Advanced Mathematical Techniques in Chemical Engineering

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Lecture No. # 36

Solution of non-homogeneous Elliptic PDE

(Contd)

We were looking into the solution of non-homogeneous partial differential equation by using Green's function method. In the earlier class, we looked into a complete solution of a parabolic partial differential equation which was non-homogeneous using Green's function method. Not only that, we looked into what will be the possible solutions, eigenvalues and eigenfunctions in the case of different types of boundary conditions; Neumann Robin mixed, etcetera. We looked into the form of the solution you will be getting.

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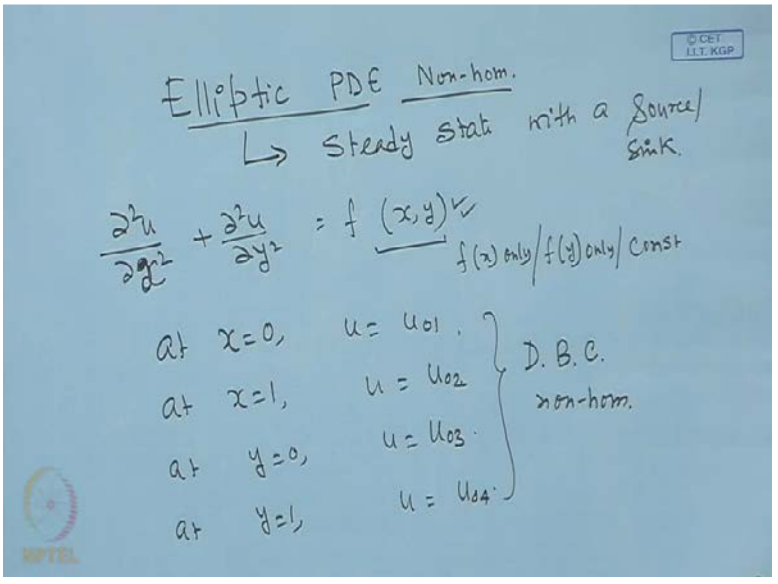

Next, we will be taking up an elliptical partial differential equation. We were told earlier that these types of equations will be occurring in a steady state chemical engineering process with a source or sink.

The governing equation may in general look something like this: del square u del x square plus del square u del y square is equal to f of x y. This non-homogeneity can be a function of x only, can be a function of y only, it may be a constant or it may be function of x and y both.

We have four boundary conditions - 2 on x and 2 on y because order to in both the directions. So at x is equal to 0, we have u is equal to u 01; at x is equal to 1, we have u is equal to u 02; at y is equal to 0, we have u is equal to u 03 and at y is equal 1, u is equal to u 04.

We have four non-homogeneous boundary conditions and for the time being we make them as Dirichlet boundary conditions. If we do that then we can identify how many sources of non-homogeneities this problem has. This problem has five sources of non-homogeneity - one in the governing equation and four in the four boundary conditions.

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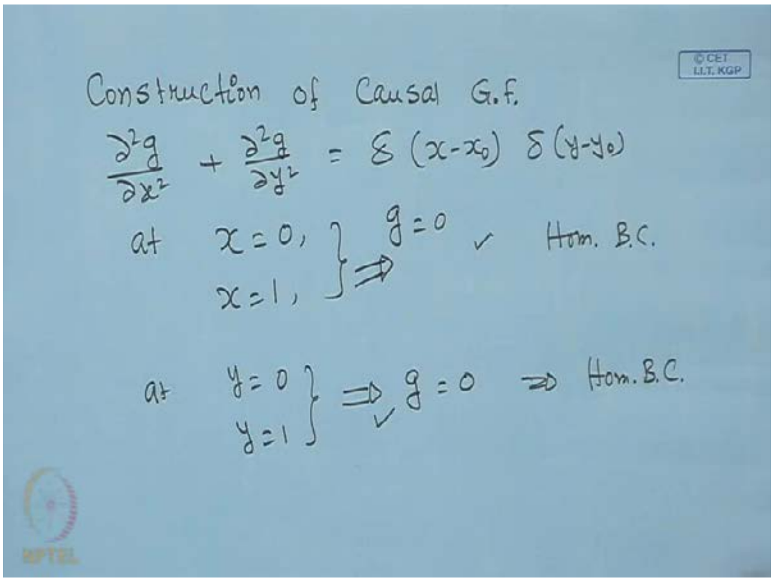

If we have this problem, we define this problem the elliptic partial non differential equation then we will be constructing next step will be the construction of causal Green's function. In this case the definition of Green's function becomes del square g del x square, plus del square g del y square is equal to delta x minus x naught delta y minus y naught.

The non-homogeneous term in the governing equation is substituted by the Dirac delta function and we force all the other boundary conditions to be homogeneous. At x is equal to 0, g is equal to 0 and at x is equal to 1, g is equal to 0. In both the cases since the boundary conditions were Dirichlet; g is equal to 0 on both the boundaries. At y is equal to 0 and y is equal to 1, we have g is equal to 0 since both the boundaries in the y direction of the original problem are Dirichlet boundary condition.

If you see that this equation is having the homogeneous boundary condition in both x direction and both conditions in y directions are homogeneous.

We can have a standard eigenvalue problem independently both in x direction and y direction. We will be using the complete eigenfunction expansion method or full eigenfunction expansion method. If you remember the parabolic problem; we had eigenvalue problem only in the boundary conditions in x direction and in the transient - it has not a boundary value problem, so boundary value problem existed only in x direction. Therefore, we consider a partial eigenfunction expansion method by defining the standard eigenvalue problem in the x direction only among the two independent direction t and x.

In this case, we have standard and independent eigenvalue problem in x direction as well as in y direction because we have eigenvalue problem in we have independent eigenvalue problem in both x direction and y direction.

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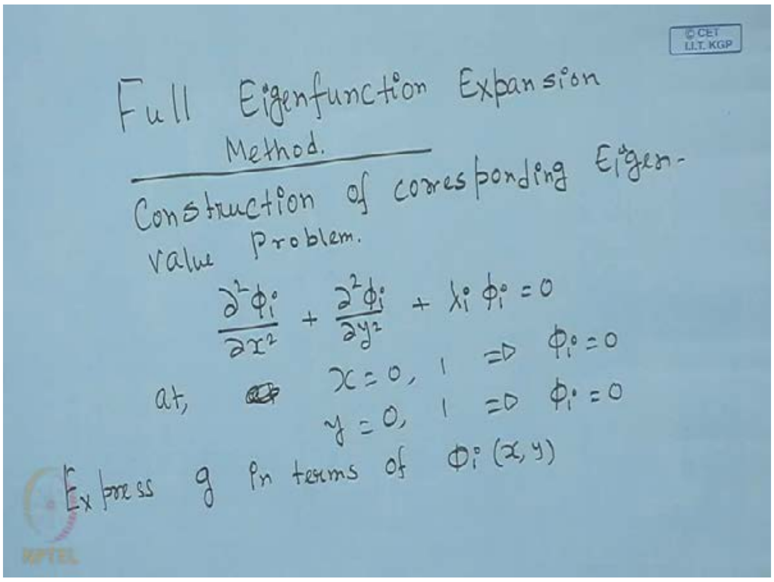

Therefore, we will be using the full eigenfunction expansion method for solution to this problem. This method is known as full eigenfunction expansion method. In the case of full eigenfunction expansion method, we construct the corresponding eigenvalue problem.

