Advanced Mathematical Techniques in Chemical Engineering

Prof. S. De

Department of Chemical Engineering

Indian Institute of Technology, Kharagpur

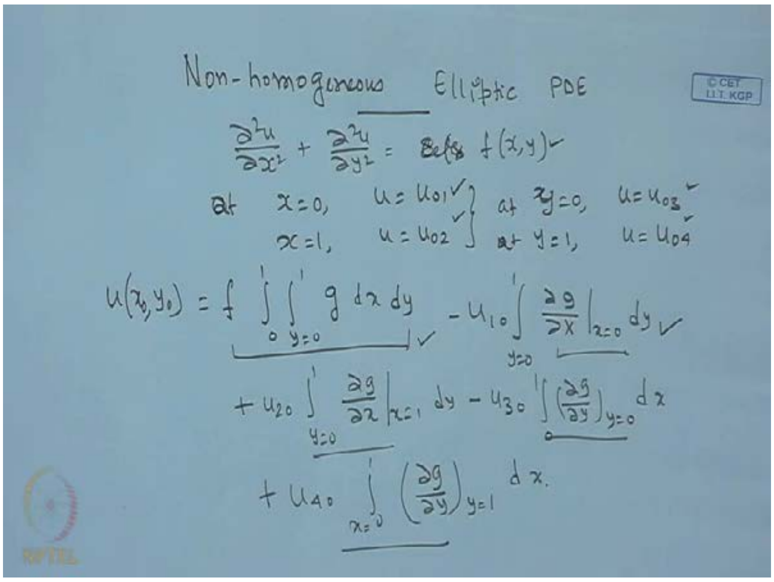
Lecture No. # 37

Similarity Solution

We will be carrying forward the problem that we had left in the earlier class. We are looking into the solution of non-homogeneous partial differential equation using Green's function method.

We are looking into an elliptical partial differential equation. We came up to a solution almost at the last step and only few steps are left. We will be solving that problem first completely and then will move forward to other problems.

(Refer Slide Time: 00:48)



Solution of non-homogeneous elliptic PDE. We considered a problem something write like this del square u del x square, plus del square u del y square is equal to delta, x is equal to a some function of x and y, at x is equal to 0; we had u is equal to u 0 1, at x is equal to 1; we had u is equal to u 0 2, at x at y is equal to 0; we had u is equal to u 0 3 and at y is equal to 1; we had u is equal to u 0 4.

We had this equation. There were five sources of non-homogeneity. We first formulated and constructed the causal Green's function. We evaluated the causal Green's function using complete eigenfunction expansion method and then we looked into the this we evaluated the eigenvalues and eigenfunctions.

We had two independent eigenvalue problem in this whole equation that was in x direction as well as in y direction.

What we did is we proofed later on that Laplacian operator; that grad square operator is a self-adjoint operator.

So therefore Green's and Since, all the boundary conditions are homogeneous, the adjoint Green's functions must be having the homogeneous boundary condition in order to have the bi-linear concomitant term vanish.

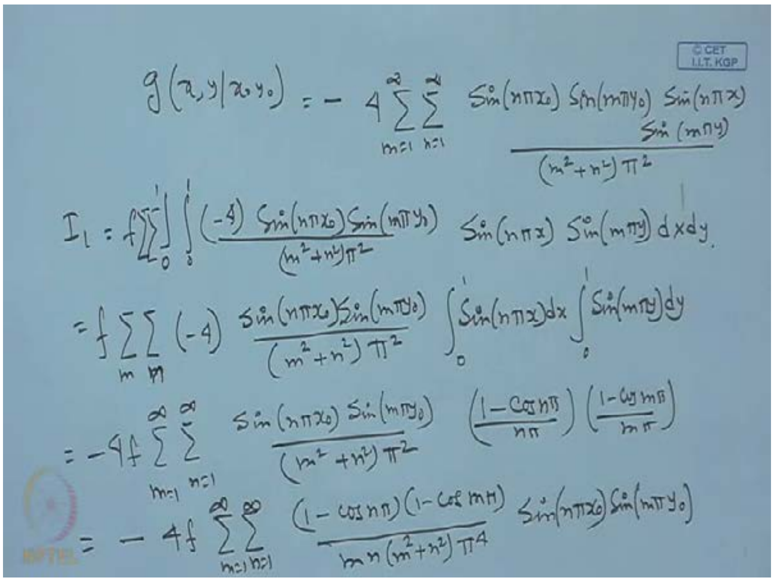
Therefore, the operator is a self-adjoint operator and we have a self-adjoint system in this particular case. The operator is self-adjoint and the boundary operator is also self-adjoint so we had a completely self-adjoint problem.

We need not go for an expression of g star and we need not connect g star with u, since g is equal to g star because L is equal to L star.

Therefore, we can directly hook up the actual equation; governing equation u with that of governing equation of g. We really did that and finally came to a point where we had the solution in this form. The solution becomes u x naught y naught is equal to f 0 to 1, y is equal to 0 to 1 g d x d y, that was the boundary condition and then we had the other terms.

These terms come because of the non-homogeneous term in the governing equation. minus u 1 0 y is equal to 0 to 1, del g del x at x is equal to 0 d y. We add one more term u 2 0 integral y is equal to 0 to 1, del g del x at x is equal to 1 d y. We add two more terms corresponding the two non-homogeneities in the y boundary - u 3 0 integral 0 to 1 over x, del g del y at y is equal to 0 times d x. Non-homogeneous boundary condition at x is equal to at y is equal to 0; u 4 0 0 to 1 over x will be del g del y at y is equal to 1 times d x.

(Refer Slide Time: 05:49)



There are five sources of non-homogeneities in this problem. The first one comes because of the non-homogeneity in the governing equation. the second these all These four terms appears because of the non-homogeneity in the boundary conditions. Therefore, the first term is basically double integral times. It is a volumetric integration time because it is it is applicable throughout the whole volume and rests of the four terms are basically the surface integral time.

They represent the boundary conditions because of the non-homogeneities in the existing on the boundary. Let us solve this problem completely.

