Course on Transport Phenomena
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Module No 2
Lecture 10
Equations for Isothermal Systems (Contd.)

This class we are going to talk about equation of motion. So what is equation of motion and how it can be derived. The valuation itself is not that important because ultimately we are going to use the race forms of equation of motion of the various forms of equation of motion, the one which is which is relevant to the system, to the geometry that we are we are trying to find out what is the velocity distribution.

But for any system, if one wants to write the equation of motion essentially what we are doing for a fluidic system is we are writing the equation of Newton's 2nd law of motion for an open system in which the fluid is allowed to enter and leave the system. So there is some momentum which is being added to the system due to the motion of the fluid and by motion, there are 2 ways by which due to the motion of the fluid, momentum can come into the system which we have seen before.

One is the convective momentum which is due to the flow of the fluid. So some amount of mass of fluid per unit time is coming, is crossing this this let us say this is the surface area, the amount of mass which crosses the surface area has our velocity at that point to. So masstroid mass times sorry mass times the velocity productivity amount of momentum which comes in due to the actual motion, the actual some amount of mass of fluid crossing the interface which is the, which is nothing but the convective flow of momentum into the control volume.

There are certain cases in which let us say the fluid is this is the control surface and the fluid is moving in this direction and there is a gradient in velocity between this point and this point. So this momentum and this momentum is this is Y, Y momentum gets transported in the X direction, if this is X direction due to viscosity which is going to manifest itself as sheer stress this surface. So the Y momentum getting transferred in the X direction must also be taken into account as a source of momentum coming into the control volume.

Since it is taking place in a direction perpendicular to that of the motion and since the principl reason why which this kind of momentum transfer takes place is molecule in nature, it is also known as the molecular transport of momentum or conductive transport of momentum. So we have to different momentum due to the flow of diff the fluid, one is the convective transport of momentum which is due to velocity and the 2^{nd} is conductive transport of momentum which is due to velocity gradient.

You remember remember Newton's law of viscosity in which the viscous transport is denoted by the velocity gradient, is expressed by velocity gradient and not by velocity itself. So convective due to velocity, conduct due to velocity gradient. This is known as also known as the molecular transport of momentum or viscous transport of momentum. So convective momentum, viscous, molecule or conductive transport of momentum.

So these are different different reasons by which different ways by which there is a change in momentum, rate of momentum, net rate of momentum coming into the control volume. This control volume can also be acted upon by different forces. So all the forces which are acting on the fluid inside the control volume, that should also be taken into account in the difference equation. And as a result of all these, there may be an unbalanced force on the control volume.

And whenever there is an unbalanced force on the control volume, its velocity may its itsmomentum may change. So the rate of change of momentum of the control volume, the rate of momentum accumulation inside the control volume can be expressed as the algebraic sum of rate of momentum coming into the control volume - rate of momentum going out of the control volume + sum of all forces acting on the system.

So this is the most general form of equation of equation of motion for a fluid which is nothing but again, Newton's 2^{nd} law for an open system.

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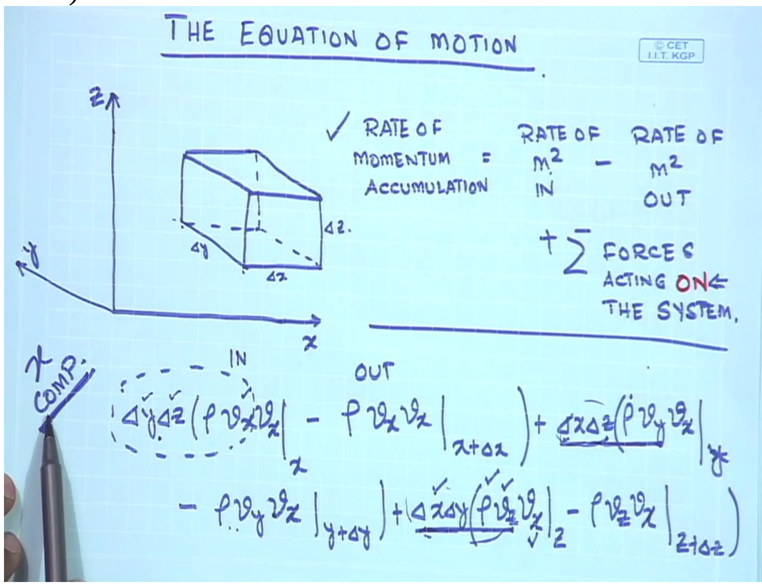

That is what I have written over here. And here you can see that if this is defined as the control volume of side del X, del Y and del Z, with the with our previous notation that any the X, essentially this is the X face, the Y base and the Z face.