The construction of corresponding eigenvalue problem will be nothing but del square phi i del x square, plus del square phi i del y square, plus lambda i phi i will be is equal to 0. so it is a Both the boundaries are having the homogeneous conditions. Therefore, the parent problem in Green's function in this the parent problem for this eigenvalue problem is nothing but the Green's function. Since the boundary conditions of the Green's functions are all homogeneous; the eigenfunction and eigenvalue problem phi i must be having homogeneous boundary condition.

That means at x is equal to 0 and 1; we have phi i is equal to 0, at y is equal to 0 and 1; we have phi i is equal to 0. We express our Green's function g in terms of eigenfunctions phi i x and y.

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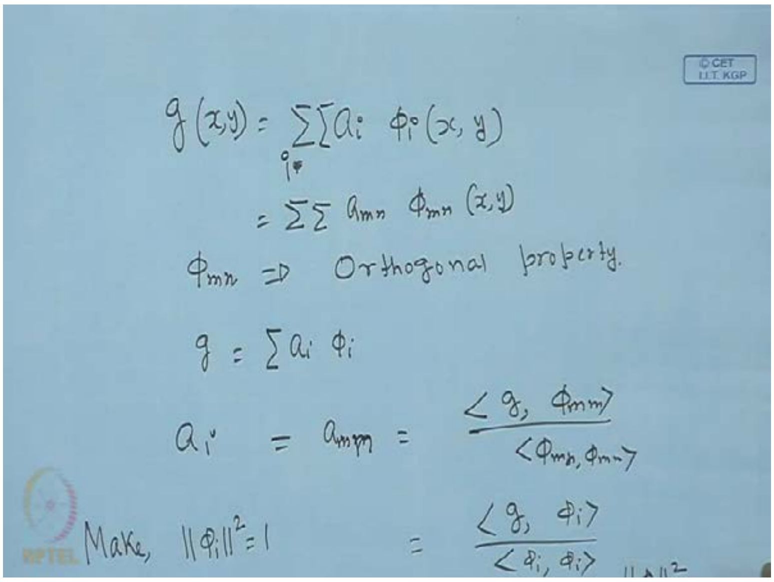

Therefore, g can be written as summation of a i multiplied by phi i x y where i is the summation is over i. In fact, there will be a double summation because we will be having two independent eigen value problem in x direction as well as in y direction; this will be a mn phi mn x y. phi mn can be obtained by using the orthogonal property. The constant a mn can be obtained; the orthogonal property of the eigenfunction so phi mn will obey the orthogonal property of eigen function.

We just write is as g is equal to summation a i phi i, just for the sake of mathematical is and handling and writing; we are just replacing the two summation by one summation.

Now, a i can be evaluated or a mn can be evaluated by taking the inner product of g with respect to phi mn or phi i; phi m n phi mn. This is because of the orthogonal property of the eigenfunction. This is identical to g phi i inner product of phi i and phi i square phi i; this is nothing but the norm of phi i square.

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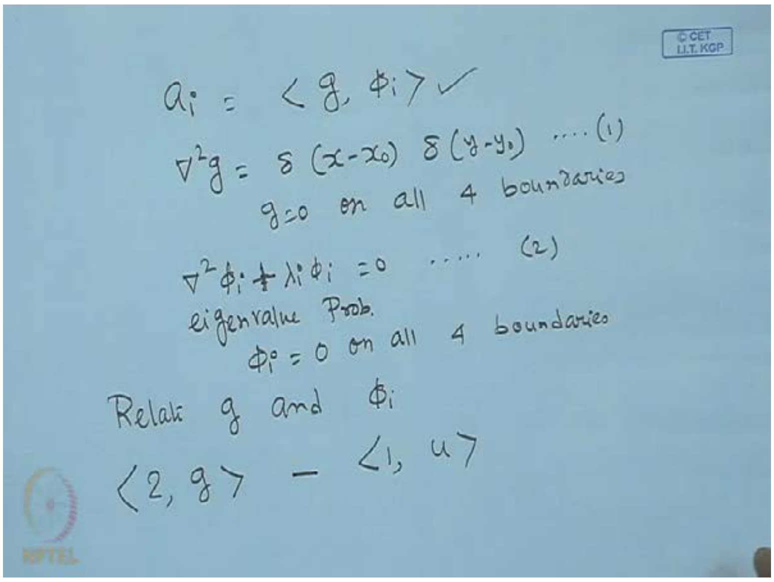

What we do is, we evaluate this. We can make this eigenfunction orthonormal; make eigenfunction orthonormal. In that case, the denominator will become 1 and the constant a i becomes inner product of g and phi I. In order to evaluate this constant what we do is we write down the governing equation grad square g - it is a two dimensional Laplacian operator, so grad square will be nothing but del square del x square plus del square del y square, delta x minus x naught delta y minus y naught, this is equation number 1; subject to g is equal to 0 on all four boundaries.

The next one is grad square phi i plus lambda i phi i is equal to 0; this is the eigenvalue problem. This is equation 2; eigenfunction phi i is equal to 0 on all four boundaries.

because Therefore, both the problems g and phi i; they are homogeneous boundaries everywhere.

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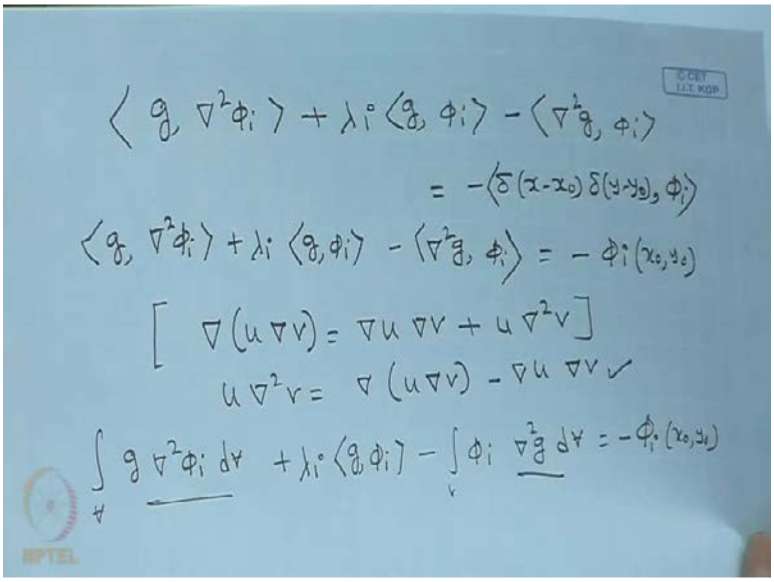

Now, what we do is we relate g and phi i. How to relate g and phi i? Take inner product of 2 with respect to g, take inner product of 1 with respect to u and then subtract. If you do that then let us see what you get; you will be getting inner product of g grad square phi I, plus lambda i g phi i,, minus inner product of grad square g phi i minus is equal to minus delta inner product of delta x minus x naught delta y minus y naught, comma phi i.

