Advanced Mathematical Techniques in Chemical Engineering

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Lecture No. # 39

Integral Method

Welcome to the session of this class. We will be starting with the integral method of solution for solving the partial differential equation. In the last lecture, we have looked into how similarity method of solution can be applied for the solution of partial differential equation.

In this class, we will be looking into the integral method of analysis. Integral method of analysis is quite common, when we are talking about the boundary layer analysis or boundary layer theory.

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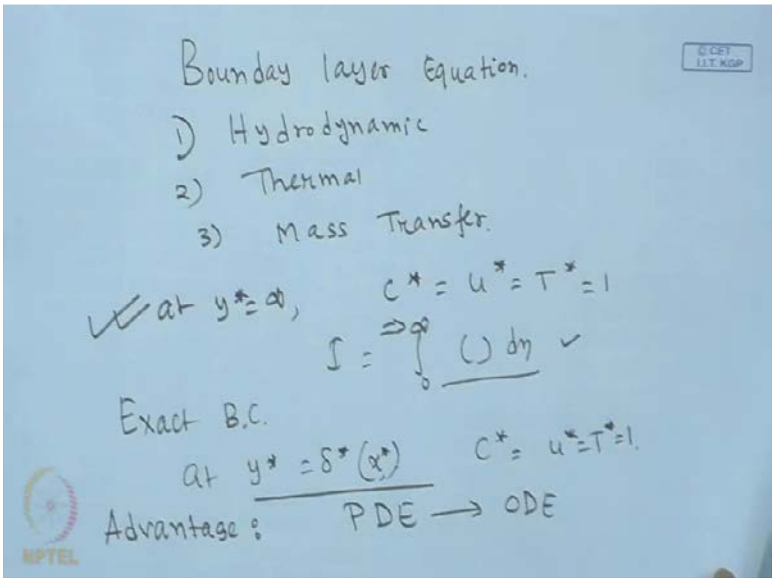

For solution of boundary layer equation, one can use the integral method of solution. In case of three types of boundary layer; hydrodynamic boundary layer, thermal boundary layer and mass transfer boundary layer; one can utilize the integral method of solution. The only difference between this solution and the similarity solution is that, if you remember at y star is equal to infinity that was one of the typical boundary condition in similarity solution, where we use to put c star is equal to u star is equal to T star is equal to 1. When it, we evaluate the definite integral from 0 to infinity in similarity parameter, we said that we have to carry out this thing numerically but infinity becomes unknown. We have to put an upper limit let us say 5 or 10 then you have to increase the upper limit from 10 to 15 and see whether the result of this integral changes or not. If it does not change over through four decimal places for some value then that will be given by the infinity.

That becomes the problem. In case of integral method analysis, we are substituting this by the exact boundary condition. What is the exact boundary condition? At y star is equal to del star c star is equal to u star is equal to T star is equal to 1. In effect, this del star becomes a function of independent variable; either x star or T in case of transient analysis.

Therefore, this is the only difference between the similarity solution and integral method solution. In case of integral method solution, we are replacing the boundary condition instead of y star equal to infinity; we are replacing it by the exact boundary condition at delta star.

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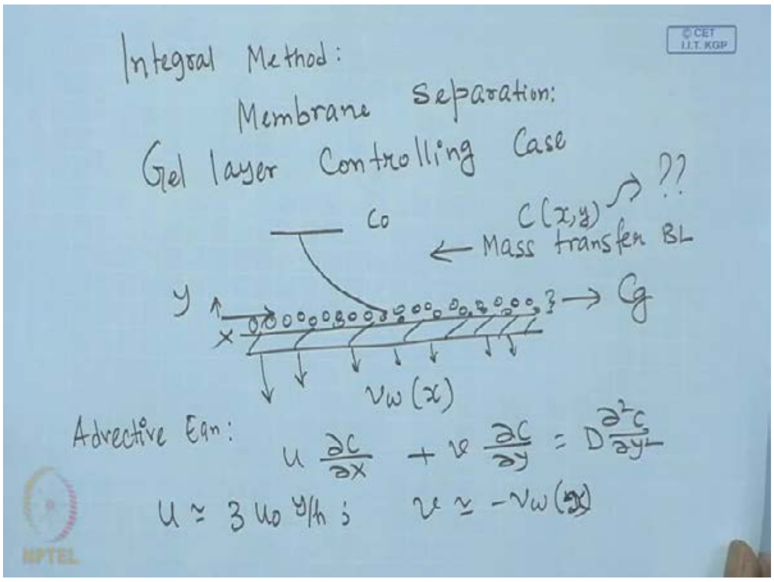

What is the advantage? The advantage is that we will reduce the partial differential equation into an ordinary differential equation that is the advantage of similarity solution.

We will take up an example of the similarity solution, the integral method of solution. The example will be talking about the same problem on membrane based separation process as we have defined earlier; this is a flow through a thin channel. We will be considering a flow through a thin channel and it is a gel layer controlling case.

Here, there is a flow through a thin channel; on this is a membrane surface and these are pores on it. We put lots of you know particles, they form a viscous layer known as the gel layer and over that there is a concentration profile, this is at y is equal to delta.

So, y starts from here and x star from there. The concentration within the gel layer is constant and it is gel layer concentration. Since, the bulk concentration varies from c naught to Cg; there exists a concentration profile within the concentration boundary layer or this is the mass transfer boundary layer.

Within the mass transfer boundary layer C is a function of x and y. You will be getting the permeate flux which will be a function of x. Therefore, our aim is to find out what is the concentration profile C as a function of x and y in the mass transfer boundary layer; with that we define that aim. We write down the governing equation within the mass transfer boundary layer.

The advective two dimensional equation becomes u del c del x plus v del c del y is equal to D del square c del y square. The same that we have done earlier, u becomes 3 u 0 y by h that is the velocity profile that we have seen earlier in the thin channel and v is equal to minus v w y, v w as a function of x.

We have discussed about this profile of velocity in the earlier class. I am just taking them as they are. We are going to solve this equation by making it non-dimensional.

