Advanced Mathematical Techniques in Chemical Engineering
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Module No. # 01
Lecture No. # 31
PDE in Cylindrical and Spherical Coordinate

Good morning everyone. In this class we will start with the partial differential equations in cylindrical and spherical coordinates. Whatever we have done in the last class, if you remember, we started the problem with the cylindrical polar coordinate - we considered a two-dimensional problem first, that was one-dimensional in time and one-dimensional in space.

Now, we have seen that in cylindrical polar coordinate, which will be typically the flow occurring in a pipe or a tube, the coordinate system will be given by the radial polar coordinate system.

We have looked into the operator, the operator in this case is 1 by r del del r r del del r. Now we have seen that for this particular case, the eigenvalue problem becomes the Bessel equation. It is in the form of Bessel equation and the solutions will be constituted by the Bessel functions.

We have already solved the problem that the eigen functions will be Bessel functions and we have proved that the Bessel functions are orthogonal to each other; using the orthogonal property one can completely solve the partial differential equations in cylindrical polar coordinate system.

Now, in today's class we will be looking into an example of cylindrical polar coordinate system where the problem is more than two-dimensional - it is 2 dimensions in space and 1 dimension in time, so therefore, it is a three-dimensional problem.

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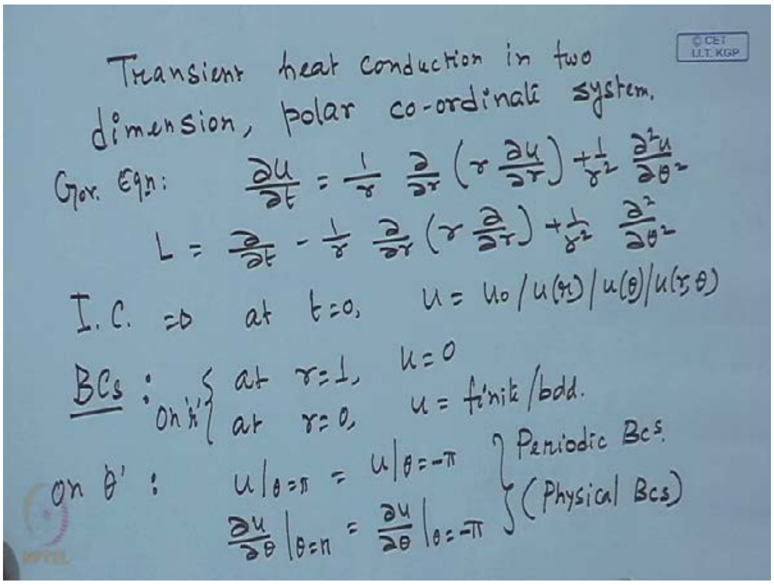

We call this problem as transient heat conduction in two-dimension, polar coordinate system. The governing equation in this problem becomes del u del t is equal to 1 over r del del r r del u del r plus 1 over r square del square u del theta square.

If you look into this problem, the operator is del del t minus 1 over r del del r r del del r plus 1 over r square del square del theta square. So, that is the operator we are talking about; therefore, this equation requires one initial condition because it is order 1 with respect to t and any two boundary conditions over r and two boundary conditions on theta.

Let us set up this problem by fixing up the initial and the boundary condition. The initial condition is at t is equal to 0, u may be a constant or it may be a function of r, it may be a function of theta, it may be a function of, in general, r and theta. So, this may be the general initial condition that u may be function of r and theta both; or it may be constant, may be function of r alone, may be function of theta alone.

Now let us set up the boundary conditions. If you look into the boundary conditions, that at r is equal to 1 we have u is equal to 0, this simply means we are maintaining a constant temperature at the wall; r is equal to 1 means, it is in the non-dimensional version in the dimensional form. This boundary condition is residing at the surface r is equal to R and where we are having a temperature, let us say we are maintaining a constant temperature, at t is equal to t 1 there.

Now, if you define a non-dimensional temperature - t minus t 1 divided by t 1, then that transformation or changing variable will be reflected in the form u is equal to 0; therefore, this means we are having a homogeneous boundary condition at the wall. At r is equal to 0, as we have seen in the earlier class that the value of u is finite or bounded, we do not know the value of temperature at the center of the whole tube, but we only know that it is assuming a finite or bounded value; therefore, this is a physical boundary condition. So these are the boundary conditions on r and on theta we will be having the periodic boundary conditions, that is, u at theta is equal to pi is equal to u at theta is equal to minus pi and del u del theta at theta is equal to pi is equal to del u del theta at theta is equal to minus pi; so, these are called periodic boundary conditions.

Again, these are the examples of physical boundary conditions; in addition to - this is also physical boundary condition (Refer Slide Time: 05:59). Thus, we have a system where we will be having the physical boundary conditions which will be governed by the physics of the problem, so with this set of initial and boundary conditions we can go ahead with the solution.

Since the operator is linear operator we can assume that the solution will be constituted by three functions - each of them will be a sole function of the independent variable.

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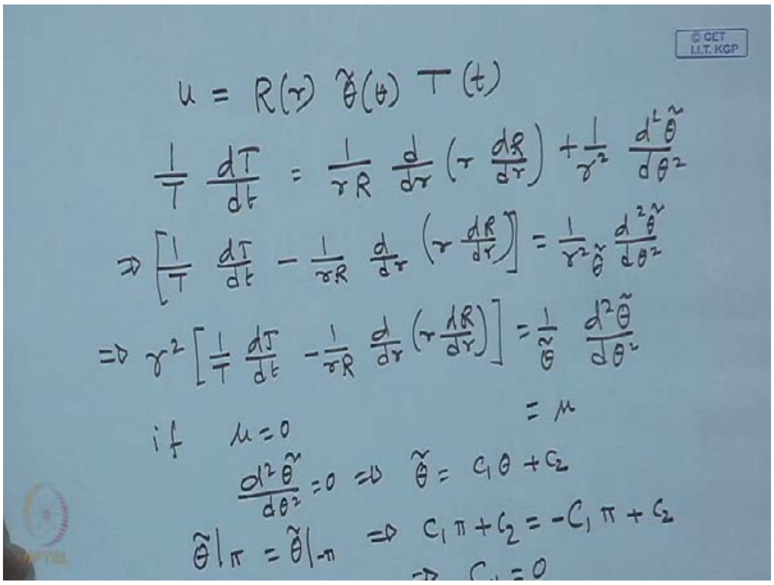

Let us assume that u is constituted of product of three terms, one is R as a function of r, theta hat that is a function of theta only and T that is a function of temperature - the function of time only.

Now, we substitute this in the governing equation as we have done earlier and divide both sides by r theta hat and T then what we get is: 1 over T dT dt is equal to 1 over r R d dr r dR d r plus 1 over r square d square theta hat d theta square.