We will be getting this 1 and then we simplify this equation one after another. This is inner product of g grad square phi I. This lambda i should be written as g inner product of g and phi i, minus inner product of grad square g phi i; this will be nothing but minus phi i at x naught y naught.

Then we utilize the identity. The identity is grad of u grad v is nothing but grad of u grad of v plus u grad square v; u grad square v is nothing but grad of u grad v minus grad of u grad of v. We utilize this identity and see what we get out of this terms. This will be nothing but inner product of g grad square phi i dv; this is a volume integral plus lambda i inner product of g and phi i, minus volume integral phi i grad square g, dv is nothing but minus phi i x naught y naught. We express these two terms by using this identity this identity and see what we get.

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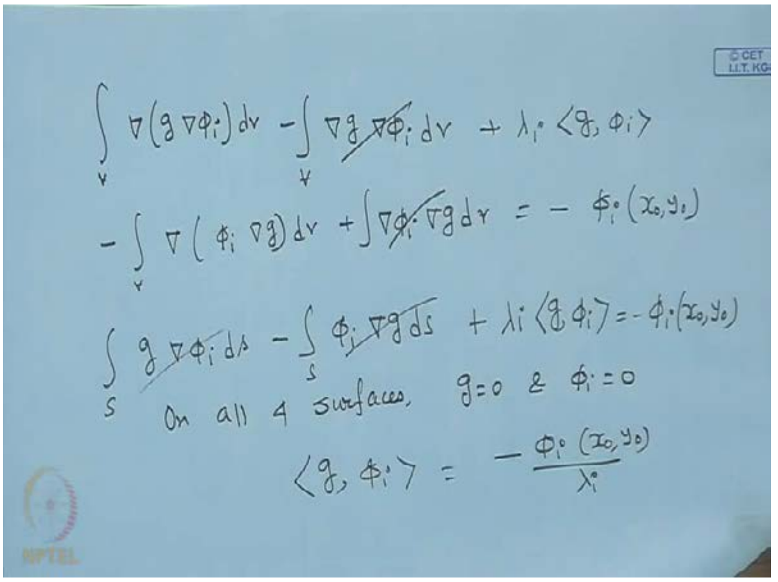

If we expand those two integrals in terms of that identity, what we will be getting is that integral gradient of g grad phi i dv; this is a volume integral, minus integral grad of g, grad of phi i dv, plus lambda i inner product of g and phi i, minus volume integral grad of phi i, grad g dv, minus minus plus grad of phi i grad of g dv is equal to minus phi i x naught y naught.

If you look into this these two terms, they are exactly same and opposite in sign. They will be simply cancelled out. now integration of This is called a Green's integral. This is a volumetric integral; this volumetric integral can be converted into surface integral. This will be surface integral - g grad of phi i ds, minus surface integral phi i grad of g ds over the surface, plus lambda i inner product of g and phi i is equal to minus phi i x naught y naught.

In this case, if you look into this equation to evaluate these surfaces - on all the four surfaces; we have homogenous boundary conditions on g and we have homogenous boundary conditions on phi i

On all 4 surfaces, g is equal to 0 and phi i is equal to 0. Therefore, these two terms will vanish because the values of g and phi i on the surfaces will be equal to 0. Inner product of g and phi i is nothing but minus phi i, x naught y naught divided by lambda i.

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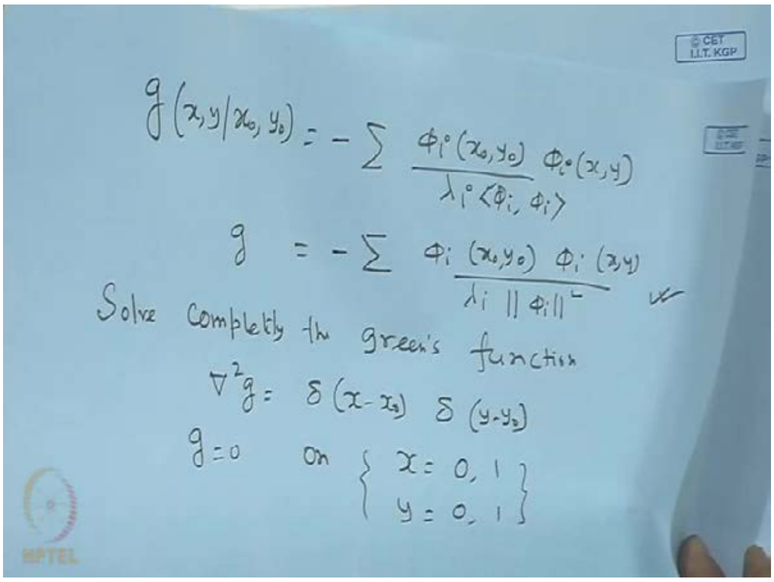

So we can We have obtained the inner product between g and phi i. so if you now write down the expression of Green's function We can write down the expression of Green's function. g of x y slash x naught y naught is equal to nothing but minus summation phi i x naught y naught, phi i x y divided by lambda i phi i phi i.

So we put the We substitute the inner product of g and phi i by this equation - by this expression, minus phi i x naught y naught divided by lambda i; this is nothing but norm of phi i. Therefore, g expression of g becomes summation phi i x naught y naught, phi i x y lambda i norm of phi i square.

Let us obtain consider the Green's function method. so equivalent eigenvalue problem let us solve this We define this theory for expression of Green's function. For this particular problem, we solve completely the Green's function. Let us obtain the expression of Green's function for this particular problem. grad square g is equal to delta x minus x naught, delta y minus y naught and g is equal to 0 on x is equal to 0 and 1; y is equal to 0 and 1.

On all the four boundaries, we will be having homogenous boundary condition. Let us solve this problem by using the complete eigenfunction expansion method.

