Advanced Mathematical Techniques in Chemical Engineering

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Lecture No. # 09

Eigenvalue Problem

Good afternoon everyone. So, we will be, as we have discussed in the last class that we have looked into the definitions and various properties of matrices and determinants and looked into so many. We recapitulated some of the old ideas and definitions of whatever we have done earlier years. In our earlier classes, the properties and various properties of the matrices and determinants which will be satisfied.

Classification of different matrices over to which one is called the symmetric matrix, asymmetric matrix, diagonal matrix, skew symmetric matrix, and various properties of them, we have already seen. Now, at the end, we have defined the Eigenvalue problem and Eigenvalues are quite important in all chemical engineering applications.

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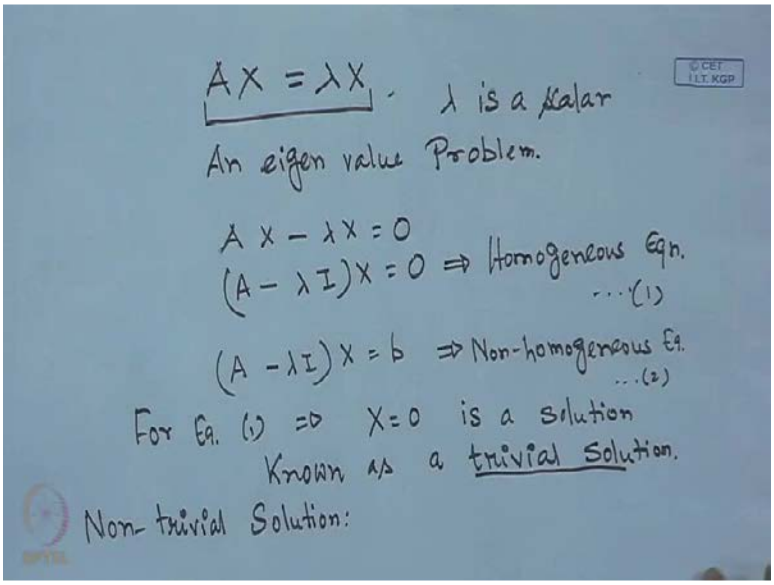

So, let us start from that point onward, if we write A X is equal to lambda X, you have already seen what is the concept of matrix and matrix is nothing but an operator, it operates on a vector x and it written a value in it maps into X; so, it a, matrix is like a function, it is an operator and if we are able to write the form of the matrix in this particular fashion A X is equal to lambda X, where lambda is a multiplier, is a scalar; we call this problem as an as a standard Eigenvalue problem. So, if you bring A X with the lambda X on the other side, it becomes A X minus lambda X is equal to 0; so, we can represent A minus lambda I should be equal to, multiplied by X, this should be equal to 0; so, therefore this equation is set of a homogenous algebraic equation.

So, this is absolutely homogeneous equation, there is no non-homogeneous term present on the other side. If we have the form of this equation is A minus lambda I times X is equal to b; this is a non-homogenous equation. Now, for 1, for equation 1, X is equal to 0 is a solution; ofcourse, if X is equal to 0, then that is a solution and this is known as a trivial solution. Therefore, the necessary and sufficient condition for, not so, we are not looking for the trivial solution, we are looking for the nontrivial solution.

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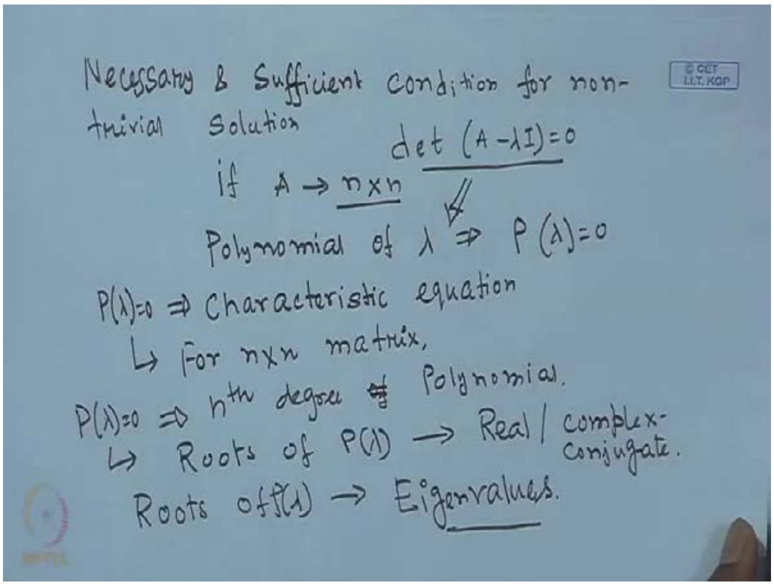

The necessary and sufficient conditions for nontrivial solution, is that determinant of A minus lambda I is equal to 0. Condition for nontrivial solution is determinant of A minus lambda I is equal to 0. And determinant of this is, if A is a square matrix of size n cross n, then this equation will gives you the determinant of A minus lambda I is equal to 0.This equation gives a polynomial of lambda; finally, it gives a polynomial of lambda that is P of lambda is equal to 0; degree of the polynomials, this is known as the characteristic equation; this P of lambda is equal to 0 is known as a characteristic equation.

If, for a matrix, square matrix n into n, then the order of this degree of this polynomial is n. So, for n into n matrix, we are talking about nth degree, P lambda is equal to 0 becomes an nth degree of, nth degree polynomial. The roots of this polynomial, roots of this polynomial of P lambda may be real, it may be complex conjugate. So, therefore these roots can occur either a complex conjugate pair and or real valued system and these roots are called Eigenvalues. Eigenvalues of the system and the corresponding solution, the solution corresponding to Eigenvalue is called Eigenvector.

