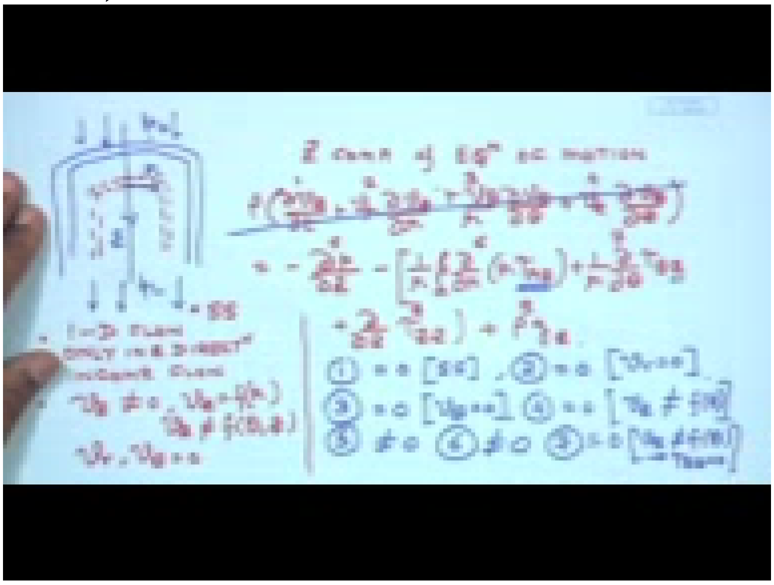

Eighth term tau z z, z component in z direction, for that to happen, velocity must be a function of z but velocity is not a function of z. So your term 8 is also equal to zero; and how about your term 9,

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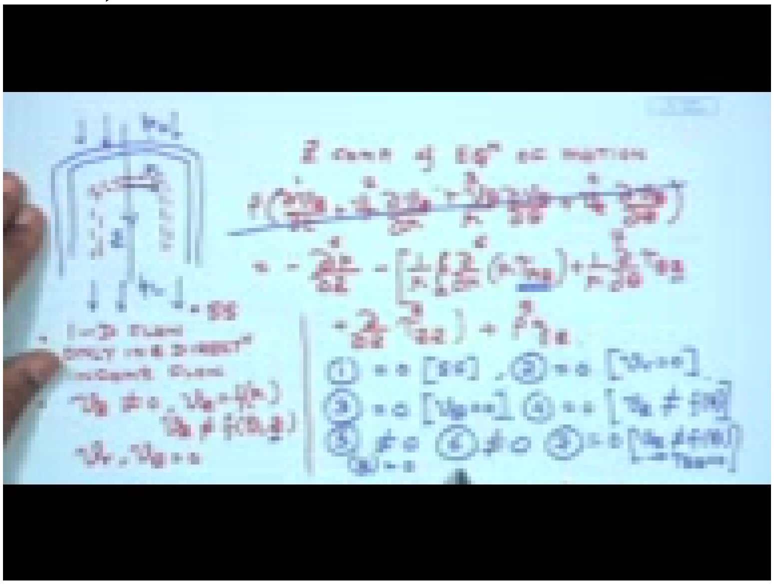

term 9 is 0? g z there is definitely a non-zero component of component of gravity in the z direction equal to plus g. So this is not equal to zero.

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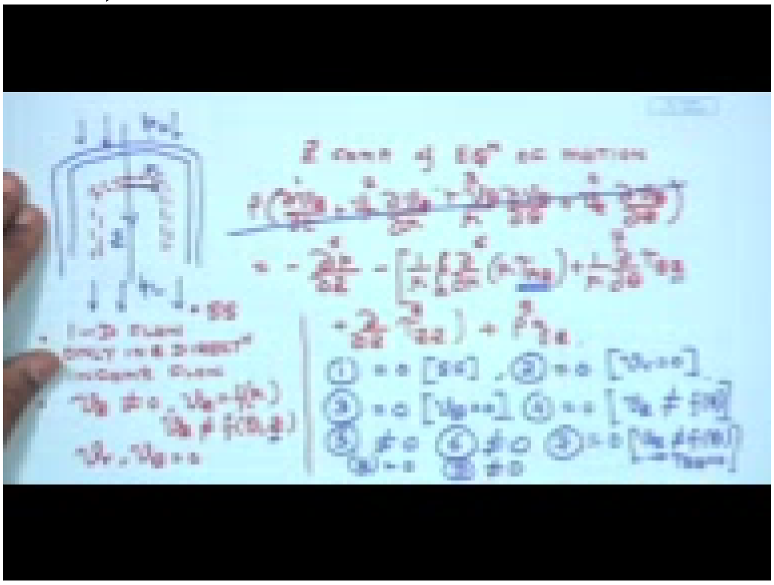