Exactly like the earlier problem we try to set up an eigenvalue problem, so we make it 1 over T dT dt minus 1 over r R d dr of r dR dr, we bring it to the other side so this becomes 1 over r square d square theta hat d theta square, there will be a theta hat in the denominator. Now we multiply both side by r square, so this becomes r square 1 over T dT dt minus 1 over r R d dr of r dR dr is equal to 1 over theta hat d square theta hat d theta square.

Now, the left hand side is completely a function of t and r, the right hand side is entirely function of theta; so, they will be equal and they are equal to some constant, let us say mu. We select a value of mu so that we will be getting a non-trivial solution; if you select mu is equal to 0, let us see what we get - you will be getting d square theta hat d theta square is equal to 0, therefore the solution is theta hat is nothing but C 1 theta plus C 2.

If we use the boundary condition theta hat at pi is equal to theta hat at minus pi then what you will be getting is C 1 pi plus C 2 is equal to minus C 1 pi plus C 2.

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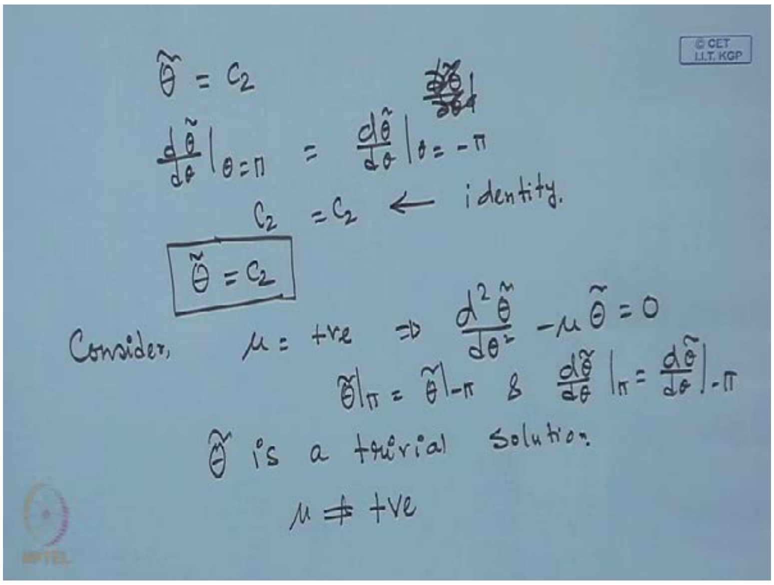

C 2 C 2 will be cancelling out so pi is not equal to 0, so what you will be getting is C 1 is equal to 0, that leads to the solution of theta, the theta hat will be giving you C 2 - a constant.

Now, if we use the other boundary condition d theta hat d theta at theta equal to pi and d theta hat d theta at theta is equal to minus pi, what you will be getting is C 2 is equal to C 2, you will be getting an identity; therefore, the solution of theta hat for mu is equal to 0, this constant equal to 0 is theta hat is equal to C 2 that means it is a constant.

Now let us consider mu is positive; if mu is positive you can go ahead with the solution that means d square theta hat d theta square minus mu theta hat is equal to 0, so it is positive. Utilize the boundary conditions that theta hat at pi is equal to theta hat at minus pi and d theta hat d theta at pi is equal to d theta hat d theta at minus pi, you will be landing up with a trivial solution; I am not going to do as we have solved this type of equation number of times in earlier classes, so please do go ahead and solve this problem yourself. You will find that you get theta hat is a trivial solution, so the solution is basically theta hat equal to 0 so mu cannot be positive; the positive constant is ruled out.

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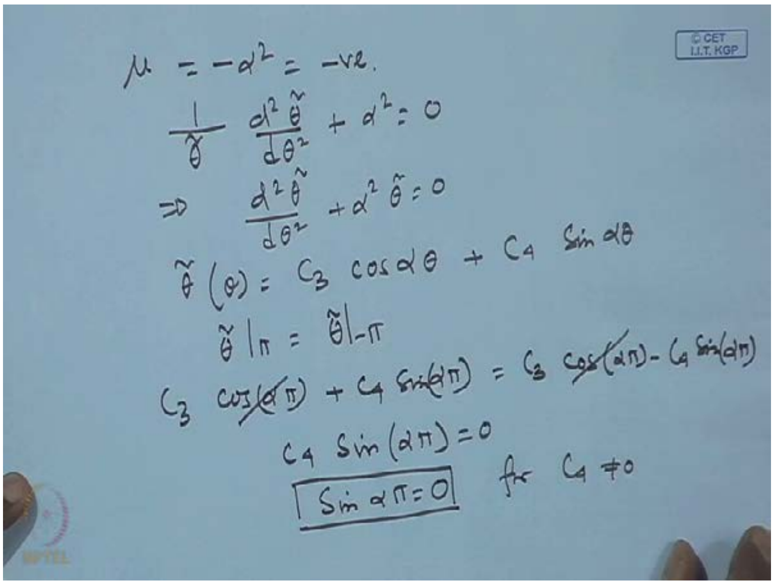

Let us look into the negative constant; if mu is negative, so minus alpha, so this value is negative. Now we write down the governing equation of theta hat and see how the theta hat varies with theta this time, so this becomes d square theta hat d theta square plus alpha square theta hat is equal to 0.

The solution is constituted by sine function and cosine function, therefore theta hat as a function of theta is nothing but C 3 cosine alpha theta plus C 4 sine alpha theta. Therefore, if we use theta hat at pi and theta hat at minus pi boundary condition, you will be getting as C 3 cosine alpha pi plus C 4 sine alpha pi is equal to C 3 cosine alpha pi minus C 4 sine alpha pi. If you look into both side of the equation, this cos term will be cancelling out, so you will be getting C 4 sine alpha pi is equal to 0 twice of that, so in order to get a non-trivial solution sine alpha pi must be is equal to 0 for C 4 not equal to 0.

