Advanced Mathematical Techniques in Chemical Engineering
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Lecture No. # 04
Vectors

Well good morning everyone. So, we will be starting whatever we have done left in the last class. We were discussing about the properties of the vectors and in the last class we have seen what is called the independent set of vectors, dependent set of vectors, dimension definition of dimension of space.

So, we start from that point onwards, in this class. The dimension of the space is basically, if there are n number of independent vectors present in the vectors or function that is present in the space, then every n plus 1 nth vector or function can be expressed as a linear combination of all this independent vectors. So, then, the dimension of this space is called it is n-dimensional space; for example, if we have a three-dimensional space then there will be existing 3 vectors which are independent and any fourth vector can be, will be dependent and it can be expressed as a linear combination of all these three independent vectors.

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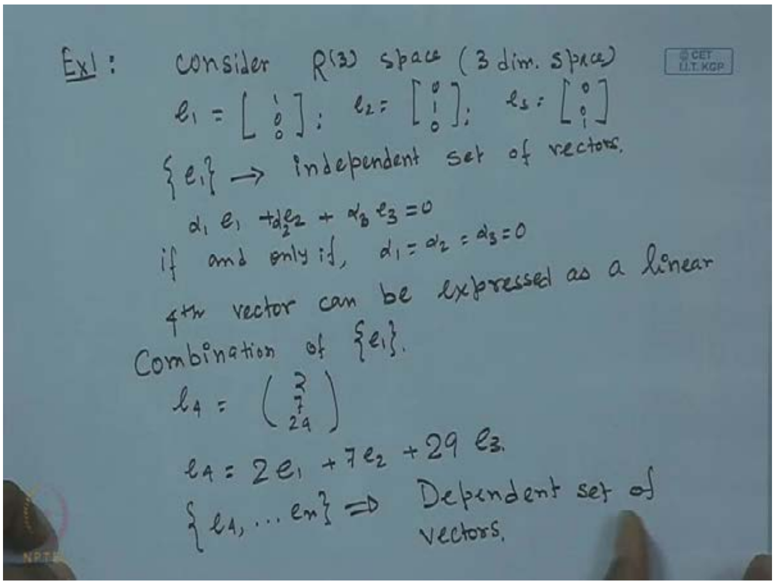

Now, in this class we will just first look into a one example; that let us consider an 3 space, it is basically a three-dimensional space where if the vectors e1 1 0 0, e 2 0 1 0 and e 3 is 0 0 1; these set of vectors three vectors will constitute an independent set of vectors; so, alpha 1 e 1 plus alpha 2 e 2 plus alpha 3 e 3 is equal to 0.

If and only if the coefficients alpha 1 alpha 2 and alpha 3 all are identically equal to 0; therefore, this three vectors are called the independent vectors, any other vector, any fourth vector in the space, any other vector can be expressed a linear combination of these three vectors e 1, e 2 and e 3, fourth vector can be expressed as a linear combination of independent set of vectors.

So, for example, if you talk about a fourth vector e 4 as 2 7 29, I mean anything then e 4 can be expressed as a linear combination of these three vectors 2 e 1 plus 7 e 2 plus 29 e 3. So, any set constituting the vectors e 4, e 5 are up to e n, they will constitute a dependent set of vectors.

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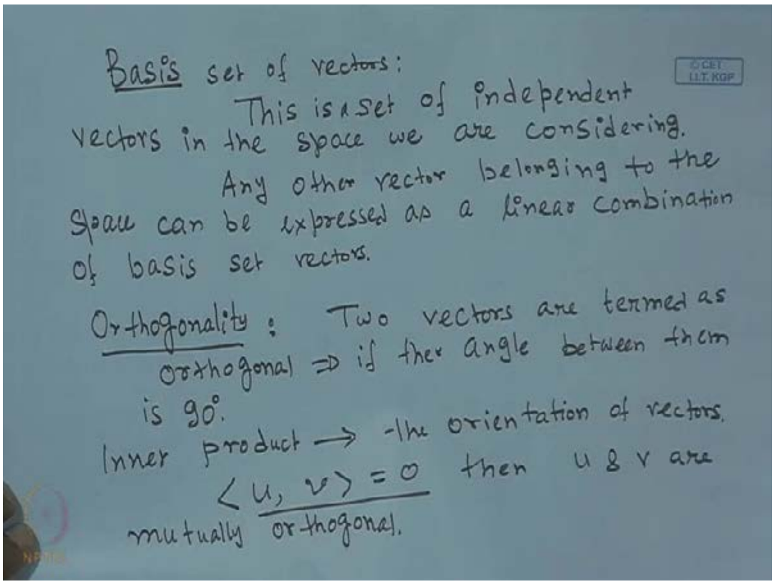

So, next we talk about something called basis of vector or basis set; it is very important, basis set of vectors: this is a set of set of independent vectors in the space we are considering.

So, therefore, the constituent of the basis set vectors are all independent vectors; so any other vector a in that space can be expressed as a linear combination of basis set vectors - that is very important.

Any other vector belonging to the space can be expressed as a linear combination of basis set vectors. Then we talk about some of the important properties called orthogonality of the vectors: two vectors are termed as orthogonal, if they are perpendicular to each other, that means, the orientation the angle between the two vectors is 90 degree.

So, two vectors are termed as orthogonal, if the angle between them is 90 degree. In other words, if you look into the property of the vector, which will be responsible to indicate the orientation of the two vectors, it is the inner product. So, this has a direct implication of the inner product of the two vectors, because inner product indicates the orientation of vectors and if you remember the inner product is nothing but the dot product of the two vectors.

So, the dot product of two vectors is nothing but the magnitude of the individual product, of the magnitude of the individual vectors multiplied by the cosine theta or theta is the angle between them.

So, if the angle is 90 degree then cosine 90 is 0. Therefore, the inner product will be turning out to be 0, that means, if inner product between two vectors u and v is equal to 0 then u and v are mutually orthogonal.