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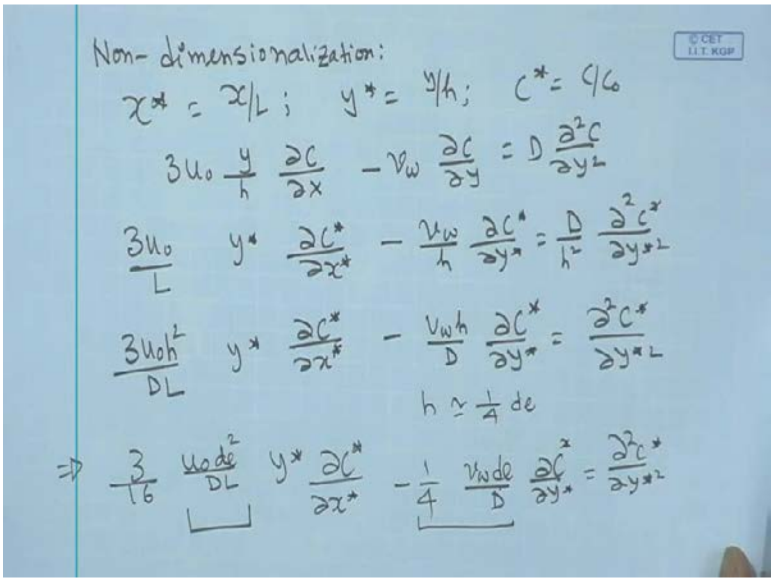

Let us make the equation non-dimensional first. We make the equation non-dimensional against these quantities: x star is equal to x by L, y star is equal to y by h, c star is equal to c by c naught. This become 3 u 0 y by h, we write down the, you put the velocity profile del c del x minus v w del c del y is equal to D del square c del y square.

Then, we substitute the non-dimensional in terms of non-dimensional variable. This becomes 3 u 0 over L y star del c star del x star minus V w over h del c star del y star is equal to D h square del square c star del y star square; multiply both sides by h square by D. This become 3 u 0 h square over D L y star del c star del x star minus V w h over D del c star del y star is equal to del square c star del y star square.

We have already seen that half height can be approximated as one-fourth of equivalent diameter. Therefore, in terms of equivalent diameter we will be having 3 by 16 u 0 D square by D L y star del c star del x star minus 1 over 4 V w d e by D del c star del y star is equal to del square c star del y star square.

We have already seen that u 0 d square d by D L is nothing but 3 by Reynold Schmith D by L and 1 by 4 V w d e by D is nothing but the non-dimensional (( )) number at the wall.

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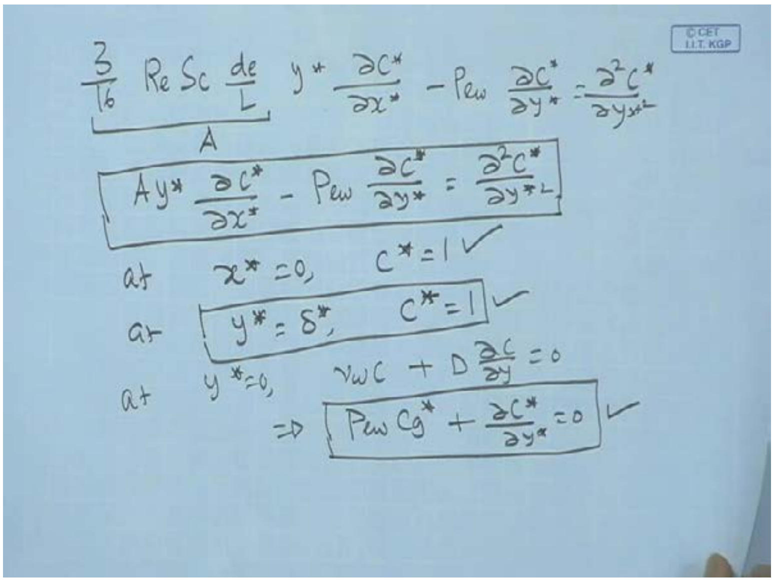

We write it down as 3 by 16 Reynold Schmith d e by L y star del c star del x star minus P e w del c star del y star is equal to del square c star del y star square. So, once we get that this is basically nothing but a constant; it depends on the operating condition. Let us write it down as A. This equation becomes A y star del c star del x star minus P e w del c star del y star is equal to del square c star del y star square. This becomes the governing equation. We have already derived this governing equation earlier.

Let us put the initial and boundary condition; at x star is equal to 0 we had c star is equal to 1, at y is equal to 0 y is equal to delta, c star is equal to c naught, c is equal to c naught, c star is equal 1 at y star equal to delta star. Let us put, at y star is equal to 0 V w c plus D del c del y is equal to 0. That was the you know boundary condition at y equal to 0.

Therefore, we make it non-dimensional; P ew c g star c at y star is equal to 0 is c g star, that is constant plus del c star del y star is equal to 0. So, these three, this is an initial the condition at x star is equal to 0; this is the condition at y star is equal to del star and this is the condition at y star is equal to 0.

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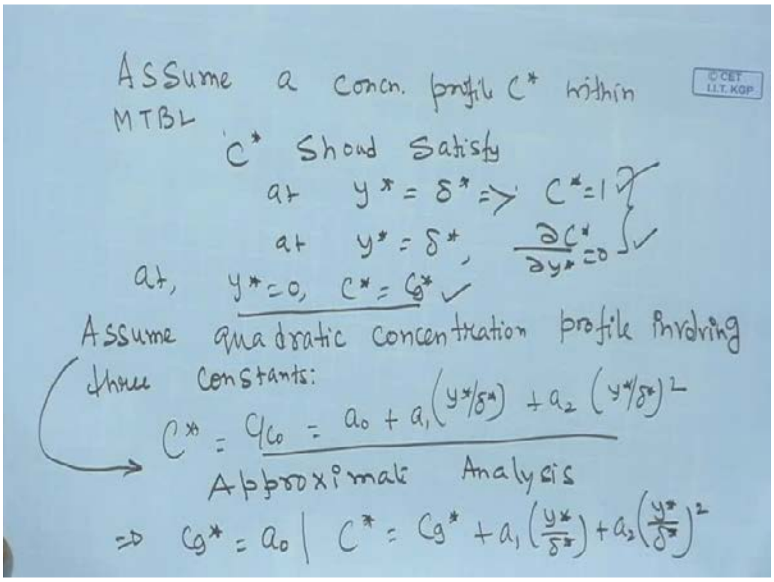

So, with this we assume a concentration profile of c star within the mass transfer boundary layer. Next, what we do is we assume a concentration profile within the mass transfer boundary layer; profile c star within mass transfer boundary layer. The necessary condition of the mass transfer boundary is that the concentration profile c star should satisfy or at y star is equal to delta star, c star is equal to 1; at y star is equal to delta star, del c star del y star is equal to 0, these are the boundary layer conditions. Any boundary layer must be satisfying these two conditions. For a thermal boundary layer y star equal to delta star, T star should have been one and del T star del y star should have been 0.