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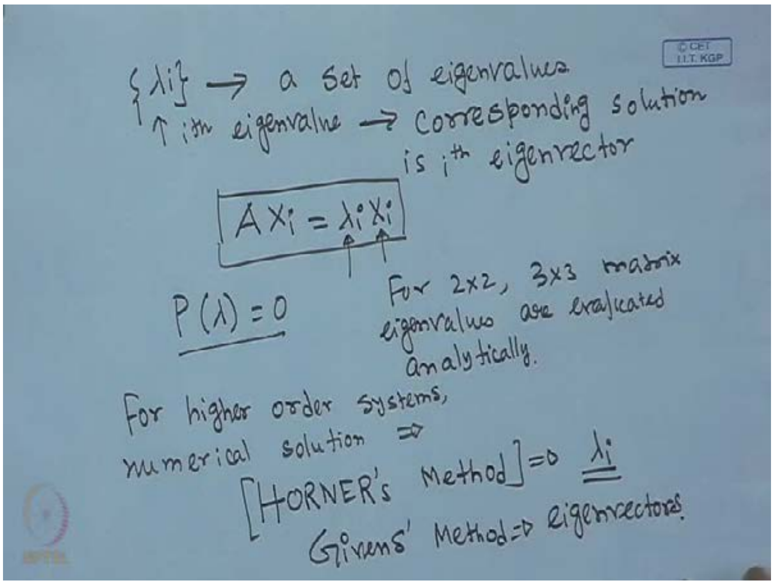

Suppose, there are lambda i is a set of real valued or complex valued Eigenvalues; i is the, lambda i is the ith Eigenvalue; corresponding to ith value, corresponding solution is known as the ith Eigenvector; solution is ith Eigenvector. So, therefore, in the equation A X i is equal to lambda i X i; so, this will be the solution of the problem corresponding to lambda I, we will be having corresponding to Eigenvalue, will be having the corresponding ith Eigenvector.

Now, if we look into the characteristic equation P lambda is equal to 0. If we are talking about a five, you know five component system, then you will be getting a five into five square matrix and degree of this polynomial will be order will be five; so, therefore you will you are going to get expect five number of roots from this polynomial or five Eigenvalues.

Now, if it is a 10 into 10 matrix, if A is a 10 into 10 matrix, we are going to get a characteristic equation of degree ten; in that case, there will be ten number of Eigenvalues present in that case. So, if it is a simple system, if it is a 2 into 2 system or 3 into 3 system, the Eigenvalues are quiet; they can be evaluated analytically, for 2 into 2 or 3 into 3 matrix Eigenvalues are evaluated analytically.

For higher order system, we have to take recourse to some numerical techniques, for higher order systems, numerical solution is required. One can take a request to Newton raphson algorithm to find out the first root, then one has to use the some kind of numerical method to evaluate all the Eigenvalues, all the roots of this equation, may be Horner's method may be one of them.

So, by using such numerical technique, one can evaluate all the root of the characteristic equation, P of lambda equal to 0; from for the corresponding lambda i one has to evaluate the Eigenvector, may be, Givens' method is there to evaluate the corresponding Eigenvectors.

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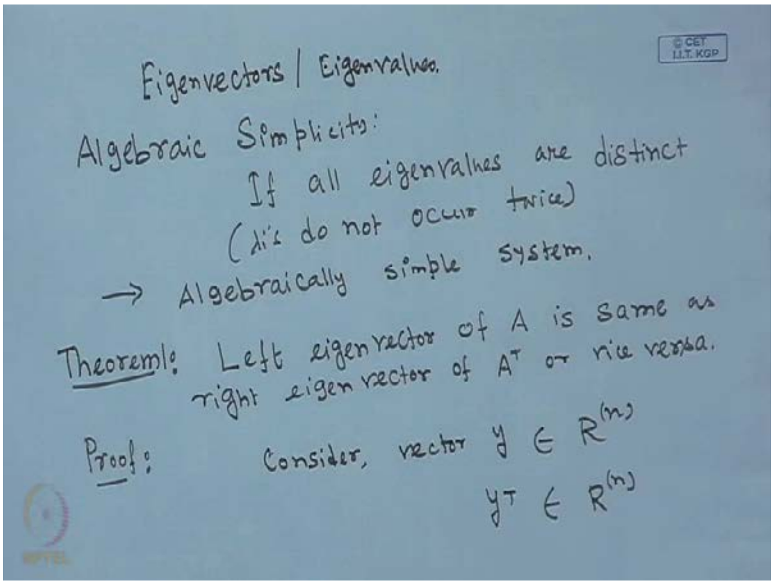

So, what are the physical significance of Eigenvalues and Eigenvectors? Eigenvalues are the Eigenvectors are basically the characteristic coordinate system.

For example, for a three-dimensional system, like any vector can be represented, as the, along the x axis, along the y axis, along the z axis; so, by the 3 unit vectors, it can be expect. Similarly, for an n-dimensional space, the Eigenvectors will represented, will represent the mutually orthogonal directions.

So, any vector can be broken down can be resolved into this Eigenvector directions and in these, every Eigenvector direction, the corresponding value, the amplitude will be corresponding to the Eigenvalues, that is the contribution coming up of the original vector the contribution of the vector into the direction of the Eigenvectors will be corresponding, will be represented by the Eigenvalues.

So, therefore, that is the physical significance of Eigenvectors and Eigenvalues. Now there is something called algebraic simplicity. If all Eigenvalues are distinct, they are not repetitive, if all of them are distinct, that means, lambda i they do not occur twice repeatedly. If all the Eigenvalues are distinct, then this is the system is called algebraically simple system.

