Advanced Mathematical Techniques in Chemical Engineering

Prof. S. De
Department of Chemical Engineering
Indian Institute of Technology, Kharagpur

Module No. # 01
Lecture No. # 30
Solution of Elliptic and Hyperbolic PDE

Good afternoon everyone. So, we were looking into the solution of partial differential equations. In the first example, you have taken about for the Cartesian coordinate and in that, we talked about the parabolic partial differential equation, solution of parabolic partial differential equation using different types of boundary conditions, homogeneous, non-homogeneous, and you know one-dimensional, two-dimensional, three-dimensional, and we have covered the parabolic partial differential equation extensively, because in chemical engineering applications, most of the transient problems are expressed in the form of parabolic partial differential equations.

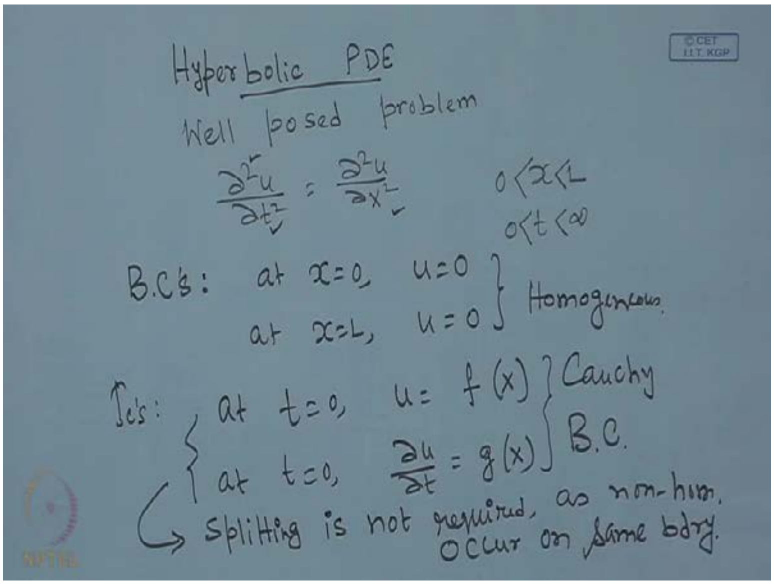
Next, we talk about the solution of elliptic partial differential equation. These conditions are also quite prevalent. These governing equations are also quite prevalent in chemical engineering applications, because the elliptic partial differential equations correspond to the steady state operations, multi-dimensional steady state operations.

Now, we will be looking into the one type of partial differential equation. That will be the hyperbolic partial differential equation, and will be solving a basic problem, and we have seen enough of the actual problem where, there are lots of many sources of non-homogeneities are present, and we have to break down the problem into sub problems. Considering, one non homogeneity at a time also when you will be having a zero initial condition, homogeneous initial condition and non-homogeneous boundary condition for that problem, again you have to divide that problem into two sub problem, considering one non-homogeneity, one time dependent part and one time independent part.

Then, you redefine the ill posed problem into well posed problem, and the initial condition of the well posed problem will become the steady state solution with a negative sign attached to it, and it is all boundary conditions are big, they become homogenous.

So, we have looked into the elliptical partial differential equations, parabolic partial differential equations. For the sake of completeness, I will be taking of hyperbolic partial differential equations. But please remember in actual main stream chemical engineering processes, the occurrence of hyperbolic partial differential equations are not quite common in some special fields they will be occurring.

(Refer Slide Time: 02:55)



So, for the sake of completeness will be looking into the solution of hyperbolic partial differential equations by using separation of variable and will be talking about a well posed problem. Only the ill posed problem can be tackled. The same way we have done exactly earlier.

So, let us consider del square u del t square is equal to del square u del x square is the governing equation, where the domain x lying between from 0 and L and domain t is of course between 0 to infinity t is greater than 0 always. Now, what we did we equate to have two initial conditions and t and two boundary conditions on t the one on x the boundary conditions are at x is equal to 0, u is equal to 0 at x is equal to L, u is equal to 0.

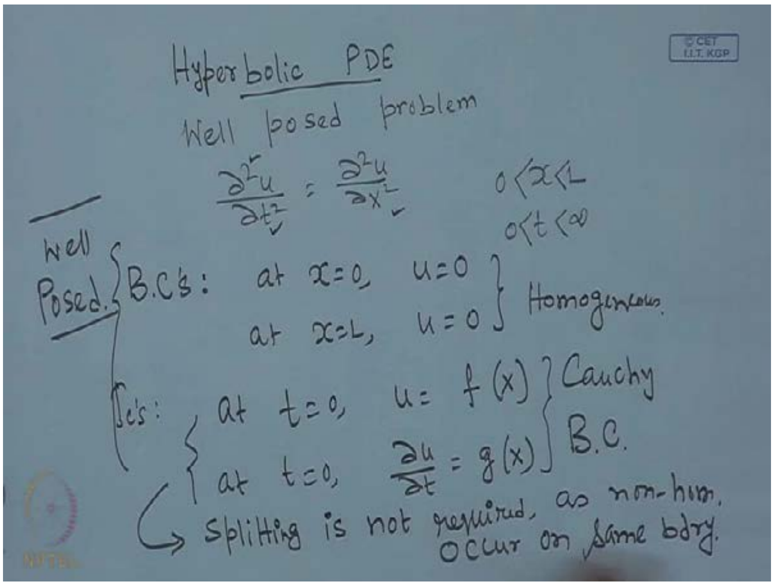
So, this boundary conditions are homogenous and we require to have two initial conditions, because it is order to with respect to t. So at t is equal to 0 , we define u is equal to some function. Let us say some prescribe function of x, and at t is equal to 0. Define del u del t some prescribe function of x. Please remember and note that these two boundary conditions are specified on the same boundary. Therefore, they are known as the Cauchy boundary condition.

Now, although these two boundary conditions are homogeneous, we need not to split it up. Splitting up that means breaking down the problem into sub problem is not required. Because this non-homogeneities occur at the same boundary, occur on same boundary.

So, we treat this problem as a well posed problem because if you just look into the nature of this problem, if you look into the initial and boundary conditions, both the initial conditions are non-homogenous.

If you remember whatever we have discussed about the well posed problem, and is ill posed problem earlier is that if the boundary conditions are homogenous, s and the initial condition is non-homogenous, then the problem is defined as a well posed problem. When the initial condition is homogenous, and one of the boundary condition becomes non-homogenous, then the problem becomes ill posed problem.

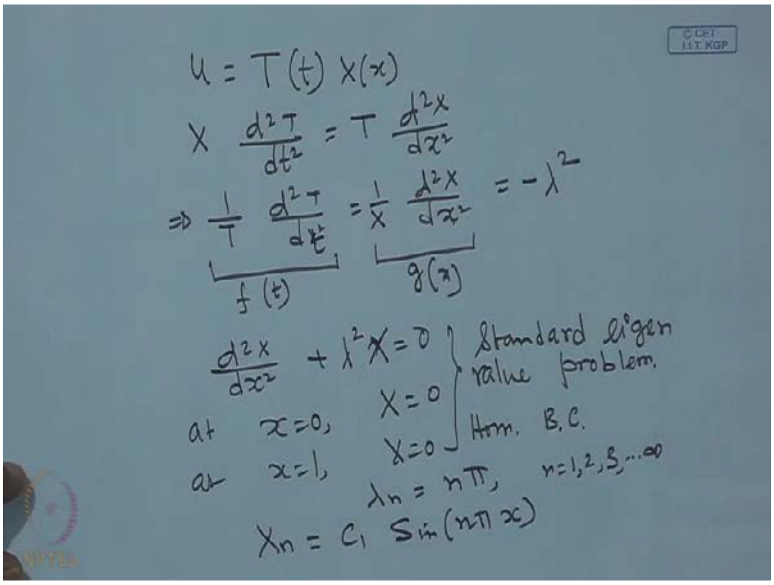
(Refer Slide Time: 06:11)



So, therefore this problem where the initial conditions both are specified and boundary conditions homogenous, then this is a well posed problem.