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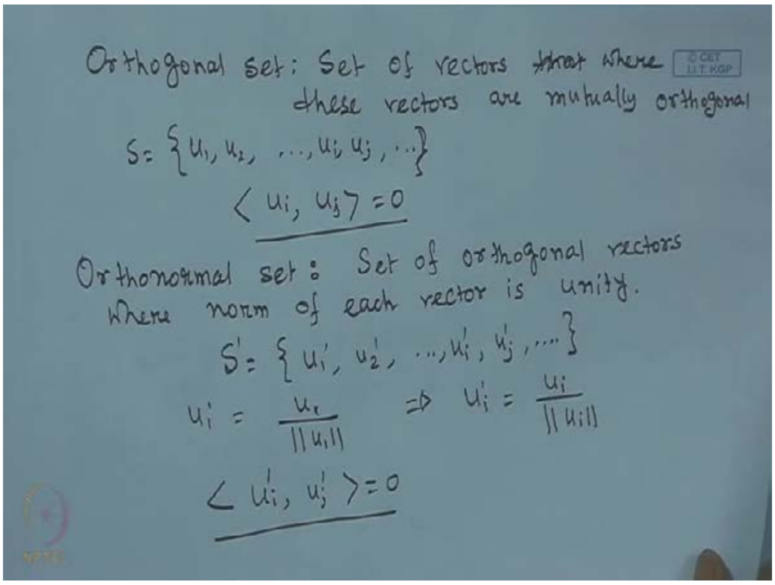

Therefore, this equation gives a criteria of orthogonality of the vector. Now, what is the orthonormal set; so what is a orthogonal set of vectors? The orthogonal set, is a set of, set of vectors where these vectors are mutually orthogonal; that means, this is a set, this set will be is constituted by u 1, u 2, u I, u j, like that; so, for any two vectors, the inner product of the two vectors will be equal to 0; then, these set of vectors is called mutually they constitute orthogonal set. Then, next one is orthonormal set; again, in this case, this is a set of orthogonal vectors where norm of each vector is unity, that means, we are talking of the set, the earlier set, it may be a different set, s prime which one is nothing but u 1 prime, u 2 prime, u i prime, u j prime, like that; so, u 1 prime is nothing but u 1 divided by norm of u 1; so, we make sure that the norm of these vector will be unity.

So, any arbitrary vector will be nothing but u i divided by norm of u i and they will be orthogonal to each other, because these vectors are orthogonal to each other; so, inner product between u i prime and u j prime will be always equal to 0.

So, the basic criteria to of the vectors which will be form in the orthonormal set is that they will be mutually orthogonal to each other, as well as their norm will be equal to one.

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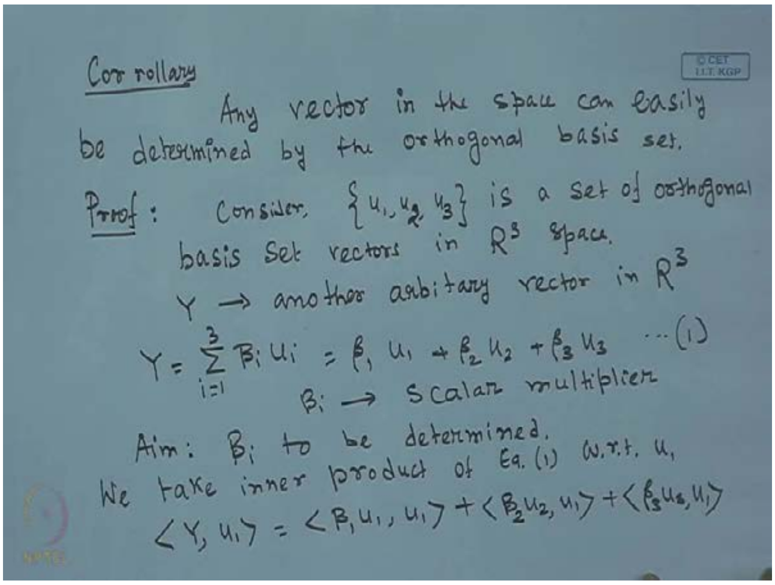

So, that particular set is termed as orthonormal set. So, what is the cor rollary? The cor rollary is that any vector in the space where we are working can easily be determined by the orthogonal basis set.

So, if we have any unknown vector in the space that can be expressed as a linear combination of all this independent set of vectors, which are basically the constituent of the basis set and if these vectors are orthogonal to each other, then you will be getting advantage of some of the properties and the whole calculation has been enormously simplified.

We will be just look into this, so let us try to proof this corollary or the interpretation. The proof goes like this, we consider u 1, u 2 and u 3 is a set of orthogonal vectors basis set vectors in R 3 space and so this is a three-dimensional space and we can identify that u 1, u 2, u 3 they are basically mutually independent to each other.

Now, we have Y and arbitrary vector arbitrary fourth vector in same R 3 space, then these vector as the definition of the independent vectors, it will be dependent and it can be expressed as a linear combination of all, in terms of the basis set vector, that means, Y can be expressed as, beta i u i where the index i runs from 1 to 3, so these will be, beta 1 u 1 plus beta 2 u 2 plus beta 3 u 3 and this beta i is are nothing but the scalar multiplier as these are the linear combination.

So, if that is the case, then our aim is to evaluate what are this individual multiplier, so that the any vector in the space can be expressed as a linear combination of the basis set vectors.

So, next is that our aim is that beta i to be determined. How this will be determined, if suppose this is equation number 1, if we take inner product of this equation 1 with respect to e 1, let us see what we get.

So, we take inner product of equation 1 with respect to u 1, let us see what we get; so this is the inner product of u 1 is equal to inner product of beta 1 u 1, u 1 plus inner product of beta 2 u 2, u 1 plus inner product of beta 3 u 3, u 1; so, let us try to simplify this equation.