We will look into the function of g. The expression of g: g x y x naught y naught. If you look into the expression of g, these will be 4 summation over m equal to 1 to n; summation n is equal to 1 to infinity.

So youThis will be sin n pi x naught, sin m pi y naught, sin n pi x, sin m pi y divided by m square plus n square pi square. That is the Green’s function.

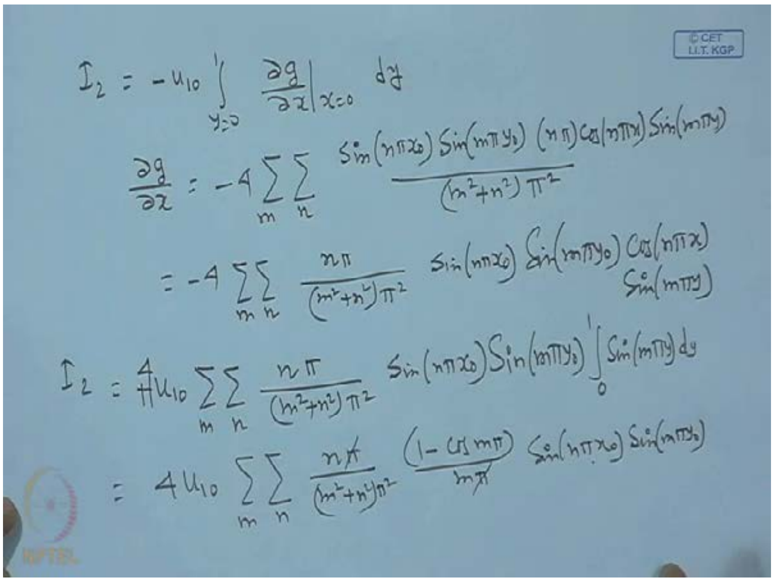
We look into these five integrals one after another. The first integral is quite complicated. f is coming out so it will be 0 to 1, 0 to 1 g g d x d y; m square plus n square pi square. We keep the summation outside and this will be sin minus 4, sin n pi x naught, sin m pi y naught, sin n pi x and sin m pi y d x d y.

Therefore, f summation over m summation over n minus 4, sin n pi x naught, sin m pi y naught m square plus n square pi square integral sin n pi x d x integral sin m pi y d y from 0 to 1 from 0 to 1. It will minus 4 f summation over m, summation over n 1 to infinity, sin n pi x naught and sin m pi y naught.

They will be treated as constant because of the integration over x and y. so this will We have already solved this one earlier. This will be cos 1 minus cos n pi over n pi, 1 minus cosin m pi divided by m pi and in denominator you have m square plus n square pi square.

That is the complete solution of the first part. Let us write it down into more you know amenable form. This will be 1 minus cosin n pi, 1 minus cosin m pi, m n m square plus n square pi to the power 4, sin n pi x naught, sin m pi y naught that is I 1.

(Refer Slide Time: 09:28)



Next, we solve for I 2. I 2 is minus u 1 0, y equal to 0 to 1, del g del x evaluated at x equal to 0 d y. What is del g del x? del g del x is nothing but minus 4 summation over m, summation over n, sin n pi x naught, sin m pi y naught del g del x. It will be n pi cosin n pi x, sin m pi y divided by m square plus n square pi square. This will be minus 4 summation over m, summation over n, n pi divided by m square plus n square pi square, sin n pi x naught, sin m pi y naught cosin n pi x, sin m pi y.

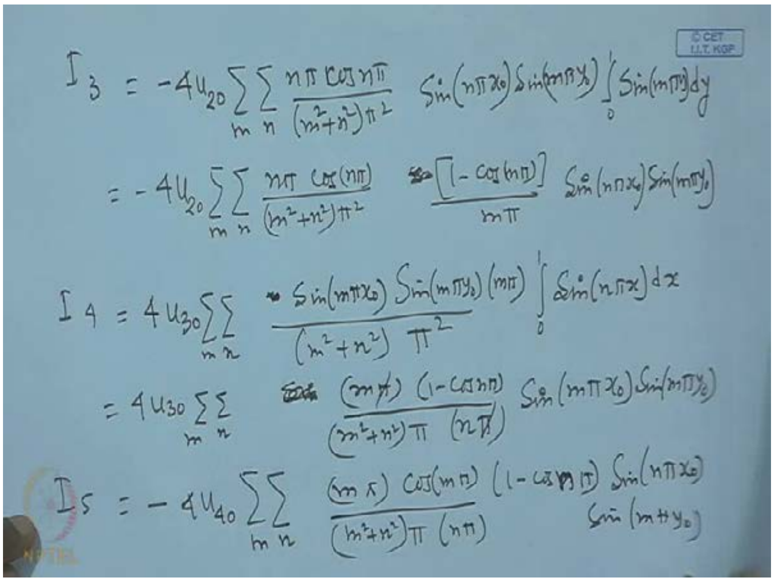
Integral of this I 2 will be minus u 1 0, minus minus plus multiplied by 4, double summation 1 over m and another over n. so if we integrate this out with respect to y.

If we integrate it out with respect to y, it will be simply integral of sin m pi y d y and del g del x is evaluated at x is equal to 0. Let me write it down. One more step m square plus n square pi square, sin n pi x naught, sin m pi y naught, cosin n pi at x is equal to 0 cos 0 will be 1.

This will be integral of sin m pi y d y from 0 to 1. This will be 4 u 1 0 summation over m, summation over n n pi m square plus n square pi square, 1 minus cosin m pi divided by m pi, sin n pi x naught, sin m pi y naught.

1 pi pi will be cancelling out. You will get n into cosin m pi divided by m into m square plus n square pi square, sin n pi x naught, sin m pi y naught.

(Refer Slide Time: 12:40)



Similarly, we can evaluate the next integral I 3. The I 3 will be del g del x evaluated at x is equal to 1. It will be u 2 0, 4 u 2 0 double summation 1 over m and another over n. so it will be 1 over and It will be evaluated at x is equal to 1, it will be cosin n pi.

cosin n pi divided by multiplied by n pi. This divided is by m square plus n square pi square, sin n pi x naught, sin m pi y naught and then integral 0 to 1 sin m pi y d y m pi y d y.