So, we have already seen that for a well posed problem, we can directly use separation of variable solution.

(Refer Slide Time: 06:42)



So, what we do next? We consider that u is a product of two functions which will be entirely function of time, and another is entirely function of space. So, if you substitute this in the governing equation, this becomes x d square T d. t square is equal to T d square x d x square so this becomes 1 over T d square T d t square is equal to 1 over X d square x d x square.

Now, the left hand side is a function of time only. The right hand side is a function of space only. They are equal, and they will be equal to some constant. Again if this constant is positive, and 0 will be getting, will be landing up with a trivial solution. So, this constant has to be a negative constant.

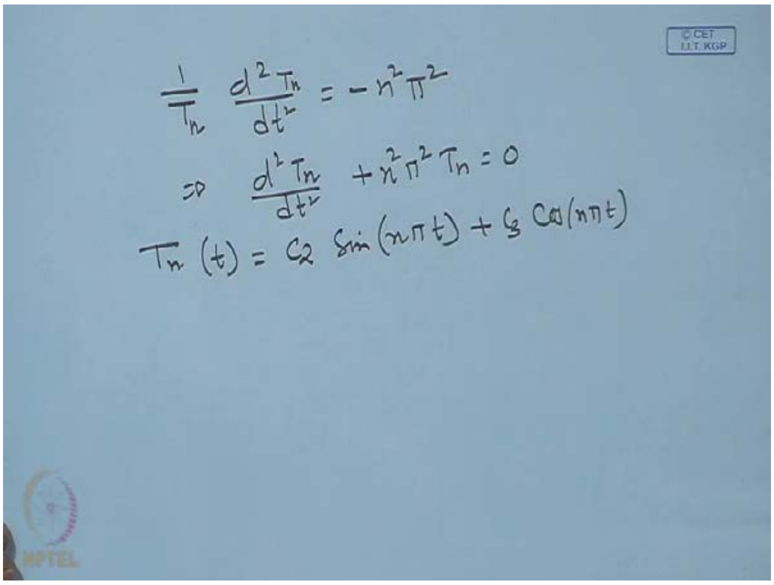
So, therefore we can formulate the standard Eigenvalue problem. Now both the ordinary differential equations are of order to so which one will be form in which direction either in t direction, or x direction will be forming the standard Eigenvalue problem.

It will be of course in the x direction will be formulating the standard Eigenvalue problem, simply because the boundary conditions are homogenous in that direction.

So we form d square X d x square plus lambda square X is equal to 0, and the boundary conditions will be at x is equal to 0 capital X is equal to 0 at x is equal to 1 capital X is equal to 0. So this is a standard Eigenvalue problem with homogenous boundary conditions.

So, therefore we know the solution of this problem lambda n are the Eigen values n pi where n is equal to 1, 2, 3 up to infinity, and Eigen functions are sin functions c 1 sin n pi x. So, we solve the spatial part completely we get the Eigen functions.

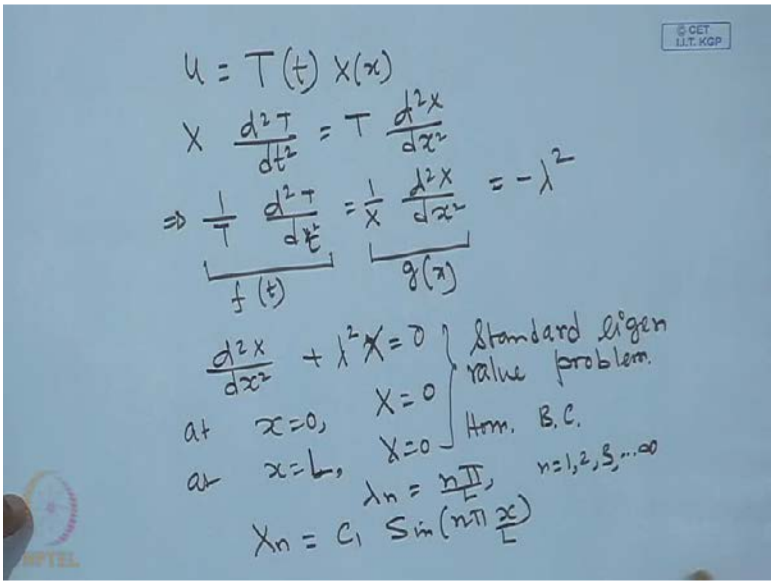
(Refer Slide Time: 09:19)



Now, let us look into the time varying part. If we look into the time varying part, this becomes 1 over T d square T d t square is equal to minus n square pi square. We put it lambda corresponding to n. So T will be replaced by there will be we consider T as nth solution. So T substitute n corresponds to nth Eigen value. So this will be d square T n d t square is equal to plus n square pi square T n is equal to 0.

If you look into the solution to this problem again the characteristic equation will be m square plus m square is equal to minus n square pi square. So plus minus i n pi will be the solution. So the corresponding equation will be the solution will be formulated by the combination of sin functions and cosine functions. So therefore T n as a function of t becomes c 2 sin n pi t plus c 3 cosine n pi t.

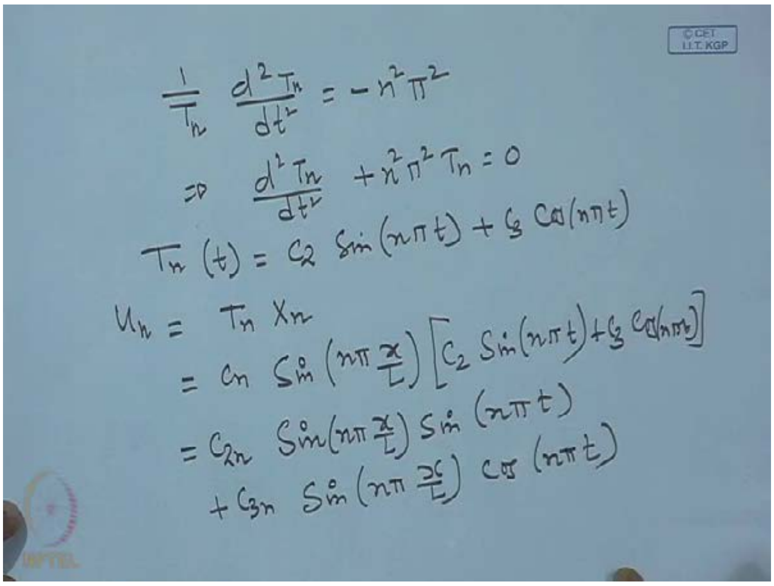
(Refer Slide Time: 10:35)



If you remember there was a slight mistake does not change the solution at all. The x if you look into the original problem, the original problem was defined as at x is equal to 0. The other boundary was defined at x is equal to L instead of 1.

So we put it x is equal to L, so lambda n becomes n pi by L and X n. Eigen functions become sin n pi x by L. Therefore will be having the Eigenvalues as n pi by L and Eigen functions as sin n pi x by L.