Also, we have a boundary condition that y star is equal to 0; c star is equal to c g star. Therefore, there are three boundary conditions that the concentration profile must satisfy. So, at least you should have, we should assume a quadratic concentration profile involving three constants. We assume a concentration profile c star is equal to c by c naught is equal to a 0 plus a 1 y star by del star plus a 2 y star by del star square. Since, this concentration profile is assumed within the mass transfer boundary layer, we call the integral method as approximate analysis.

We apply these three boundary conditions and see what we get. At y star is equal to 0; c star is equal to c g star. Therefore, c g star is nothing but a 0; the last two terms will varnish. Therefore, the concentration profile looks something like this, c star is equal to c g star plus a 1 y star by del star plus a 2 y star by del star square.

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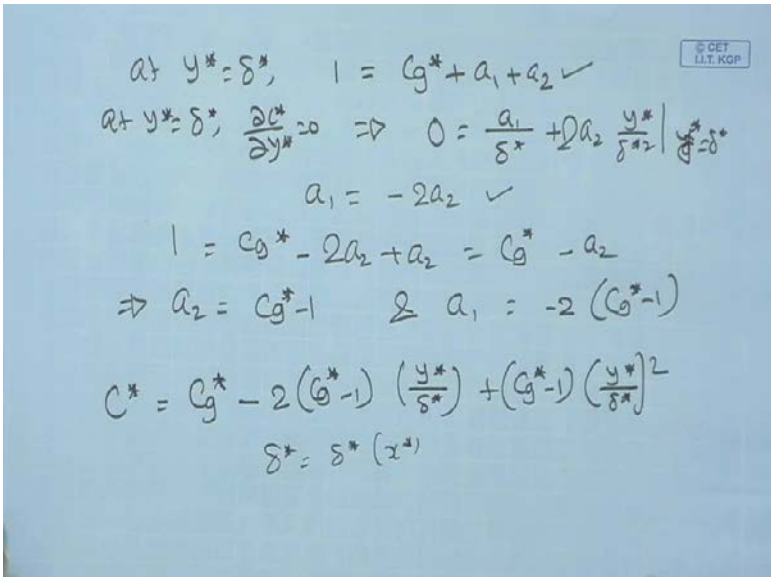

That goes the concentration profile. We have two constants to be evaluated, a 1 and a 2. Let us utilize the two boundary conditions that the concentration profile must satisfy. At y star is equal delta star, we had c star is equal to 1. So, this, therefore, 1 is equal to c g star plus a 1 plus a 2 and at y star is equal to delta star, del c star del y star is equal to 0. We have 0 is equal to a 1 by delta star plus 2 a 2 y star by del star square evaluated at y star is equal to delta star.

One delta star will be cancelling over here and one will be cancelling from there in the denominator. So, this becomes a 1 becomes minus 2 a 2. We will combine these 2 boundary conditions and get a 1 and a 2. We will be getting 1 is equal to c g star minus 2 a 2 plus a 2 is equal to c g star minus a 2. Therefore, a 2 becomes c g star minus 1 and a 1 becomes minus 2 c g star minus 1, we get the concentration profile c star as c g star minus 2 c g star minus 1 y star by del star plus c g star minus 1 y star by del star square.

That gives the concentration profile. In this concentration profile everything is known, except how del star varies as a function of x star, that is left behind. By using the integral method of solution we are exactly going to do that.

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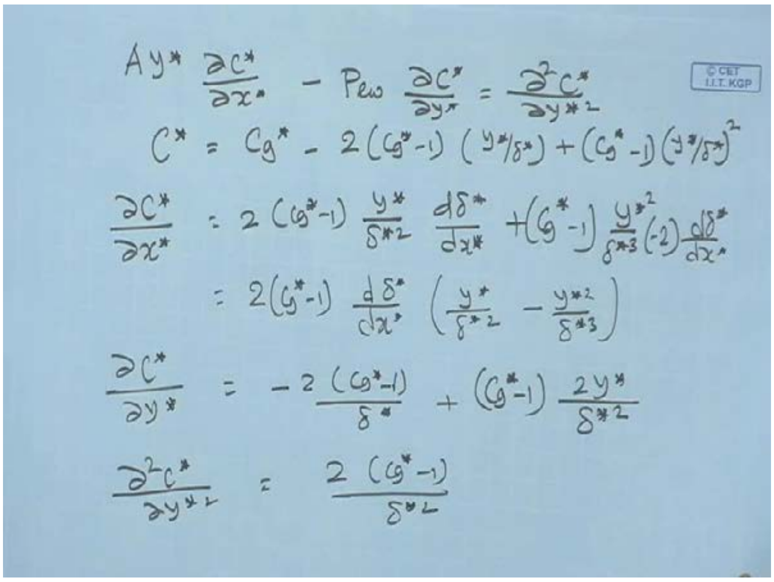

Let us look into the governing equation. The governing equation becomes A y star del c star del x star minus P ew del c star del y star is equal to del square c star del y star square. We have the concentration profile as c star is equal to c g star minus 2 c g star minus 1 y star by del star plus c g star minus 1 y star by del star square of that.

Sodel c star what do we do next? Exactly, like the earlier problem we evaluate different derivatives that is del x star of c star, del y star of c star, del square del y star square of c star from this governing, from this profile and substitute back to the governing equation; del c star del x star is nothing but 2 c g star minus 1 y star del star square, it will be minus del star square, so minus into minus, plus d del star d x star plus c g star minus 1, similarly, y star square by del star cube minus 2 d del star d x star. It takes 2 c g star minus 1 d del star d x star common, this becomes y star by del star square minus y star square by del star cube; that goes for del c star del x star.

Then we use derive, we take the derivative, del c star del y star is equal to minus 2 c g star minus 1 over delta star plus c g star minus 1 2 y star by del star square and del square c star del y star square will be 2 c g star minus 1 over del star square. We take the derivative of this equation; this becomes 2 c g star minus 1 del star square.

We will write down the governing equation. We just substitute everything there and see what we get.