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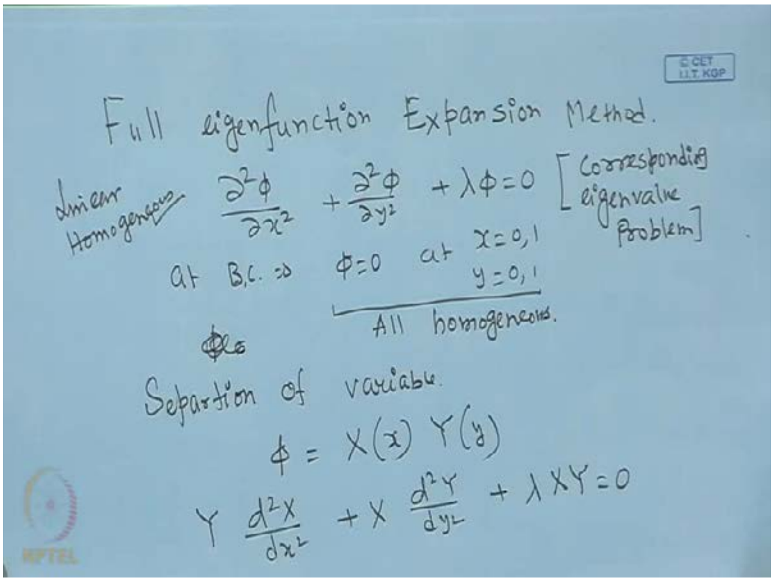

Use the full eigenfunction expansion method. We will be having del square phi del x square, plus del square phi del y square, plus lambda phi is equal to 0. This is the corresponding eigenvalue problem.

All the eigen the boundary conditions are that on all the four boundaries phi is equal to 0 at x is equal to 0 and 1; y is equal to 0 and 1. phi becomes We use the separation of variable because this equation is linear homogenous boundary conditions; all homogenous. We use the separation of variable method to solve this problem.

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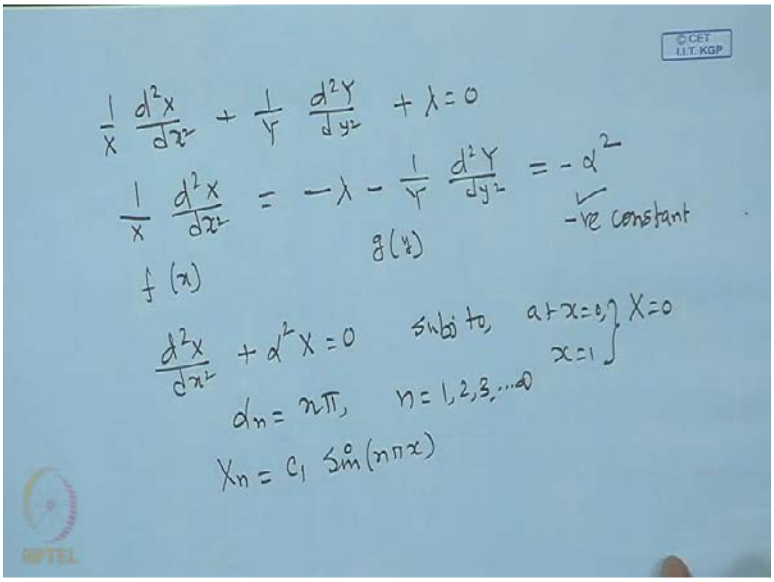

We assume that phi is a product of two functions: capital X which is a which is entirely a function of x and capital Y which is entirely a function of y. We substitute in the governing equation; this will be Y d square X dx square, plus X d square Y d y square, plus lambda XY is equal to 0. We divide both sides of this equation by XY. What we will be getting is: d square 1 over X d square, X dx square plus 1 over Y, d square Y dy square, plus lambda is equal to 0.

We write 1 over X d square X dx square take the both the terms on the right hand side; this becomes minus lambda minus 1 over Y d square Y dy square.

If you examine this equation; the left hand side is a function of x alone and the right hand side is a function of y alone. They are equal and they will be equal to some constant and this constant has to be can be 0, can be positive and can be negative. We have seen earlier that if this constant is 0 and positive we will be getting a trivial solution but we are looking for non-trivial solution. Therefore, this constant has to be a negative constant.

We will be having d square X dx square, plus alpha square capital X is equal to 0, subject to, at x is equal to 0 capital X is equal to 0, at x is equal to 1 capital X is equal to 0.

We know the solution to this problem. The eigenvalues of these problems are: n pi where the index n runs from 1, 2, 3; up to infinity and eigenfunctions are sign functions - c 1 sin n pi x.

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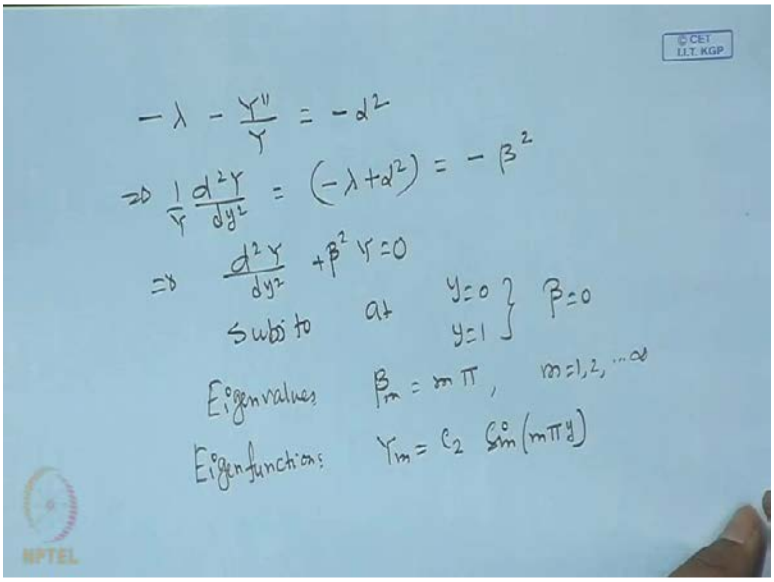

Let us solve the y dimensional problem - that problem in the y direction. This becomes minus lambda minus Y double prime divided by Y prime Y is equal to minus alpha square. This becomes d square Y dy square, 1 over Y is equal to minus lambda plus alpha square is equal to a constant. Again this constant can be positive, this constant can be negative and this constant can be 0. We have seen earlier that if this constant is 0 and positive, we will be getting a trivial solution. Therefore, this constant has to be a negative constant in order to have an eigenvalue problem.

d square Y dy square plus beta square Y will be is equal to 0, subject to, at y is equal to 0; at y is equal to 1 your beta is equal to 0. You know the solution to this problem. Again, the solution remains the same. beta m the eigen values beta m are nothing but m pi where the index m runs from 1 2 infinity and eigenfunctions are Y m is nothing but c 2 sin m pi y. These are the eigenfunctions. We have different subscript m because this is an independent eigenvalue problem.