So, what is an algebraically simple system? A characteristic equation, where all the Eigenvalues are distinct. Then, next we will be proposing will be proving some of the axioms or theorems, which we will be using quite often in our course. So, the first theorem that we are going to prove state and prove, is that, theorem one left Eigenvector of a matrix A is same as right Eigenvector of corresponding matrix transpose matrix A transpose or vice versa.

Now, to prove this thing, let us consider a vector in n-dimensional real space. Consider vector y belongs to n-dimensional real space; so, ofcourse, the transpose vector also belongs to n-dimensional real space.

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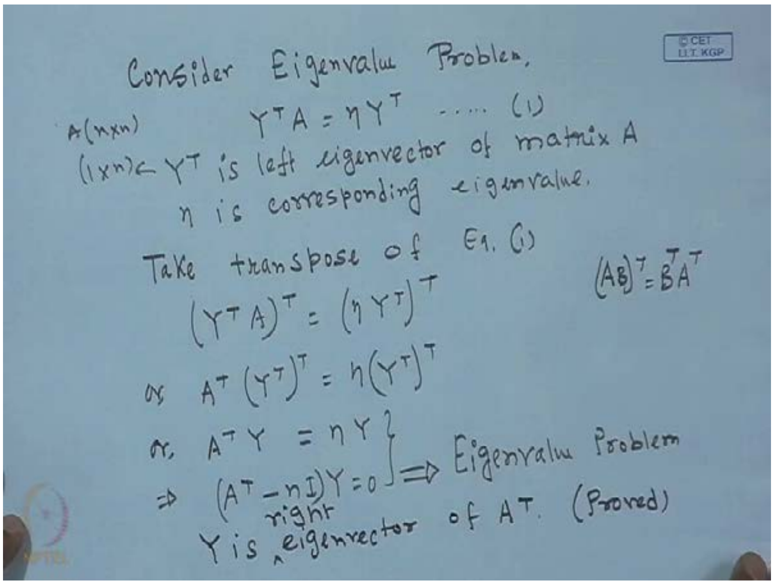

Once, we define this vector y and y transposes, then we consider the Eigenvalue problem Y transpose A is equal to eta Y transpose; suppose, this equation number one. In this particular equation, Y transpose is left Eigenvector of matrix X and eta is corresponding eigenvalue. The size of Y transpose is 1 cross n, the size of A will be n cross n, so since the number of rows number of columns of A matching with the number of rows of Y transpose; therefore, they are confirmable and the matrix multiplication is allowed.

Now, let us take the transpose of equation one; if you take the transpose of equation one, this becomes Y T A transpose of that eta Y transpose of that, you remember we have already taught in the earlier class that, AB transpose is nothing but B transpose A transpose. So, therefore this becomes A transpose transpose of Y transpose eta is being a scalar, so it does not matter; so, it becomes Y transpose and transpose of that.

Now, transpose of a transpose matrix is nothing but vector is basically the same one; so, basically A transpose Y is equal to eta Y. So, again if you look into this equation, this is again an Eigenvalue problem; the form of the equation is again an eigenvalue problem. Here, the corresponding vector is A corresponding matrix is A transpose and the Eigenvector is Y. So, Y is Eigenvector, it is basically Eigenvector of A transpose and Y is right Eigenvector of A transpose, because y is occurring in the right direction, so it will be right Eigenvector of A transpose; so, therefore that completes the proof.

So, we have proof that left Eigenvector of a matrix is nothing but the right Eigenvector of transpose matrix, Y is right at Eigenvector of the transpose matrix; so, we have proved our theorem first theorem.

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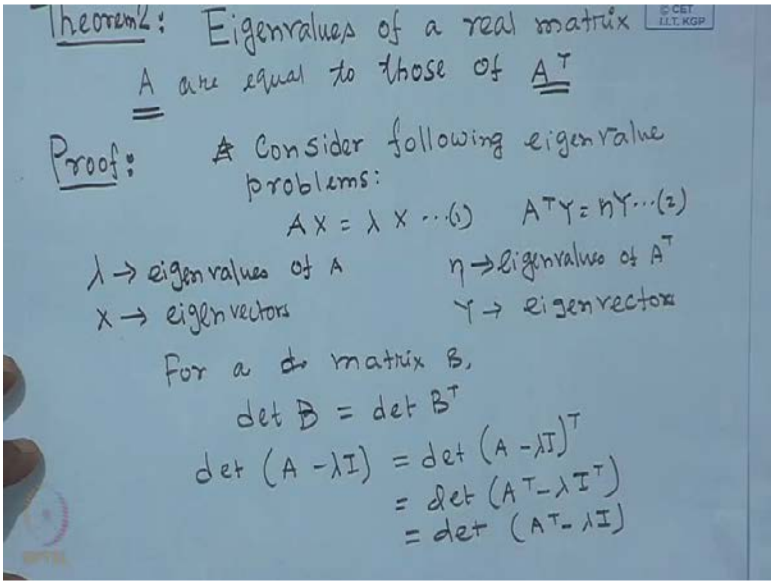

Then, we go the next theorem; this theorem says that Eigenvalues of a real matrix is identical to those of transpose matrix. Eigenvalues of a real matrix A are same identical equal to those of A transpose; so, you have to proof that Eigenvalues of A and Eigenvalues of A transpose both are identical.

So, we assume the associated Eigenvalue problem, consider following Eigenvalue problem, will consider a pair of such problems. First one is A X is equal to lambda X and second one is A transpose Y is equal to eta Y; so, in this case X is the lambda, is the Eigenvalues of A. And corresponding Eigenvectors are X, in the second case, in the second problem eta are the Eigenvalues of A trans transpose and the Y are the corresponding Eigenvectors.