Therefore, this will be having value u to 0 m n n pi cosin n pi divided by m square plus n square pi square, 1 minus cosin m pi, divided by m pi sin n pi x naught, sin m pi y naught. That is the value expression of I 3 and we will be getting I 4.

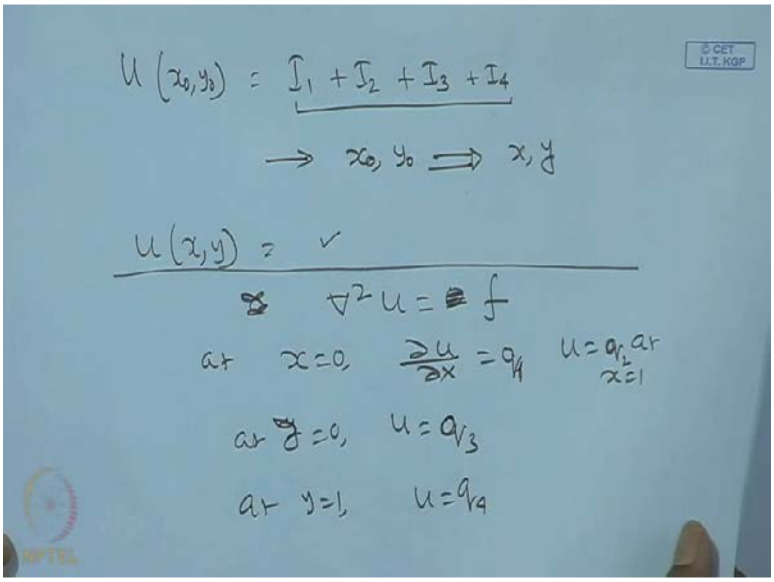
So, I 1, I 2, I 3, I 4 and I 5. It will be minus u 3 naught. minus minus plus 4 u 3 naught m n, m square 1 over sin m pi x naught, sin m pi y naught divided by m square plus n square multiplied by pi square. It is integral of del g del y evaluated at y is equal to 0 d x.

del g del y we just multiply del g del y will be nothing but m pi cosin m pi y naught evaluated at 0; that will be 1. cos 0 will be 1 and then we will be having integral 0 to 1 sin n pi x d x.

4 u 3 0 summation m, summation n sin; this will be m pi. 1 minus cosin n pi divided by m square plus n square pi square into n pi so pi pi will be canceled out. It will be sin m pi x naught, sin m pi y naught. Similarly, we will be having I 5.

I 5 will be, I guess it will be minus 4 u 4 naught summation over m, summation over n. We will be having m pi cosin m pi divided by m square plus n square pi square divided by multiplied by n pi in the denominator.

(Refer Slide Time: 17:08)



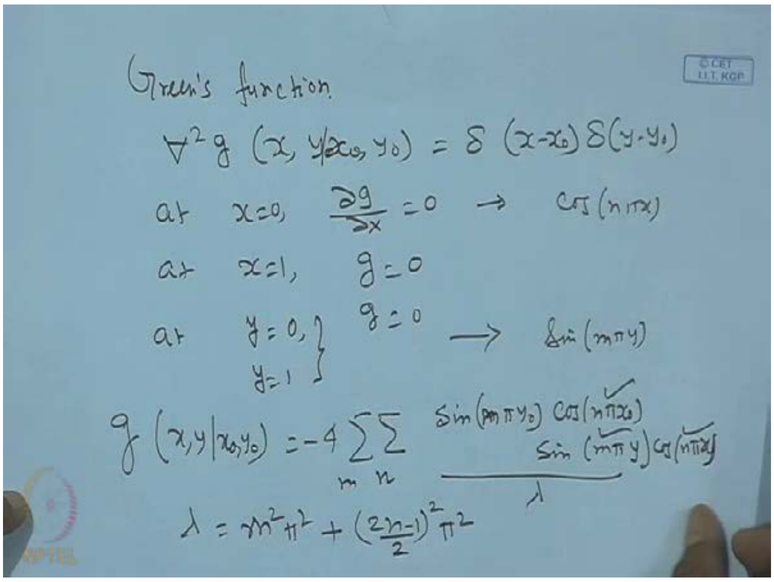
It will be 1 minus cosin n pi 1 minus cosin m pi n pi multiplied by sin n pi x naught, sin m pi y naught. Therefore, we can add all these contributions and we will be getting the complete solution. Complete solution will be obtained - u as a function of x naught y naught; we will simply sum up all the solutions: I 1, I 2, I 3 and I 4.

If that is the case, you will be finding that the right hand side will be having only summation of some constant term multiplied by summation of some variable multiplied by sin m pi x naught, sin n pi sin sin n pi y naught and sin n pi x naught.

Therefore, we can change the variable from x naught y naught to normal variable x and y. We will be getting u as a function of x and y. You will be getting the complete solution of elliptic partial differential equation using the full eigenfunction expansion method; by using Green's function procedure.

One more thing I would just like to talk here. If the boundary conditions are different, if instead of boundary if instead of Dirichlet boundary condition - for the same problem; two dimensional problem, grad square u is equal to 0 is equal to f; if you have that at x is equal to 0 instead of u, del u del x will be is equal to 0, u is equal to 0 , y is equal to 0, x equal to 0, del u del x is 0 and u equal to 0 at x is equal to 1. y is equal to 0 at y is equal to 0 u is equal to let us say q 1; this is let say q 2, this is q 3 and at y is equal to 1 you have u is equal to q 4.