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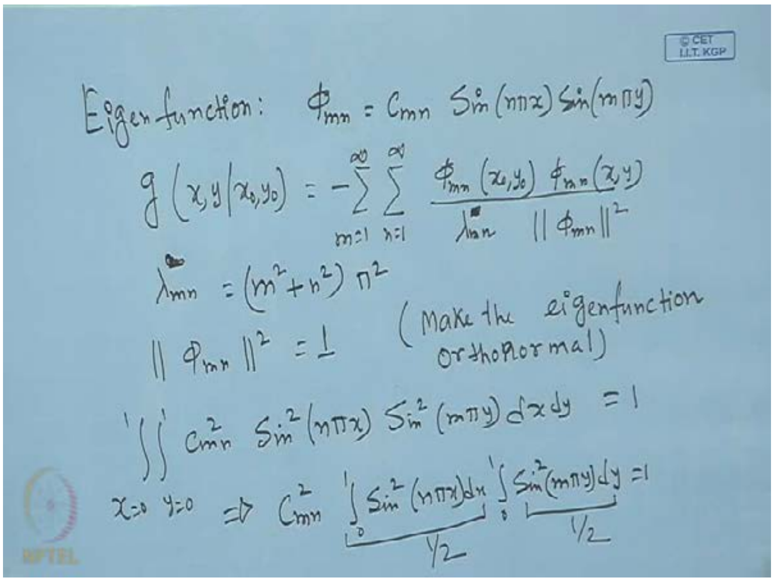

We will be in a position to get the eigenfunction phi as a function of n and m. We write down the eigenfunction. eigenfunction phi mn is nothing but a constant C mn sin n pi x and sin m pi y. We can construct the expression of Green's function. This will be g x y x naught y naught is equal to minus - it will be a double summation; 1 index over m and another index over n. Both m and n run from 1 to infinity. This will be phi mn x naught y naught, phi mn x y lambda mn, norm of phi mn square. What is lambda mn? lambda mn is nothing but lambda mn square. this will be nothing but m square lambda mn is nothing but m square plus n square times pi square.

What is phi mn square norm of phi mn square is equal to 1? We force norm of phi mn square is equal to 1 so that the denominator; this term becomes 1 and that simplify our calculations. This means make the eigenfunction orthonormal. If we make the eigenfunction orthonormal, let us see what we get. It will be nothing but double integral C mn square sin square n pi x, sin square m pi y, dx dy is equal to 1, so x from 0 to 1 and y from 0 to 1. We can carry out this integral since they are in the product terms productform, so you can carry out integration independently. 0 to 1 sin square n pi x dx, 0 to 1 sin square m pi y dy is equal to 1. We have already seen the half value of this integral is half so C mn is nothing but 2.

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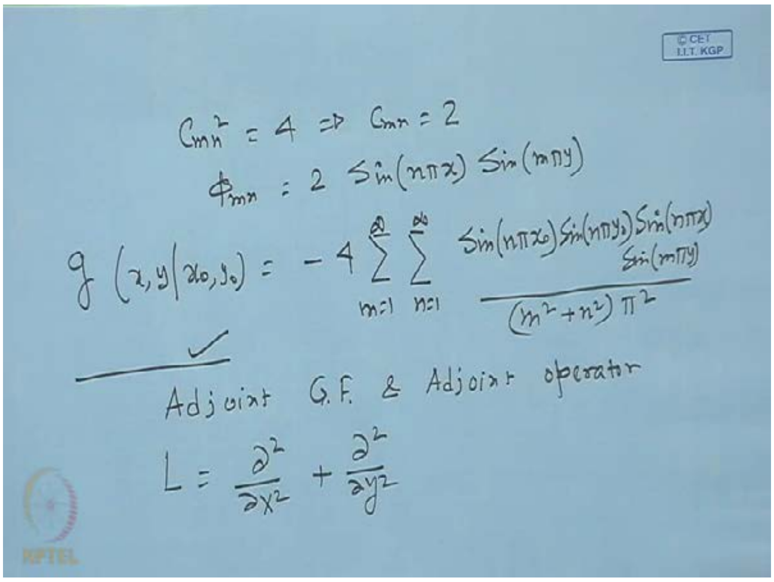

C mn square is 4 and C mn is nothing but 2. We get the phi mn as 2 sin n pi x sin m pi y. I get the expression of Green's function now completely. g x y as a function of x naught y naught is nothing but minus 4 double summation 1 over m and another over n. This becomes sin n pi x naught, sin n pi y naught, sin n pi x, sin m pi y divided by m square plus n square pi square.

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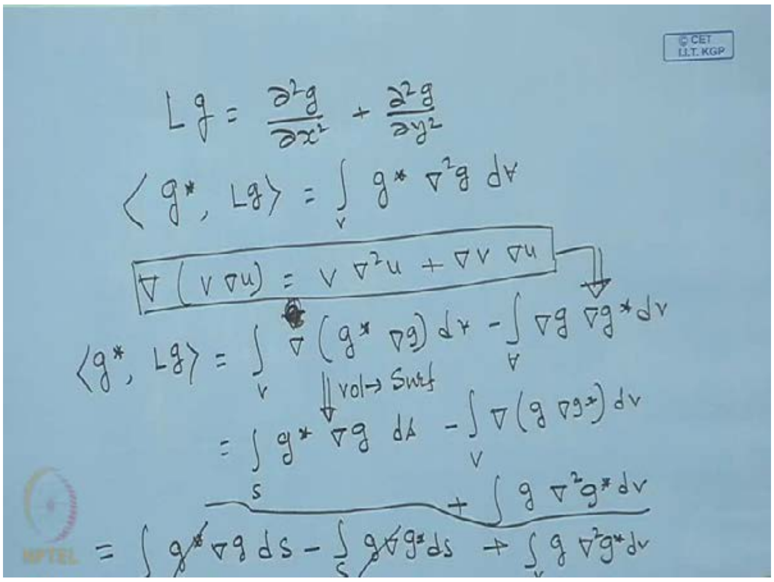

We obtain the Green's function now. Next what do we do? We look into the adjoint Green's function and see whether the operator is self adjoint or not. Next, I will take a diversion; we will look into the adjoint Green's function and adjoint operator. If you look into our operator L, this is a Laplacian del square del X square plus del square del y square. so write Lg is nothing but del square g del x square plus del square g del y square.

If we evaluate this inner product g star Lg that is nothing but volume metric integral g star grad square g dv.

Again we utilize the identity, grad of v grad u is nothing but v grad square u plus grad of v grad of u. We evaluate the inner product of g star LG is equal to volume integral g star grad square, we just put it as grad of g star grad g dv, minus volume integral grad g grad g star dv. This can be substituted as grad of g So, utilize this identity to evaluate this, to simplify to breakdown this one into two terms; g star grad square g plus grad g grad u.

What we get is volume integral. We write this as in the form of surface integral. This surface integral becomes g star grad of g ds and grad of g and grad of g star we obtain we write it from here. This becomes volume integral of grad of g grad of g star dv, minus minus plus, g grad square g star dv. We are utilizing this to simplify this term. We convert the volume integral into surface integral from the first term.

Next, what do we do? We again convert the volume integral into surface integral. What we will be getting is surface integral g star grad of g ds, minus s g grad of g star ds, plus volume integral g grad square g star dv.

