Advanced Mathematical Techniques in Chemical Engineering
Prof. S. De
Department of Chemical Engineering
Indian Institute of Technology, Kharagpur

Lecture No. # 03
Onto, Into, One to One function

Well, good afternoon everyone. So, we have to look on, like to, continue the discussion of One to One function, whatever we have, we are doing in the last class. So, in the last class what we did, let me summarize once again, that we have defined the One to One function, Into function and Onto function. And after the definition, we came across some of the examples, how to categorize some of the functions, whether they belong to the category of One to One, Into or Onto function.

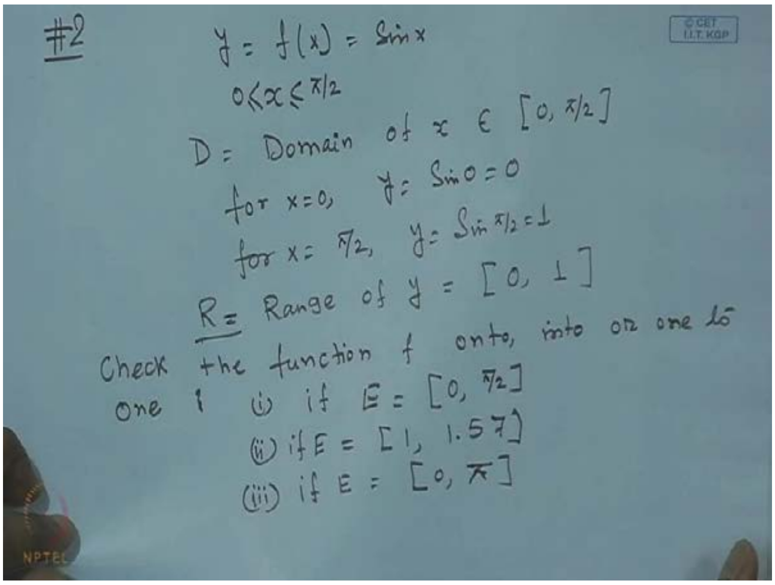
So, it is, they, whenever we are writing y is equal to f x, f is nothing but a map; so, what this map does? This map takes any value of x belonging to the domain of d and operates on it and maps it in the domain of y. So, therefore, depending on the functional form, this, and the domain of x, this y is equal to f x presents a range of y in the domain of e; so, this range, whether this range is a subset of the domain of y that is e, whether this range is equal to the set e and the range is denoted by r; whether r is equal to the set e or whether r is a subset of e, depending on that, we can categorize the function, whether it is Onto or Into.

If it is equal, then you call this function as Onto; if r is a subset of e, we call this function as Into; if for any two values of, if for a particular value of x in the d domain, there exists a unique value of f or y in the e domain, then we call it is a One to One function.

If there exists more than one value in the domain of x and we written the same value in the domain of y, after doing the mapping by the function f, then the function f is not One to One function.

So, we have already looked into couple examples, in the last class; we will look into some more examples, in this class, as well.

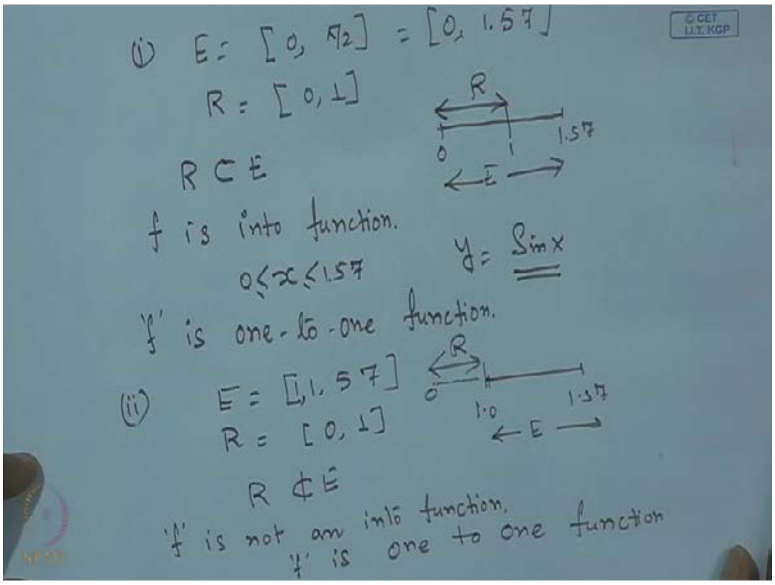
(Refer Slide Time: 02:45)



So, the next example goes like this; we have a function y is equal to f x is equal to sin x and the x in radian lying in the domain 0 to pi by 2. Therefore, the domain of x is, D is the domain of x and this belongs to a set 0 to pi by 2. And once we put the value of the maximum value and the minimum value of x in the function, that will give you the range of function f; so, for x is equal to 0 y is equal to sin 0 and that will be equal to 0; for x is equal to pi by 2, y is equal to sin pi by 2 and that will be equal to 1.