(Refer Slide Time: 19:13)

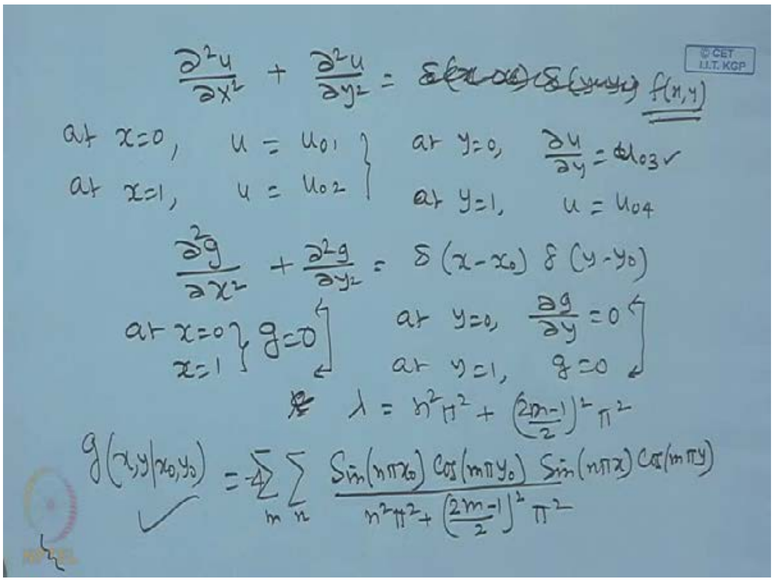


Corresponding Green's function can be written as grad square g x y x naught y naught is equal to delta x minus x naught delta y minus y naught. At x is equal to 0, we should have del g del x is equal to 0; at x is equal to 1, we should have g is equal to 0; at y is equal to 0, we should have g is equal to 0 and at y is equal to 1, we should have g is equal to 0.

Therefore, the eigenfunctions in this case will be the cosin functions cos n pi x and eigenfunctions in in the y direction will be sin m pi y. Therefore, Green's function here becomes minus 4 summation over m, summation over n sin m pi y naught cosin n pi x naught, sin m pi y cosin n pi x divided by lambda. lambda will be - in this case, n square pi square plus 2 n minus 1 by 2 square pi square.

In this case, the eigenfunctions will be constituted by the cosin functions in x direction because you have a Neumann boundary condition in the x direction. Eigenfunctions will be constituted by a cosin function in the x direction; sin function in the y direction. This will be cosin function and x y and this will be m n. The eigenvalues will be constituted by m square pi square plus 2 n minus 1 by 2 whole square pi square because these are the eigenvalues in the x direction and these are the eigenvalues in the y direction.

(Refer Slide Time: 21:50)



If we have a problem like del square u del x square plus del square u del y square is equal to delta x minus x naught, delta y minus y naught; actual problem we have, let us say Dirichlet boundary condition. and u u is equal to u 0 1, u is equal to u 0 2; this is at x is equal to 0 and this is at x is equal to 1. We have this; then at y is equal to 0 we have let say del u del y is equal to 0. Let us say, u 0 3 is constant well flux condition and at y is equal to 1 u is equal to u 0 4.

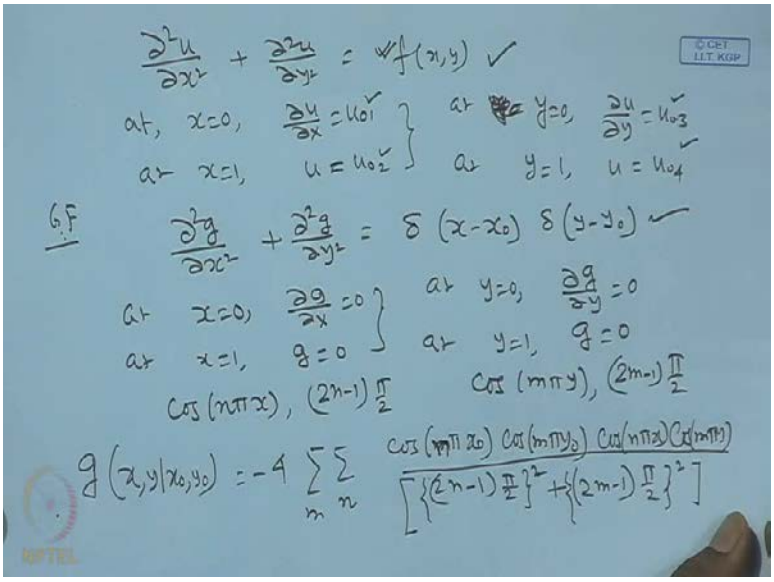
The corresponding Green's function solution will be governing equation with del square g del x square plus del square g del y square is equal to; this will be some function of x and y.

This will be delta x minus x naught, delta y minus y naught. that will be We are replacing the non-homogeneous term in the governing equation by the direct delta function. The boundary conditions on u on g will be: at x is equal to 0 and at x is equal to 1, g is equal to 0; at y is equal to 0, del g del y is equal to 0 and at y is equal to 1, g is equal to 0.

Therefore, eigenfunctions in the y direction will be the cosin functions. Eigenfunctions in the x direction will be the sin function. lambda square lambda will be given; by this will be n pi. This will be the eigenvalues. n square pi square plus 2 m minus 1 by 2 whole square pi square will be the eigenvalues. The Green's function will be g x y x naught y naught. This will be nothing but 1 summation over m, another over n minus 4 n square pi square plus 2 m minus 1 by 2 whole square pi square. In the numerator we will be having cosin functions; cosin m y naught are the eigenfunctions.

sin n pi x naught cosin m pi y naught, sin n pi x cosin m pi y as the Green's function. The rest of the procedure is exactly the same that we have done earlier. If we have the Neumann boundary condition on in both x and y, then you will be having the cosin functions as the eigenfunctions.

(Refer Slide Time: 24:49)



If you are talking about a problem like del square u del x square, plus del square u del y square is equal to f of x y; at x is equal to 0, del u del x is equal to u 0 1; at x is equal to 1, u is equal to let us say u 0 2; at x at y is equal to 0, del u del y is equal to u 0 3 and at y is equal to 1, we have u is equal to u 0 4. In that case, the Green's function will be del square g del x square plus del square g del y square is equal to delta x minus x naught delta y minus y naught.