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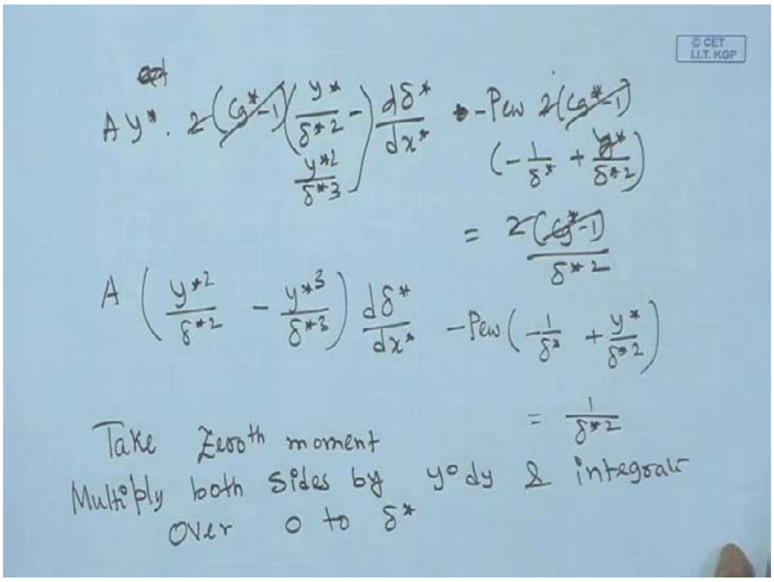

Before that we utilize the governing equation. At A, we substitute the derivative A y star, del c star del x star is nothing but 2 c g star minus 1 y star by del star square d del star d x star plus P ew and del c star del y star minus P ew del c star del y star is we take 2 c g star minus 1 common, 2 c g star minus 1 common, this becomes minus 1 by del star plus y star by del star square is equal to del square c del y star. This becomes 2 c g star minus 1 over del star square. We cancel both side by 2 c g star minus 1. It goes off and this becomes A y star square.

This will be, d del star d x star y star del star square minus, we had one more term, y star square by delta star cube, this will be in common. So, A What we will be getting is A y star square del star square minus y star cube del star cube d delta star d x star minus P ew minus 1 by delta star plus y star by delta star square is equal to 1 over delta star square.

What we do is we take zeroth moment of this equation, that means, multiply both sides by y to the power 0 d y and integrate over 0 to del star, that is the boundary layer thickness.

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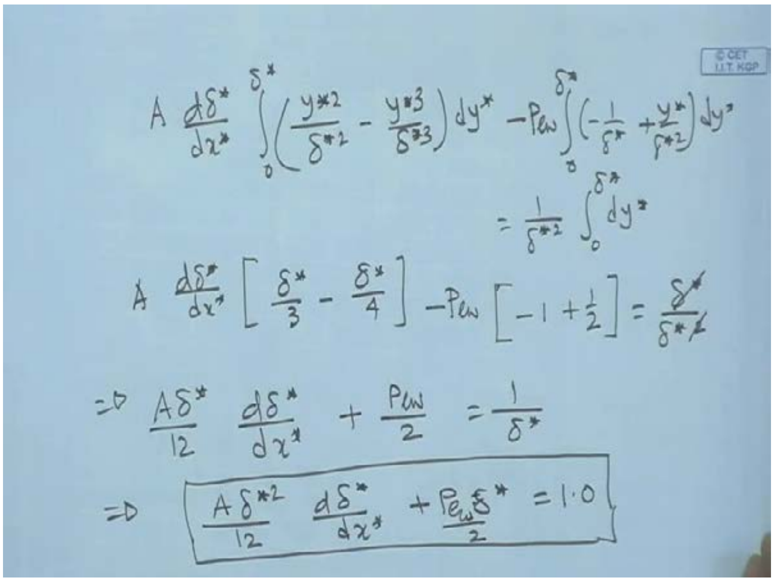

We multiply both sides by y to the power 0 d y. A in d delta star d x star, this is a sole function of x, it will be taken out of the integral sign 0 to del star y star square divided by del star square minus y star cube divided by del star cube d y star minus P ew, that is a function of x only, it will be minus 1 by del star plus y star by del star square 0 to del star d y star is equal to 1 by del star square d y star 0 to del star.

We carry out this integration, this becomes A d del star d x star and this will be delta star cube divided by 3, it will be delta star by 3 minus delta star by 4, after integration minus P ew, after integration it will be minus 1 delta star, delta star will be cancelled out, it will be plus y square by 2, it will be half but delta star square will be cancelled out and you will be having a delta star over here in the numerator, you will be having delta star square in the denominator, one will be cancelled out. This becomes A delta and this will be delta star by 12 A delta star by 12 d delta star d x star, this will be minus half so minus minus plus P ew by 2 is equal to 1 by del star. You will be having A delta star square by 12 d delta star d x star plus 1 plus P ew delta star by 2 is equal to 1; this becomes the governing equation.

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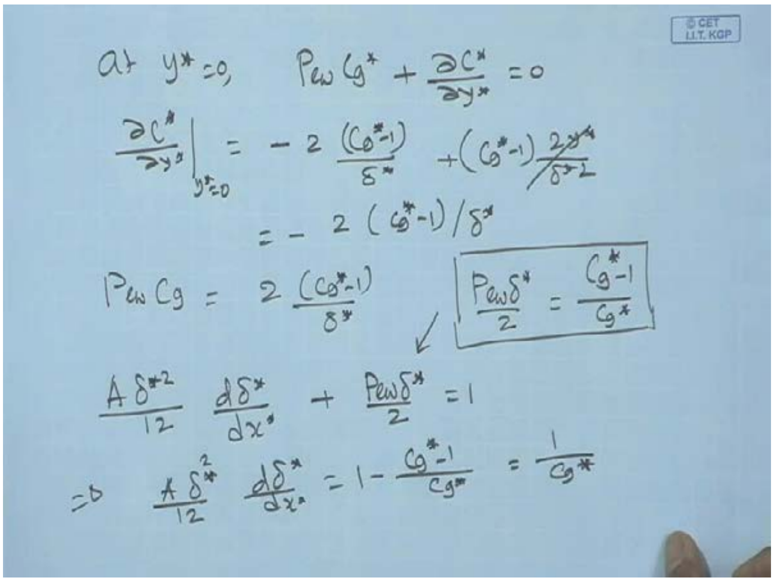

P ew is a function of x and delta star will also be the function of x. This equation can still be simplified by the boundary condition at x, at y is equal to 0. If we look into the boundary condition at y is equal to 0; at y star is equal 0 P ew c g star plus del c star del y star is equal to 0.

If you look into the derivative of del c star del y star, it becomes minus 2 c g star minus 1 over del star plus c g star minus 1 2 y star by del star square. Evaluated at y star is equal to 0, this term will go and you will be simply having minus 2 c g star minus 1 over delta star.

So, P ew so substituted over here, so P ew c g is equal to 2 c g star minus 1 over delta star. We will be having P ew delta star by 2 is nothing but c g star minus 1 divided by c g star.

Therefore, we substitute this in the governing equation; A del star square by 12 d del star d x star plus P ew del star by 2 is equal to 1.