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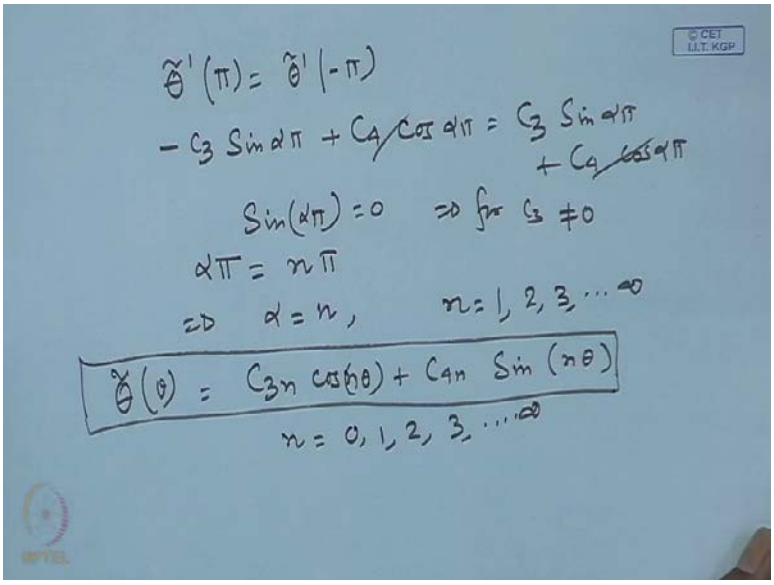

Now let us use the other boundary condition, that is, the d theta hat d theta is equal at pi and theta is equal to minus pi and see what we get. If we use other boundary condition, theta hat prime evaluated at pi is equal to theta hat prime evaluated at minus pi, you get minus 3 sine alpha pi plus C 4 cosine alpha pi is equal to C 3 sine alpha pi plus C 4 cosine alpha pi and C 4 cosine alpha pi will be cancelled out from both the sides, you will be again getting sine alpha pi is equal to 0.

So from both the boundary conditions on theta you get the same equation, alpha pi. Therefore for a non-trivial solution, for C 3 not equal to 0 in the other boundary conditions we came to a conclusion sine alpha pi is equal to 0 for C 4 is not equal to 0. In this case you are getting sine alpha pi is equal to 0 for C 3 is not equal to 0, therefore the eigen functions in this problem will be constituted by both C 3 is not equal to 0 and C 4 is not equal to 0; therefore combination of sine alpha theta and cosine alpha theta both will be formulating the eigen function.

The general solution of this is nothing, but as we have already seen earlier, alpha pi is equal to n pi so therefore alpha is nothing but n where the index n runs from 1 2 3 up to infinity.

So let us write down the eigen function in this case, C 3 n cosine n theta plus C 4 n sine n theta; now if you put n is equal to 0 then this term (Refer Slide Time: 15:56) will vanish because sine 0 is 0 but cos 0 is 1, therefore you will be getting C 3 n, that is a constant, and if you remember that for the value of mu equal to 0 we had a constant value of theta; therefore, if we write the index from 0 1 2 3 up to infinity that will include all the cases that mu is 0 and mu is negative.

So, theta hat theta is constituted by cosine n theta and sine n theta is an eigen function for this particular problem with the index n running from not 1 2 3 but from 0 1 2 3 up to infinity, therefore it includes the term for the situation where mu is equal to 0.

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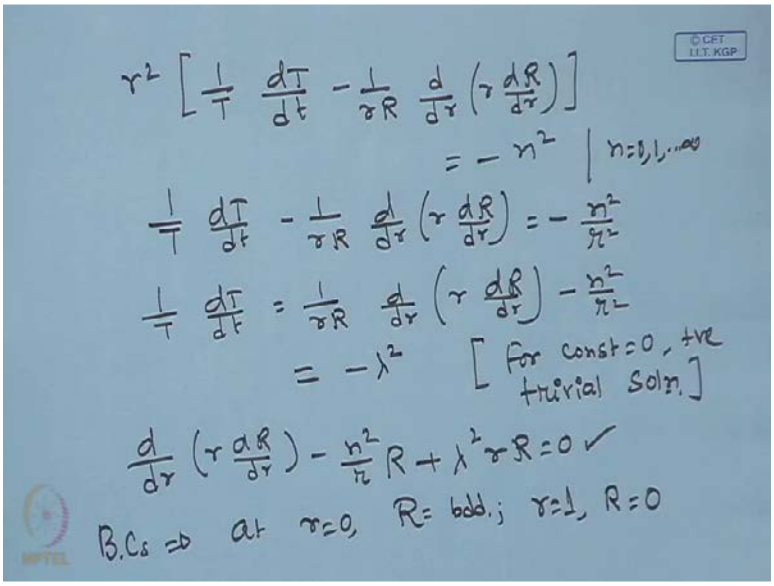

We obtained the eigen function in the theta direction; now let us look into the eigen function in the r direction. If you look into the solution of the other part - r square 1over T dT dt minus 1 over r R d dr r dR dr is equal to minus alpha square, as we have found out that alpha is equal to n, so it will be minus n square. Now, n is from 0 1 2 up to infinity so what we get is 1 over T dT dt minus 1 over r R d dr of r dR dr is equal to minus n square divided by r square; we bring it on the other side, so 1 by T dT dt is nothing but 1 over r R d dr r dR dr minus n square r square.

Now the left hand side is entirely a function of time, the right hand side is entirely function of space in r, they are equal therefore they will be equal to some constant, let us say this constant is a negative constant. We can prove that if this constant is 0 or positive we will be getting a trivial solution, but we are not looking for a trivial solution, we are looking for a non-trivial solution. Hence, we solve the r varying part d dr of r dR dr minus n square divided by r multiplied by R plus lambda square r R is equal to 0.

Now, let us set up the boundary condition on R. The boundary conditions on R must be satisfying the boundary condition of the original problem in the r direction or coordinate system.

Therefore, the boundary conditions will be at r is equal to 0, R is bounded, that means it is finite; at r is equal to 1, R is equal to 0, so these two boundary conditions. If you remember, the original problem used to satisfy in the r direction so R varying part must satisfy these two conditions as well.

Now if you see the nature of this equation, this equation is nothing but the Bessel equation of order n, so the solution of this equation will be constituted of the functions nth order Bessel function J n r and Y n r.

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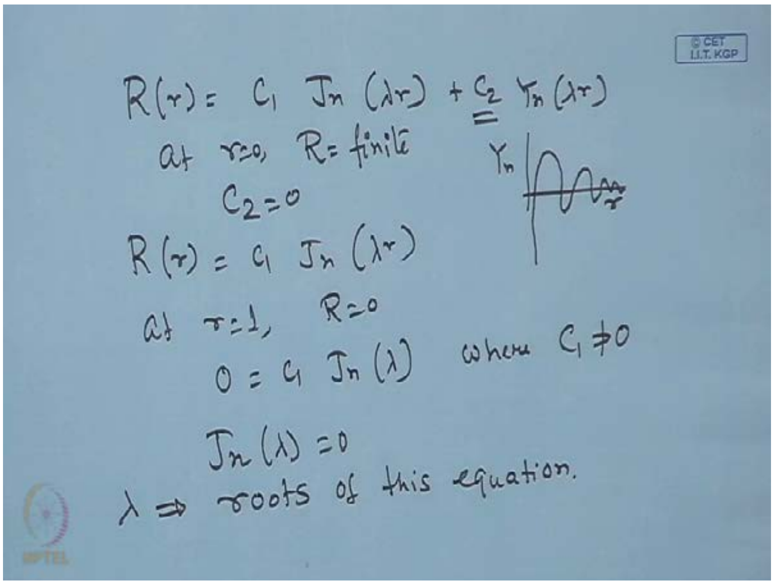

Therefore let us write down the solution of R varying part, the solution is C 1 J n lambda r plus C 2 Y n lambda r.