At x is equal to 0, del g del x is equal to 0; at x is equal to 1, g is equal to 0; at y is equal to 0, del g del y will be equal to 0 and at y is equal to 1, g is equal to 0.

Therefore, in this case cosin n pi x and cosin m pi y are the eigenfunctions; with the eigenvalues 2 n minus 1 pi by 2 and with the eigenvalues 2 m minus 1 pi by 2. Therefore, g as a function of x y x naught y naught should be minus 4 summation over m, summation over n cosin m pi n pi x naught cosin m pi y naught cosin n pi x cosin m pi y divide by 2 n minus 1 pi by 2 square of that plus 2 m minus 1 pi by 2 square of that, then bracket end.

That will be the expression of Green's function. One will be getting the solution exactly the same way that we have done before. Similarly, if one of the boundary conditions of the robin mixed boundary condition; we will be getting the transcendental equation as the eigenvalues. We will be getting the sin functions as the eigenfunctions in the corresponding direction where the robin mixed boundary condition exists.

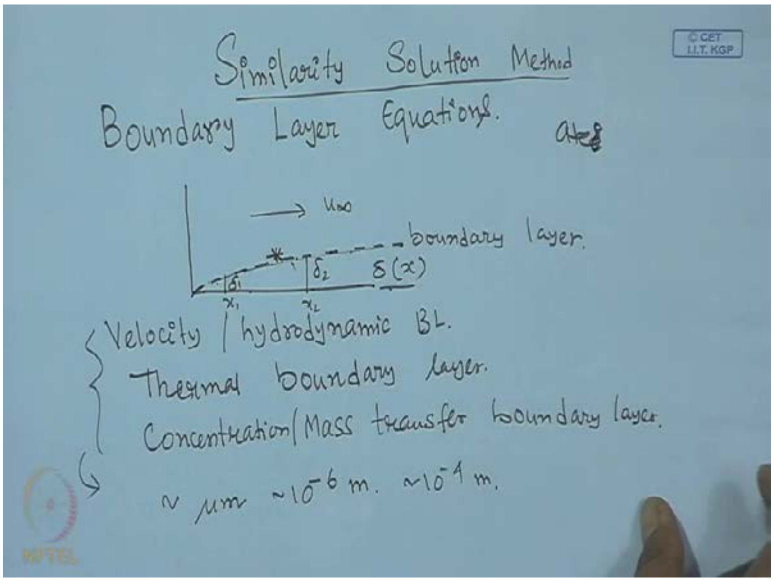
The other boundary condition, the other direction and the eigenvalue problem can be evaluated as we have done earlier. We will be able to get the expression of g. Once you get the expression of g, the rest of the calculation follows exactly the same way as we have done earlier. We connect u with g and completely solve the problem. In a general case like this, there will be five sources of non-homogeneity because in the original problem there are in a five sources of non-homogeneity corresponding to each of these terms. We will be getting the five terms in the final solution.

Out of these five terms, one will be the volumetric integral corresponding to the non-homogeneous term appearing in the governing equation; the other four will be the surface integral terms corresponding to the non-homogeneity in the boundary conditions. The way we have solved the earlier first problem in this class, exactly, the same way these problems have to be solved and completed by the using the Green's function method.

We have come to the end of solution of non-homogeneous partial differential equations using Green's function methods. This is a most general form and all the we have to consider all the boundary condition to be non-homogeneous. Combinations of all such of all solve sorts of boundary conditions; Dirichlet, Neumann, both Neumann, Dirichlet Robin mixed and Robin mixed Neumann, all this.

We have seen how the Green's function will be evaluated by full eigenfunction expansion method; then it must be hooked up or linked with the original differential equation. We will be getting the entire solution by the analytically if the problem is linear and non-homogeneous.

(Refer Slide Time: 29:59)



We have completed that part. Next, we move to another method of solution - mathematical tool that we are going to use quite often and several times. We use this technique for solution of chemical engineering problems. Many of such problems can exits. I will name some of these. I will first name about the method that we are going to discuss now - it is the similarity solution method. This method is quite popular when we are solving the boundary layered equations.

When you are solving the boundary layer equations this becomes quite important. Let us first see what these boundary layers are: Boundary layers are the locus in a flow field where the velocity components the values of velocity will be 99 percent of the bulk velocity considering the flow is over a flat plate. In the freeze stream, the velocity is u infinity. At the boundary of the plate the velocity is 0 because of the no slip boundary condition. As you go up, the velocity keeps on increasing and beyond a particular point it becomes u is equal to u infinity. One can identify a locus in the flow field where every point of this locus will be having a velocity 99 percent of free stream velocity. Thus, that particular locus is called the boundary layer.

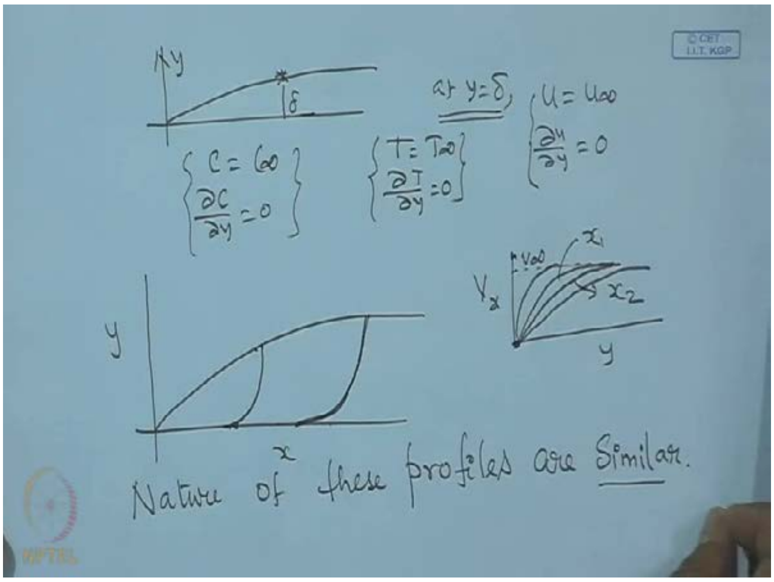
Now this boundary layer can be If you are talking about momentum transfer, we can have a velocity boundary layer. This is also called hydrodynamic boundary layer. Whenever we will be having a heat transfer case, then there will be a profile and there is a mismatch of you know bulk temperature and the well temperature. There is a profile existing in the flow field where temperature at every point will be 99 percent temperature of the freeze stream. That boundary layer is called the thermal boundary layer.