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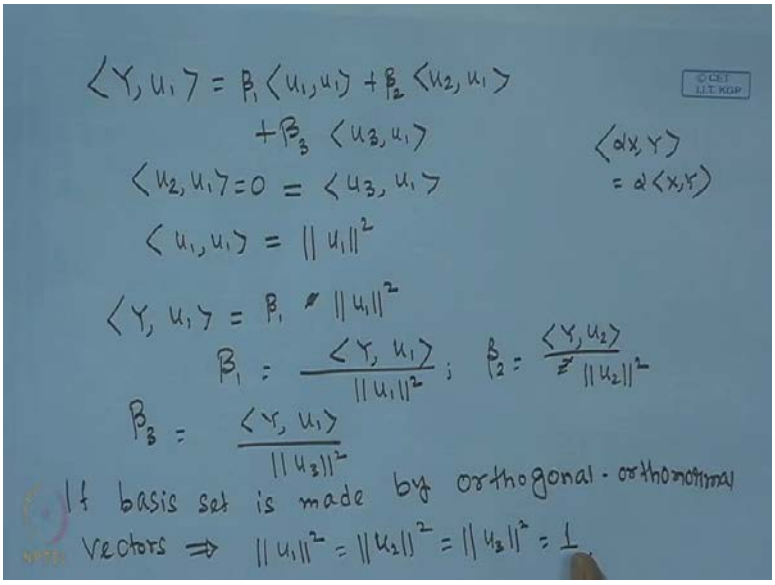

So, inner product of Y and u 1 will be nothing but we invoke the property of the inner product that, inner product of alpha x Y is nothing but alpha inner product of x and Y, where x and Y are the two vectors and alpha is the scalar multiplier.

So, therefore, beta 1 can be taken out and it will be inner product of u 1 and u 1 plus beta 2 inner product of u 2 and u 1 plus beta 3 inner product of u 3 and u 1. Now, since use u vector is a set of orthogonal basis set vector, therefore u inner product of u 2 and u 1 will be equal to 0 by definition, because there mutually orthogonal to each other inner product of u 3 and u 1 will be equal to 0.

On the other hand, if you look into this, we can identify in the whatever you have studied in the last class that inner product of u 1 and u 1 is nothing but the norm of u 1.

So, therefore, this equation boils down to inner product of Y and u 1 is nothing but beta 1 norm of u 1 square; so, beta 1 can be identified as following quantity inner product of u and Y and u 1 divided by norm of u 1 square.

Similarly, we can identify what is beta 2 this is nothing but inner product of Y and u 2 and norm of u 2 square and beta 3 is nothing but inner product of Y and u 1 divided by norm of u 3 square. So, if we are considered the basis set is constituted by the orthogonal vectors u 1, u 2, u 3. Now, if we would have met the basis set is constituted by orthogonal- orthonormal vectors u 1, u 2, u 3 then all these norms would have been equal to one.

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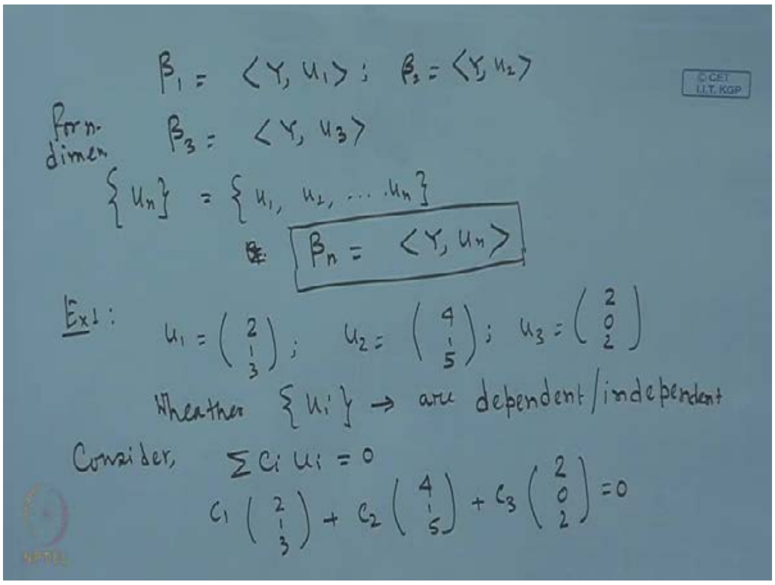

So, that, that simplifies the holder up things; if basis set is made by orthogonal- orthonormal vectors, then it has two simultaneous properties, that in this set the individual vectors are orthogonal, as well as they are norm is equal to 1; that means, u 1 square is equal to norm of u 2 is equal to norm of u 3 square equal to 1; so, in that case, the denominators of all these scalar multipliers will be equal to 1 and the whole thing has been tremendously simplified and you will be getting, beta 1 is equal to inner product of Y and u 1, beta 2 is nothing but inner product of Y and u 2 and beta 3 is equal to nothing but inner product of Y by u 3.

If you have n-dimensional space, then let us say, u n for n dimensional, these for three-dimensional one, for n-dimensional space we can generalized this one that u n is constituted by n number of basis set vectors, which are mutually orthogonal and orthonormal. Therefore, any i th vector or any n th vector with the coefficients beta n will be expressed as the inner product of Y and u n; that means, if Y is an vector that can be in that space, in that n dimensional space, apart from the basis set vectors it can be expressed as a linear combination of all the independent, all the basis set vectors and the corresponding coefficient can be evaluated by taking simply, the inner product of the other vector and the n th vector of the basis set.

So, in this way, we can identify, we can expressed the a vectors in terms of linear combination of the basis set vectors. So, for that suppose you will be having a basis set vectors, which does not ensure that you will be getting an orthonormal and orthogonal set of vectors.

So, there is a definite method available to convert all the basis set vectors into mutually orthogonal and orthonormal vectors, but before in, before looking into that particular method and technique is known as the Gram-Schmidt Orthogonalization technique, but before going into that, we just look into some of the examples to illustrate the properties of the vectors that whatever we have learnt till now.