So, therefore, the subset, therefore, the set R in the y domain is called range of y and this range is from a set lying between 0 and 1. Now, we have to check the function, check the function f Onto, Into or One to One, depending on several situation; number one, if E is lying between 0 and pi by 2; number two, if E is lying between 1 and 1. 5 7 and third is if E lying between 0 to pi that is 3. 14.

(Refer Slide Time: 05:27)



For these three separate cases, we have to check, whether this function f is Onto Into or One to One. Now, we have already determined the values of the set are and for different three cases, we have to compare whether R is a subset of E or not. So, let us consider the first case; case one, E is 0 to pi by 2 that is 0 to pi is 3.14 divided by 2 is 1.57, whether my subset R is from 0 to 1. So, clearly, if we have an axis like this, let us say this 0, this 1.57, so 1 must be somewhere here, so the subset, this is the set R and this is the set E and of course this R is a set which is nothing but a subset of set E.

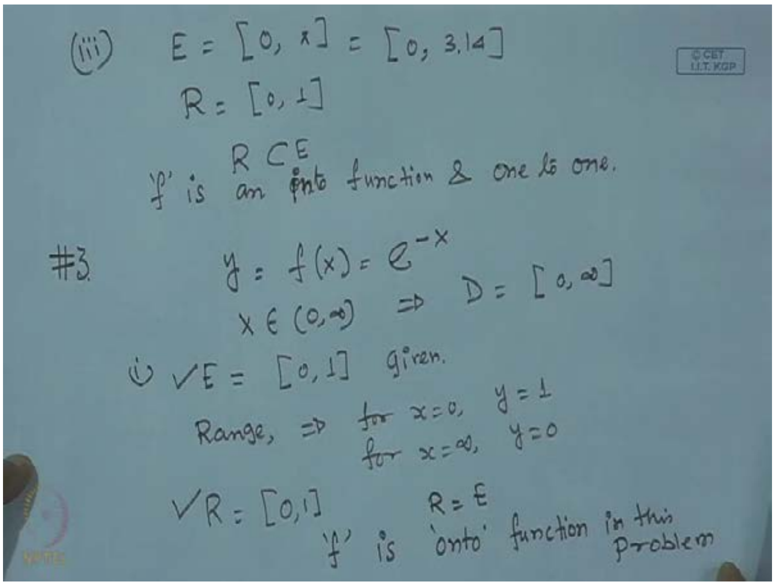
So, therefore, R belongs to E and it is a subset of E and we come to a conclusion, that f is an Into function; although the sin function is a periodic function, but therefore, it will repeat its value for different periods of pi, but since the domain of x is from 0 to pi by 2; so, therefore, it will be assuming unique value corresponding to every value of x in the domain D from 0 to 1.57, that is the domain of x 0 to 1.57.

So, for every value in this domain 0 to 1.57, sin x will be having a unique value, it will not be having a periodic value; periodic, means, after a particular value of x, it will be assuming the same value, that is not the case in this a particular problem, because the domain of x is not from more than pi 3.14, it is from 0 to 1.57. So, within this range of 0 to 1.57 for every value of x, there will be exist a unique value of y, that is why the function f is One to One function.

Next, we look into the part two; in part, two your E is lying between 1 and 1.57 and range of y is basically from 0 to 1. So, we cannot say that E is a subset because this portion, let us say 1 to 1.57, this is E and from 0 to 1, this is R; so, we cannot say whether R is a subset of E or not; so, R is definitely not a subset of E. Therefore, f is not an Into function, neither it is… And on R is equal to E, so it is not an Onto function, but it is One to One function.

Still this relation holds because for every value of x, there exist a particular value of y in E domain; so, f is One to One function.