You substitute this here, what we get is A del square 12 d del star d x star is equal to 1 minus c g star minus 1 divided by c g star. So, it will be getting c will be cancelled out in the numerator; it will be simply c g star.

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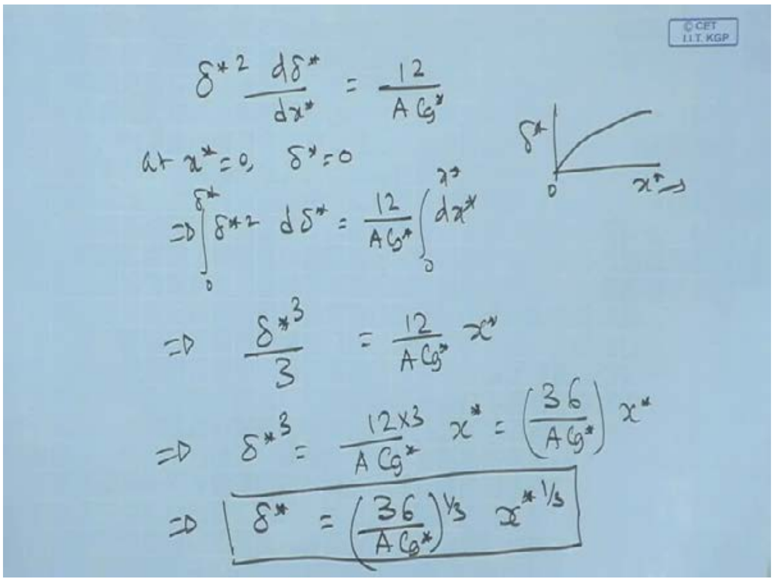

The governing equation of del star square will be nothing but del star square d del star d x star is equal to 12 A c g star. You carry out the integration and integration will be from 0 to let us say, x star in this and delta star will be here. At x is equal to 0 my delta star is equal to 0; with this we evaluate the boundary condition. We will be getting that delta square d delta star is equal to 12 by A c g star d x star.

Once we get that we carry out this integration from 0 to del star from 0 to x star delta star cube by 3 is equal to 12 A c g star x star; delta star now becomes 12 by 12 into 3 A c g star x star; this becomes 36 A c g star x star and this is delta star cube. Therefore, delta star becomes 36 by A c g star to the power 1 upon 3 and x star to the power 1 power upon 3.

If you now recall that this is the functional form that how del star is varying as a function of x star in case of similarity solution. In case of some a similarity solution delta star was a function of x star and the functional variation was x star to the power 1 upon 3. In integral method of solution also, we have found out that delta star is varying as a function of x star to the power of 1 upon 3 with the coefficient 36 A c g star rest to the power 1 upon 3.

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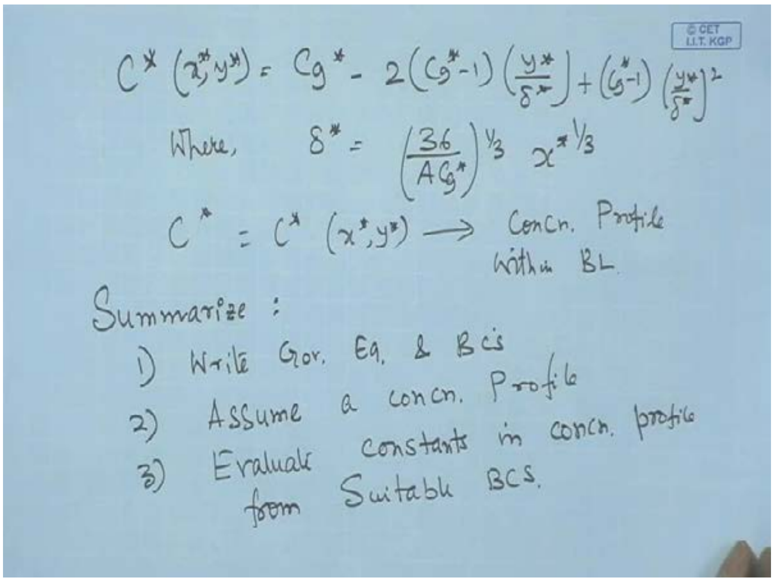

That completes the solution. If you look into the concentration profile which is a function of x star and y star; this becomes c g star minus 2 c g star minus 1 y star by del star plus c g star minus 1 y star by del star square.

Where, my del star can be explicitly written as 36 divided by A c g star rest to the power 1 upon 3 x star to the power 1 upon 3. We substitute delta star as a function of x star over here, you will be getting completely c star as a function of x star and y star. We get the concentration profile within the boundary layer.

This example clearly demonstrates that how integral method of solution can be applied to boundary layer analysis in chemical engineering applications. Let us summarize the different steps. First, you write down the governing equation and boundary conditions. Next, we assume a concentration profile that is why it is known as approximate integral method. Once you assume a concentration profile then third step is to evaluate constants in concentration profile from suitable boundary conditions.

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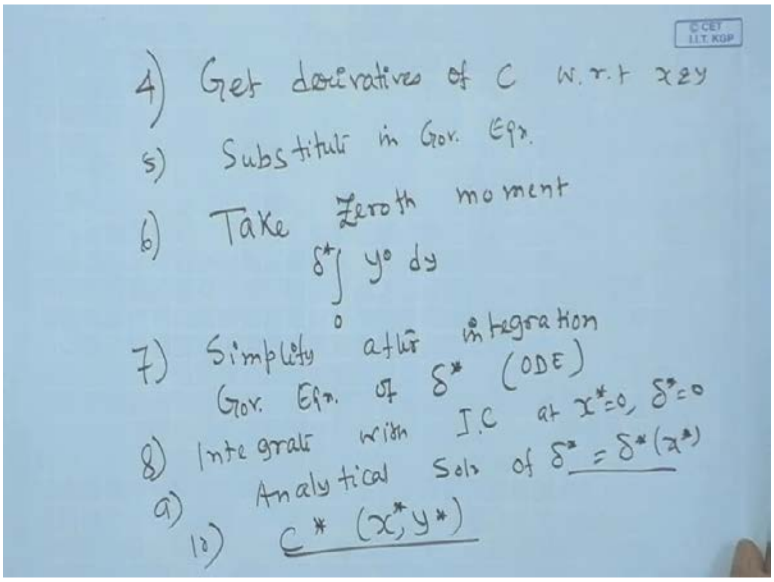

Among the suitable boundary condition, the boundary layer conditions must be satisfied that at y star is equal to delta star, c star is equal to 1 and del c star del y star is equal to 0. Corresponding to velocity on thermal boundary layer you will be having the corresponding conditions, get the derivatives of concentration with respect to x and y; substitute them in the governing equation; after that take the zeroth moment, that means, you multiply both sides by y to the power 0 d y and then integrate across the boundary layer thickness 0 to delta.