Now we have two boundary conditions on R, we utilize them - at r is equal to 0 your R is finite, if you remember the variation of Y n as a function of r, Y n is infinite at r is equal to 0, so therefore it increases from infinity and then it oscillates about the r axis with a diminishing magnitude; therefore to have a finite value of R at r equal to 0, the associated constant must be is equal to 0; so C 2 is equal to 0 we have the solution as C 1 J n lambda r.

Now, we utilize the other boundary condition - r is equal to 1, R is equal to 0 so 0 is equal to C 1 J n lambda where C 1 is not equal to 0, if C 1 is equal to 0 then we are going to get a trivial solution, therefore the other option is J n lambda is equal to 0.

Now, what is this lambda? Lambda is basically the roots of this equation; again you have standard readymade program in C or FORTRAN where you can compute the roots of this equation or 0s of this equation to find out the eigenvalues.

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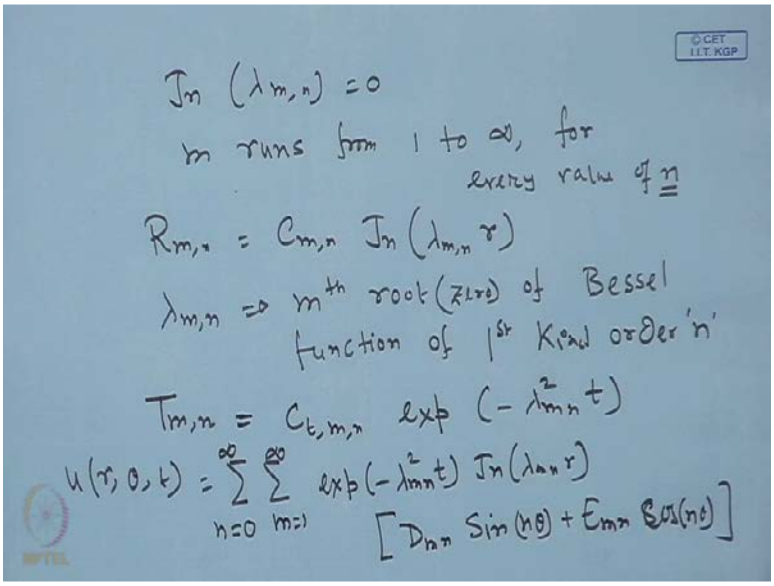

So, eigenvalues of this problem - we write it as J n lambda m n is equal to 0, so eigenvalues are lambda m, so m runs from 1 to infinity for every value of n. What is this n? These n are basically whatever we have found by solving the theta varying part, so we write R m n is equal to C m n J n lambda m n times r. What is lambda m n then? Lambda m n is nothing but m^{th} root or 0 of Bessel function of first kind of order n.

So, that solves the problem; then we get the expression of T m n. If you look into the time varying part you can easily find out that its solution is some constant multiplied by exponential minus lambda m n square times t.

We get the complete solution of u r theta t as 2 summation 1 is over n and if you remember that n varies from 0 to infinity, it is not from 1 to infinity, m varies from 1 to infinity, exponential minus lambda m n square t J n lambda m n r multiplied by D m n sine n theta plus E m n cosine n theta; because, theta varying part contains both the terms sine and cosine and the associated constants are not equal to 0; therefore, we will be getting this solution as a general solution to this problem.

Now, the only thing that is left is that one has to find out the constants D m n and E m n. We have the initial condition at t is equal to 0 unutilized, so we can utilize the initial condition.

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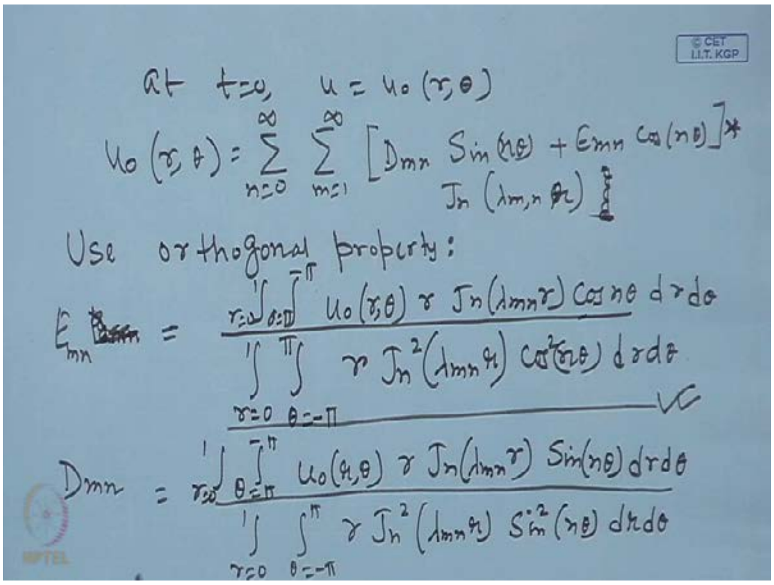

We use the initial condition at t is equal to 0, u is equal to u naught as a function of r theta. In general, it may be a constant, it may be a prescribed function of r, a prescribed function of theta or it may be a prescribed function of r and theta.

If you write it down then u r theta becomes summation of n is equal to 0 to infinity, summation m is equal to 1 to infinity, you have D m n sine n theta plus E m n cosine n theta multiplied by J n lambda m n r.

Now, we use the orthogonal property of sine functions, cosine function and Bessel function; the eigen functions, sine functions and cosine functions are orthogonal to each other as well as the Bessel functions are orthogonal to each other. We have already proved that earlier in the last class - that Bessel functions are orthogonal functions. So, using the orthogonal property one can evaluate the constants D m n and E m n, we have done these thing lots of time earlier so I am not going to do it once again.

I am just writing the final expression, at r is equal to 0 to 1 integral theta is equal to minus pi to plus pi - that is domain of variation of theta, r J n square lambda m n r cos square n theta dr d theta - that is the denominator, in the numerator we have r is equal to 0 to 1 and theta is equal to pi to minus pi u 0 r theta r J n lambda m n r cosine n theta d r d theta, this is the value of E m n because it is associated with cosine n theta so it will be cos square n theta so this is not the D m n this is the expression of E m n. (Refer Slide Time: 27:08).