Now, we have already seen that for a matrix from the property of the determinant that determinant of a matrix is equal to the determinant of the transpose matrix. So, therefore we utilize the property for a matrix B, determinant does not change whether the matrix is transpose or normal; so, determinant of B is identical to determinant of B transpose. So, therefore, we write determinant of A minus lambda I should be equal to determinant of A minus lambda I transpose.

So, you just consider a matrix A minus lambda I as B; so, determinant of the matrix A minus lambda I is identical to determinant of matrix A minus lambda I transpose. So, we just open up this transpose operator here, so this becomes determinant of A transpose minus lambda being a scalar multiplied by I transpose, but I transpose in essence I is an identity matrix, it remain same.

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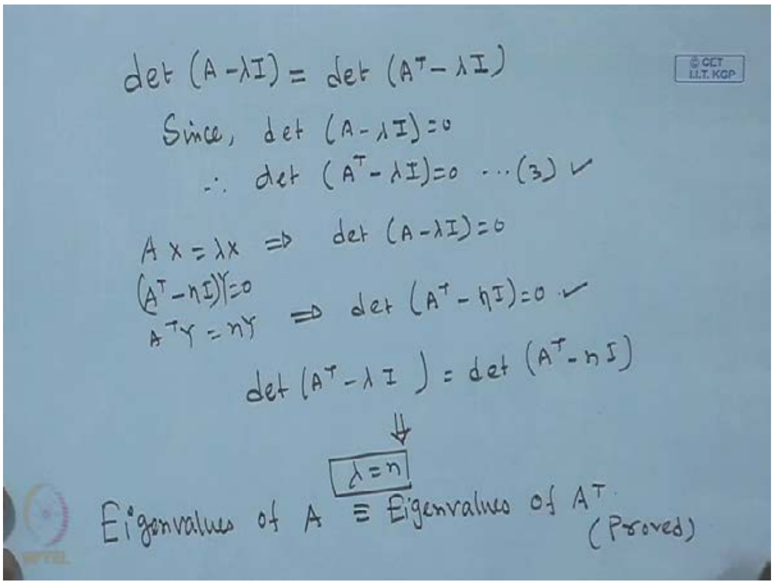

So, determinant of A transpose minus lambda I; so, therefore, from this equation, we can write down that determinant of A minus lambda I should be is equal to determinant of A transpose minus lambda I. Now, since determinant of A minus lambda I is equal to 0, we can, we can say that determinant of A minus A transpose minus lambda I should also be equal to 0. From this equation determinant of a lambda equal to 0; so, therefore determinant of a transpose minus lambda I should also be equal to 0.

Now, look into equation number two and equation number three, from the equation number three, what we have got. So, whenever we have written that, A X is equal to lambda X, that simply means, determinant of A minus lambda I is always 0 and whenever we write A transpose minus eta I is equal to 0 or A transpose Y is equal to eta Y this implements determinant of A transpose minus eta I is equal to 0.

So, from this we can compare this equation and this equation; so, if you compare this equation and this equation, what we will get that determinant of A transpose minus lambda I is equal to determinant of A transpose minus eta I; that simply tells us that lambda is equal to eta.

So, if you remember what is lambda? Lambda is at the Eigenvalues of the matrix A, eta are the Eigenvalues of matrix A transpose, since they are identical; so, we can say that Eigenvalues of matrix A are identical with the Eigenvalues of matrix A transpose; so that completes the proof of the second theorem.

So, we move over to the next theorem and all these theorem are quite useful and helpful for solving the chemical engineering problems, that we will see later on.

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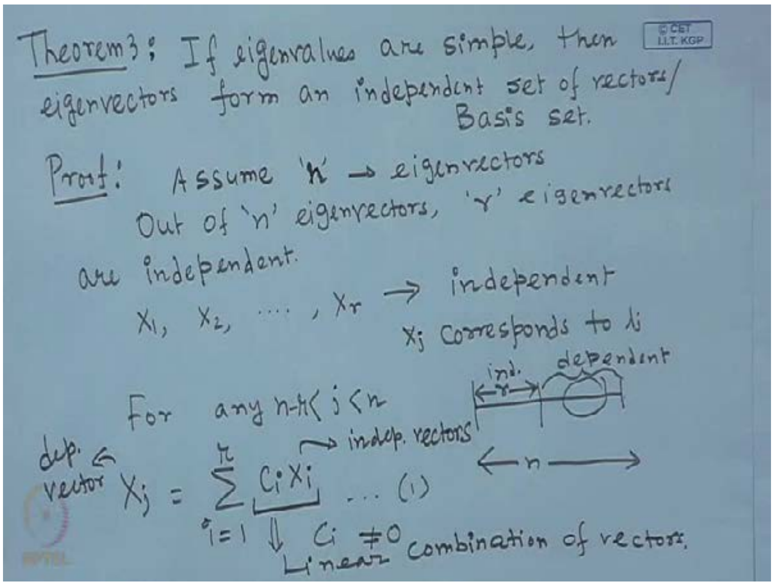

So, next theorem goes like this, this says if Eigenvalues are simple; simple, means, there is no repetition of the Eigenvalues, they are distinct. If Eigenvalues are simple, then Eigenvectors form an independent set of vectors or a basis set. So, that simply, that means, if for the simple Eigenvalues for distinct Eigenvalues, the Eigenvectors form an independent set of vectors of the basis set vectors; that means, that was, I was telling in the beginning of this class, that is the physical significance of the Eigenvectors and Eigenvalues.

The Eigenvectors represent individually orthogonal and independent directions, so any vector in the space can be represented as a linear combination of the Eigenvectors; so that is the physical significance of Eigenvectors and contributions in each direction of the Eigenvectors will be dictated by the corresponding values of corresponding Eigenvalues.

The, let us prove this, that Eigenvectors, they form, they are mutually or independent to each other; so that, they will be form in the independent as if they are represented by the independent axis or this axis will be basically, they are independent directions.