So what I have then out of the governing equation is this, this would be zero and only these three terms, term 5, term 6 and term 9

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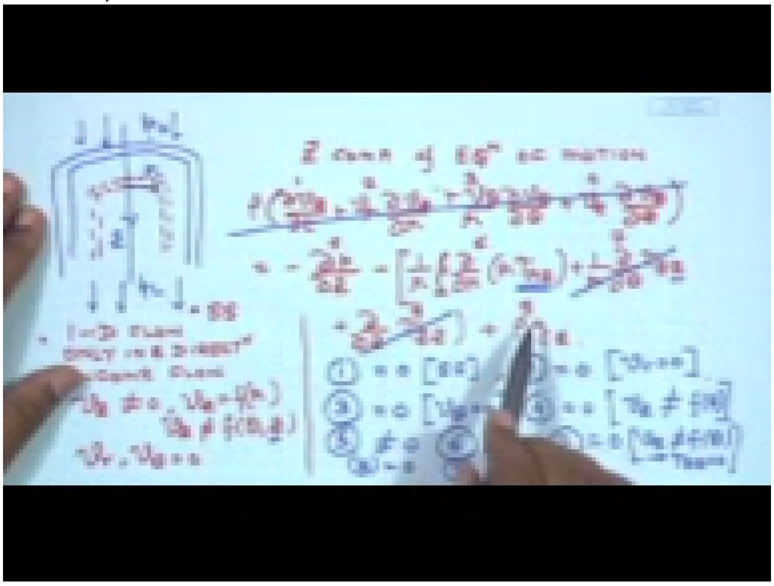

would remain in the expression and when that happens the, this equation takes the form as minus del p del z plus 1 by r, sorry minus 1 by r del del r of r tau r z plus rho g and

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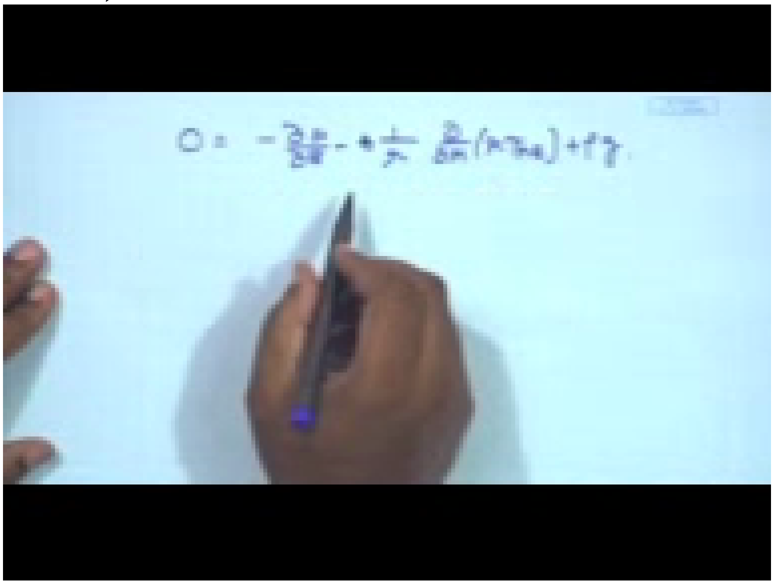

since you have, your velocity is a function of z only, the velocity is a function of z only, essentially I can write this expression