So you understand that the big question that I have just described, would be rate of momentum accumulation, that is rate of momentum accumulation in the control volume must be equal to the rate of momentum in - rate of momentum out + some of all forces acting on and this is important, acting on the on the system. So we would see that let us assume let us see how the convective momentum comes into the system.

If I only look at this part, the face del Y del Z, del Y del Z and let us assume that the velocity here is VX, so the amount of mass which comes in through this face, through this face must be equal to del Y del Z we X meter square metre per second metre cube per second multiply it with rho and what you what you get is what you get is kg per second. So this term up to this part tells you what is the mass which is crossing the X face having area del Y del Z.

So del Y del Z rho VX is the mass which is crossing the X face evaluated at X. Now in order to obtain the momentum, the X momentum because I am writing the X component of the equation of motion and the other components can be evaluated in a similar fashion. So if I am writing the

X component, I am only talking about the X component of momentum. This is the amount of mass which is coming in.

So the amount of X momentum must be equal to rho VX, another VX. So up to this is the mass of the fluid which is coming in through the face at X. So multiplying it with another VX gives me the momentum, rate of rate of momentum coming into the control volume. And when we talk about the momentum that is going out, so this would be again a VX but it is evaluated at X + delta X.

Similarly when we talk about the X del X del Z face, so del X del Z face, this is the Y face, the amount of mass which is coming in through the Y face must be equal to the area times rho times the velocity the Y component of velocity. So the amount of mass which is coming in is equal to this much. But this amount of mass has an next component of velocity. So in order to obtain the X component contribution of this much of mass, this must be multiplied with VX.

I will go through it once again. The underlined terms simply tell tell simply tells you the mass which is crossing, which is entering the control volume through the Y face having area del X del Z. So this is the mass which is coming into the control volume through the Y face and this mass this mass of fluid has an X if I would like to find out what is the X component contribution to momentum I will simply have to multiply it with VX.

So mass coming in multiplied by the component of velocity in the X direction at that point would give me the amount of momentum that comes in to the amount of X momentum, that is important, amount of X momentum coming into the control volume through the face at Y. Here also if you see, this is the mass coming in through the through the X face and I multiplied it with the component of velocity at that point. So this is the total amount of momentum coming in through the X face.

This is the total amount of momentum coming in through the Y face. And if it is at Y, if it is at evaluated at Y, then this has to be multiplied by VX and at Y + delta Y. So this would give you the X momentum that goes out of the system, out of the control volume through the face at Y. Similarly I can write the Z component, the amount of mass pages coming in through the Z face is the area, density and velocity of Z component.

Now this amount of mass has some X component of momentum associate it with it. In order to obtain that, I simply multiply it with VX. So mass is evaluated by the component of in in in the direction and this amount of mass contains some amount of X momentum, the value of which is obtained by multiplying it with the with the X component of velocity at that point. So similarly, this is to be multiplied by VX and it is evaluated at Z + delta Z.

So this is in, out, all are X momentum. X momentum in through X face, out through X face. X momentum in through Y face, out through Y face, X momentum in through Z face, out through Z face. So these 6 terms tells us the X component contribution of momentum to the control volume.