So, the proof goes like this, we assume there are r number of Egenvectors of a matrix and out of these r Eigenvectors, there are n number of Eigenvectors; in a system there are n number of Eigenvectors and out of these n number of Eigenvectors, out of n Eigenvectors r number of Eigenvectors are independent; that means, rest Eigenvectors from n minus r numbers they are algebraic, they are independent vectors; so, we write X 1, X 2, up to X r these are Eigenvectors are independent.

Now, in this notation X j corresponds to lambda j, that means, for the Eigenvalue lambda j, the corresponding Eigenvector is X j. So, therefore, for any j lying in between n minus r to n, what is that, all Eigenvectors are independent n minus or eigenvectors are dependent; that means, any Eigenvectors, so we have said that the total number of Eigenvectors are n out of these n, r number of vectors are independent.

So, what is this set lying in between n minus r and n is this set; that means, it is a dependent set of Eigenvectors. Therefore, any Eigenvectors present in this set it can be expressed as a linear combination of all the independent Eigenvectors X 1, X 2, X 3, up to X r.

So, that we have already proved earlier that, any if there are n number of, n number of independent vectors present in a space, any other vector can be represented as a linear combination of these n independent vectors. So, therefore, X j can be written as a linear combination of all the other independent Eigenvectors and this index i runs from 1 to r, because r number of independent Eigenvectors present in your system; this is equation number one but corresponding C i must not be equal to 0.

So, why we write this equation? This equation will represents a linear combination of vectors C i X i are nothing but the linear combination of vectors. So, this is a dependent vector and X i is these are all independent vectors.

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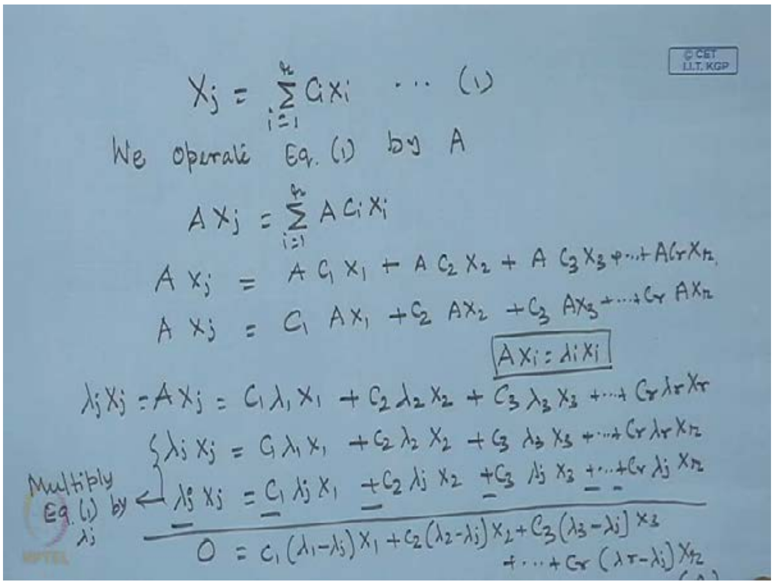

So, let us go to the next step; so, we just open up this equation C i X i i is equal to 1 to r; we this, let us say equation number 1, we take, we operate this equation by the matrix A. We have already seen earlier that matrix is like a function, it is an operator; so, we operate equation 1 by matrix A, like, it is like differentiation is an operator. We are taking differentiation on both sides; so, it is like, we are taking, we are operating equation 1 by A; so, if that is the case, it will be A X j is equal to summation of A C i X i or i is equal to 1 to r.

Now, we open up this summation series; so, this becomes A X j is equal to A C 1 X 1 plus A C 2 X 2 plus A C 3 X 3 up to A C r X r. Now, each of them, so since C 1 is a scalar multiplier, this will be A X j is equal to C 1 A X 1 plus C 2 A X 2 plus C 3 A X 3 up to Cr A X r. And we have seen we know that A Xi is nothing but lambda i X I, that is the standard Eigenvalue problem, for the corresponding value of the lambda i the Eigenvalue, the corresponding Eigenvector is X i.

So, therefore we can write A X j is equal to C 1, A X 1 is nothing but lambda 1 X 1, so we write lambda 1 X 1 plus C 2, A X 2 we write lambda 2 X 2 plus C 3, A X 3 is lambda 3 X 3, likewise Cr A X r is nothing but lambda r X r. Now, so that is 1 equation we get, now we multiply equation 1 by lambda j, let us and so we can also write, A X j as lambda j X j; so, we get this equation lambda j X j is equal to C1 lambda 1 X 1 plus C 2 lambda 2 X 2 plus C 3 lambda 3 X 3 up to C r lambda r X r.

Next, what you do, we multiply equation number 1 with lambda j and see what we get; so, if we multiply equation number 1 by lambda j, we will be getting lambda j times X j is equal to C 1 lambda j X 1 plus C 2 lambda j X 2 plus C 3 lambda j X 3 up to Cr lambda j X r; this will be getting by multiplying equation 1 by lambda j, then we subtract this equation from that equation; so, we do a subtraction here and see what we get.

So, if we really do the subtraction, the left hand side will be equal to 0 and let us see what we get in the right hand side, this will be C 1 lambda 1 minus lambda j X 1 plus C 2 lambda 2 minus lambda j X 2 plus C 3 lambda 3 minus lambda j X 3 up to C r lambda r minus lambda j X r.