Simplify after integration in y, what you will be getting? You will be getting the governing equation of delta star which is nothing but an ordinary differential equation, then integrate it out with initial condition at x star is equal to 0 del star is equal to 0, then what we will be getting is that you will be getting an analytical solution of delta star as a function of x star. The final step is that once you know the del star as a function of x star analytically, then you substitute that in the concentration profile and from the concentration profile you will be getting concentration within the mass transfer boundary layer as a function of x star and y star. In fact, this is the similar type of approach which should be taken for calculation of temperature profile within the thermal boundary layer and velocity profile within the hydrodynamic boundary layer. We can successfully utilize the integral method of solution for the solution of boundary analysis in chemical engineering problem.

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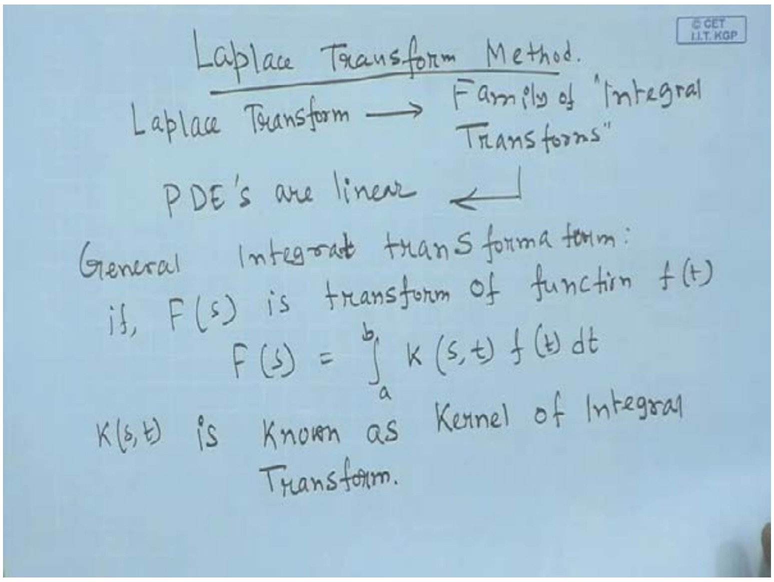

We complete the integral method of solution. Next, we will look into the Laplace transform method. This method belongs to the family of integral transform, Laplace transform belongs to family of integral transform.

Integral transforms can be applied if and only if your governing partial differential equation or ordinary differential equations are linear for linear governing equations only, the integral method of solution can be, can be integral method of integral transform methods can be applicable.

General integral transformation can be written in this form, F s is the transform of function f t, then F s can be written as a to b k of s t f of t d t this k of s t is known as Kernel of integral transform.

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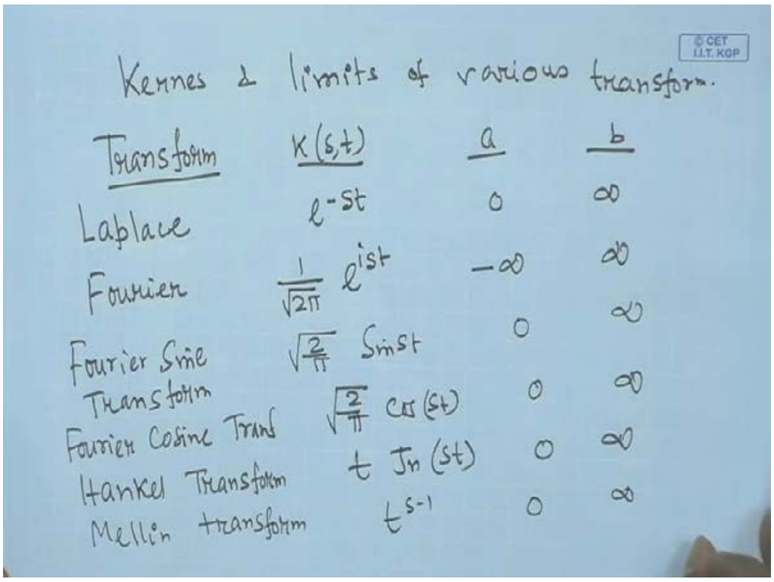

And this kernels and limits of various transforms are given below. Now we just put transform in a tabular form then k of s t that is the Kernel lower limit a the upper limit b. So, if it is a Laplace transform then Kernel is e to the power minus s t this from 0 to infinity. Fourier is 1 over root over 2 pi e to the power i s t minus infinity to plus infinity, then Fourier trans sin transform, this is becomes root over 2 over pi sin s t 0 to infinity. Fourier cosine transform, this becomes root over 2 over pi cosine s t 0 to infinity. Hankel transform, t nth order Bessel function s t 0 to infinity. Mellin transform, t to the power s minus 1 0 to infinity. So, these are the various kernels lower limit and upper limit of the of the transform for various types of transform.

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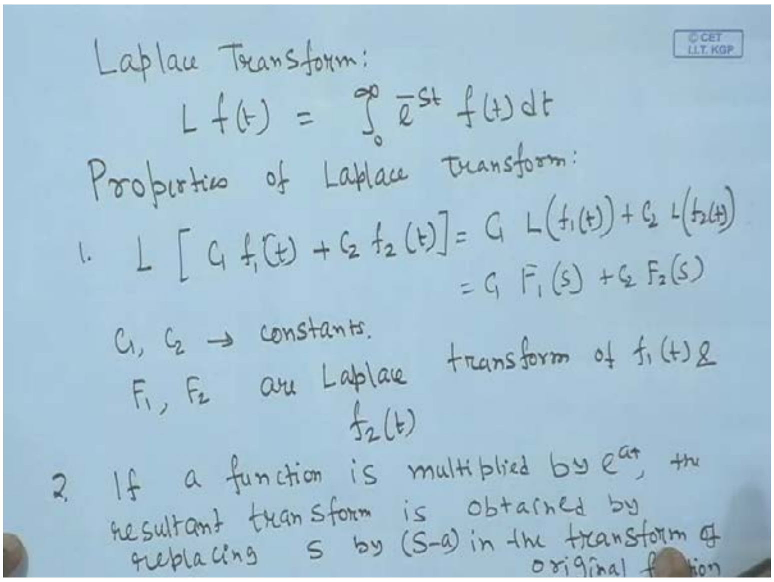

And then let us concentrated on Laplace transform and how this transform can be utilized for the solution of linear partial differential equation. L is called the Laplace operator L of f t is nothing but 0 to infinity e to the power minus s t f t d t so e to the power of minus s t is the kernel. Now, let us look into the various properties of Laplace transform the first property is Laplace of c 1 f 1 t plus c 2 f 2 t should be is equal to c 1 L f 1 t plus c 2 L f 2 t. So, this is nothing but c 1 F 1 s plus c 2 F 2 s, where c 1 c 2 are constants F 1 F 2 which are the function of s are Laplace transform of f 1 t and f 2 t.