Whenever there is a concentration failed; there is a mismatch of bulk concentration and the wall concentration, we keep it different. There is a concentration gradient. Therefore, there exists a profile in the flow field of the concentration such that at every point on this; there exists a locus where on every point there will be concentration which will be equal to the bulk concentration. This is known as concentration or mass transfer boundary layer.

This boundary layers are typically occurring in the order The thickness of these boundary layers are typically occurring in the order of micron 10 to the power minus 6 meter or it might be slightly more - 10 to the power minus 4 meter but not larger than this. All these momentum transfer, heat transfer and mass transfer; they are occurring within this thin boundary layer, that is why the boundary layer calculations are quite important in a all chemical engineering applications because heat, mass and momentum are basically getting transported within this thin boundary layer.

Therefore, we did the boundary layer equation. If you look into the you know profiles of velocity or temperature or concentration within the boundary layer; we will see that. First of all there are two characteristics of this boundary layer. The thickness of this boundary layer is increasing as you go along the length. Therefore, this thickness is here delta 1, at this location here it is delta 2.

(Refer Slide Time: 34:31)



In general, delta is a function of x. it is an increasing function of x but with a decreasing slope; that is number one characteristic. Number two characteristic is that the conditions on h of the boundary layer everywhere will be must be satisfying two conditions: one will be that at y is equal to delta at the edge of the boundary layer. At the edge of the boundary layer, at y is equal to delta, u may be u free stream and del u del y should be is equal to 0. The y gradient of velocity at the edge of boundary layer must vanish. Similarly, at y is equal to delta T is equal to T infinity and del T del y must be equal to 0, that is for the thermal boundary layer. For concentration or mass transfer boundary layer, C is equal to C infinity del C del y must be equal to 0 at the edge of boundary layer or at y is equal to delta.

These conditions are compulsory conditions and they are necessary. These are called the boundary layer conditions.

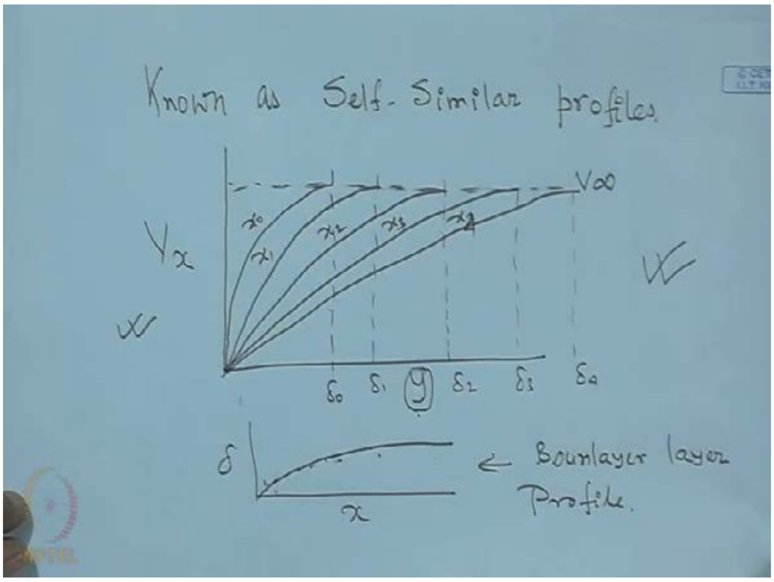
If you look into the profiles of dependent variable as a function of x and y within the boundary layer; the profiles will be If it is a velocity boundary layer; velocity is 0 over here and it is maximum there. The profile will be something like this.

So there increase There exists a profile within the boundary layer. This is at the location x where the velocity is 0 and velocity is high there. If you plot velocity V x as a function of y at y is equal to 0, it is having a high value and y is equal to infinity. It is the other way round. At y is equal to 0 this is having a no slip boundary condition; this value will be equal to 0. At y is equal to delta, it will be having a value. Let us say this is v infinity.

Similarly, when you for this is for a particular x location. If you look into another x location, a velocity profile will be something like this. It will be 0 then it increases and it increases and goes up to value v x is equal to infinity. This will be at a particular location x 1. We will be having another profile of v x; this is a particular location x 2, this will be another location x 3 and this will be another location x 4.

Therefore, these profiles the nature of these profiles are similar in nature. These profiles are similar in nature and they are known as self-similar profiles.

(Refer Slide Time: 38:04)



If you plot the value of velocity - let us say as a function of y, you will be having the free stream velocity at a particular level. This is v infinity. There is the free stream velocity and it reaches infinity at different values of delta. Delta keeps on increasing. This is a particular x location, this is another x location, this is another x location and this will be another x location.

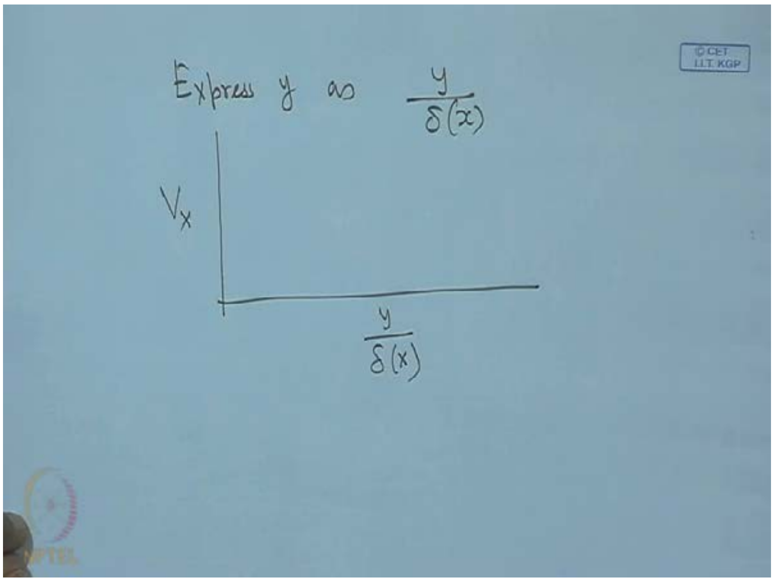
There may be another profile here; this is x naught. As x increases, the velocity will be attaining the free stream velocity at different values of y.