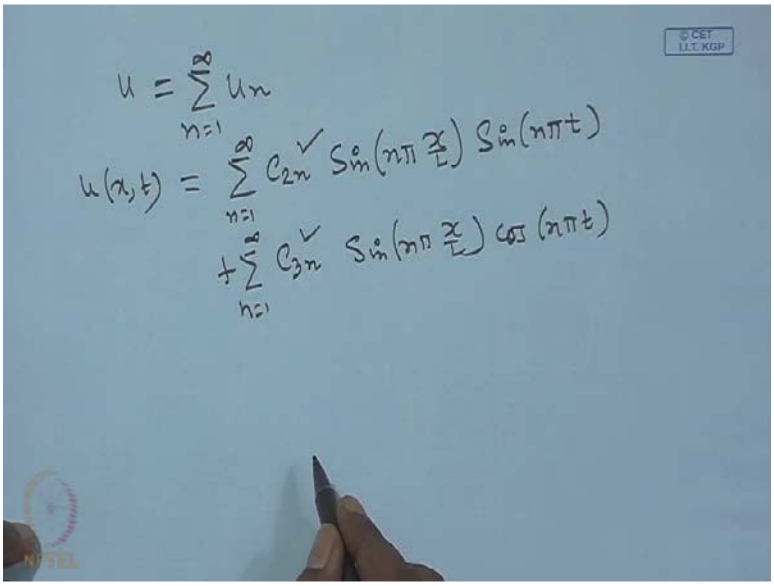
(Refer Slide Time: 11:10)



So we will be getting the corresponding nth solution that will be nothing but multiplication of T n and X n. So, this will be c 2 multiplied by c 1. So, it will be c n sin n pi x by L multiplied by c 2 sin n pi t so this will be does not matter plus c 3 cosine n pi t.

So, the solution will be c 2 n c n and c 2. It will be multiplied, new constant will be c 2 n sin n pi x by L times sin n pi t and plus c 3 n is sin n pi x by L cosine n pi t.

(Refer Slide Time: 12:29)

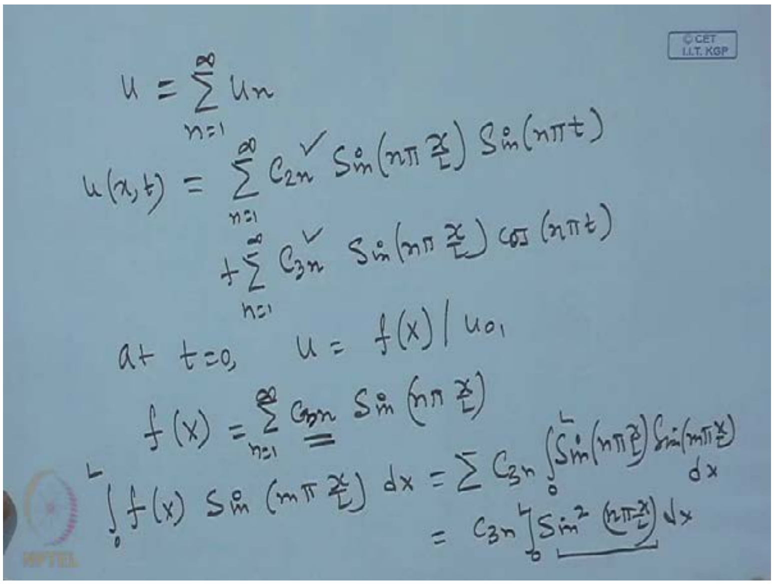


Now, you have 2 constants to be evaluated. So, this is the nth coefficient nth solution. Now the solution u will be composed as a linear super position of all the solutions. So u will be nothing but linearly superposed summation of u and n is equal to 1 to infinity. So, u is x t it becomes summation of c 2 n sin n pi x by L sin n pi t, plus summation c 3 n n is equal to 1 to infinity sin n pi x by L cosine n pi t.

Now only two things, the other complete solution is ready. The two constants c 2 n and c 3 n need to be evaluated. Now, these 2 constants will be evaluated. If you remember we have in the original problem , we had 4 boundary conditions. Two conditions on t is equal to 0 and two conditions on x.

Now, we have already utilized the two boundary conditions on x and we arrived at the solution of Eigen corresponding the Eigen functions sin n pi x by L.

(Refer Slide Time: 13:50)

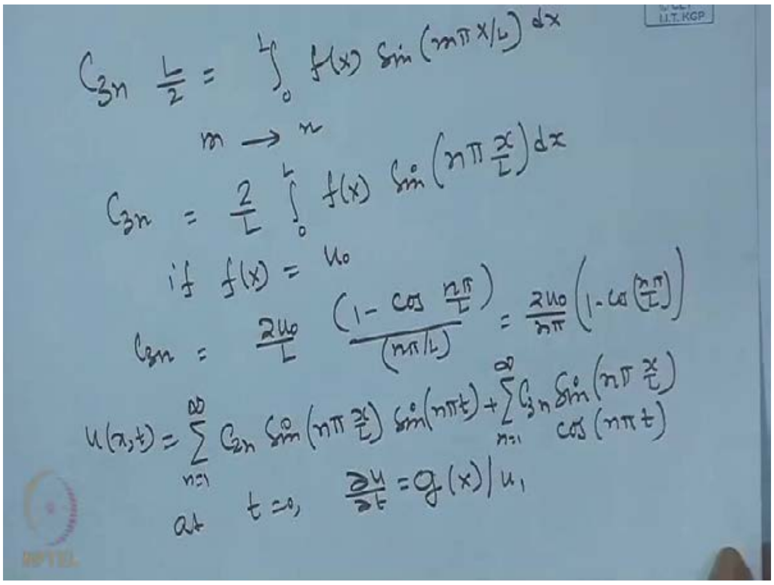


Now, will be utilizing the left over two initial conditions. This where at t is equal to 0 u is equal to some of some prescribe function of x or it may be a constant as well u 0 1

So, if we put in generally some constant function of x f x. So f x will be is equal to when Iput t is equal to 0. This part becomes 0. The whole equation becomes 0. So only c 3 n will survive summation of c 3 n sin n pi x by L and cosine 0 becomes 1. So, we have now, we can estimate the constant c 3 n by multiply both side by sin m pi x by L d x and integrated out over the domain of x from 0 to L.

So f of x sin m pi x by L d x from 0 to L and summation c 3 n integral sin n pi x by L sin m pi x by L d x from 0 to L so if we open up this summation all the terms will vanish because of the orthogonal property of the Eigen functions only one term will survive that will be c 3 n when m becomes equal to n sin square n pi x by L integral d x 0 to L and we know the solution of this. This will be 2 by L in this particular case, so c 3 n we can obtained.