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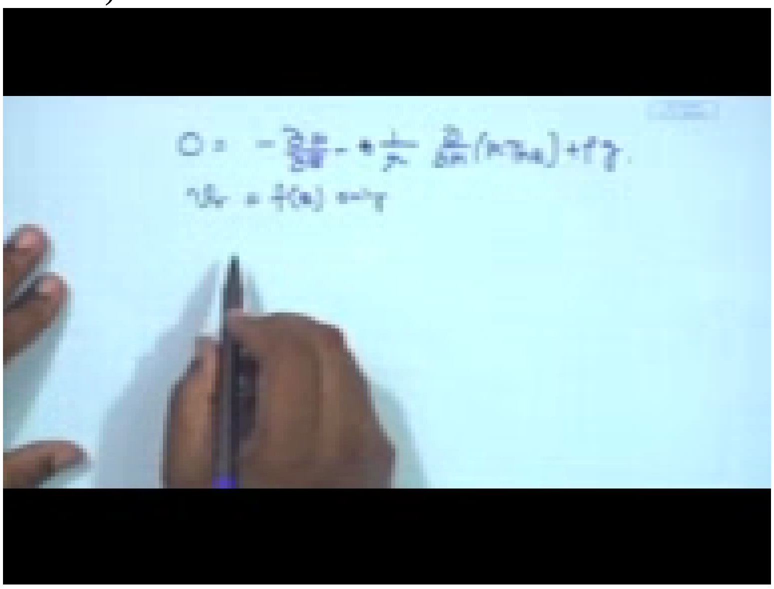

as zero equals minus del p del z minus 1 by r d d r of r tau r z. There is no need to write a partial one now. And this governing equation, if you look at,

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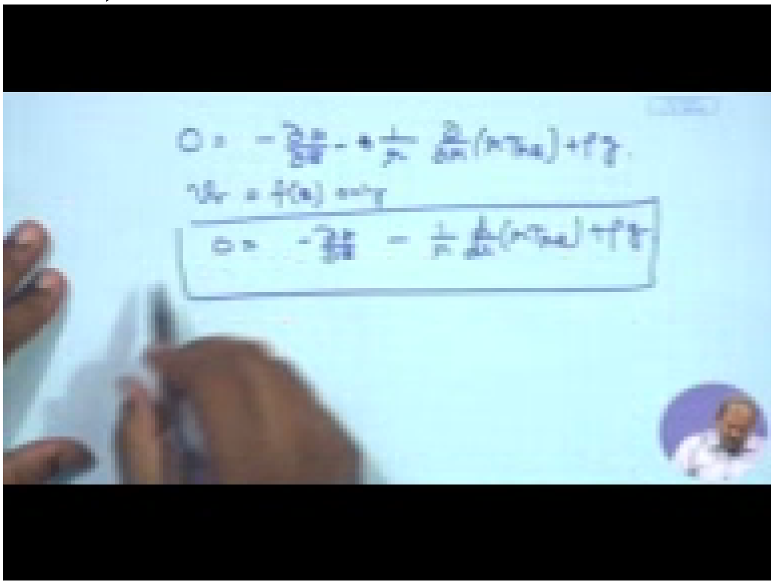

it is identical to the one that we had derived in the previous class by taking, by writing a shell, by imagining a shell and getting, getting all the momentum in and out terms of this and this was solved using the condition that velocity is zero, v z is zero at small r equal to capital R which is the no-slip condition and

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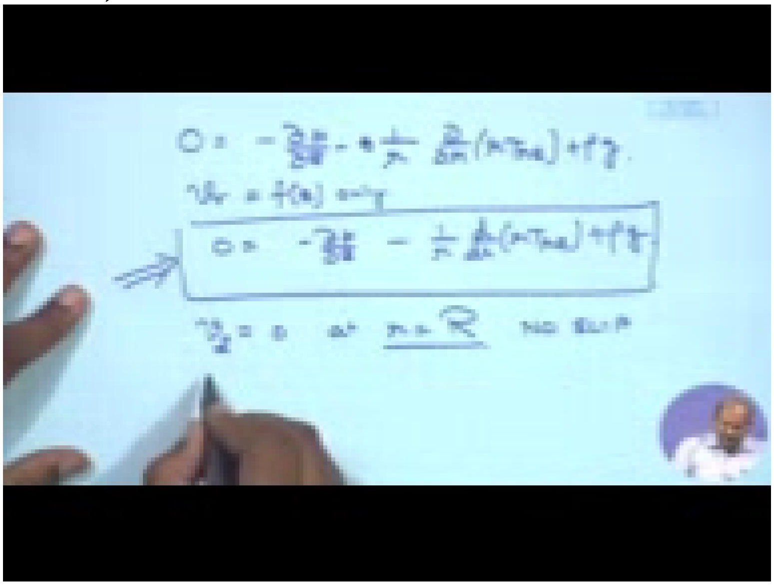