This is corresponding to delta 0, this is corresponding to delta 1, this is corresponding to delta 2, this is corresponding to delta 3 and this will be corresponding to delta 4. If I plot delta as a function of x, the plot looks something like this; this is the boundary layer profile.

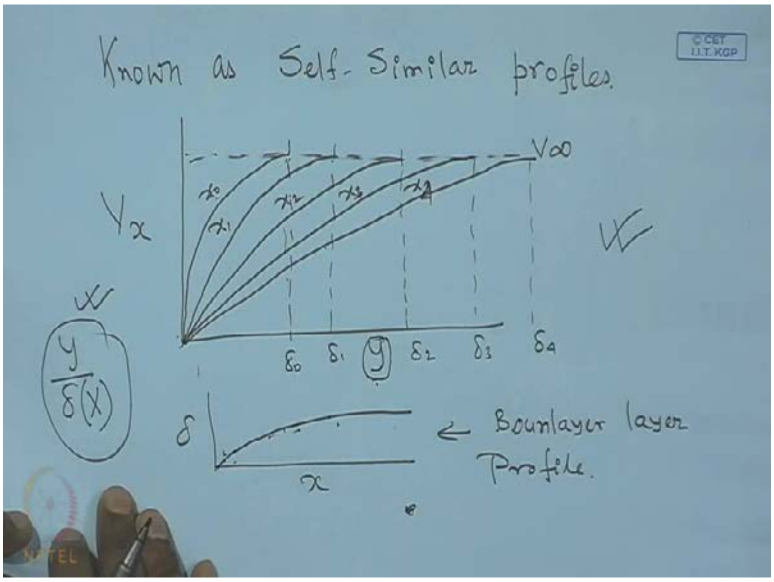
Therefore, we will be getting the velocity profile as a function of y for different x location. If you just look into this curve, all these profiles are self similar; they are similar profile. now if we now plot If you normalize this variable - the y with respect to delta, which is nothing but a function of x.

That means if you plot V x as a function of y by delta, where delta is eventually function of x then let us see what is the nature of this plot looks like and how this part looks like.

(Refer Slide Time: 40:52)



(Refer Slide Time: 41:17)



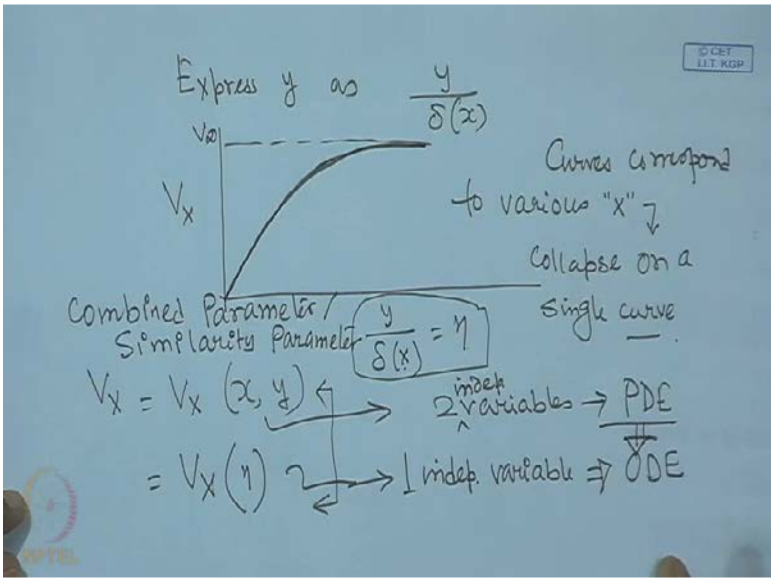
If we express y as y by delta which is nothing but a function of x and if we plot V x as a function - as this quantity y by delta, that means we are normalizing this axis by the corresponding deltas. This is delta, delta delta; but each delta is a function of x.

What we have. Since, we are not differentiating the values of x 0 x 1 x 2 x 3 x 4; we are not differentiating with respect to at what location we are talking about, because we are now normalizing and we are plotting V x as a function of y by delta which is in fact a function of delta. In this quantity if we plot velocity as a function y by delta; we are not at all differentiating at what x location we are talking about.

Since, the curves are similar then what we can expect out of this. If we really do that then all these curves will be superposing on each other; they will collapse on a single curve.

Why they will collapse on a single curve? Because we are dividing it at y is equal to delta. You are normalizing again a delta which is eventually a function of x.

(Refer Slide Time: 42:32)



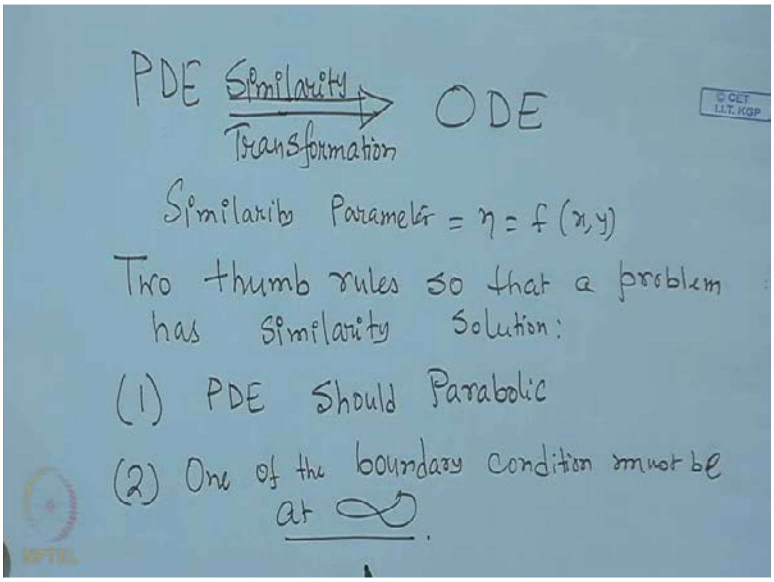
This curve and all these curves will super impose on one another and we will be getting a single curve. In this case we will be getting a single curve. so by doing a Therefore, all these curves will collapse. Curves correspond to various x; collapse on a single curve.