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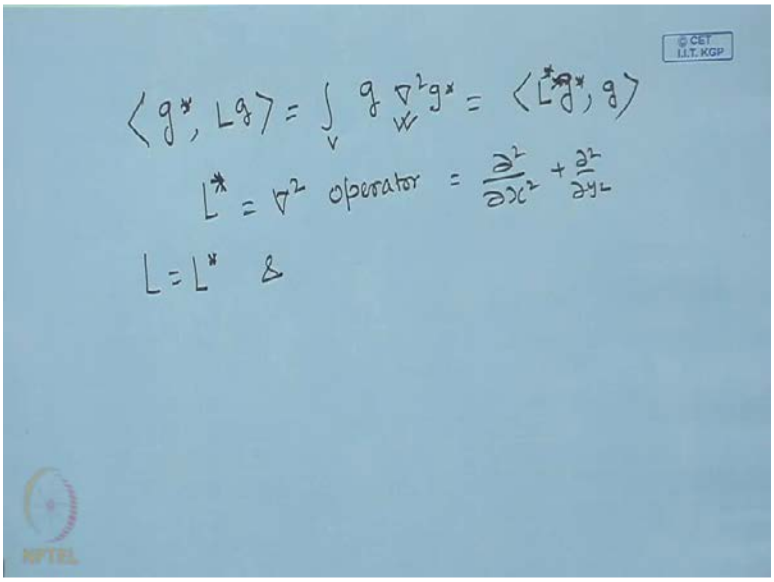

Since the g is having homogenous boundaries on all the boundary conditions all the boundaries; this term will vanish. We make the We assume the boundary conditions on g star to be homogenous on all the four boundaries. Therefore, this term will also be vanished. So, bilinear concomitant term is gone. What we will be getting out of this is inner product of g star Lg, is nothing but volume integral g grad square g star. Therefore, this is nothing but inner product of L star g star, comma g.

What is L star? L star is nothing but the grad square operator that means, by two-dimensional problem this is del square del x square plus del square del y square.

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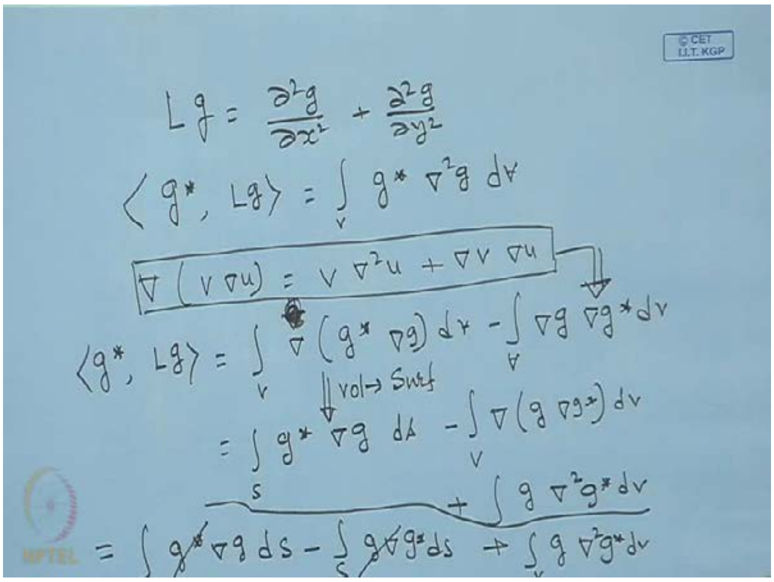

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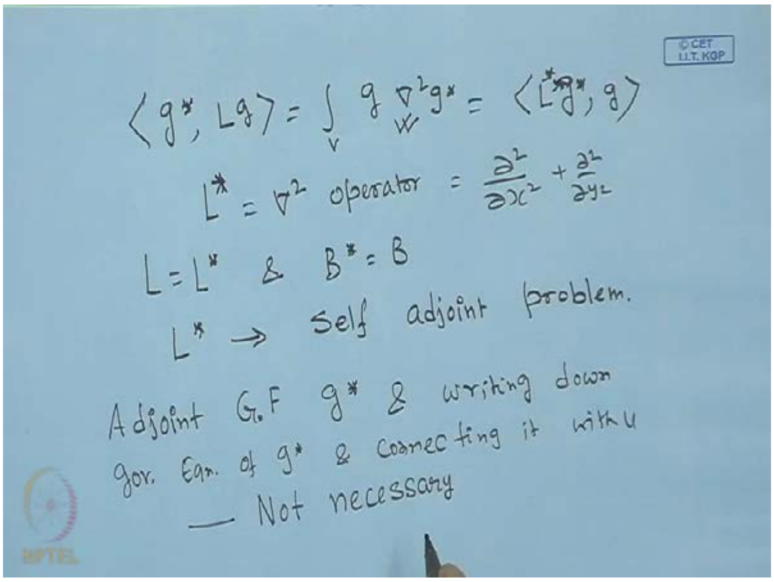

L is equal to L star. We have already seen that B star - in the earlier problem, that in order to make this bilinear concomitant part to be homogeneous, to vanish all the boundary conditions on g star have to be equal to 0 or homogeneous. Therefore, the boundary operator also says that B star is equal to B because the boundary conditions on g star are equal to 0 or homogeneous. The boundary conditions on the g are homogeneous. Therefore, the operator is a self adjoint operator. It is a self adjoint problem. We have B star is equal to B and L star is equal to L.

We need not go for the adjoint Green's function, evaluation of adjoint Green's function g star, writing down governing equation of g star and connecting it with u is not necessary for this boundary problem, simply because this elliptical problem the problem itself is self adjoint; L is equal to L star and B is equal to B star. Therefore, g star and g-they will have identical expression, they will have identical boundary conditions and they will have identical governing equation. The last two steps are not required for this particular problem, simply because the operator the problem itself is self adjoint problem. We need not go for evaluation of expression of g star, then writing down the governing equation of g star and connecting it with u.

We will do that thing which we have done till now. We can connect we have already got the expression of Green's function g; we connect the governing equation of g with the original problem u and can proceed for the solution of the problem. Therefore, let us look into the final solution.

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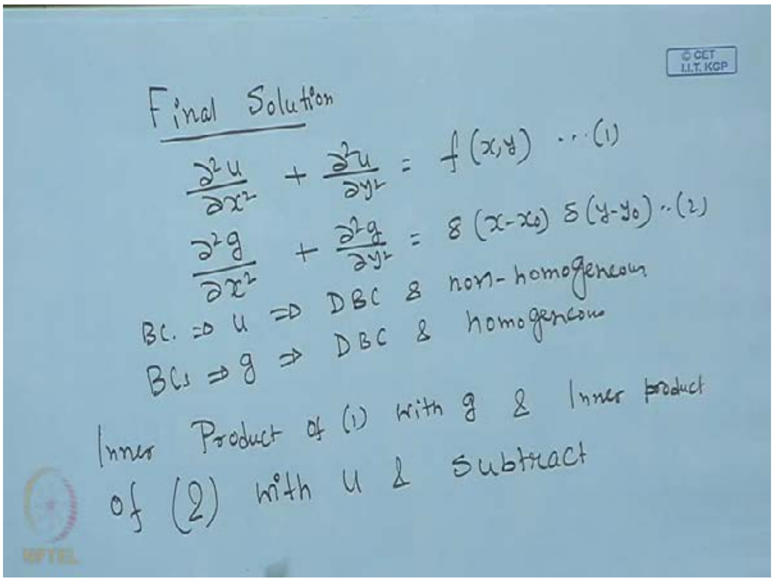

The final solution is: let us look into the actual problem. del square u del x square plus del square u del y square is equal to f of x y; this is the original problem.