So, first example that goes is that, suppose there are three vectors is given 2 1 3, let say u 1 is 2 1 3, u 2 is 4 1 5 and u 3 is 2 0 2; now the point is, question is that whether u i are dependent or independent?

So, for that, if you, if you considered the equation summation of c i u i is equal to 0 then we can obtain this following equations; so that means, c 1 2 1 3 plus c 2 4 1 5 plus c 3 2 0 2 is equal to 0.

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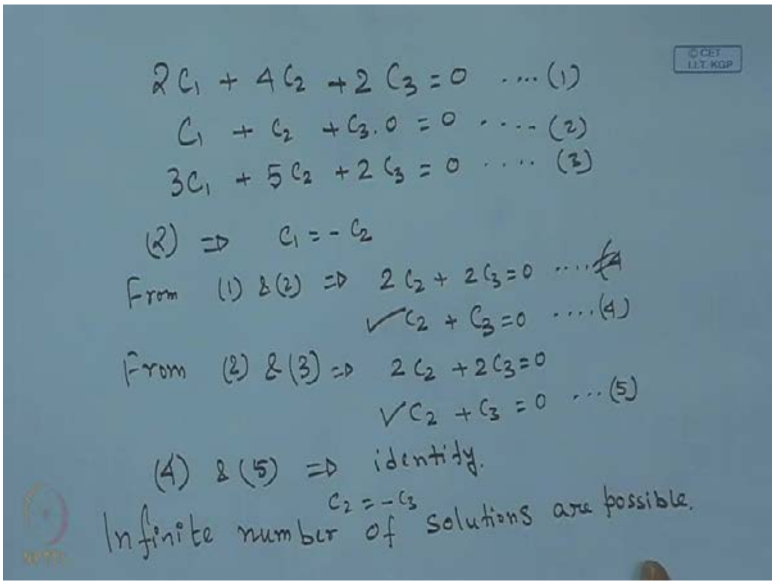

So, let us consider this equation, whether, where the coefficients c 1, c 2, c 3, are all scalar multiplier. So, you will be constituting three equations out of this 2 c 1 plus 4 c 2 plus 2 c 3 is equal to 0.

Then, c 1 plus c 2 plus c 3 multiplied by 0 is equal to 0; and, 3 c 1 plus 5 c 2 plus 2 c 3 is equal to 0. So, this is the first equation, this is equation number 2, this is equation number 3.

Then, from in, let us try to find out what are the 3 equations 3 unknowns; let us try to find out what are the values of these scalar multipliers c’s. So, from equation 2, we will be getting c 1 is equal to minus c 2. And, from 1 and 2, if that is the case by combining these two we can get, you just put c 1 equal to minus c 2, so it will be minus 2 c 2; so, it will be 2 c 2 plus 2 c 3 is equal to 0; this is equation, so you just simplify it further, so it will be divided by two.

So, it will be c 2 plus c 3 is equal to 0 this is equation number 4. Similarly, from equations 2 and 3, what we get is that you express c 1 everything in terms of c 2 substitute c 1, so it will be minus 3 c 2; so, it will be 2 c 2 plus 2 c 3 will be equal to 0. So, c 2 plus c 3 equal to 0, so this is equation number 5.

So, if you look into, you will be having two equations and two unknowns, but the point is if you look into this equation 4 and 5 they are identical and they represent an identity.

Now, the solution is c 2 is equal to minus c 3 and there will be infinite number of solutions can be present in this system, there will be no definite unique answer.

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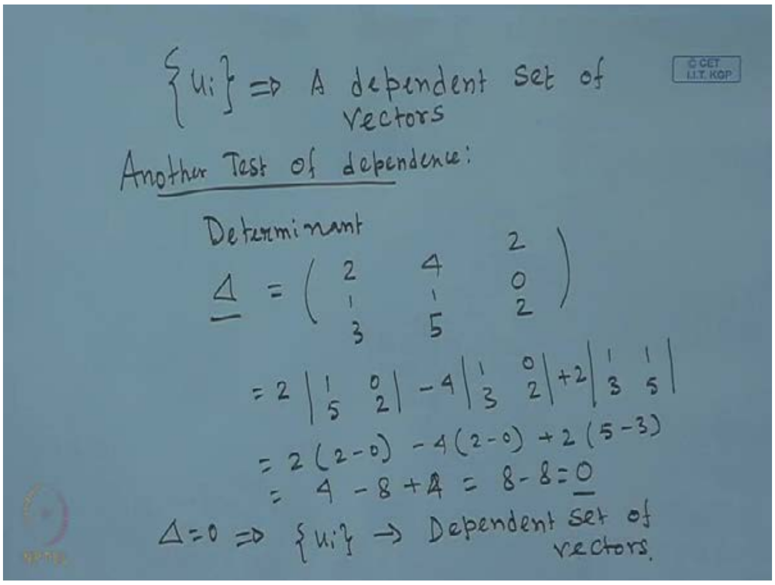

So, infinite number of solutions are possible and therefore these set of vectors are dependent set of vectors, they are no longer independent set of vectors. So, conclusion is u i constitute a dependent set of vectors. There is another test to check whether the set of vectors are dependent or independent, that test is that we evaluate the determinant produced by these 3 equations; basically, the elements the coefficients of these 3 equations will be the elements of this determinant.

So, if this determinant will be 2 1 3 4 1 5 2 0 2. Now, if you would like to find out what is the value of this determinant, just open up this determinant 2 so it is 1 5 0 2 minus 4 it is 1 3 0 2 plus 2 1 3 and 1 5. So, this becomes 2 multiplied by 2 minus 0 the cross multiplication minus 4 2 minus 0 plus 2 5 into 1 minus 3 into 1. Therefore, this will be 4 minus 8 plus 2 into 2 is 4. So, therefore, this will be 8 minus 8 equal to 0.