Therefore, what it means is that initially, we had V x as a function of x and y. and now what we can do is that we can Therefore, there will be separate curves as a function, if you plot V x verses y as different positions of x.

Since, we are getting a single curve as V x, as a function of y by delta; we call this a new parameter eta. We will be getting V x as a sole function of eta because the whole characteristic can be represented as a single curve.

What this means? This means that here we are talking about 2 independent variables. and therefore and In this case, we have one independent variable. Whenever we will be having 2 independent variables, the governing equation is partial differential equation. When we will have only 1 independent variable, the governing equation is ordinary differential equation. therefore by and This parameter eta which is a combined parameter of y and x delta is essentially function of x. This combined parameter of y and x is known as a combined parameter or similarity parameter.

(Refer Slide Time: 45:28)



So once we get. That means in terms of similarity parameter, the partial differential equation can be boiled down into an ordinary differential equation. That is the beauty of similarity transformation. Partial differential equation gets transformed and ordinary differential equation by using similarity transformation.

ODE being easier to solve helps us know the solution. People do the similarity transformation to simplify the mathematical TDS ness of the problem.

The similarity parameter is also known as a combined very well parameter and it is a function of x; a combined form of x and y. There are two thumb rules. One has to remember whether a problem will be having a similarity solution or not. Two thumb rules so that a problem has similarity solution. What are this thumb rules?

Thumb rule number one: PDE; partial differential equation should be a parabolic partial differential equation.

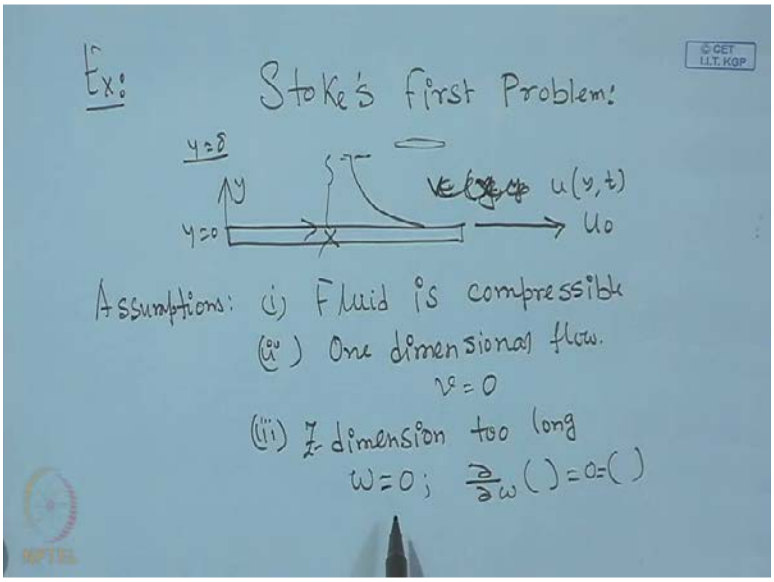
Second, one of the boundary condition must be at infinity; this is very important, one of the boundary conditions must be at infinity. Why this is required? The boundary condition is required to be an infinity so that the velocity field or concentration of velocity profile or concentration profile or temperature profile does not get distorted because of imposition of any other boundary condition at the edge of the boundary layer. Let it get developed; it will die down or reach the prescribe value asymptotically.

Therefore, there should the self similarity of the profiles are maintained. Because of that one of the boundary condition needs to be at infinity.

These two thumb rules has to be met in order to a problem have a similarity solution that means if these two conditions If a problem admits a similarity solution, it must satisfy these two thumb rules.

If these two thumb rules are there, then a problem may have or may not have a similarity solution. It is a sufficient condition but it is not a necessary condition.

(Refer Slide Time: 48:44)



Let us look into an actual problem. We will be taking up an example. The first example: we will be talking about the Stoke's first problem.

In this problem there is a pool of stationary liquid. We keep a plate at time t is equal to 0 in the pool of stationary liquid. This is y axis and this is x axis. At time t is equal to 0, it starts moving in the positive x axis with a uniform velocity u naught.

What happens? The fluid adjacent to the plate will be x will be because of no slip boundary condition it will be experiencing a velocity u naught and because of the viscous forces the velocity keeps on decreasing. At a particular y will be having the with the velocity decreases down to value of 0.

Therefore, at this place, at this time and at this location the velocity will be equal to 0. We call this a sort of boundary layer. Therefore, we consider the velocity is 0 there and the fluid element stain at this location does not experience the moment of the wall at y is equal to 0.

Therefore, this y is equal to delta. Let us solve the velocity profile as a function of y and t.

We have some assumptions. The assumptions are: fluid is compressible, that means density is constant. Second, one dimensional flow; since the velocity is in the positive x direction, the velocity in the y direction does not exist.

v, the y component velocity is 0. Third is z dimension too long. Therefore, w the z component velocity becomes 0 and all the derivative with respect to w should vanish. Therefore, under these assumptions one can write down the equation of continuity, one can write down the equation of motion and can derive the expression of velocity expression as a function of y and t. Once we formulate the governing equation, we will set up the boundary conditions and we will examine that whether this problem will admit a similarity solution or not.

I stop here in this class. I will take up this problem in the next class and completely solve this problem. We will be looking into the basically the application of similarity solution in actual chemical engineering problem.

We will finish this problem in the next class. Thank you very much.