(Refer Slide Time: 16:00)



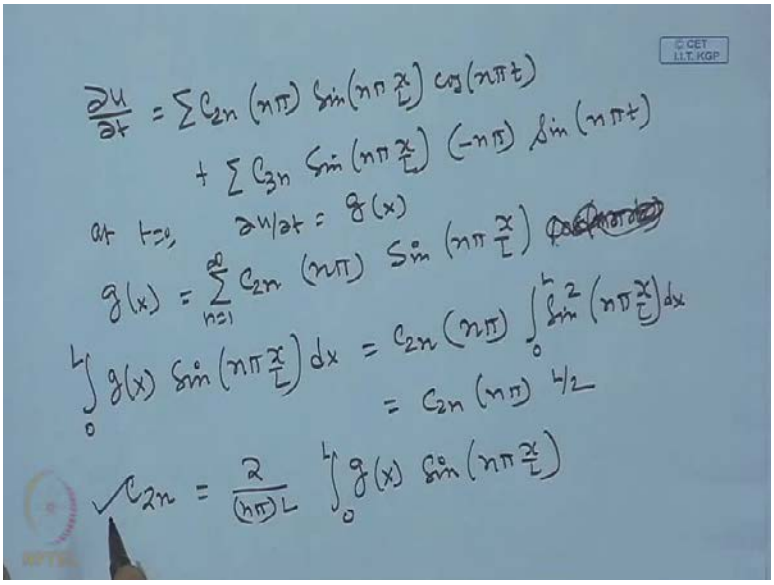
So this becomes L by 2. So, this becomes L by 2. So, we will be getting c 3 n multiplied by L by 2 is integral 0 to 1 o to L f of x sin m pi x by L d x and changing the index from m to n will be getting c 3 n is equal to 2 by L 0 to L f of x sin n pi x by L d x. Now for if f x is a known function we can integrate out if f x is constant.

Let us say u 0 then c 3 n will be nothing but 2 u 0 by L 1 minus cosine n pi by L divided by n pi by L. So, L L will be cancelled out. So it will be 2 u 0 over n pi 1 minus cosine n pi by L, so that will be the value of the constant c 3 n. Similarly the value of the constant c for n can be evaluated from the other boundary condition.

For that we have to write down the expression of u x t u x t was summation of c 2 n sin n pi x by L sin n pi t plus summation c 3 n sin n pi x by L cosine n pi t. So, if you remember the other boundary condition is from other boundary condition was at t is equal to 0 del u del t is equal to 0.

So del u del t is equal to some function. Write g of x if you remember for a general case it is g of x it may be a constant, as well it may be u 1 if it is a constant.

(Refer Slide Time: 18:25)



Now then we can differentiate this expression with respect to t. So what will be getting is del u del t is nothing, but summation c 2 n n pi sin n pi x by L cosine n pi t differentiated of that so n pi has taken out plus summation of c 3 n sin n pi x by L cosine n pi t. So, minus n pi will be there into sin n pi t. Now we put the boundary condition at t is equal to 0 del u del t will be is equal to g of x.

So g of x when you put t is equal to 0. This term the second term will vanish, but the cos 0 becomes 1 so c 2 n will survive c 2 n n pi sin n pi x by L cosine n pi t.

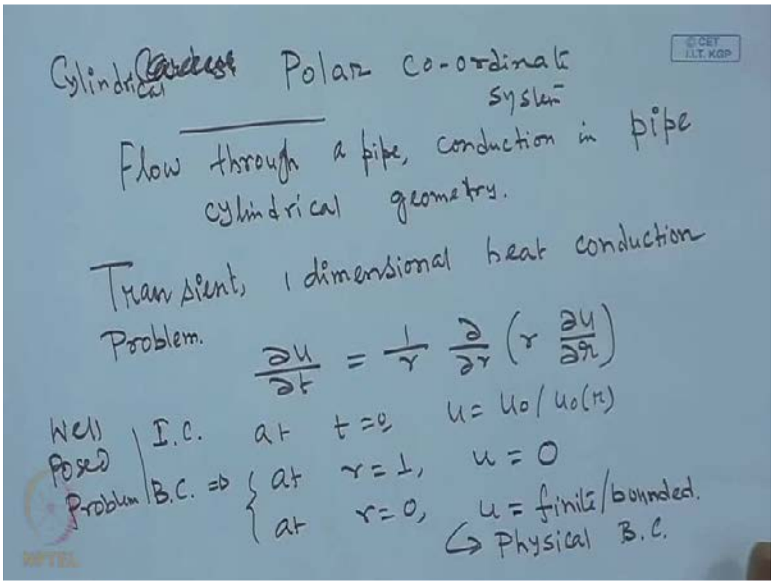
Now, we multiply both side by ah cosine n pi t will be 1. So, we are putting t is equal to 0. So it becomes cos 0. So this cosine n pi t becomes 1. So you will be getting this equation. We multiply again by both side by sin m pi x by L d x and integrate over the domain of x from 0 to L, and we carry out the open up this summation series. All the terms will vanish because of the orthogonal property of the sin functions. Only one term will survive that is m is equal to n.

So, therefore you will getting integral g of x sin n pi x by L i. Change the index from m to n d x from 0 to L is equal to c 2 n into n pi sin square n pi x by L d x integral 0 to L, and this will be L by 2 so c 2 n into n pi L by 2. So you can calculate evaluate c 2 n by 2 divided by n pi times L integral 0 to L g of x sin n pi x by L.

So, we will be evaluating the c 2 n value. So one can get the c 3 n. As we have done earlier, we can get the c 2 n. If you know the functional form of g x either constant or may be something x or x square e to the power k x, we can evaluate this integral by integration by parts.

So that will completely solve and specify the problem. So that way a basic problem or a well posed hyperbolic problem can be solved, and will be we are almost at the end of the solution of partial differential equation in cartesian coordinate. What we have done? We have taken up all source of boundary conditions, homogenous, non-homogenous. We have looked into the different types of differential equations, like parabolic hyperbolic elliptical and there are various variants.

(Refer Slide Time: 22:02)



Now, will be moving over to the cartesian polar coordinate system. We looked into the cartesian coordinate system. Now will be looking into the polar coordinate system, cylindrical polar coordinate system.

So, this is cylindrical polar coordinate system. This will be quite popular. They will be apparent whenever we are solving a cylindrical polar coordinate. For example, flow through a pipe conduction, in a pipe may be solid, may be angular. That means whenever the geometry is cylindrical in nature for cylindrical geometry. This equations are prevalent. So we will be solving a transient heat conduction problem in one-dimension transient, one-dimensional heat conduction problem. So this becomes del u. The governing equation become del u del t is equal to 1 over r del del r r del u del r.

So this is the governing equation. This is transient one-dimension in time. Another dimension in space, and del 1 by r del del r r del del r is the operating condition, operating as the operator in the cylindrical polar coordinate system.

Now the boundary conditions is specified as the initial condition boundary conditions. The initial condition is that at t is equal to 0. u is equal to either constant or it will be a specified variation or function with respect to r the boundary conditions are at r is equal to 1. That means these are all non-dimensional term r is equal to 1 means it is on the wall r star is equal to r by capital R. So, it is on the wall. We will be having a constant heat, constant temperature condition. u is equal to 0. So, the other boundary conditions need not to be specified. You do not know the fact, the other boundary, the other boundary is at r is equal to 0.

So, what will be the condition at r is equal to 0. The condition at r is equal to 0 is that u the temperature must be a finite quantity, that means it is bounded.

So, if you look into this problem, that u is finite is basically we do not have any specification on this boundary. This boundary, we are assuming that is apparent from the physics of the problem.