We write down the Green's function. We need not go for the adjoint Green's function because the operator elliptic operator is a self adjoint operator. del square g del x square plus del square g del y square is equal to delta x minus x naught and delta y minus y naught. This non homogeneity in the governing equation is replaced by the Dirac delta function. Boundary conditions on u are Dirichlet and non-homogeneous. Boundary conditions on g are all Dirichlet and homogenous. Therefore, what we do is we connect with this equation 1; this is equation 2.

We take inner product of 1 with g and inner product of 2 with u and subtract. Let us see what we get.

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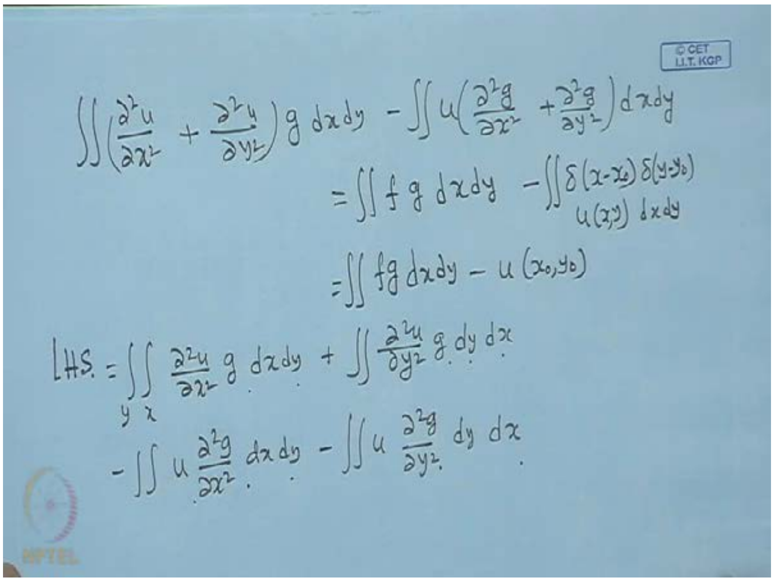

Double integral del square u del x square plus del square u del y square g dx dy - that is the first term, minus double integral u del square g del x square plus del square g del y square; dx dy.

That is the left hand side. In the right hand side we have double integral f times g dx dy minus delta x minus x naught delta y minus y naught u x y dx dy. This will be double integral f times g dx dy minus u x naught y naught. That is the right hand side and this is the left hand side. Let us look into the left hand side first and then we will look into the right hand side.

Left hand side; let us see what we get. Left hand side we have y x del square u del x square g dx dy plus del square u del y square g dy dx minus u del square g del x square dx dy minus u del square g del y square dy dx. These four terms are present on the left hand side. Next what we do is we carry out the integration by parts. In the first case, we will do the integration with respect to x first then y, do the integration with respect to y first then x, do the integration of with respect to x first then y, y first then x and see what you get.

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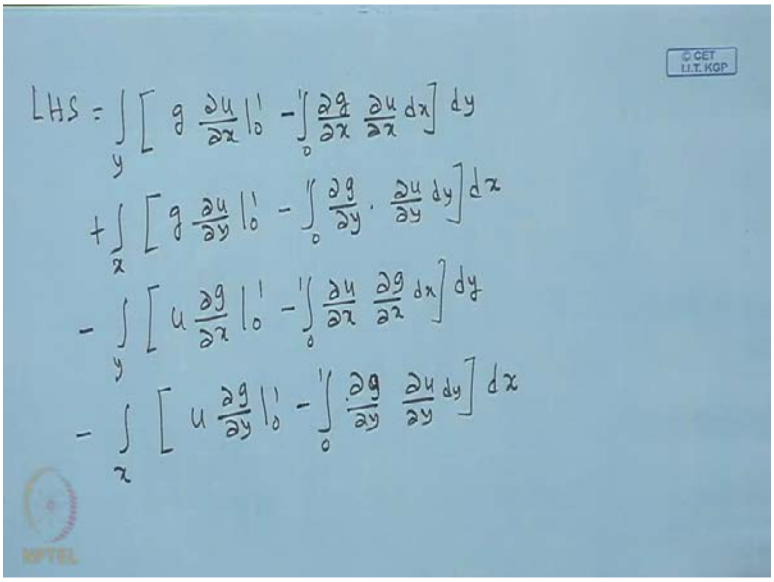

We evaluate the left hand side integral of y. First function g; first function integral so for second function del u del x from 0 to 1 because x varies from 0 to 1, minus differential of first function that is del g del x integration of second one del u del x dx from 0 to 1 dy that is the first one. Second term will be integration over y first; we put the integration of x later on. It will be g del u del y from 0 to 1 minus integral 0 to 1 del g del y del u del y dy and the end we will be having dx minus the third integral that is over x first, y remains same; this becomes u del g del x from 0 to 1 minus integral 0 to 1 del u del x del g del x dx; dy will be outside.

Again, we will be having the 4th term that is minus integration over y first; x will be out. It will be u del g del y from 0 to 1 minus integral 0 to 1 del u del g del y, del u del y and dy dx.

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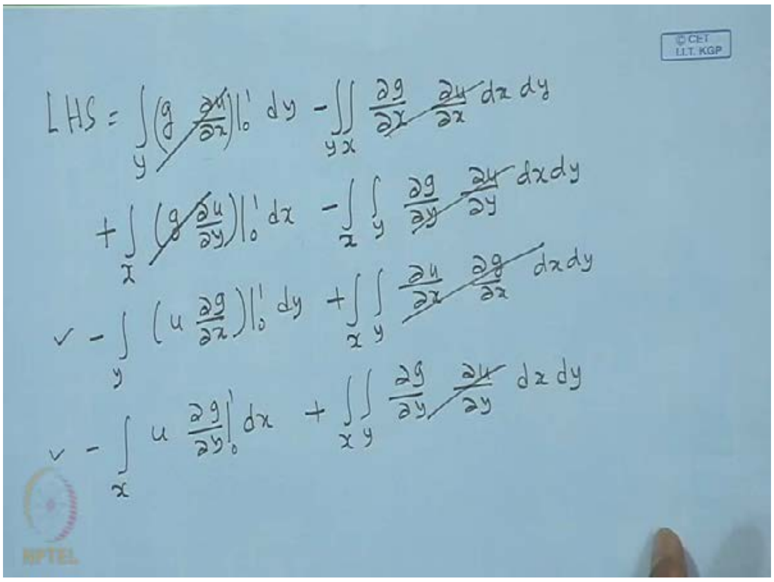

Next, what we do is we open up these brackets and see what we get. We write down the individual terms. Left hand side is equal to integral over y, g del u del x 0 to 1. This is the whole term times dy minus double integral y x, del g del x, del u del x and dx dy plus integral over x, g del u del y from 0 to 1 dx then minus over x over y, del g del y, del u del y dx dy minus integral over y u del g del x from 0 to 1 dy minus minus plus x y, del u del x, del g del x, dx dy minus integral over x u, del g del y dx; this is from 0 to 1 minus minus plus x y del g del y and del u del y dx dy.