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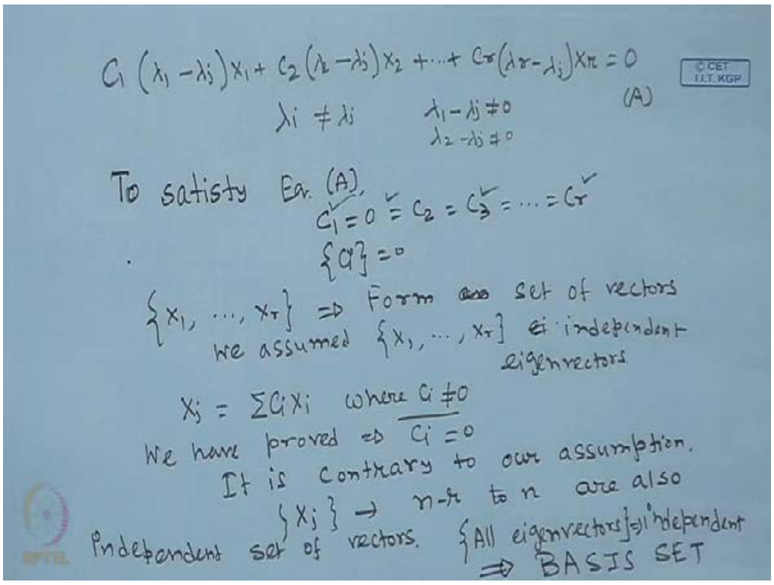

So, let us say, this is equation number 1; so, we get an equation number 1 an equation number a; let us write down equation number a, once again for convenience lambda 1 minus lambda j X 1 plus C 2 lambda 2 minus lambda j X 2 likewise up to C r lambda r minus lambda j X r is equal to 0.

Now, in this equation, we have already, this problem is for algebraically simple problem. So, therefore lambda i is not equal to lambda j; so, therefore lambda 1 is not equal to lambda j, lambda 2 is not equal to lambda j; so, lambda 1 minus lambda j is not equal to 0; so, lambda 1 minus lambda j is not equal to 0.

Similarly, lambda 2 minus lambda j is not equal to 0, lambda r minus lambda i is not equal to 0. So, therefore, in order to satisfy this equation A, to satisfy equation A, we must have that all the corresponding coefficients must vanish each and individual; that means, to satisfy equation a compulsorily, we should have C 1 equal to 0, C 2 equal to 0, C 3 equal to 0, up to C r all of them should be individually equal to 0.

So, therefore, this simply proves that, whatever we have done in the earlier classes, that X 1 for the C i; each of this C i will be equal to 0. Then, Xi, X 1 to X r, they form, they form a set of equations, set of vectors and we have assumed, these are, these are Eigen independent Eigenvectors.

So, therefore, X j any other vector can be represented as summation C i X i, where C i is not equal to 0; so that was our earlier assumption, when we started that we have taken up any Eigenvector from the dependent set lying between j the n and n minus r j, being the, being lying between n minus r and n; so, X j can be expressed as a linear combination of these vectors; so, therefore we assumed that Cy was not equal to 0, but in this case whatever we have proved that it goes to the contrary of our assumption, that each of them will be individually equal to 0.

So, we have proved that C i must be equal to 0; so, it contracts, contradicts of our assumption, so it is contrary to our assumption; so, this is called a negative proof. So, what is the implication? The interpretation is that, the rest Eigenvectors the X j Eigenvectors which a set lying between n minus r to n; they are not dependent set of vectors, they are also independent set of vectors, X j are also independent set of vectors.

So, therefore, all the Eigenvectors are the independent set of vectors; therefore, all Eigenvectors are independent, all Eigenvectors are independent and they constitute the members, they are the members of the basis set. So, they form a basis set, this completes this proof. So, we have seen that all the Eigenvectors of a matrix A are basically the members of the basis set and basis set vectors any other vector in the space can be represented as a basis set vectors.

So, therefore Eigenvectors of a matrix are always independent, each of them is in independent, they are the members of basis set vector. So, any other vector in the space can be represented as a linear combination of all these Eigenvectors.

So, therefore, as we have said a few minutes back, regarding the physical interpretation of the Eigenvectors, that Eigenvectors are nothing but the directions which are mutually independent to each other and any other vector in the space can be represented as a linear combinations of these independent vectors.

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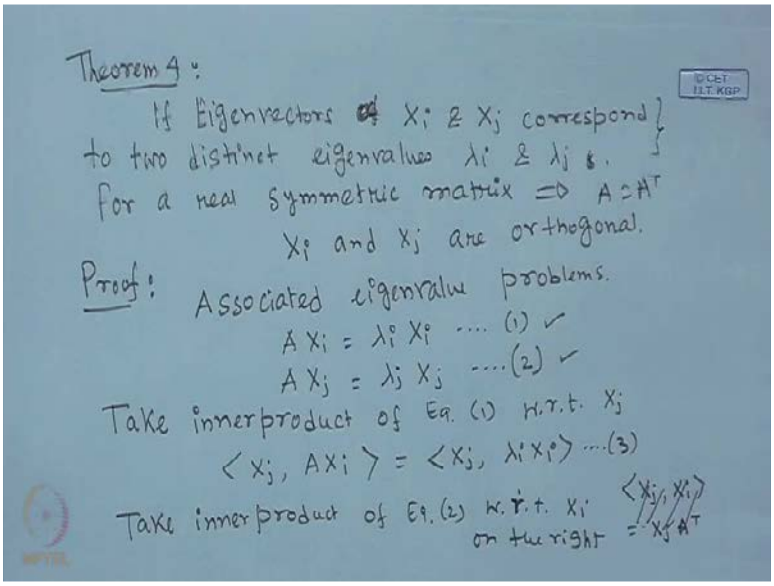

So, next we go to the fourth theorem; theorem number 4, this says Eigenvectors of X i, Eigenvectors X i and X j corresponds to, if Eigenvectors X i and X j correspond to two distinct Eigenvalues lambda i and lambda j; then if Eigenvectors X i and X j correspondent to distinct Eigenvalues; lambda i and lambda j they are orthogonal to each other, for a real symmetric matrix.