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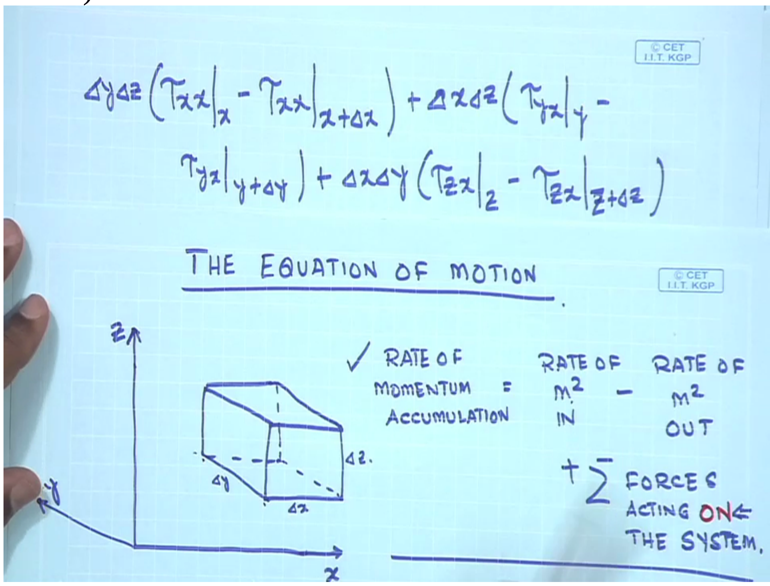

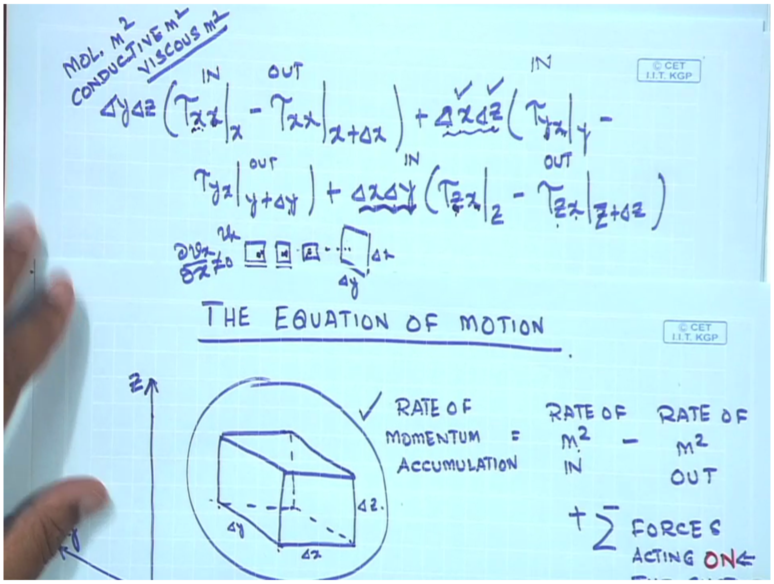

Next we go into the next we go into this I will leave this figure in you to see and here we are trying to find out what is the contribution of molecular momentum, molecular momentum or conductive momentum or viscous momentum. Now if you see the, let us let us start with the Y face. The area of the Y face is del Y del Z, the shear stress, the X component of momentum gets transported in the Y direction.

So the X component of momentum gets getting transported in the Y direction must act on an area which is equal to del X del Z. So Y direction has this much of surface area associated with it. So the X component contribution in the Y direction, this is the in term must be equal to del X del Z multiplied by the shear stress acting on the Y face. The one that goes out would also be equal to same thing but evaluated at Y + del Y.

Let us try to again workout this. X component of momentum getting transported in an in the Z direction. So if X component of momentum gets transported in the Z direction, in order to obtain the total amount of momentum coming in, the viscous momentum which is coming in through the Z face, I must multiply this tao ZX with the area of the Z face which is del X del Y. So X component getting transported in the Z direction multiplied by the area of the Z direction, this gives me the rate of viscous momentum in through the Z face.

And this is the rate of momentum of X momentum, rate of entry, rate of X momentum getting out of the Z face, so these are the in and out terms. So what is Tao XX? Tao XX is the X component of momentum getting transported in the X direction. This is slightly slightly unusual because you get the X component gets transported in the X direction, that is only happen when let us say these are the packets of fluid and this is the X face which is delta Y times Delta X.