So, therefore this determinant is equal to 0, if the determinant is equal to 0 then there will be infinite number of solutions present into the set of a equations involving the coefficients c 1, c 2, and c 3. So, this indicates that the basis set vectors, the vectors that we are talking about they constitute a dependent set of vectors.

That is a quick way of evaluating to determine, whether the vectors are dependent or independent.

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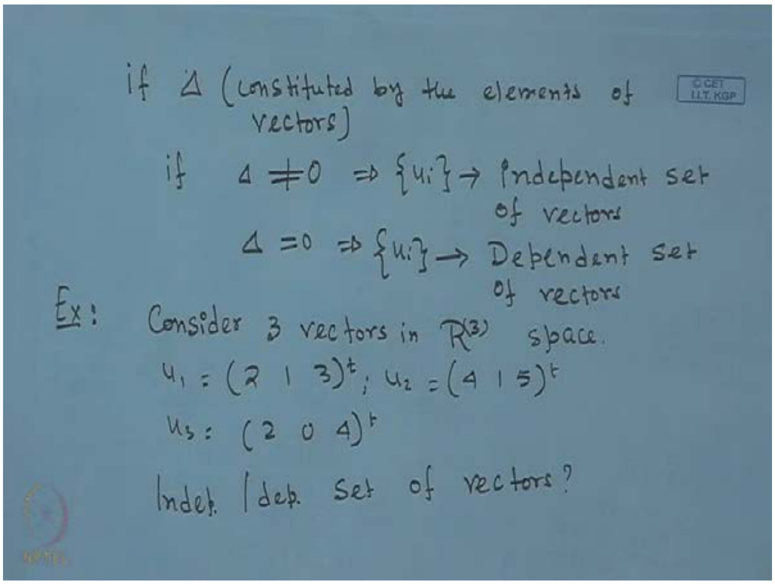

So, let us try to summarize whatever we have done; if determinant, the determinants are basically constituted by the elements of vectors; if determine this constituted by the elements of vector.

Now, if determinant is not equal to 0 then vectors constitute independent set. If determinant is equal to 0, then this set of vectors are dependent set of vectors. So, given a problem, we can identify the determinant of the constituted by the corresponding coefficients of the elements of the vector and we can go about it.

Next example, is again for the R 3 space. These are u 1 is 2 1 3 transpose, u 2 is 4 1 5 transpose and u 3 is 2 0 4 transpose, then we just change the some of the elements, the last element of the last example and you have to check whether they constitute independent or dependent set of vectors; so, we check both the methods and like to determine what is the answer for this problem.

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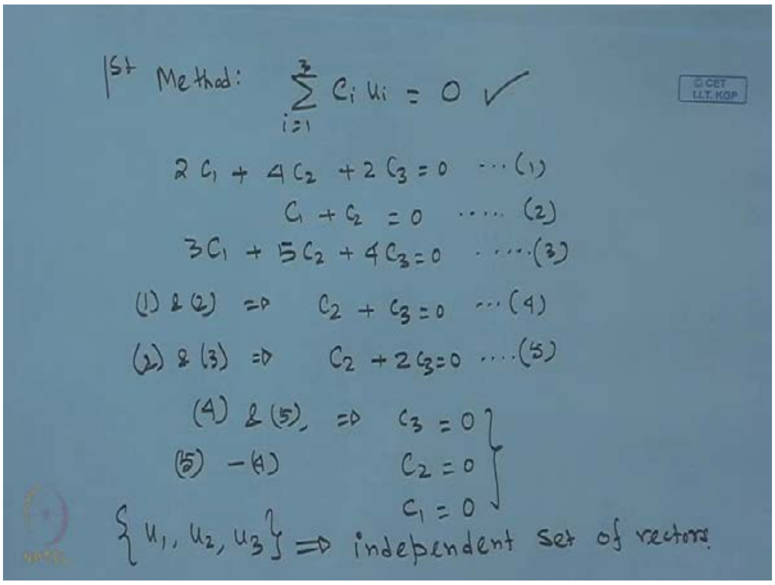

Then, let us first method, using the first method, we can express all the vectors as a linear combination of the of the 3 individual vectors, summations c i u i and equal to 0. You consider this equation; so, these will be, 2 c 1 plus 4 c 2 plus 2 c 3 is equal to 0, that is equation number 1; then, c 1 plus c 2 equal to 0 that is the equation number 2 and the third 1 will be 3 c 1 plus 5 c 2 plus 4 c 3 equal to 0, that is equation number 3.

So, combining 1 and 2 what we will be getting is that, we will be getting c 2 plus c 3 is equal to 0, as we have already done earlier, but combining 1 and 2 and 3 what we can obtain is that will be getting, c 2 plus 2 c 3 equal to 0, that is equation number 5.

So, if you now combine these 2 equation to solve this 2 unknowns, from 4 and 5 will be getting c 3, we just subtract 4 from 5, so we subtract 4 from 5, we will be getting c 3 equal to 0 if c 3 equal to 0, then from equation number equation number 4 c 2 will be equal to 0.

If c 2 equal to 0 then from equation number 2, c 1 will be equal to 0. So, in this case, in order to satisfy this equation, all the individual coefficients have to be equal to 0 individually.

So, therefore will be getting a unique solution for this particular case.; they are does not exist an infinite number of solution. In this case, all the individual coefficients scalar multipliers will be equal to 0, in order to satisfy this equation. Therefore, the set constituted by u 1, u 2 and u 3 are independent set of vectors; so, that is the conclusion of this analysis.