For a real, if this is the case, then for a real symmetric matrix X i and X j are orthogonal and symmetric matrix, means, A is equal to A transpose. Therefore, if we have, so what is the proposition in this theorem? In this theorem, it is proposed that if we have a real symmetric matrix, that means, A is equal to A transpose then 2 Eigenvectors X j and X i they correspond to, if they correspond to two distinct Eigenvalues lambda i and lambda j, then X i and X j form an orthogonal set; that means, they are orthogonal to each other.

So, therefore, we proof this theorem and the proof goes like this, let us say lambda i and lambda j are two distinct Eigenvalues, corresponding Eigenvectors are X i and X j. And let us write down the associated Eigenvalue problems, with the linked, with the Eigenvalues lambda i and lambda j.

So, we write down the associated Eigenvalue problems, A X i is equal to lambda i X i, this is equation number 1; A X j is equal to lambda j X j, this is equation number 2; these are the associated Eigenvalue problems for lambda i and lambda j. Then what we do, we take the inner product of equation 1 with respect to X j.

Take inner product of equation 1 with respect to X j; if we take the inner product, we will getting X j, A X i is equal to inner product of X j lambda i X i. So, if we remember that we have proved this equation in the last class, X inner product of X j and X i should be X j A transpose.

So, therefore, so this will be inner product of X j. And we write it in the other form, we will take that later on, first we take the inner product of equation 1 with respect to X j and we will be getting equation number 3.

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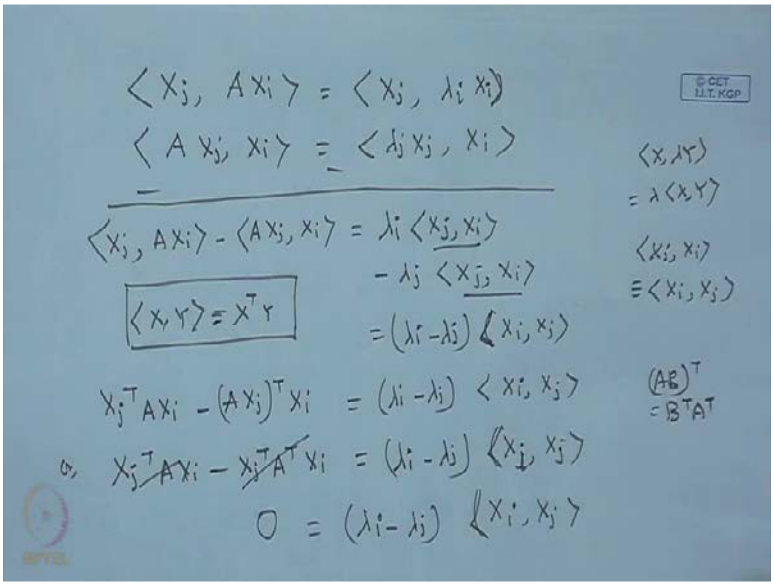

Then we take inner product of equation 2 with respect to X i on the right; so, let us see what we get. Now from these 2, we will be getting, inner product of X j A X i is equal to inner product of X j lambda i X i. Second, the inner product of the second equation with respect to X I, so A X j, X i is equal to inner product of A X j is nothing but lambda j X j the inner product of lambda j X j and X i.

Now, we subtract these 2 equation; if you subtract these 2 equations, what we get, it is that inner product of X j and A X i minus inner product of A X j, X i is equal to we can take, we invoke the property of the inner product that X lambda inner product between X lambda Y is nothing but lambda inner product of X and Y, that is, if lambda is a scalar; so, similarly, we can take lambda i out, so this becomes X inner product between X j and X i minus same here we took lambda j out so it will be lambda j multiplied by inner product of X j and X i and we have already proved earlier that inner product of X j X i is identical to inner product of X i X j; that means, these two are identical and we can write lambda i minus lambda j multiplied take an inner product X i and X j can be taken as common.

Then we utilize the formula that whatever we have derived earlier that inner product between X and Y should be is equal to X transpose Y, that we have already proved probably in the last lecture. So, utilizing that, we will be doing X j transpose A X i minus A X j transpose full X i is equal to lambda i minus lambda j inner product of X i and X j.

So, just open up this transpose, so what we get X j transpose A X i is nothing but X j transpose A transpose X I, we utilize the formula AB transpose is nothing but B transpose A transpose, so X j transpose will be coming in the front, so it will be X j transpose A transpose X i is equal to lambda i minus lambda j inner product of X j and Xi and X j.

Now, if you see the quantity on the left hand side is that the left hand side will be having the identical quantities one with the negative sign; so, these 2 will vanish. Now, we have already seen, so what we get from here is that 0 is equal to lambda i minus lambda j inner product of X i and X j.

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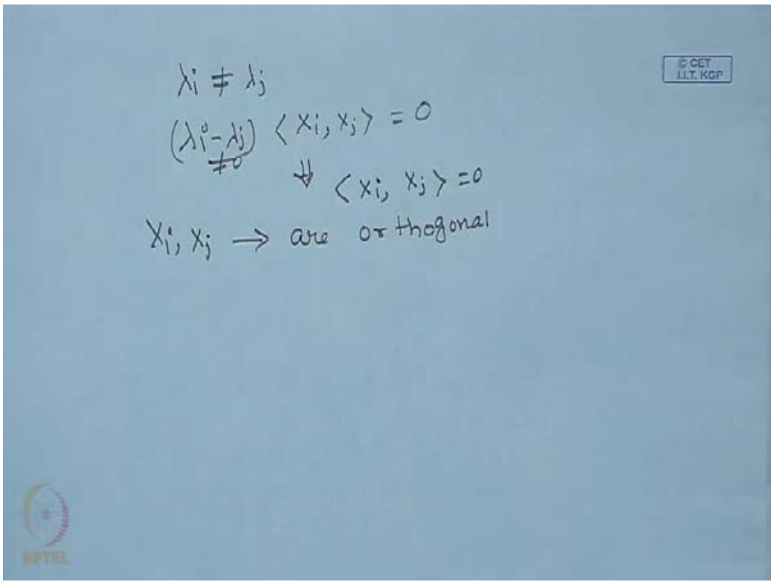