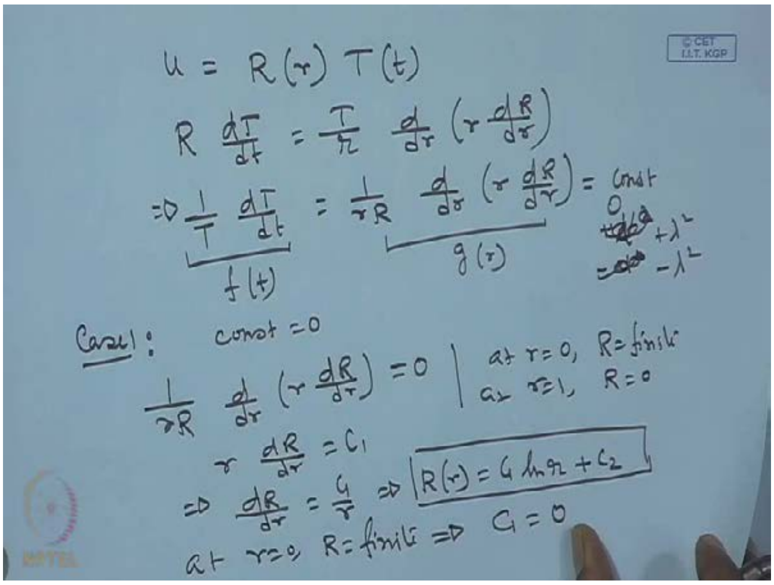
So, this is known as a physical boundary condition. So, if you look into the original problem, the original problem would have been rho c p del t del t is equal to k by r del del r or del u del r so del u del t is nothing but alpha by r del del r or del t del r.

Now, the boundary conditions and initial conditions where at t is equal to 0 temperature t was is equal to t naught at r is equal to capital R t was is equal to t 1 and r is equal to 0 t was equal to finite.

Now we make everything non-dimensional. We make the non-dimensional temperature as t minus t 1 divided by t 1. So, this boundary condition at r is equal to 1 becomes homogenous, and the initial condition will be non 0 initial condition, and since we do not have any control over this boundary. So, it will be replacing it a by a physical boundary condition and therefore the problem becomes a well posed problem.

The non zero initial conditions that means non-homogenous initial condition, and zero and homogenous boundary condition at the boundary at r equal to 0. It is become finite or bounded.

(Refer Slide Time: 26:59)



So, we solve this problem by using separation of variable. Consider u is a product of two function. One is completely entirely function of time, another is entirely function of space. So we put this equation, the governing equation this becomes R d T d t is equal to T over r d d r r d R d r. So, you will be getting 1 over T d T d t is equal to 1 over r R d d r r d R d r.

Now, the left hand side is completely a function of time. The right hand is entirely a function of space, and they are equal. They will be equal to some constant, and this constant can be 0. It can be positive, it can be negative. So, let say plus alpha is lambda square and this will be minus lambda square, you put everything in lambda, let say.

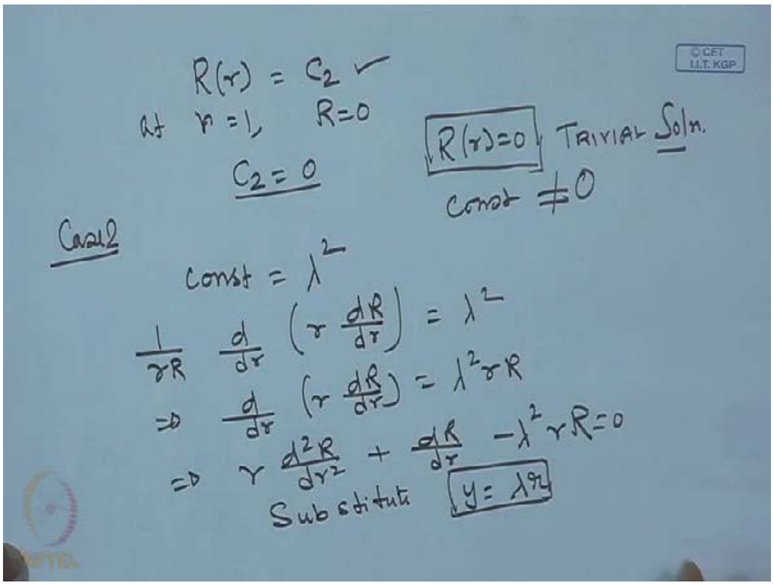
So if the constants, so we examine these three cases and look into the r varying part. So case 1, constant is equal to 0 so therefore it becomes r Rr d d r r d R d r is equal to 0. So will be having, we can solve this problem and let us look into the boundary condition that at r is equal to 0 capital R is finite at r is equal to 1 capital R is equal to 0.

So it must be satisfying the boundary conditions on r on the original problem. So what will be this equation r r will be cancelling out so 1 integration we give you r d R d r is equal to c 1. So this becomes d R d r is equal to c 1 by r and solution becomes c 1 l n r plus c 2.

So, we have these two. This is a solution where c 1 and c 2 are the corresponding constants of integration. Now when you put at r is equal to 0 capital R is finite, then this term becomes a problem.

So this becomes infinite. So the solution is the corresponding constant must be equal to 0. In order to have a finite value of r, so at r is equal to 0 capital R is finite that simplify leads to a condition that c 1 is equal to 0, because it cannot be r cannot be infinite at any location not even r is equal to 0.

(Refer Slide Time: 30:17)

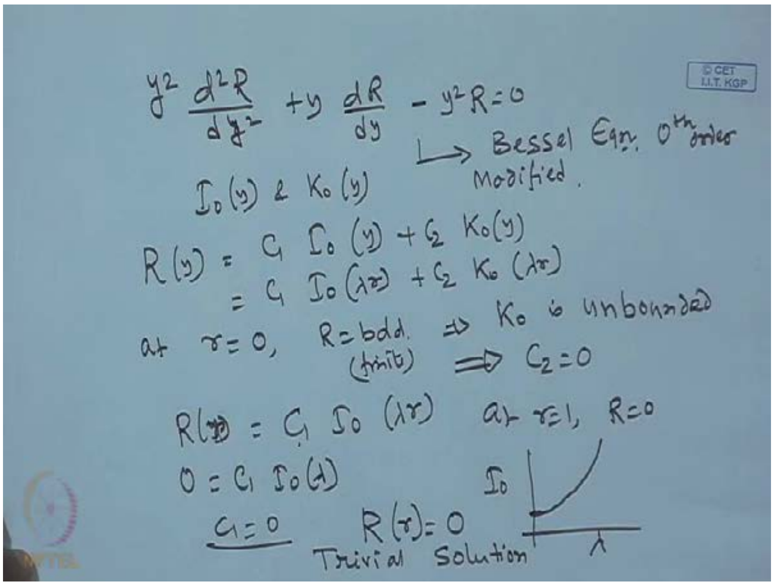


So the associated constant must be equal to 0. Therefore to have r bounded. So, R is equal to r solution is a constant solution. Now let us look into the other boundary condition that at r is equal to 1 capital R is equal to 0. Now as I said earlier the solution at the differential equation is valid throughout the whole volume of the system as well as on the surface. But the surface condition boundary condition must be valid on the boundary need not to be satisfied within the control volume.

So, therefore to make at capital R is equal to 0 at r equal to one that c 2 must be equal to 0. So, the solution is r is equal to 0. So, this is nothing but a trivial solution. So that is one in case two. So what is this indicates? That means the constant we are talking about it cannot be is equal to 0.

Now let us look into case 2. The constant is positive constant is plus lambda square. So we have one over r R d d r r d R d r is equal to lambda square. So, d d r of r d R d r is equal to lambda square r R. So, you will be having r d square R d r square plus d R d r minus lambda square r R is equal to 0.