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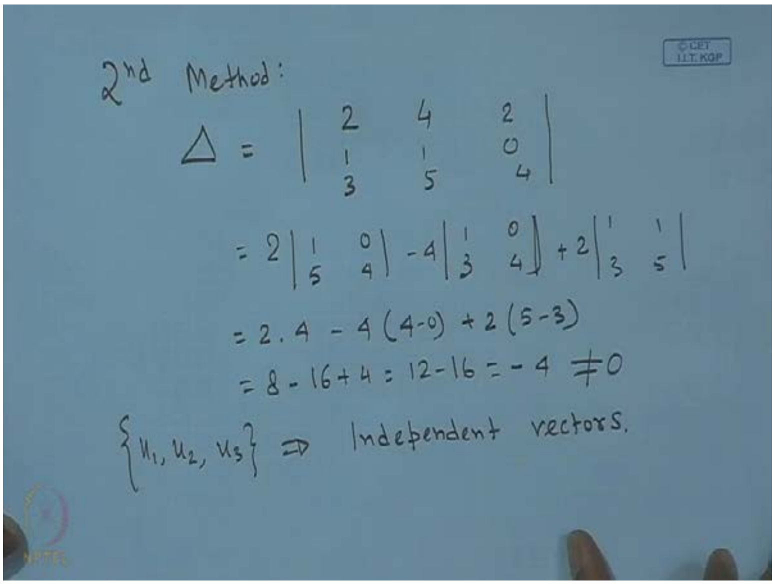

Now, let us look into the method number 2; it is a second method, by looking into the coefficient determinant, so the determinant is constituted by this coefficients that is 2 by this elements, 2 1 3 4 1 5 2 0 4 and these turns out, if you would like to simplify 2 1 5 0 4 minus 4 1 3 0 4 minus plus 2 1 3 1 5, this will be 2 4 minus 0. So, it will be 2 into 4 minus 4, this will be, 4 minus 0, so it will be 4, plus 2 5 minus 3. So, it will be 8 minus 16 plus 2 into 2 is 4; so, it will be 12 minus 16 is equal to minus 4 and this is not equal to 0, in this particular case. So, therefore, u 1, u 2, u 3, they are independent set of vectors.

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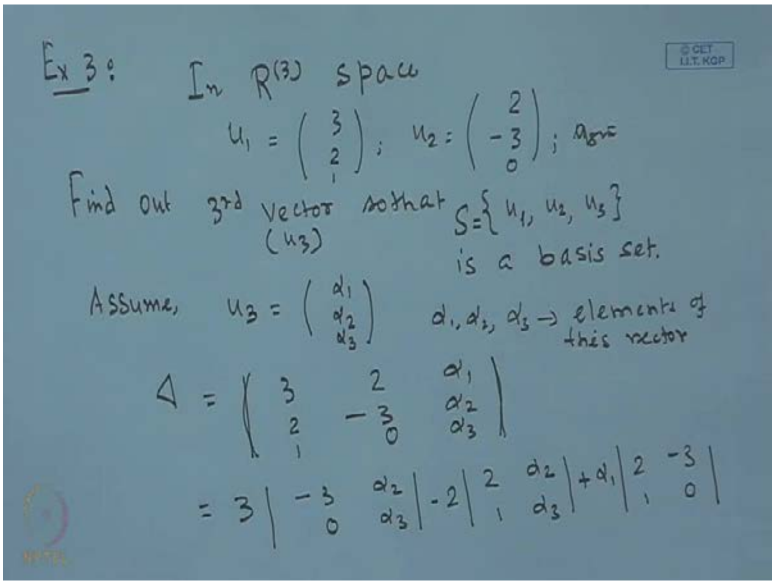

So, from both the methods we come to the same conclusion. Now, i have another problem example 3, maybe we have two vectors available in a in R 3 space, we have 2 vectors available, u 1 is equal to 3 2 1, u 2 is equal to 2 minus 3 0; so, therefore, these 2 are available and we have to find out third vector in the space; so that, that vectors along with u 1 and u 2 will constitute a basis set vectors so, u 1, u 2, u 3 is a basis set.

Find out the third vector, so that s is a basis set, so that we have to two vectors are given, we have to find out the elements of the third vector, such that u 1, u 2 and u 3 constitute the basis set vectors. So, in this problem, first we do not know what is the element of the third vector, therefore, we assume vector number three that is u 3 is nothing but alpha 1 alpha 2 alpha 3, where alpha 1, alpha 2, alpha 3, are elements of this vector that we are going to determine. If that is the case, then we constitute the corresponding determinant of this problem this will be 3 2 1 2 minus 3 0 alpha 1 alpha 2 alpha 3.

So, just evaluate this determinant, if we evaluate this determinant 3 minus 3 0 alpha 2 alpha 3 minus 2 2 1 alpha 2 alpha 3 plus alpha 1, so it will be, 2 1 minus 3 0.

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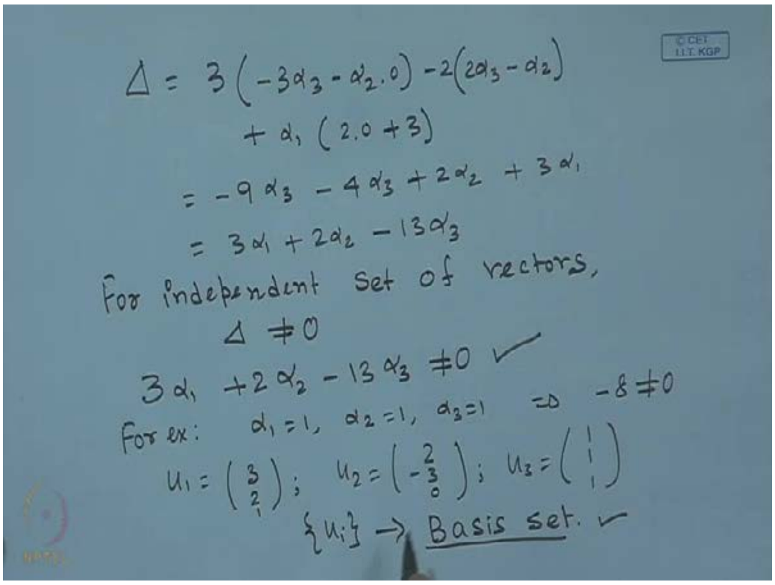

So, we just keep on simplifying this, so determinant is equal to 3 minus, 3 into alpha 3, so this becomes, 3 alpha 3 minus alpha 2 into 0 minus 2 2 into alpha 3 minus 1 into alpha 2 plus alpha 1 2 into 0, minus into minus plus, 3 into 1 so it will minus into minus plus, so it will be 3.