Now, lambda i the Eigenvalues being simple in nature, that means, they are distinct; distinct, means lambda i is not is equal to lambda j, they are not repeated roots they are distinct roots. So, therefore, to satisfy this equation inner product of X i and X j should be equal to 0; therefore, so since, lambda is not equal to lambda j, this quantity is not equal to 0; therefore, to satisfy this equation only option left is that inner product of X i and X j must be equal to 0. So, therefore, that simply means, that the Eigenvectors X i and X j are orthogonal.

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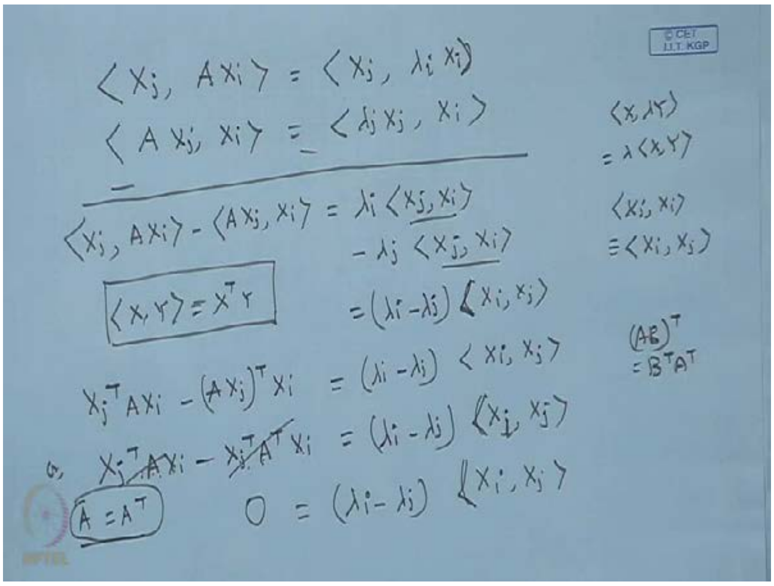

So, just one more clarification is that, in this equation whenever we are omitting the left hand side, the we are putting the left side is equal to 0; if you look into this equation X j transpose A X i minus X j transpose A transpose Xi, but since it is a symmetric matrix A is equal to A transpose. Therefore, since A is a symmetric matrix these two quantities become identical and they will be subtracted.

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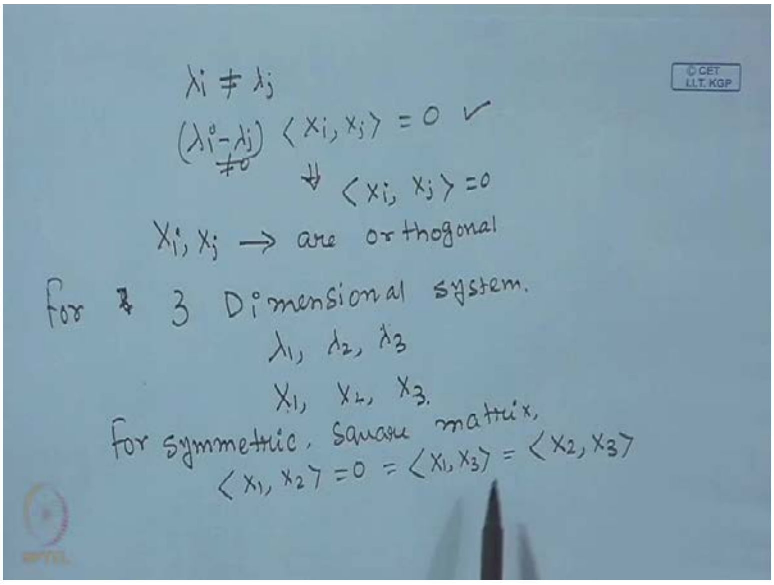

So, therefore this whole left hand side will become 0 and you will be getting this equation; since, lambda i and lambda j are not equal, they are distinct, they will be not equal to 0. So, in order to satisfy this equation, X i inner product between X i and X j must be equal to 0; so, they are orthogonal to each other.

So, therefore if we have three-dimensional system, for three-dimensional system we have three distinct Eigenvalues, lambda 1 lambda 2 and lambda 3. Corresponding to three distinct Eigenvalues, you will be having three distinct Eigenvectors X 1, X 2, X 3.

Now, this simply means, that for the symmetric matrix for symmetric square matrix, because Eigenvalue problems are defined on square matrix; only for symmetric square matrix X 1 inner product of X 1 and X 2 is equal to 0, is equal to inner product of X 1 and X 3 and equal to inner product of X 2 and X 3.

This, simply says that, the Eigenvalue, Eigenvectors are orthogonal to each other, if we have a symmetric matrix; the other is not true that if the matrix is not symmetric that we cannot say, the Eigenvectors are not orthogonal to each other, that may not be the case, if the matrix is not symmetric. But in that case the Eigenvectors of a matrix A and the Eigenvectors of matrix A transpose they form an orthogonal set.

So, that we will prove in the next class and after that proof we will be able to solve take up any chemical engineering problem and solve them by this Eigenvalue, Eigenvector method.

We stop in this class at here and we will take up from this point onwards, in the next class. Thank you very much.