(Refer Slide Time: 32:23)

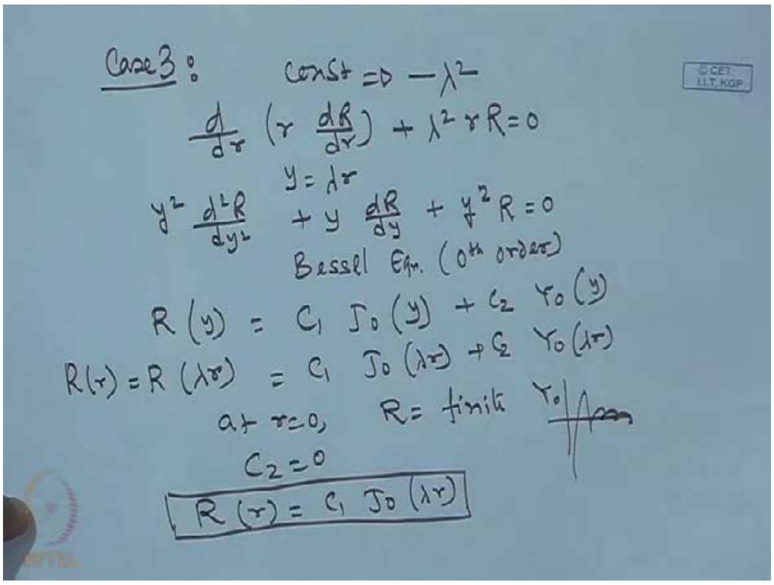


Now, put y is equal to substitute y is equal to lambda r, and then try to simplify this equation. Express this equation in terms of y. So, what you will be ultimately getting, I am just omitting couple of steps in between, will be getting y square d square R d y square plus y d R d y minus y square R is equal to 0. Now, if you look into this equation, you can identify, this equation is nothing but the Bessel equation of 0th order. So, this is 0th order Bessel equation. So, the solution is constituted. This is 0th are the bessel equation, but it is a modified Bessel equation because this is a minus signe. There so the solution will be comprised of I 0 y and K 0 y. Therefore the solution becomes R function of y c 1 I 0 y plus c 2 K 0 y, so y is lambda r, so c 1 I 0 lambda r plus c 2 K 0 lambda r.

Now at r is equal to t. Let us look into the boundary conditions R equal to 0, R was bounded and if you look into the property of I0 functions, and K0 functions , that K 0 is unbounded, that we have already seen earlier. Therefore to make this r 0 bounded or finite. These are same bounded or finite the associated constant must be equal to 0. This indicate c 2 must be equal to 0. So R of y becomes y is lambda, r R of r becomes c 1 I 0 lambda r.

Now, let us put the other boundary conditions that at r is equal to one capital R becomes 0. So, 0 equal to c 1 I 0 lambda. Now if you remember the variation of I 0 as a function of lambda will be some positive. Then it is increasing like this. So, I0 lambda is always positive it is ever positive function. So, in order to satisfy this equation I0 cannot be 0. So, c 1 must be equal to 0 if c 1 is equal to 0. Let us see the set of the function r. This become 0. So, again we are going to get a trivial solution which we are not looking for. Therefore lambda cannot be a positive quantity. So, what is left behind is that lambda is a negative quantity.

(Refer Slide Time: 35:12)

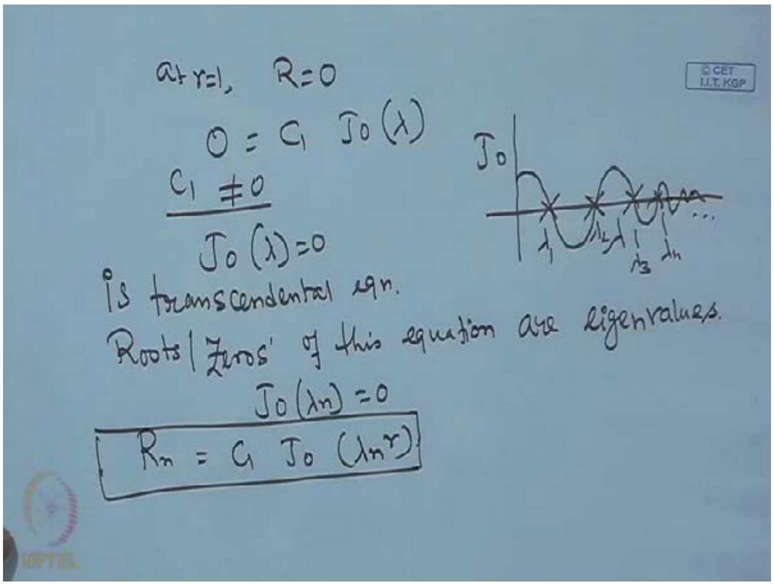


Case 3: lambda is negative constant is negative constant is negative that means it is minus lambda square. So if you look into the solution of the equation d d r of r d R d r is plus lambda square r R is equal to 0.

So again, we put y is equal to lambda r. So, it becomes y square d square R d y square plus y d R d y plus y square R. So the form of the equation is the Bessel equation 0th order and the solution will be constituted of J 0 and y 0. So, c one J 0 y plus c 2 Y 0 y. So, this becomes R lambda r is equal to c one J 0 lambda r plus c 2 Y 0 lambda r.

So this is nothing but R of r. Now, let us look into the boundary conditions. At r is equal to 0 capital R is finite. That means if you look into the property of Y 0 function Y 0 is infinite at r is equal to 0. So, this is Y 0. So to make it a finite value the associated constant must be equal to 0 therefore R of r is nothing but c one J 0 lambda r.

(Refer Slide Time: 37:20)



Now let us look into the where use the other boundary condition the other boundary condition is at r is equal to 1 capital R is equal to 0 so 0 is equal to c 1 J 0 r is equal 1 so J 0 lambda.

Now, if you look into the formulation of J 0, the property of the J 0 J 0 as a function of lambda J 0 starts from a positive quantity, and it cuts x axis infinite number of time with a diminishing amplitude. That means it cuts x axis infinite number term times. So, at every point at these are the every location where the J 0 lambda is 0 is a root are the 0. So, these are the solutions are the Eigenvalues for this particular problem.

So, in order to have a non-trivial solution c 1 must not be equal to 0 that means J 0 lambda is equal to 0 and J 0 lambda cuts intersects the x axis. The lambda axis in infinite number of times and each of the values lambda 1 lambda 2 lambda 3 lambda 4 like up to lambda n they becomes the Eigenvalue.

So J 0 lambda equal to 0 is the transcendental equation roots or zeros of this equations are Eigen values. So, what are the Eigen functions? So, we can write J 0 lambda n is equal to 0 lambda n is nth root of this solution and the Eigen functions are R corresponds to n is nothing but c 1 J 0 lambda n r.

So these are the Eigen functions. So, this proves that in order to have a non-trivial solution, we must be having the constant must be equal to a negative constants, and will be getting the Eigenvalues as the roots of the 0th order Bessel function, and the Eigen functions are the bessel functions. Eigen functions are the J0 functions. So, then will be will be getting the corresponding Eigenvalue problem.

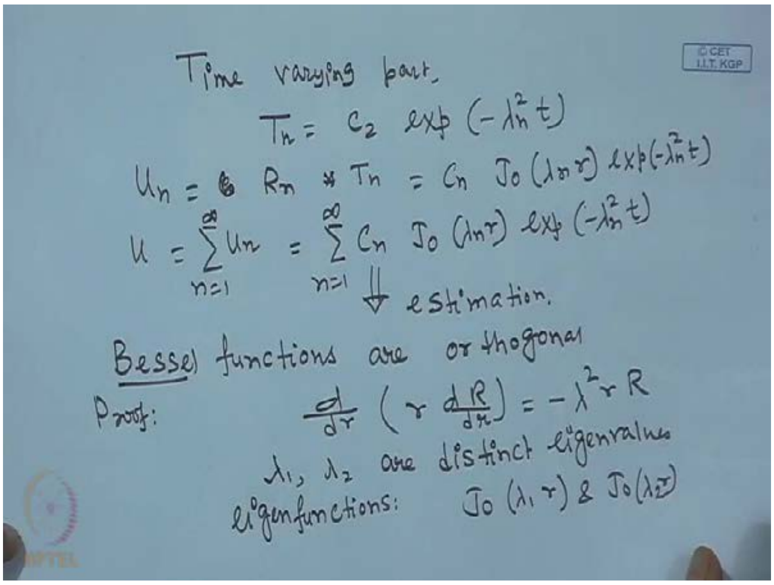
Now, we will be looking into a property of this Eigen functions, the Bessel functions that Bessel functions are orthogonal to each other or not. So that we will be evaluating final constant of using separation of variable, and the solution of this cylindrical polar coordinate system becomes complete.