So, therefore, this becomes minus 9 alpha 3 minus 4 alpha 3 minus 2 alpha 2 plus 3 alpha 1. So, will be getting 3 alpha 1, this will be minus minus, that means, plus, plus 2 alpha 2 minus 13 alpha 3.

Now, in order to get an independent set of vectors, my criteria is that - for independent set of vectors, the criteria is the determinant cannot be equal to 0. Therefore, 3 alpha 1 plus 2 alpha 2 minus 13 alpha 3 should not be equal to 0. Now, we can assume any value corresponding to alpha 1, alpha 2, alpha 3, such that this in equality is satisfied.

For example, if you consider alpha 1 is equal to 1, alpha 2 is equal to 1, alpha 3 is equal to 1, you will be getting 3 plus 2 is 5, 5 minus 13 there is minus 8, minus 8 cannot be equal to is, it is not equal to 0. Therefore, 1, 1, 1, in along with the other 2 vectors u 1 3 2 1, u 2 2 minus 3 0 and u 3 as 1 1 1, constitute a basis set vectors are independents set of vectors, that means, u i is basically a basis set. similarly one could have used any three values of alpha 1, alpha 2, alpha 3, such that this inequality is satisfied not necessary 1, 1, 1, need to be taken; one can take any other combination of alpha 1, alpha 2, alpha 3, but keeping in mind that this inequality has to be satisfied, so that, that vector along with the first two vectors u 1 and u 2 will constitute a basis set of vectors.So, that indicates how one can determine the dependent and independent set of vectors.

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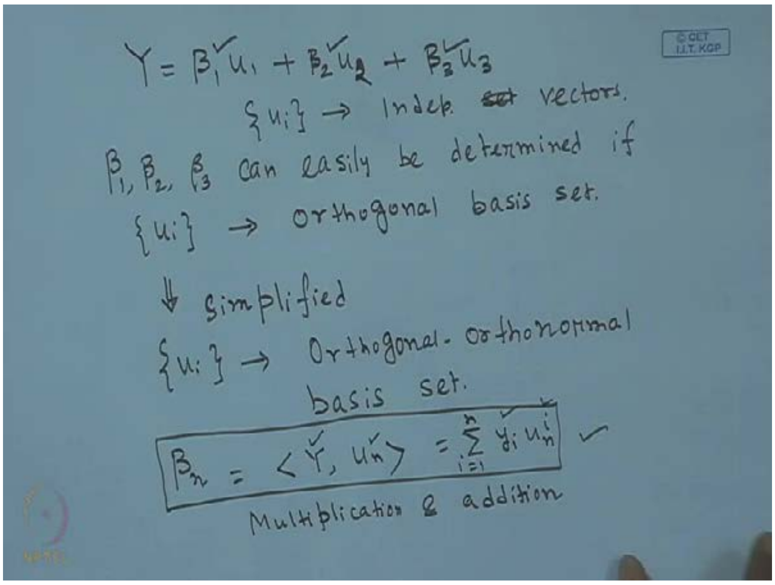

Next, we move to the topic on Gram-Schmidt Orthogonolization. As we have said that any fourth vector in the space can be expressed as a linear combination of the 3 basis set vector in R 3 space of further, but any space it can be expressed as linear combination of the basis set vectors.

Let us consider, an R 3 space which will be easy to comprehend; therefore, any fourth vector can be expressed as a function of, as a, as a linear combination of the basis set vectors; where u 1 to u 3, they constitute the independent vectors and we have seen earlier in this class itself, how to find out or determine the coefficients beta 1, beta 2, beta 3, if and only if these basis set vectors are mutually orthogonal to each other. Beta 1, beta 2, beta 3, can easily be determined If u i from the orthogonal basis set. The calculation will become further simplified, if u i constitute a set of orthogonal - orthonormal basis set.

So, therefore, i am just given example of 3 dimensional space, if you have n-dimensional space then the whole thing becomes very easy. So, the coefficients, if you remember beta n can be expressed as the inner product of Y and n th vector in the space, there is the u n, so we know this vector, we know this vector, so this will be simply summation of Y i u n i. So, the coefficient can be determined for a larger dimensional problem, by simply you identify the corresponding elements of this two known vectors and multiply them and then add them up. So, this involves the simple operations like multiplication and addition, so for a larger spaced problem we will write a small program in computer, so that the elements of this 2 vectors, known vectors, will be arranged in an array and we will identify that corresponding i th element of the two vectors will be multiplied and then being added up for all the subsequent terms and the addition will be give you the coefficient beta n.

Similarly, we could be able to put it in a loop and only couple of lines programs you will be able to calculate all the undetermined coefficients betas in this equation.

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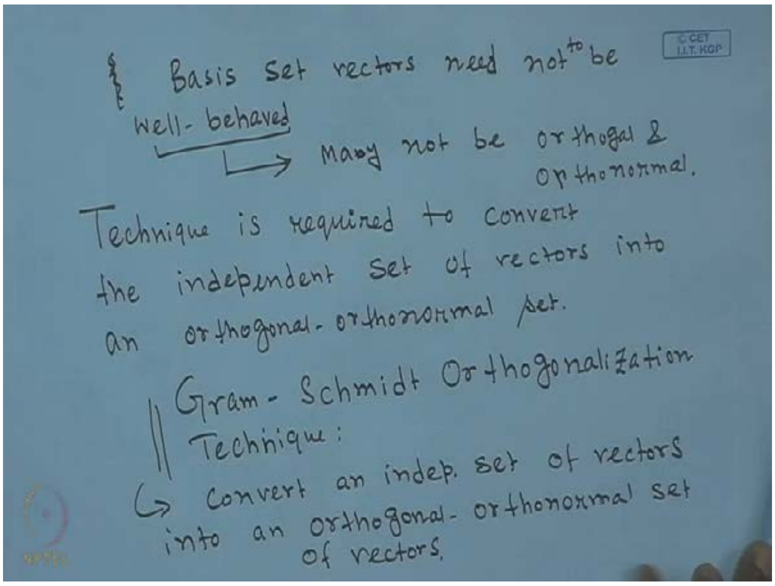

But the point is, all the independent set of vectors may not be well behaved set of vectors, that means, basis set vectors need not to be well behaved. What do meant by well-behaved vectors, that means, these vectors may not be orthogonal and at the same time orthonormal.