So these packets are coming towards the towards this this this point and there is a variation on velocity between these 3 packets. So since there is a variation in velocity in the X direction, variation in VX in the X direction, that means del VX by del X is not 0. Then by Newton's law, by over our understanding of viscosity, there must be a stress between these 3 packets of fluids which will also be transmitted on this face.

So this kind of stress where the principle directional of motion and the direction in which the momentum gets transported are identical, they are commonly called as the normal stress. So Tao XX is nothing but the normal stress exerted by the fluid on the X face due to its, due to a variation in velocity of the X component. So Tao XX is nothing but the normal stress exerted by the flowing fluid on the X face.

So this is going to be the in term and this is going to be the out terms. So these 6 terms in total would give you the amount of momentum which is coming in to the control volume as a result of viscous transport of momentum.

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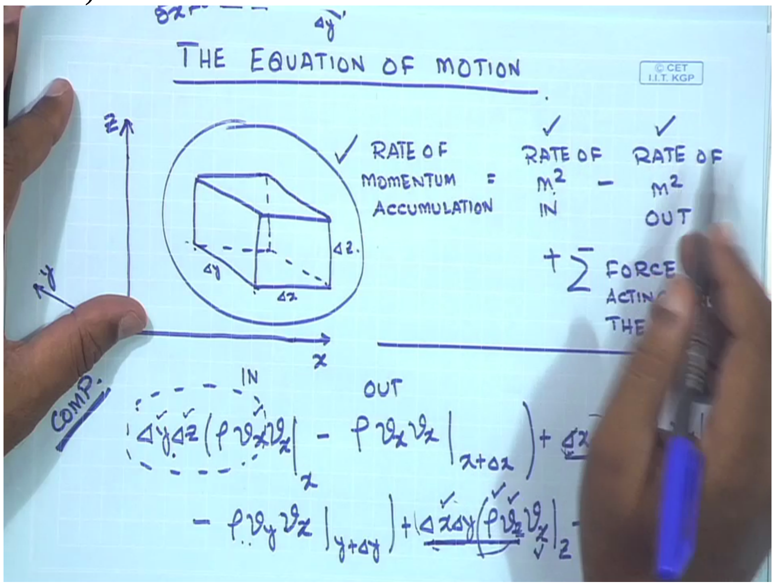

So we have correctly identified in our equation of motion, the rate of momentum in and the rate of momentum out, both for convective motion and the conductive motion.

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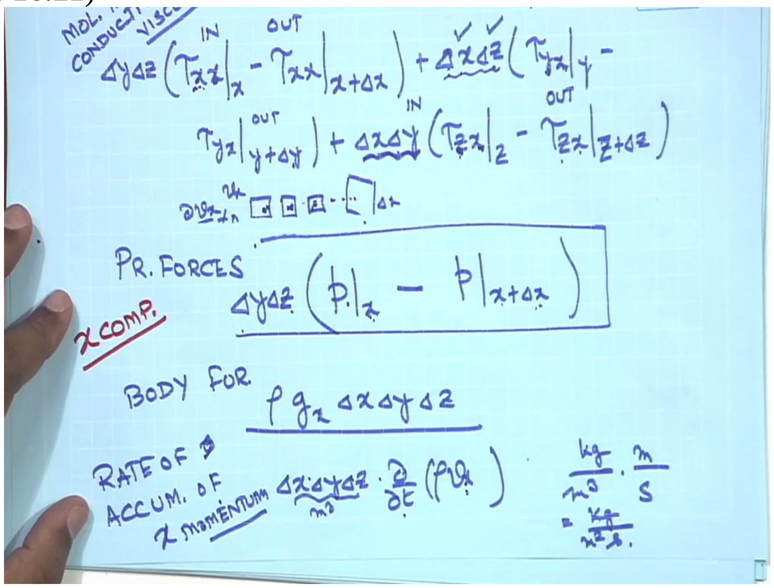

So what is left right now is to identify what is going to be the pressure forces and what is going to be the body forces. So let us assume that the pressure at X is P at X and the pressure at X + del X is P X + del X. So this pressure is acting on the X face, the difference in pressure is acting on the X face, the other other pressure difference is between not contribute to the X component of momentum, X component of force.