your tau r z has to be finite at r equals zero. So once you

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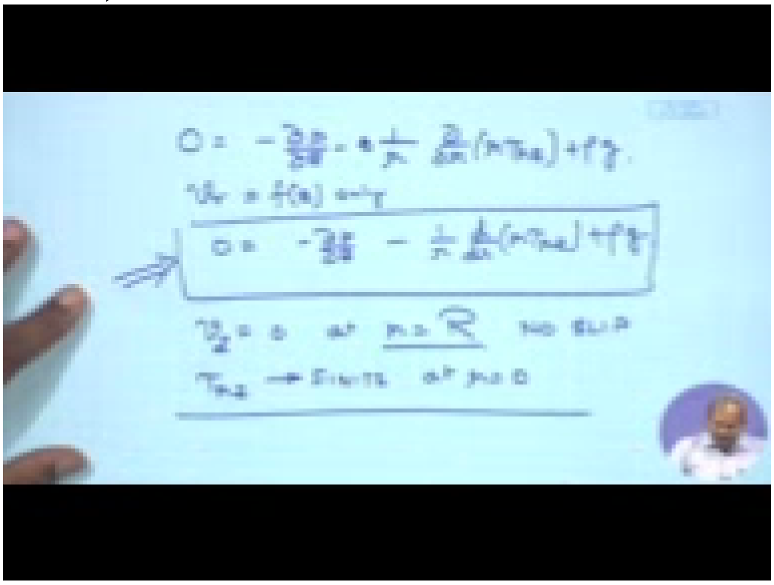

reached this point then the rest of the solution would be identical to the one you had done before. Now what you then see here is

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you do not need to think of a shell right now. You do not need to individually evaluate and balance all the, all the mechanism by which momentum can come into the control volume, the forces and so on. All you need is simply choose the expression, the right expression of Navier–Stokes equation in the correct component form. So the table, the table that is provided from that, choose the equation. And then look at the equation and think carefully, should the term remain based on my understanding and statement of the problem? Is there a temporal term? Is it unsteady state process or a steady state process?

Then look at the component, each component of the velocities; are all v x, v y, v z in the system or is it a one D flow therefore only v x is present and I can safely neglect any term containing v y and v z and, and, and only have the term containing v x? In some cases it is not the velocity but it is the velocity gradient which is important. Is it, is it one D flow where you have v x, non zero v x but v x is not a function of x, v x is a function of some other dimension, let's say z. So v x may not be zero but del v x del x is zero therefore the term which is v x del v x del x that term would be equal to zero so what you would see in most of the cases, most of the simple problems the entire left hand side of the Navier–Stokes equation which represents convective transport of momentum would, can be set to zero and then you come to the right hand side.

The first term is the pressure gradient term. Is there any input pressure in the system? If not, then that term can also be dropped. The last term is the gravity term, the body force term. Is there anybody force term present in the system? If so, choose the right component of gravity in order to evaluate what is the force which is the body force which is acting on the control volume, on the fluid. Once you are done then what you have is essentially the viscous contribution of momentum transport and there you would see that from looking at the two subscripts let's say, tau y x, it is the y, it's the x component of momentum getting transported in the y direction because of the gradient in velocity v x that exists in y so x component can get transferred in y only if there is a variation in velocity in the y direction. Do we have that?

So look at the subscript and think whether or not there should be any stress in the prescribed direction due to the motion, due to the principal direction of motion. So there also you would see that many of these, some of these sheared stress contribution to the overall momentum transfer can be neglected, Ok. And after you clear, after you cancel all the terms that are not relevant to the problem at hand, what you are left with is the governing equation. That governing equation can then be used; can then be integrated, that's a differential equation, it can be integrated with relevant boundary conditions to obtain what is the velocity distribution. The functional form of velocity as, as a function of the x, y, z and in some cases time as well.

This differential approach of using Navier–Stokes equation would give rise to a compact velocity expression in many of the cases, not in all but in many of the cases. So in subsequent lectures we would see applications of Navier–Stokes equation in slightly more complicated geometry and how to solve them. In some cases we will get a nice compact analytical solution, in some cases we may not but the concepts behind the, the Navier–Stokes equation and the ease of using Navier–Stokes equation to obtain the governing equation to obtain the governing equation for a given flow condition will always supersede any other method, specially the shell momentum balance. That's all I wanted to convey in this lecture. Thank you