So, for that what you have to do? You have to add up a technique to make the basis set or independent set of vectors to be orthogonal and orthonormal.

So, the mathematical technique is required to convert the independent set of vectors into an orthogonal - orthonormal set; that this particular technique is known as Gram - Schmidt Orthogonalization.

So, by using this technique one can convert an independent set of vectors into an orthogonal - orthonormal set of vectors.

So, next we look into the details, derivation or the technique for getting the Gram-Schmidt Orthogonalization technique.

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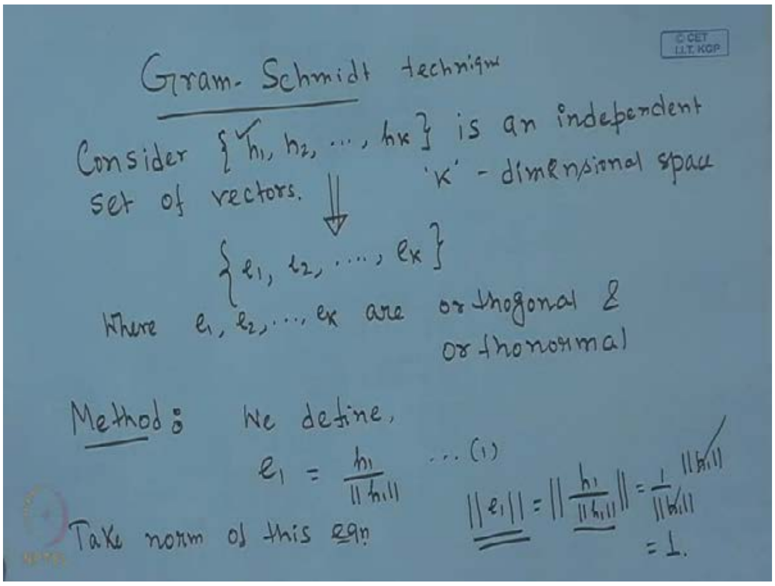

So, let us look into that and see how an independent set of vectors can we converted into orthogonal - orthonormal set of vectors. So, for that, let us consider a k dimensional space, where if the set h 1, h 2, h 3 up to h k is an independent set of vectors; it is basically you are talking about a k dimensional space.

We try, will be getting a set from this set; this set are not orthonormal, so will be getting another independent set of vectors. From these vectors will be getting e 1, e 2 and up to e k set, where e 1, e 2, e k are orthogonal and orthonormal both.

So, let us look into the details of this method, so we start with the first vector h 1, so we define e 1 from this vector itself, such that e 1 is h 1 divided by norm of h 1. So, if you look into the definition of this vector e 1, we can, we can take the norm of this equation, what will be getting is that you will be getting norm of e 1 is nothing but norm of h 1 divided by norm of h 1 and norm of h 1 is nothing but a scalar, so it comes out of this norm, so it becomes. You just remember the properties of a norm, so this be nothing but norm of h 1 divided by norm of h 1, this two are the scalars they will be canceling out, so the magnitude or norm of e 1 turns out to be 1. Therefore, this transformation ensures that the norm of vector e 1 is you will be unity; so, we made the e 1 orthonormal.

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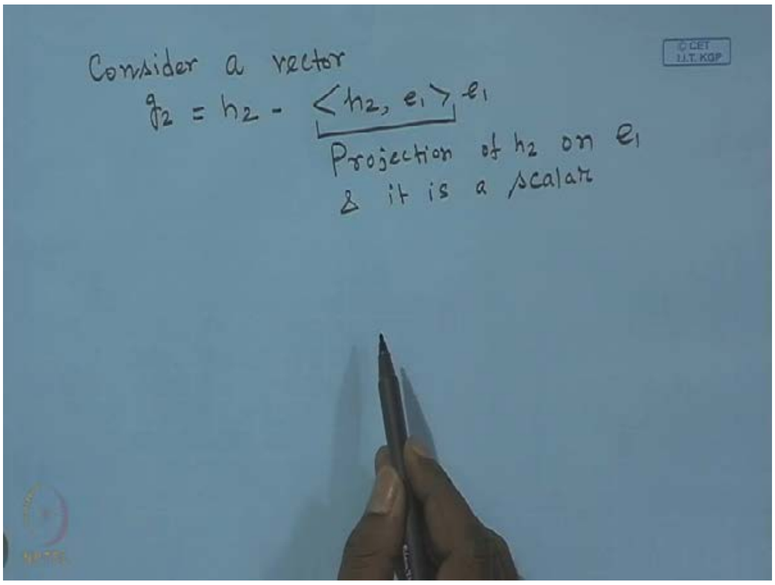

Next, we take consider a vector g 2, such that g 2 is equal to h 2 minus inner product of h 2 and e 1 multiplied by e 1. So inner product is a scalar, so it is basically we are formulating a vector so which is nothing but the vector h 2 minus some scalar multiplier of the vector e 1 that whatever we have derived just now. So, what is the inner product of h 2 an e 1 physically signifies, it is nothing but a projection of h 2 on e 1 and it is a scalar; so, this is a scalar multiplier.

So, by doing a mathematical operation on this will be going to the next step that will be a stepping stone for formulation of the next vector e 2 of our target set, which will constitute the set of orthogonal an orthonormal basis set. So, will proceed in the next class to complete this formulation, thank you very much.