Now, let us write down the expression of D m n, the expression of D m n goes like this - r is equal to 0 to 1 theta is equal to minus pi to plus pi r J n square lambda m n r sine square n theta dr d theta r is equal to 0 to 1 theta is equal to plus pi to minus pi u 0 r theta r J n lambda m n r, this will be sine n theta dr d theta. So, that is the expression of D m n and E m n one can get, and knowing the prescribed function of u 0 r theta you can evaluate this integral analytically or one can take request to the numerical techniques to evaluate this integral using Trapezoidal rule or Simpson's one third rule.

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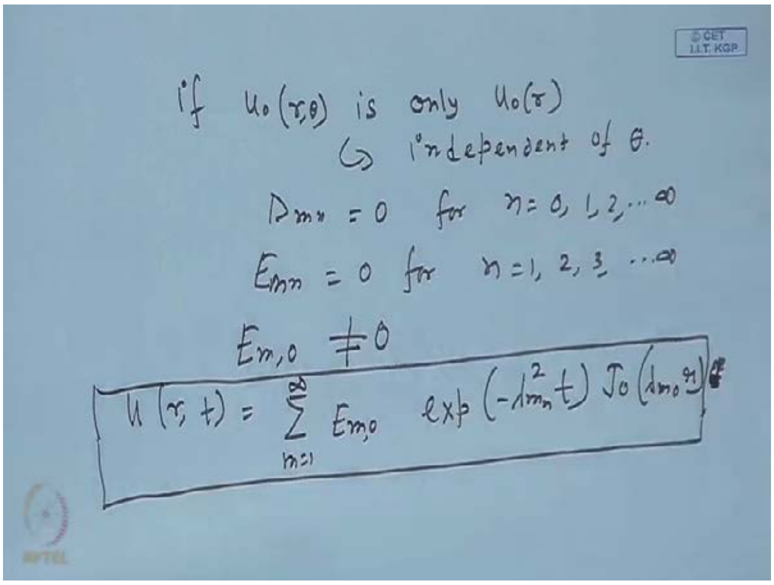

Now, the point is if we can do a slight simplification - if u 0 r theta is only u 0 r, that is, it is independent of theta then you will be having D m n is equal to 0 for n is equal to 0, 1, 2 up to infinity. E m n is equal to 0 for n is equal to 1, 2, 3 up to infinity, so the difference is, here n starts from 1, 2, 3 and here n starts from 0, 1, 2, 3.

So, E m 0 is not equal to 0, the solution boils down to u r t is equal to summation m is equal to 1 to infinity E m 0 exponential minus lambda m n square t J 0 lambda m 0 r, so, that is the solution in this particular case if it is independent of theta. If it is dependent on theta then we will be having both the constants intact and in terms of the formulation, we have just done now, that will give you the complete solution.

Let me summarize, we have seen how the problem of cylindrical polar coordinate can be solved in two-dimensional case and can be solved in three-dimensional case, that gives us a fair idea. The difference between the two cases is - in case of two-dimensional, the boundary conditions on theta will be provided by the periodic boundary conditions, so these are the physical boundary conditions they are dictated by the physics of the problem.

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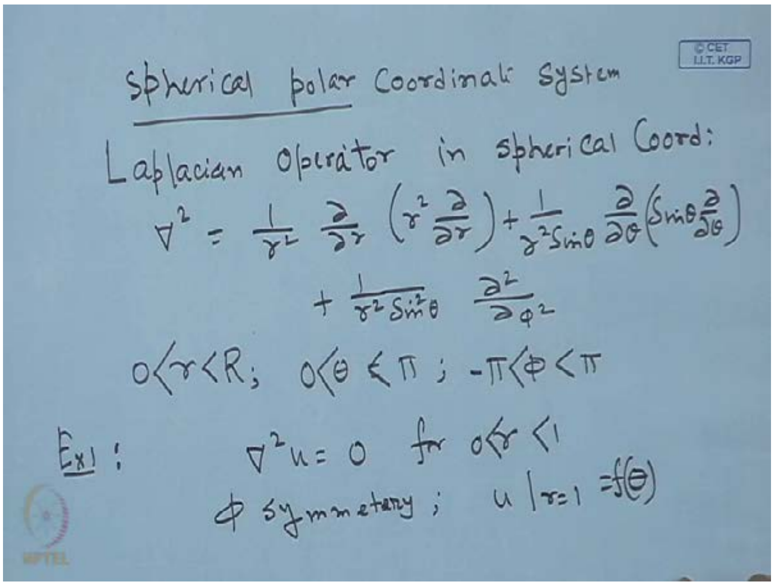

Next, we look into the spherical polar coordinate system and these examples in chemical engineering are many. Spherical polar coordinate system - whenever we are talking about the heat transfer characteristic in a sphere or in a solid sphere or in hollow sphere, these equations will come in place.

Now let us look into the Laplacian operator in spherical coordinate. Grad square is nothing but 1 over r square del del r r square del del r plus 1 over r square sine theta del del theta sine theta del del theta plus 1 over r square sine square theta del square del phi square. This is the operator in spherical polar coordinate system, so r varies from 0 to R, theta varies from 0 to pi and phi varies from minus pi to plus pi.

Now we take up one example first. The first example is solution of grad square u is equal to 0 for r lying between 0 and 1; maybe it is a non-dimensional case and we have phi symmetry. If we have a phi symmetry that means phi varying parts will be equal to 0, so we will be having only 2 dimensions in r and theta, and we put the boundary condition that u at r is equal to 1 should be is equal to, in general, some function of theta.

This function of theta may be constant, may be 0, or in general, it may be some function of theta at r is equal to 1. Now since we have the non-homogeneity at f of theta, it is independent of phi, therefore u is independent of phi and we have also the phi symmetry.

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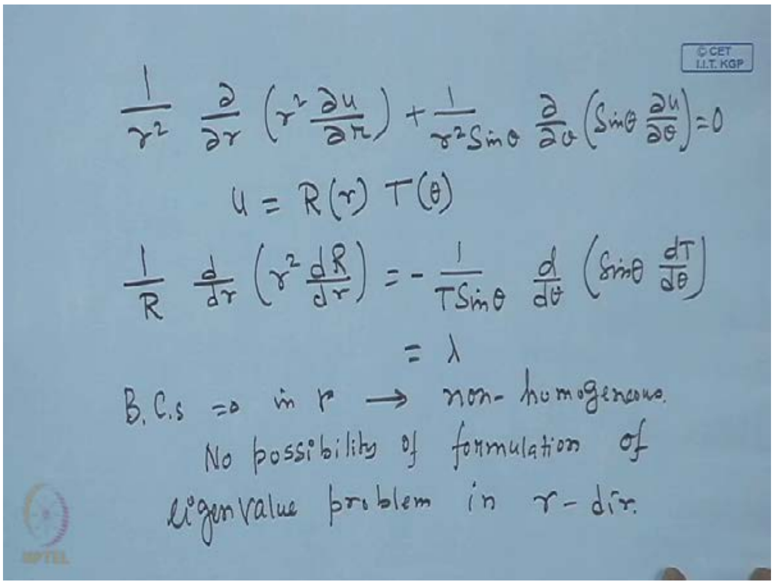

So we write down the two-dimensional problem as 1 over r square del del r r square del u del r plus 1 over r square sine theta del del theta sine theta del u del theta is equal to 0. Now since this problem is linear, the operator is linear, the boundary conditions etcetera are linear, therefore we break down this problem into 2 parts - that is one is the r dependent part, another is the theta dependent part.