Only the pressure over on acting on the X face and on the X + del X face, they have contribution in the X direction. So I need not have to consider P at Y or P at Z and P at Y + del Y and Z + del Z. So this is acting on an area which is equal to del Y del Z. So the pressure force, the net pressure force acting on the control volume is given by whatever be the pressure at X - the pressure at X + del X and multiplied by del Y del Z.

Please remember I am pointing out once again that we are writing this equation for the X component, X component of momentum, X component of the of the forces, be it pressure or beat other body forces. Since they are, all the other components, Y and Z components can be written in a similar fashion and you do not have to write each one of those X those components separately. So we have identified the momentum, we have identified the pressure forces, the only thing that is remaining is the body force which is acting on it.

So what is the body force? Body force must be equal to the M, the total mass of the system multiplied by the component of gravity if gravity is the only body force, component of gravity in the X direction since we are writing this equation for the X component of equation of motion. So what is the mass of the control volume? It would simply be equal to del X del Y del Z that is the volume multiplied by rho which makes it kg multiplied by G which is GX where GX is the X component of the body force.

So the body force term can simply be expressed as del X del Y del Z rho GX. So the body force here is simply going to be rho GX del X del Y del Z. This is, this would be the body force term. And the rate of accumulation of momentum inside the control volume would simply be del X del Y del Z. And del del T of rho VX. So rho VX would simply be rho would be equal to kg per metre cube.

VX is metre per second so this would simply be equals kg per metre square per second and what we would get here is then the momentum changes, so if you, this is metre cube and what you get is in kg metre per second and del del T of that. So this is the accumulation of rate of accumulation of this is important, X momentum in the control volume is going to be del X del Y del Z del del T of rho VX.

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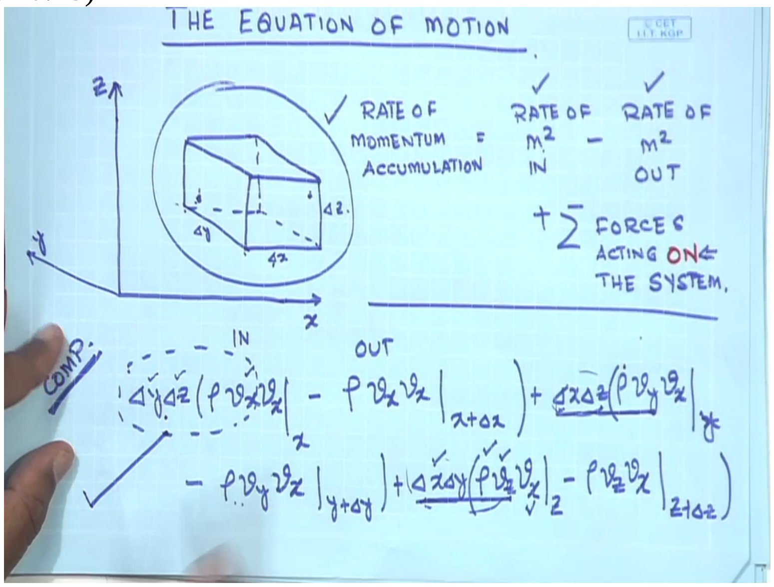