The second property is that, if a function f of t is multiplied by e to the power a t, the resultant transform is obtained by replacing s by s minus a in the transform of original function. That is the second property, it should satisfy. Therefore, we just write it down in a neat form compact form.

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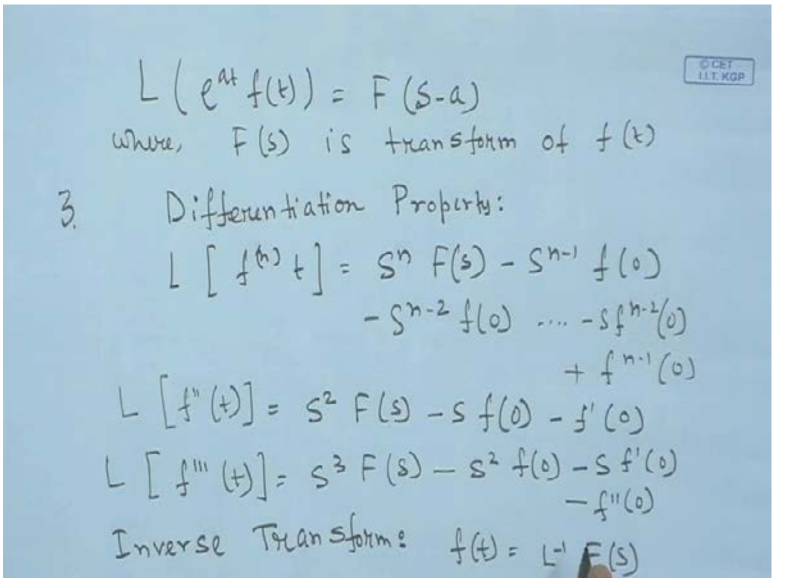

L of e to the power a t f of t should be is equal to F of s minus a, that means, if F of s is the Laplace transform or f t, then Laplace transform of e to the power a t of f t is nothing but I just replace s by s minus a in the transform, where, so in this equation where F s is transform of f t.

Third important property is the differentiation property Laplace of f of n t n means nth order of differentiation is nothing but, s n s to the power n F s minus s to the power n minus 1 f of 0 minus s to the power n minus 2 f of 0 minus s f to the power n minus 2 0 plus f n minus 1 0. So, Laplace transform of f double prime t is nothing but s square F of s minus s f evaluated at t is equal to 0 minus del f del t evaluated at t is equal to 0. Laplace of f triple prime t is nothing but s cube of f s minus s square of f of 0 minus s f prime at 0 minus f double prime evaluated at 0.

Next, we get the once we get the transform, then we get an inverse transform what is inverse transform f of t? Can be obtained by taking the Laplace inverse of F s.

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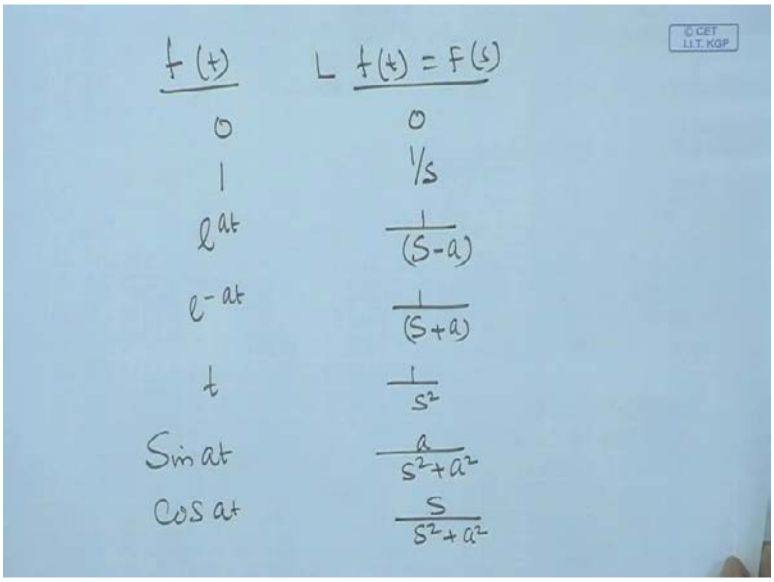

So, we can get from s to t domain; now, let us get some of the f of t and what are the corresponding Laplace transform in s domain? So, for 0 this is 0, for 1 this is 1 over s, for e to the power a t this will be 1 over s minus a, for e to the power minus a t this will be 1 over s plus a, for t it will be nothing but 1 over s square, for sin a t this will be a divided by s square plus a square, for cosine a t this will be s divided by s square plus a square.

So, these are some of the typical you know Laplace transform of some simple function in s domain from t domain. So, these functions are quite common, in fact in the book of Carsick or any such you know fundamental books, which are applicable for first year all courses, under graduate courses. There are (( )) of some functions are available in the tabular form where the Laplace transform are given.

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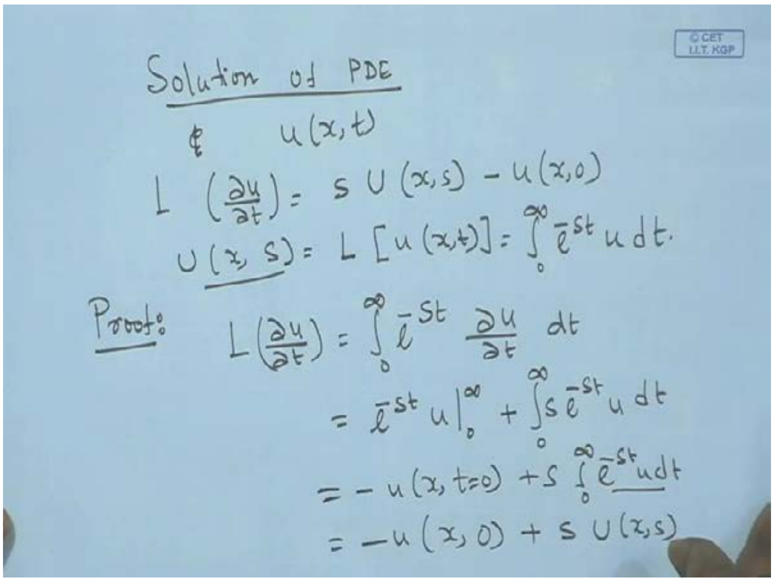

Now, let us with this background of Laplace transform, let us see, how this Laplace transform can be utilized for the solution of partial differential equation. Now, Laplace of if u is function of x and t then Laplace of del u del t is written as s U x and s minus u x evaluated at time t is equal to 0, where capital u is basically the transform function where we have transform t into s domain. So, U x s is nothing but Laplace of U x t, so this will be nothing but 0 to infinity e to the power minus s t u d t.