So we assume two sub-problems that is entirely a function of r and entirely a function of theta, they can be multiplied and the product will be constituted in the solution. So we separate the variable as we have done earlier, 1 over r d dr of r square dR dr is equal to minus 1 over T sine theta d d theta sine theta d T d theta. The left hand side is entirely a function of r and the right hand side is entirely a function of theta, so they will be equal to some constant lambda.

Now the boundary conditions, if you see, the boundary conditions in r is non-homogeneous, so we cannot formulate an eigenvalue problem in the r direction, that has to be made very clear; so no possibility of formulation of eigenvalue problem in r direction because the boundary condition is not homogeneous on that boundary.

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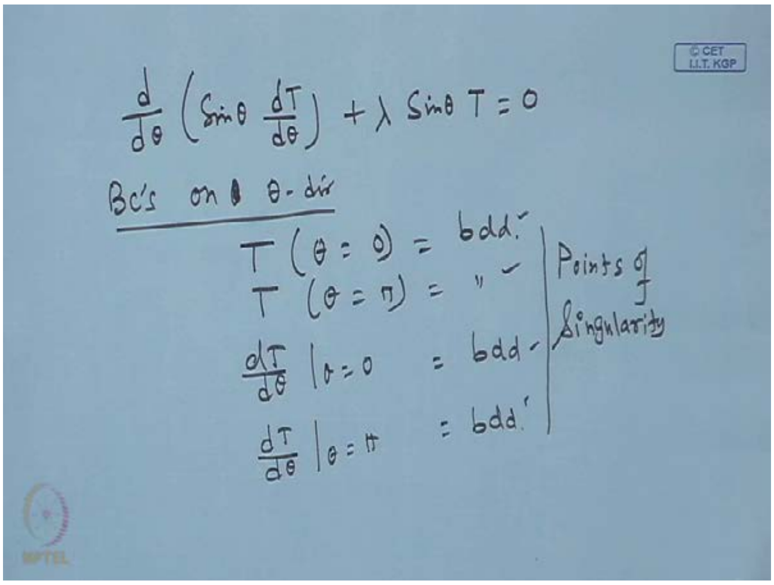

Let us look into the theta varying part - d d theta of sine theta dT d theta plus lambda sine theta T should be is equal to 0. Now let us fix up the boundary conditions on theta direction; we do not know exactly what boundaries are present, therefore we put the conditions so that they become physical, so we use the physics of the problem and impose the physical boundary conditions, so the solution does not become singular. If the solution becomes singular then the solution does not exist at those points, so based on that we fix up the physical boundary conditions on theta.

If you write down T at theta is equal to 0 - this is bounded, T at theta is equal to pi - this is bounded, dT d theta at theta is equal to 0 - it is bounded, dT d theta at theta is equal to pi - again it is bounded. From where these boundary conditions are coming - these boundary conditions are coming since these are the points of singularity.

Since the solution exists at these points, therefore we will be having a finite value at those points and this becomes the singular points, we impose that the solution becomes bounded at those points; therefore, these are the perfect examples of physical boundary conditions that we have talked earlier.

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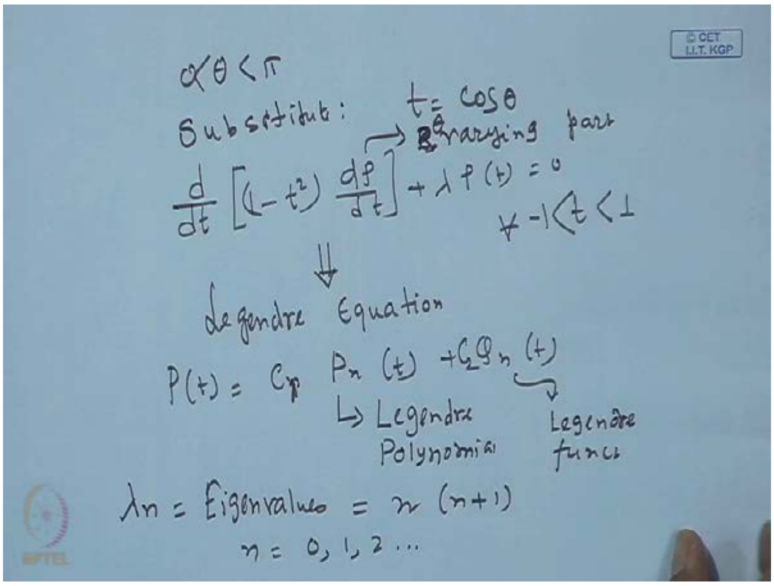

Now let us solve this theta varying part - so theta varies from 0 to pi and we substitute t is equal to cosine theta, if we do that the form of the theta varying part becomes d d t of 1 minus t square dp dt plus lambda P t will be is equal to 0 for t varying from minus 1 to plus 1. If you remember this is nothing but the Legendre equation and Legendre polynomial is the solution of this Legendre function.

So solution is P of t is nothing but C 1 P n t plus C 2 Q n t. Now, P n is nothing but the Legendre polynomial and this is Legendre function; this p is nothing but the R varying part. We identify, this R varying part is basically equal to R, I just wrote it in the form of p, so it does not matter.

If you look into this problem, we have seen earlier - the eigenvalues lambda n is the eigenvalues to this problem. The eigenvalues we have discussed - the properties of the Legendre polynomial, Legendre function earlier, the eigenvalues are n into n plus 1 and where the index n runs from 0 1 2. There is one mistake - this p is not R varying part it is the theta varying part (Refer Slide Time: 40:05). We are solving the theta so we make it theta varying part because we are solving the eigenvalue problem in the theta direction not in r direction, because r direction itself is boundary condition, is homogeneous so we cannot formulate the eigenvalue problem in the r direction. This is the governing equation of theta varying part; after substituting t is equal to cosine theta which will be in the form of the Legendre equation, the solution will be in terms of Legendre polynomial and Legendre function.