Now it must be mentioned here, there in an actual problem this Eigen values and Eigen functions can be evaluated numerically. So, there are standard sub routines like photon photon recipe of photon recipe is in c they in those books. There are standard sub routines available one can get back into the those standard sub routines those are available in IMSL those are available in the Matlab

So, one can get n number of roots of the Bessel function, and one can compute the Bessel functions for different values of r etc etera the recurs formula etcetera all are available Now all these things can be numerically done. You need not to write any sub routines these sub routines are standard sub routines are available online ok.

So the alternative method is that at the end of any standard heat conduction text book o, r any standard mathematical book , the Bessel functions. They are the roots of the Bessel function are computed first five roots or of first ten roots, and various values of Bessel functions are also computed for different values of the arguments.

(Refer Slide Time: 41:33)



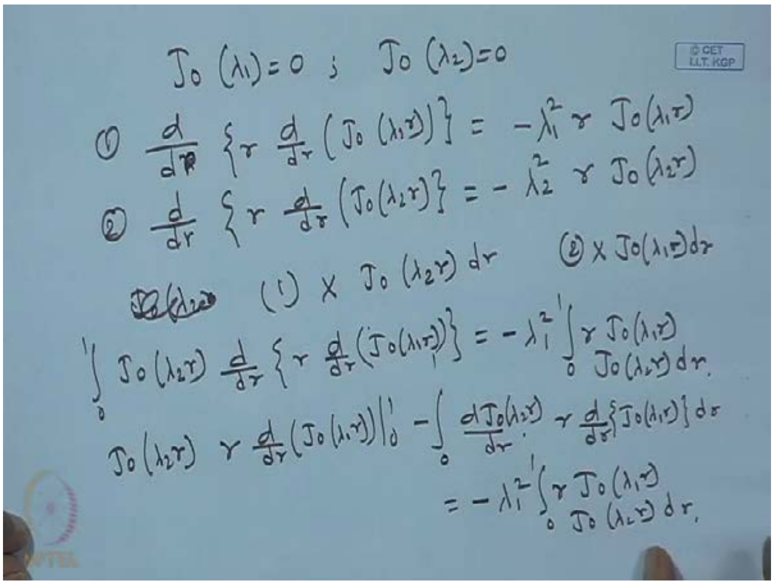
Next, we look into the solution at the t varying time varying part. If you look into the time varying part, T becomes some constant. c 2 exponential minus lambda n square t. So, we correspond we write t substitute n to denote for the corresponding nth Eigenvalue.

So, u n will be r n multiplied by T n. So, this will be c 2 multiplied by c 1. So, it will be a new constant c n J 0 lambda n r exponential minus lambda n square t. So, we can construct the complete solution by using principle of linear superposition.

So, n is equal to 1 to infinity so summation of C n J 0 lambda n r exponential minus lambda n square t n is equal to 1 to infinity. So, that gives the complete solution now in order to get still one thing is incomplete. So, that is the evaluation of this final constant the c m estimation of this constant

If you remember for the trigonometric functions in the earlier cases in Cartesian coordinate, we use the orthogonal property of the Eigen functions. Now in this case we have to prove that the Bessel functions are also the 0th order of the Bessel functions are also orthogonal to each other. For that if we can proof that this Bessel function are orthogonal to each other, then we can use the orthogonal property of the Bessel function and we can estimate the value of c n. For that let us proof that Bessel functions are orthogonal. Let us the proof goes like this if you write down the governing equation r d d r of r d R d r is equal to minus lambda square r R.

(Refer Slide Time: 44:36)



Now, let us consider lambda 1 lambda 2 are distinct. You know Eigenvalues and the corresponding Eigen functions are J 0 lambda 1 r and J 0 lambda 2 r. Now they are the lambda 1 lambda 2 are Eigenvalues. We have these equation J 0 lambda 1 is equal to 0 and J 0 lambda 2 is equal to 0. So, because they are Eigenvalues they must be satisfy this equation completely.

Now if you since J 0 and J 0 lambda r and J 0 lambda 2 r , they satisfy the equation. So, first equation will be getting d d r r d d r of J 0 lambda 1 r is equal to minus lambda 1square r J 0 lambda 1 r and the second one is d d r of r d d r of J 0 lambda 2 r is equal to minus lambda 2 square r J 0 lambda 2 r.

So, we take the inner product of this equation with J 0 lambda 2 r, and take the inner product of this equation with respect to J 0 lambda 1 r and then we subtract it.

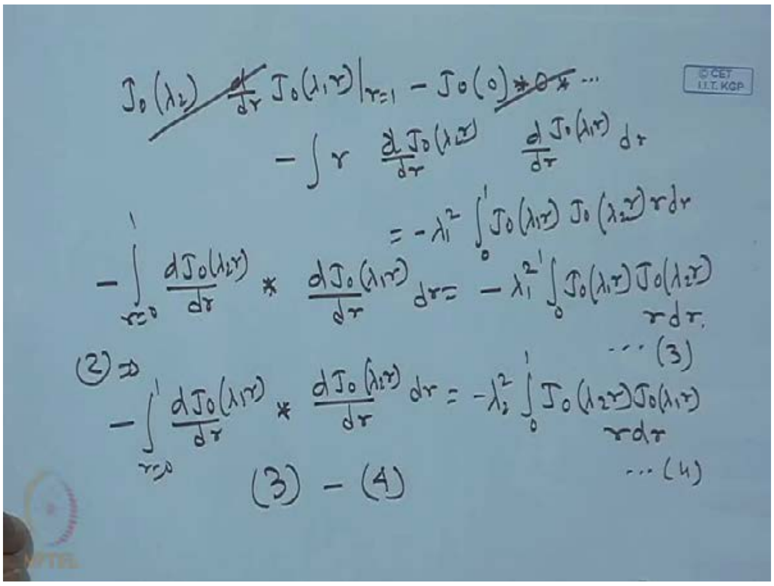
So, we multiply this one by J 0 lambda 2 r. So, J 0 so this is equation number 1 this is equation number 2. We multiply equation 1 by J 0 lambda 2 r d r and multiply equation 2 with J 0 lambda 1 r d r integrate over the domain of r and then subtract.

So what will be getting is 0 to 1 J 0 lambda 2 r. So, we multiply this equation by J 0 lambda 2 r d r multiply this equation by J 0 lambda 1 r d r . Then integrate over the domain of r and then we subtract.

So d d r of r d d r of J 0 lambda 1 r minus we just before subtracting we just do it . Then will subtract then it will be simpler for you. So this become minus lambda 1 square are lambda 1 square because lambda 1 is constant, it is taken out of integral, so it becomes r J 0 lambda 1 r J 0 lambda 2 r d r.