So, we can proof this thing and the proof goes like this Laplace of del u del t is nothing but 0 to infinity e to the power minus s t del u del t d t. So, this becomes e to the power minus s t u zero to infinity, first function, this is the first function integral of second function minus differentiation of the first function 0 to infinity, minus into minus, plus e to the power minus s t integration of the second function u d t.

So, I take e to the power minus infinity is 0 and at time t is equal to 0, so this becomes minus, so first term will be 0, minus e to the power 0 is 1, so u x at t is equal to 0 plus s 0 to infinity e to the power minus s t u d t, so what is this, this is nothing but capital U as a function of x and s. So, this will be nothing but minus u x t is equal to 0 plus s U of x and s so that is the Laplace of del u del t

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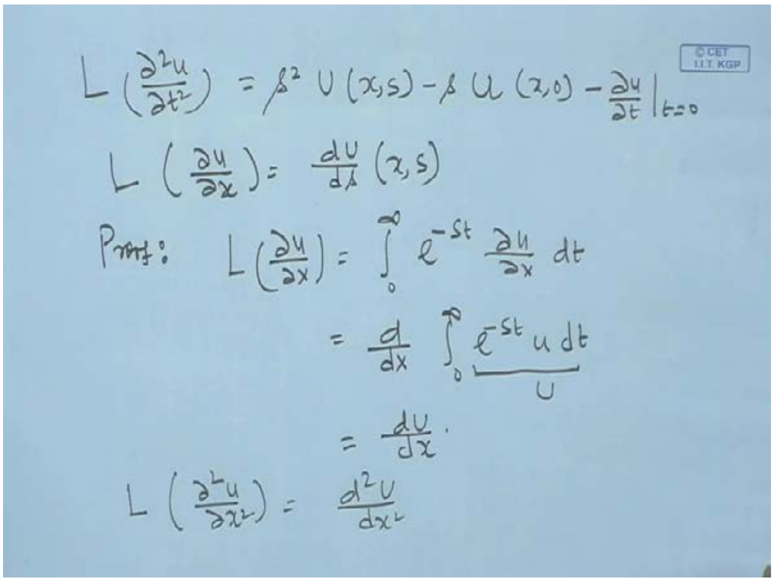

Similarly, one can proof that Laplace of del square u del t square is same as s square U x s minus s u x at 0 minus del u del t at t is equal to 0. Similarly, one can proof Laplace of del u del x is nothing but d U d s, where x this is a function of u is a function of x and s. So, this we can proof Laplace of del u del x is nothing but 0 to infinity e to the power minus s t del u del x d t, so this integral is over t so we can take del u del x over we can take d d x outside.

So, outside it becomes d d x 0 to infinity e to the power minus s t u d t, so what is this this will be nothing but capital U, the Laplace transform of f of t of small u therefore this is nothing but d u d x.

Similarly, one can prove that Laplace of del square u del x square is nothing but d square u d x square, so with this you can get different derivative with respect the Laplace transform of various derivatives are presented in the s domain from t domain.

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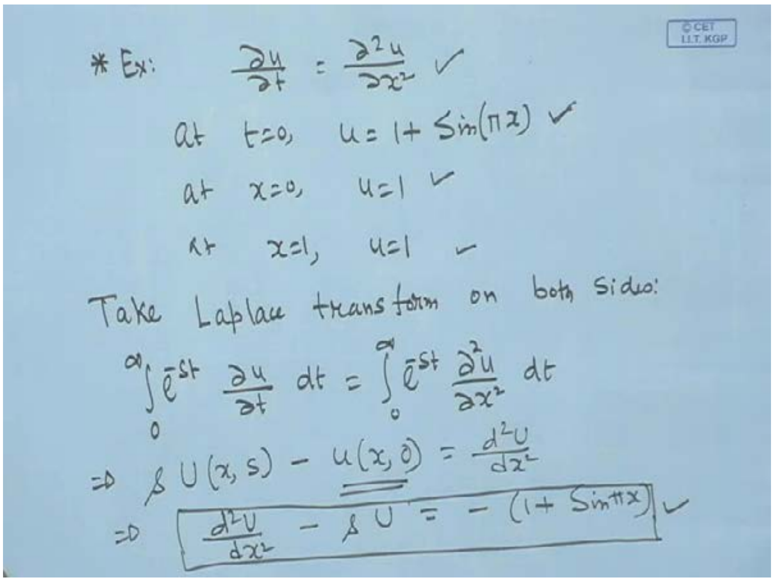

With this, we will be in a position to formulate the first example to take up the first example that is del u del t is equal to del square u del x square at t is equal to 0 u is equal 1 plus sin pi x and at x is equal to 0, we have u is equal to 1 at x is equal to 1, we have u is equal to 1. See this governing equation is a linear and homogeneous, we have a non-homogeneous term as an initial conditions and both the boundary conditions are non-homogeneous.

So, therefore we are going to solve this equation using Laplace transform. So, what we do? We take Laplace transform on both sides, if we do that it will be multiplied by e to the power minus s t and integrate from 0 to infinity del u del t d t from 0 to infinity e to the power minus s t del square u del x square d t.

So, this becomes s U of x and s minus u x at time t is equal to 0 is equal to d square u d x square, we have already proved that and what is u at time t is equal to 0 that is the initial condition. So, therefore we will be getting the governing equation of d square u d x square minus s u is equal to minus 1 plus sin pi x.

So, what is essentially the message is that by using this the integral transform, we are able to get down the partial differential equation into an ordinary differential equation. If you look into the similarity transformation integral method, as well as the Laplace transform, we have seen that in all the cases the partial differential equation has boil down into an ordinary differential equation and the solution of ordinary differential equations are quiet simpler.

So, we take up this problem in the next class and solve this problem completely. So, I stop here, in this class; next, we will talk about more detail the solution of this problem or then I will be taking up couple of more examples of how to of application of Laplace transform for solution of partial differential equations, which will be more common in our transient chemical engineering processes, thank you very much.