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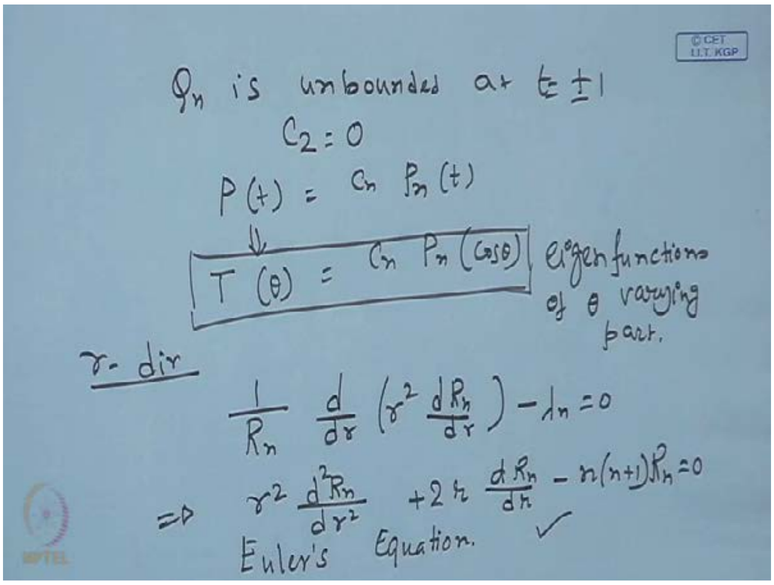

Therefore, we can solve the theta varying part completely and if you look into the property of Legendre function, the property Q n is unbounded, the Legendre function is unbounded at t is equal to plus minus 1; therefore, the associated constant with this must be equal to 0, so we will be having the P of t is C n P n t - are the eigen functions in terms of T. The theta varying part, T of theta should be is equal to C n P n cosine theta, so this is the theta varying part and that is the eigen functions in the theta varying part.

Eigenvalues are n into n plus 1 and the index n runs from 1 to infinity. Now let us go back to the r direction and see how we can solve that part in the r direction, we cannot formulate the eigenvalue problem in the r direction simply because we have the non-homogeneous boundary condition there, so we write it at R subscript n corresponding to nth eigenvalue so d dr of r square dR n dr minus lambda n is equal to 0. If you just open up this differentiation, this becomes r square d square R n d r square plus 2 of r dR n d r minus n into n plus 1 R n is equal to 0.

So, lambda n is nothing but n into n plus 1; therefore, this is the equation. Now if you remember, if you recognize this equation, this equation is nothing but Euler's equation. When we were discussing about the special ordinary differential equations, we discussed about the Bessel equation, we discussed about the Legendre equation, we discussed about the Euler's equation, so this is an application where Euler's equation will be coming into play and we have to look for the solution of this equation.

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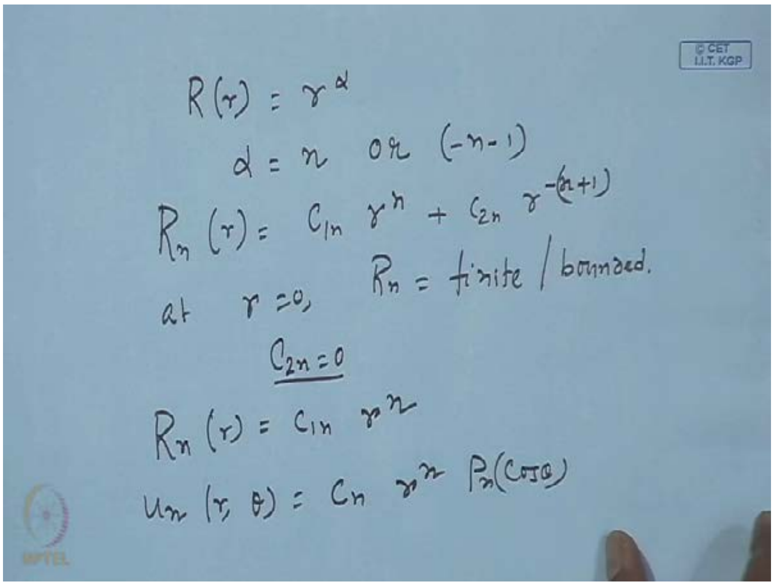

The solution of this equation will be the form of r to the power alpha, so you will be getting a quadratic. As we have seen earlier, this alpha becomes either n or minus n minus 1, so these are the two roots of the quadratic; solution becomes R n of r is C 1 n r to the power n plus C 2 n r to the power minus n plus 1.

Now, if you remember, we have the boundary condition at r is equal to 0, this R n was finite or bounded. In order to satisfy this boundary condition, the associated constant has to be equal to 0, therefore R n is equal to r is nothing but C 1 n r to the power n. Therefore, u n r theta becomes combination of C 1 n and d theta, so it will be some constant, let us say C n, multiplied by r to the power n P n cosine theta. This is the solution corresponding to nth eigenvalue; the complete solution can be obtained if you sum up all the contributions.

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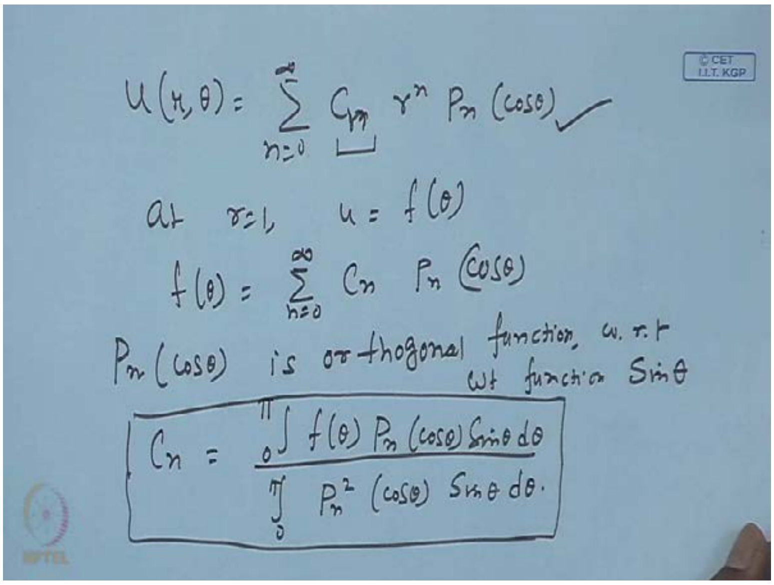

So, u as a function of r and theta becomes summation n is equal to 0 to infinity, C 1 n r to the power n, may be, C n r to the power n p n cosine theta, so that is the solution. Now we have integration constant C n which has to be evaluated. If you remember, we have one condition that still remains unutilized. What is that condition? That condition is at r is equal to 1, u r theta is non-homogeneous; so, at r is equal to 1, u is equal to f of theta. Therefore, we utilize this boundary condition and evaluate the constant C n; so f of theta is nothing but n is equal to 0 to infinity. Whenever we evaluate r this becomes 1, so this will be nothing but C n P n cosine theta and we can prove that P n cosine theta is orthogonal function with respect to weight function sine theta.