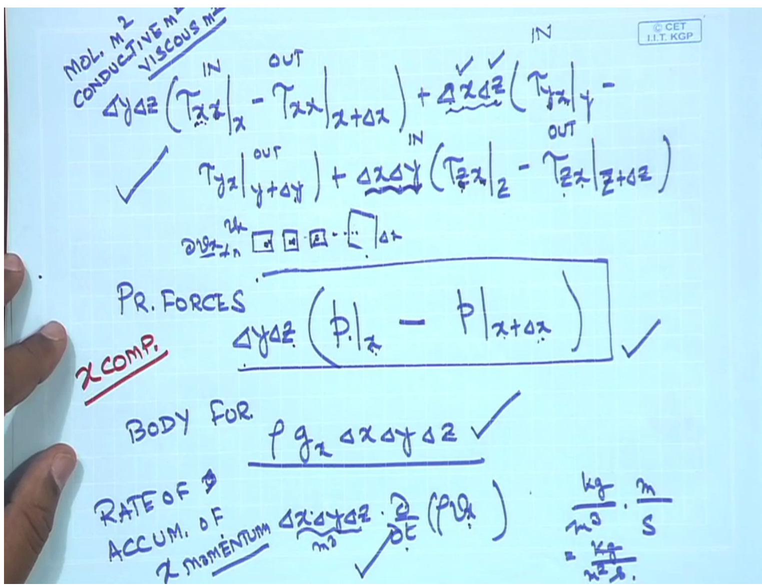

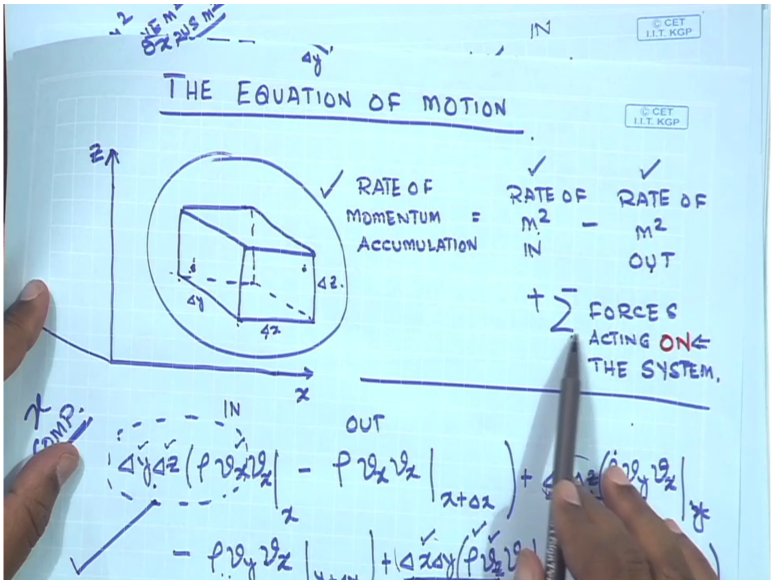

So if I take the convective momentum conductive momentum, the X component of the pressure force, the X component of the body force and the rate of accumulation of the X momentum, then according to the equation, the 1st equation which we have written, rate of momentum accumulation must be word to rate of momentum in - rate of momentum out + sum of all forces acting on the system. So this equation so all these terms together can be can be written and I am not going to write the entire, the full form of the expression.

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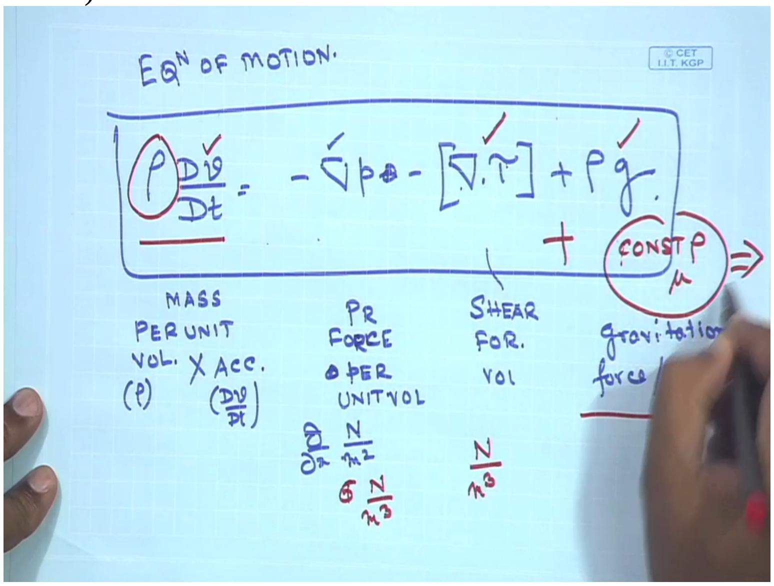

What I am going to do is I am going to simply give you the compact form of this equation which is rho this is after some simplification. This equation in that pretence notation is is is essentially the known as the equation of motion where the rho times dividity is essentially mass per unit volume times acceleration. So this is rho and DV DT where here, this is pressure, force per unit volume because pressure is Newton or force per unit area.