Therefore, if you look into this term; these terms are same, equal and opposite in sign; they will simply cancel out. What will we have? We will have four bilinear concomitant terms and let us evaluate this bilinear concomitant term. Now, g at x is equal to 1, so this term means g at x is equal to one, del u del x at x is equal to 1 minus g at x is equal to 0 and del u del x at x is equal to 0. That simply means, we have the boundary conditions g at 1 and g at 0 is equal to 0; this term will go.

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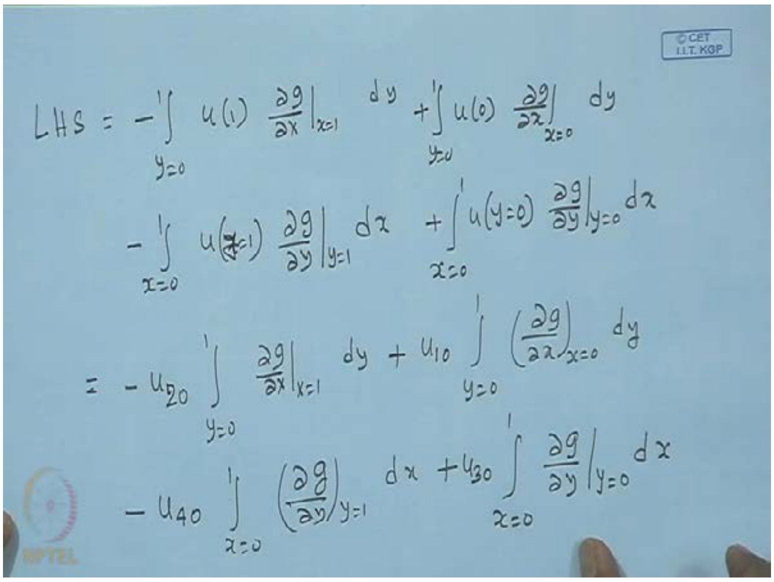

Similarly, if you look into this term g del u g at y is equal to 0 y is equal to 1, del u del y at y is equal to 1 minus g at y is equal to 0, del u del y at y is equal to 0. We know the boundary conditions in y direction on g are also homogenous. So g at y is equal to 1 equal to 0, g at y is equal to 0; it is also 0, so this term is gone.

What are these two terms? Only these two terms we will be leaving behind. Let us write down these two terms. More explicitly left hand side is nothing but minus over y - y means from 0 to 1. It is u at 1 del g del x at x is equal to 1; this is x equal to 1, dy minus minus plus, so y is equal to 0 to 1, u at 0 del g del x at x equal to 0 dy. We have minus x is equal to 0 to 1. We have u at x is equal to 1, u at y is equal to 1 and this will be del g del y at y is equal to 1 dx minus minus plus u at y is equal to 0 and del g del y at y is equal to 0 dx and this is from x is equal to 0 to 1.

We already had the non-homogeneous boundary conditions on u. u at 1 is equal to u 1 naught, it will be minus u; u at x is equal to 0. It was u 1 naught; this is u 2 naught. This is u 2 naught integral y is equal to 0 to 1, del g del x evaluated at x is equal to 1, dy plus u at x is equal to 0; this will be u 1 0 y is equal to 0 to 1, del g del x evaluated at x is equal to 0 dy. We have u at y is equal to 1, so it is u 4 0, x is equal to 0 to 1, del g del y at y is equal to 1 dx plus; this is u 3 0 that will be out. It will be x is equal to 0 to 1 and del g del y at y is equal to 0 dx.

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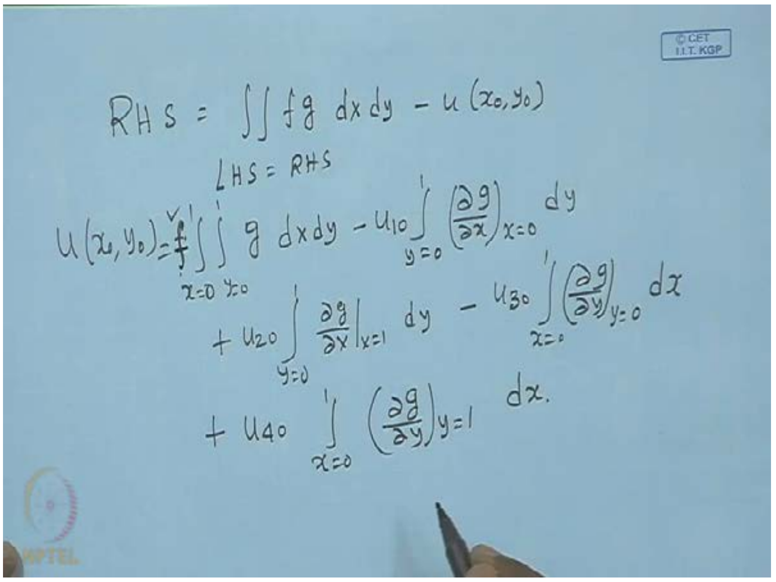

Left hand side will be having four terms. This four terms will be corresponding to four non-homogeneous terms on the boundary conditions. If you look into the corresponding equation now let us look into the right hand side this is left hand side and Let us look into the right hand side; right hand side is nothing but double integral f times g dx dy minus u x naught y naught.

What we do, we take the we use left hand side is equal to right hand side and we take u x naught y naught on the other side. We put u x naught y naught is equal to integral over x from 0 to 1 and y from 0 to 1. f may be common or if it is a constant function f is taken out. If f is a non-function it can be kept inside and it will be integrated by parts, so g dx dy. This will be minus u 1 naught integral y is equal to 0 to 1, del g del x at x is equal to 0, dy minus minus plus so it will be u 2 naught, del g del x at x is equal to 1 dy then it will be minus u 3 naught x is equal to 0 to 1; this is from y is equal to 0 to 1, del g del y at y is equal to 0 times dx minus minus plus u 4 0 x is equal to 0 to 1, del g del y at y is equal to 1 times dx.

You have five terms in the right hand side. If you look into this more carefully, the first term is double integral corresponding to the volumetric integral or non-homogeneous term present in the governing equation. There are four single integral terms or surface integral terms. These four terms corresponds to the four non-homogeneities present on the boundary.

What I will do is I will stop the class at this point and I will take this up in the next class and solve atleast couple of these integrals analytically in order to demonstrate how to solve this problem completely. Once, we are able to solve these problems completely that will give you a complete demonstration of solution of elliptical partial differential equation non-homogenous using the Green's function method.

I will take up this problem in the next class. Thank you very much for your kind attention.