Therefore, if you remember in the Cartesian coordinate, the sine function and cosine functions are orthogonal function with respect to weight function 1; in cylindrical coordinate the Bessel functions are orthogonal function with respect to weight function r; in the spherical polar coordinate the eigen functions are orthogonal to each other with respect to weight function sine n theta.

Therefore, we write down the complete solution; we multiply both sides by P m cosine theta d theta and integrate over the theta, that is, 0 to pi. All the terms will vanish except m is equal to n, it will be 0 to pi f of theta P n cosine theta sine theta d theta, because it is a weight function, so 0 to pi P n square cosine theta sine theta d theta.

That gives the estimation of the integration constant C n; once we get the estimation of integration constant C n we can put it back here and we will get the complete solution in r and theta.

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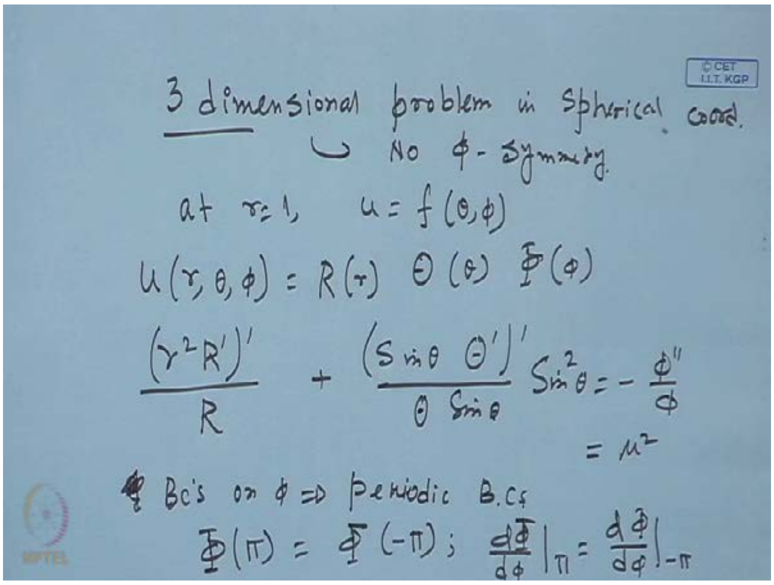

Next we look into one more example, this example deals about the spherical coordinate but it is not two-dimensional, it is a three-dimensional problem in spherical coordinate - no phi symmetry in this case. The boundary condition - at r is equal to 1, you have u is equal to f theta and phi in general. So, it may be a function of theta alone, it may be a function of phi alone, it may be a constant, it may be function of theta and phi both. In the earlier case if you remember, we had the condition that r is equal to 1, u is equal to f of theta.

So, we constitute the solution as a function of r theta phi by multiplying 3 functions which are functions of r alone, theta alone, and phi alone. I am just writing them R as a function of r, theta as a function of theta and capital phi as a function of phi.

If you substitute in the governing equation of Laplacian in spherical polar coordinate system and then separate the variables you will be getting this. I am not solving this for you, please do it by yourself; we have done this type of things lots of times earlier.

Differentiation is with respect to the variable, so, plus sine theta theta prime, prime of that divided by theta sine theta sine square theta is equal to minus phi double prime divided by phi. Here the prime is with respect to small r, here the prime is with respect to small theta, and here the double prime is differentiation with respect to small phi.

Now, we put it as a constant; so the left hand side is a function of r and theta only, right hand side is a function of phi, so we put it as a constant. We have 4 boundary conditions on phi, these are the periodic boundary conditions; so 4 boundary condition means they will be equal - periodic boundary conditions - so boundary condition phi we define as capital phi at pi is equal to capital phi at minus pi; d phi d phi at pi is equal to d phi d phi at minus pi; so these are the periodic boundary conditions, these are the physical boundary conditions; we use the periodic boundary conditions here.

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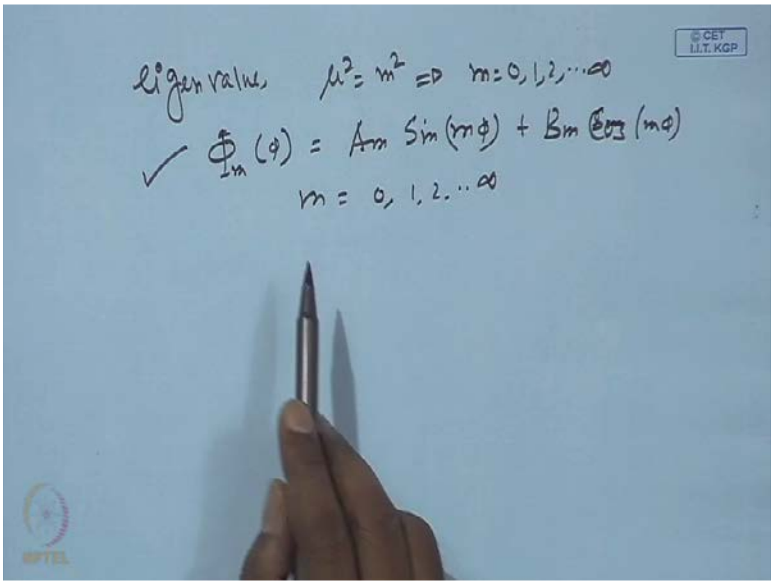

So, we can find out the eigenvalues and eigen functions in the phi direction; if you do that the eigenvalues become mu square is equal to m square, that we have already seen; and m is equal to from 0, 1, 2, 3 up to infinity; we have already seen this thing in earlier case and therefore, phi m phi is nothing but A m sine m phi plus B m cosine m phi, where the index m runs from 0, 1, 2 up to infinity.

In case of three-dimensional cylindrical polar coordinate, we have a situation like this with periodic boundary conditions and we have seen there as well as the eigen functions are constituted by the sine function and cosine function. So, in this case also, we will be getting the eigen function which will be constituted by the linear combination of sine function and cosine function and the index m runs from 0, 1 to infinity instead of 1 to infinity.

So, that solves the phi varying part completely, then we will be looking into the theta varying part and r varying part and we will be constructing the solution completely in the next class.

I stop it here, in the next class I will be completing this problem and then we will look into the solution of partial differential equations, but those are non-homogeneous using Green’s function method.

Thank you very much.