Then you have another del del X in front of it due to Tao. So this is pressure force per unit volume, this is sheer force per unit volume and this is gravitational force per unit volume. So how did we get to this equation from all the previous equation. Remember what we have done for the case, for the X component of equation of motion ocean Mark I have identified all the terms in there.

Now these, all these terms can be expressed, can be can be put into the equation which is rate of momentum accumulation is equal to rate of momentum in - rate of momentum out + sum of all forces acting on the system. So we both sides can then be divided del X del Y del Z and in the limit del X del Y del Z tends to 0 then one can get a differential equation which is the X component of equation of motion.

Hence in a similar fashion, one can write the Y component of equation of motion and the Z component of equation of motion. All these 3 equations can be added to obtain the compact equation of motion once once you express them in terms of in in a vector tension notation. So no new concepts are involved beyond what I have taught you in this part. So you can see the text and you can yourself see the symbol occasions that are made which are only algebraic in nature without the involvement of any additional concepts.

So why did not derive the entire equation in this. I have given you enough pointers for the fundamental development of equation, fundamental development of the equation and then I have told you how to combine these 3 equations in vector tension notation and what you get is the equation of motion in in in equation of motion when considering all the 3 directions. I would like to draw your attention to this equation once again because each term of this equation are essentially, this is mass per unit volume which is rho times acceleration.

So this is force per unit volume. This term is then force per unit volume. Here you would see that it is del of P. We has units of Newton per metre square which is force per unit area and because of this, you have another method dimension added to the denominator. So this is also going to come as force per unit volume. Similarly sheer stress is Newton per metre square.

You have a del operator on it. So this, unit of this is also going to be Newton per metre cube. And similarly in the in this also rho times de is nothing but the gravitational force per unit volume. So this tells you what is going what is commonly known as the equation of motion. And everything possible has been taken into account in here.

So if you if in this equation, you introduce the concept of introduce the restriction of constant rho and constant mew, if this is added to it what you get is

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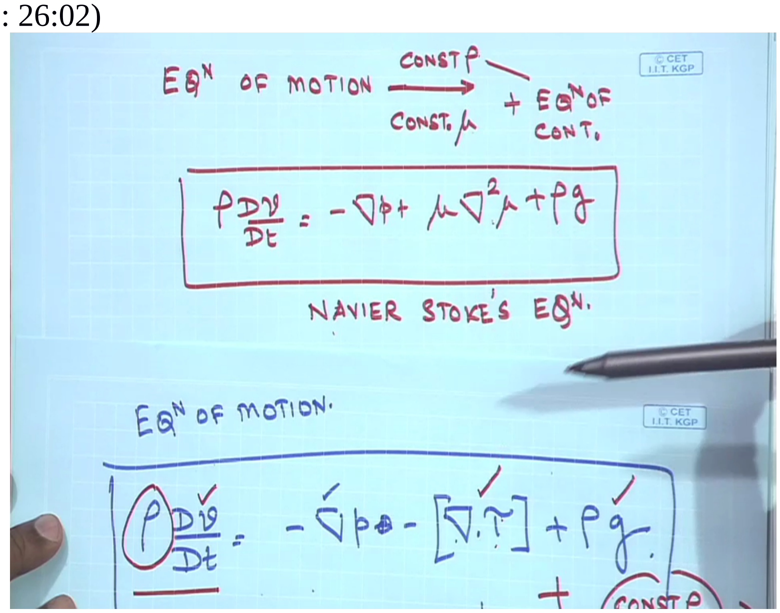