So we just see what you get out of this. So, first function integral of second function. So, first function J 0 lambda two r integral of the second function will be nothing but r d d r of J 0 lambda 1 r from 0 to 1 minus differential of the first function. That means d J 0 lambda 2 r d r multiplied by integral of the second function r d d r J 0 lambda one r.

(Refer Slide Time: 48:40)



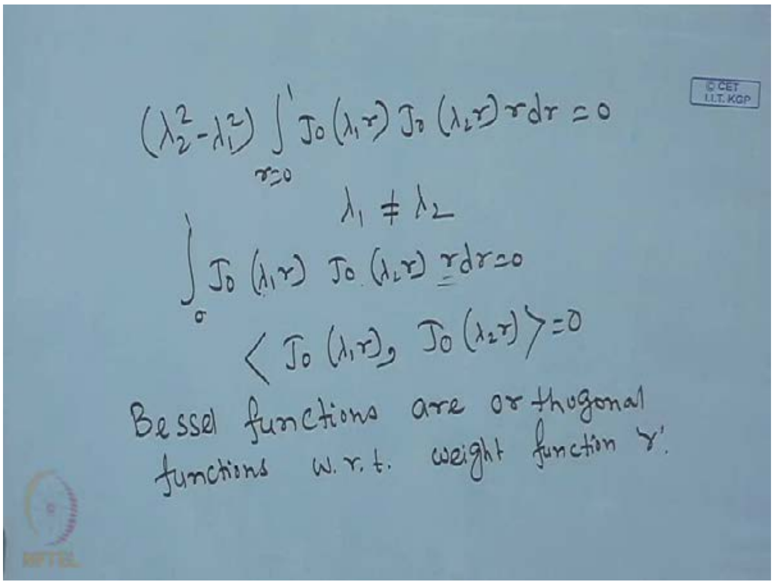
So, 0 to r d r is equal to minus lambda 1 square 0 to 1 r J 0 lambda 1 r J 0 lambda 2 r d r So, just see this bilinear concomitant term. It will be first term will be J 0 evaluated at r is equal to 1. So, J 0 lambda 2 r will be 1. So, d d r of J 0 lambda 1 r evaluated at r is equal to 1 minus J 0 at 0 so J 0 at 0 multiplied by r is 0. So multiplied by whatever it is so it becomes 0 it does not matter. minus r d J 0 lambda 2 r d r multiplied by d J 0 lambda 1 r d r d r is equal to minus lambda 1 square integral J 0 lambda 1 r J 0 lambda 2 r r d r from 0 to 1

So, we have already seen that J 0 lambda 2 is 0. So this whole term becomes 0. It is multiplied by 0. So, whole this term will become 0. So, what is left is minus d J 0 lambda 2 r d r multiplied by d J 0 lambda 1 r d r is equal to minus lambda 1 square 0 to 1 J 0 lambda 1 r J 0 lambda 2 r r d r.

Similarly from the second equation, if you simplify the second equation will be this may be equation number 3. From the second equation we will be getting minus integral d J 0 lambda 1 r d r multiplied by d J 0 lambda 2 r d r d r is equal to minus lambda 2 square 0 to 1 J 0 lambda 2 r J 0 lambda 1 r r d r.

So this is equation number 4, then we subtract 3 4 minus 3. So, this becomes if you look into this integral this becomes the inside the integral this argument, becomes same as this argument. When you subtract there in same, when you subtract the same becomes it will they that they sine becomes this plus sine becomes plus. So, they will be equal and opposite, so they will be cancelled out.

(Refer Slide Time: 51:19)

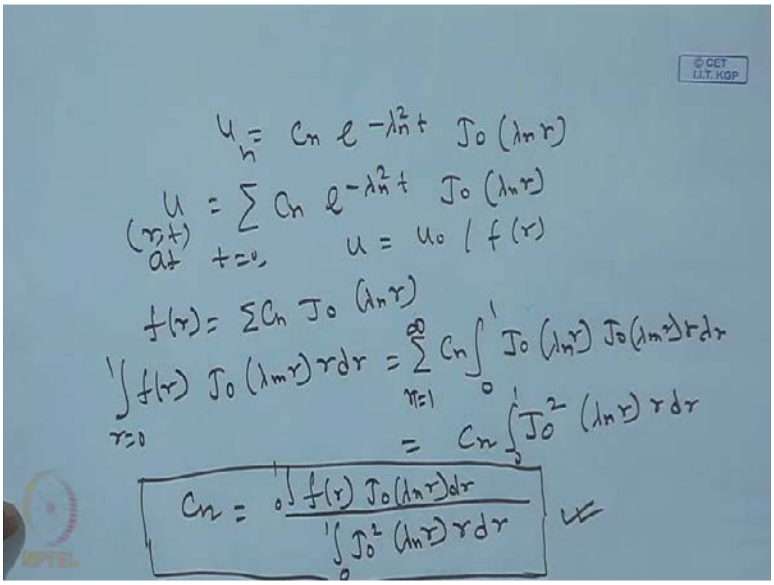


And what will happened on the right hand side? In the right hand side, you will be getting lambda 2 square minus lambda 1 square integral J 0 lambda 1 r J 0 lambda 2 r r d r is equal to 0 from r is equal to 0 to 1. Now lambda 1 and lambda 2 are not equal. They are distinct. That simply means in order to satisfy this equation, this equation has to be satisfied.

That means inner product of J 0 lambda 1 r and inner product of J 0 lambda 2 r. They are 0. That means Bessel functions are orthogonal to each other with respect to the weight function r.

So write Bessel functions are orthogonal functions with respect to weight function r . So once we proved, we are in a position to solve this problem, the transient one-dimensional heat conduction problem in cylindrical coordinate you will be able to complete it.

(Refer Slide Time: 52:51)



So if you remember if you look into the final solution, the final solution was u is equal to u n was c n e to the power minus lambda n square t J 0 lambda n r, and the complete solution will be constituted by summation of c n e to the power minus lambda n square t J 0 lambda n r. So, we use the initial condition at t is equal to 0 u is equal to u naught or some function of r generally some function of r. So, this will be f of r is equal to summation of c n J 0 lambda n r. So, you multiply both side by J 0 lambda n r r d r. So, this becomes integral and integrate over the domain of r from 0 to 1 f r J 0 lambda m r r d r, and you multiply again both side by J 0 lambda n r lambda m r r d r and integrate over the domain of r from n equal to one to infinity.

Now in this case if you open up this summation all the other terms will vanish. Only the term when because orthogonal property of the Bessel functions, that we have proved just few minutes back. So, will be getting, only one term will survive on the right hand side that is m is equal n so this becomes J 0 square lambda n r r d r from 0 to one , and you will be getting this equation. So, c n can be evaluated by J 0 square lambda n r r d r from 0 to 1 f of r J 0 lambda n r d r we change the index m into n and we get the complete solution of c n.

So, we evaluate the constant c n and that will give you the complete solution. We put it but there and we can get u as a function of r and t. So, knowing the function J f f of r, we can evaluate this integration integral by parts, or one can take the numerical sub routines, may be Simpson's one third rule or trapezoidal rule. To evaluate these two finite integrals to obtain the value of c n for every value of lambda n. So, that completes our solution of one dimensional transient heat conduction problem in cylindrical polar coordinate system. In the next class, I will be taking up a multi-dimensional problem in the cylindrical polar coordinate system.

So, I stop it here at this class. So, you continue in the next class for solution of multi-dimensional cylindrical polar coordinate system, and how to solve with, solve them using separation of variable type of solution. Thank you very much.