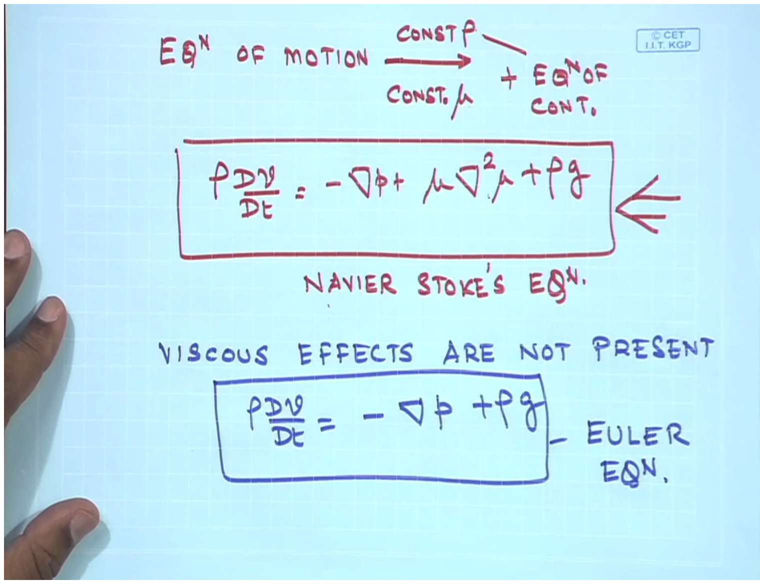

So equation of motion when you add constant rho and constant mew, the equation simply and + equation of continuity, if you add the equation of continuity and since now you have constant rho, this is the previous equation simply boils down to, this is known as the famous Navier Stoke's equation. Again, if you look at these 2 equations, the only change is it is a constant rho and constant mew, so your Tao can now be replaced in term and Tao can be replaced in this form.

And you also have equation of continuity. So the equation of motion for the special case when the row and mew of the fluid are constant, then what we get is Navier Stoke's equation and this is the one which we are going to use in all our subsequent studies of fluid motion. There is one more symbol creation that can be that can be thought of, is if viscous effects are absent so if viscous effects are not present, that means we are dealing with an invicid fluid, in that case, rho DV DT would would simply be equals - del of P + rho G.

This is special form of Navier Stoke's equation is known as the well-known Euler equation. So what we have then done in this class and the previous class is introduced the concept of different derivatives, introduced the concept of equation of continuity, we have seen what is equation of motion. The equation of motion for the special case where the density and viscosity of the fluid are constant, would revert to an equation, would simplify to an equation which is more more commonly known as the Navier Stoke's equation.

But the all these equations, the equation of motion or the Navier Stoke's equation are nothing but the the the statement of the Newton's 2nd law for an open system. And another fundamental, another fundamental relation in fluid mechanics is the special case of Navier Stoke's equation where there is no effect of, effect of viscosity is absent. So for a fluid of very low viscosity or another was truly foreign invicid fluid, fluid with no viscosity, idealised condition, what you get is from Navier Stoke's equation is all is known as the Euler's equation.

And Euler's equation has so many users, fundamental uses in in fluid mechanics. It is the starting point from where you can start to obtain the the Bernoulli's equation. So a less equation is for invicid fluid, Navier Stoke's equation is the more common viscous fluids in in which case rho and mew are assumed to be constant. If they are not, then one has to go back to the fundamental equation which is valid both for study and unsteady, constant rho or or variable rho, so constant mew and variable mew, Newtonian or non-Newtonian.

So equation of motion in itself is a complete expression that gives you the the entire physics behind the motion of fluids and the momentum transfer. Now all these equations, the Navier Stoke's equation as well as equation of motion are available in different coordinate systems. What would be the components equation for X component in Cartesian, cylindrical and spherical components or similarly for other components, in all 3 coordinate systems, are available in your textbook.

So what we would do is, in the next class we would see these components and we will try to figure out how to use those for all the problems that we have dealt with so far using a shell momentum balance. And and I am sure you will all of you will see with me that the use of these white components of Navier Stoke's equation for the problems that we have dealt before would essentially simplify our life a lot. We would be able to much more conveniently handle problems of momentum transfer if we start with the right component of the Navier Stoke's equation.

And from our understanding they would cancel the terms which are not relevant and what would be left with is a compact governing equation that we should be able to solve using appropriate boundary conditions. That is what we are going to do in the next class.