(Refer Slide Time: 09:28)

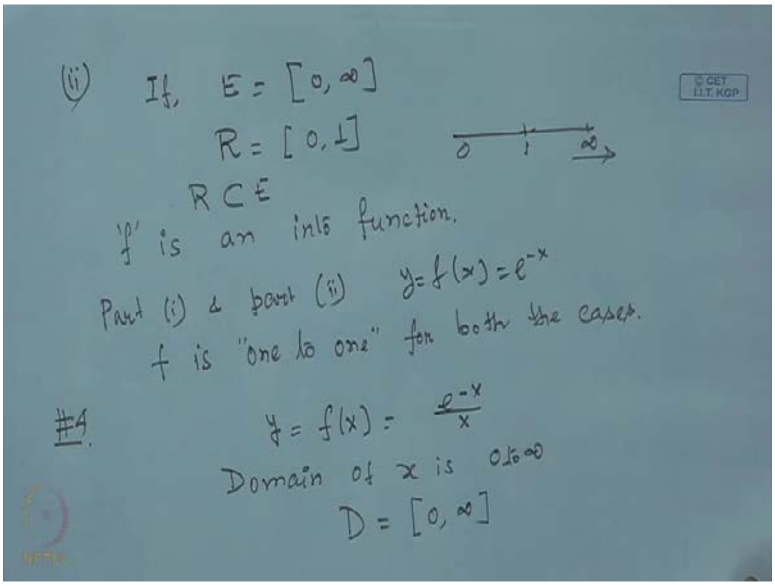


Next, you look into the third part of the problem; the third part says that, E is from 0 to pi, the set E is from 0 to pi, means, 0 to 3.14. And in this case, the range is from 0 to 1 and of course 0 to 1 is a subset of E; so, therefore, R is a subset of E and hence f is an Into function and of course it is One to One; this is an Into function for this particular case and it is as well as One to One function.

Now, let us go to the next problem; the next problem is again, we consider a function y is equal to f x and f x is an exponentially d k function e to the power minus x and x belongs to the domain 0 to infinity; therefore, domain is not, D is nothing but from 0 to infinity. Now, we have to check the whether f is Onto, Into or One to One for the domain of e from 0 to 1, this given.

If that is the case first step will be, let us find out what is the range of y? The range will be obtained by putting the maximum value and the minimum value of x in the expression; so, for x is equal to 0, y is equal to 1; for x is equal to infinity, y is equal to 0. So, therefore the range is varying from 0 to 1 and in this particular case, the R and E both are equal, these two sets are identical; therefore, we can say that f is an Onto function, in this problem.

(Refer Slide Time: 12:21)

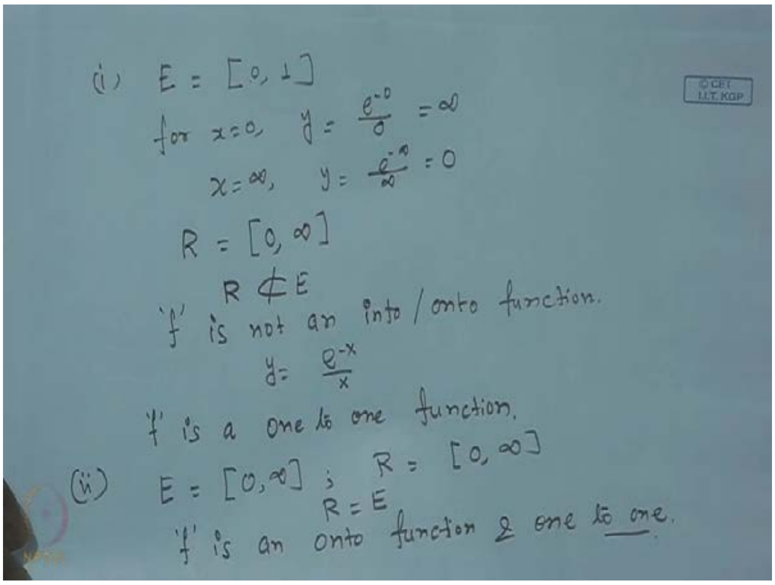


So, next example I can take from part two of this problem; in this part, we have to do the same thing, if the set E is defined as 0 to infinity, it is not 0 to 1 in the earlier case; so, this is the case, then what is the range of this problem? The actual range of this problem is from 0 to 1; so, of course 0 to 1 is a subset of 0 to infinity, in the real space

So, therefore R belongs to the set E; therefore, f is an Into function. So, likewise we can depending on the problem, we are dealing with, we can identify what is the definition or what is the nature of the function, whether it is Onto Into or One to One. In both the cases, in part one and part two, for every value of x, so f x is equal to e to the power minus x, for every value of x, there will be a unique value of y existing in the domain of x; therefore, the function f is One to One, for both the cases.

So, let us move to the next problem; so, in this case our y is equal to f of x and in this case, it is e to the power minus x by x. So, there will be a non-singularity in this case and the non-singularity is at x is equal to 0. And with the domain of x is given as 0 to infinity; so, therefore our domain is basically from 0 to infinity, so we can define our D, D is from 0 to infinity.

(Refer Slide Time: 15:05)

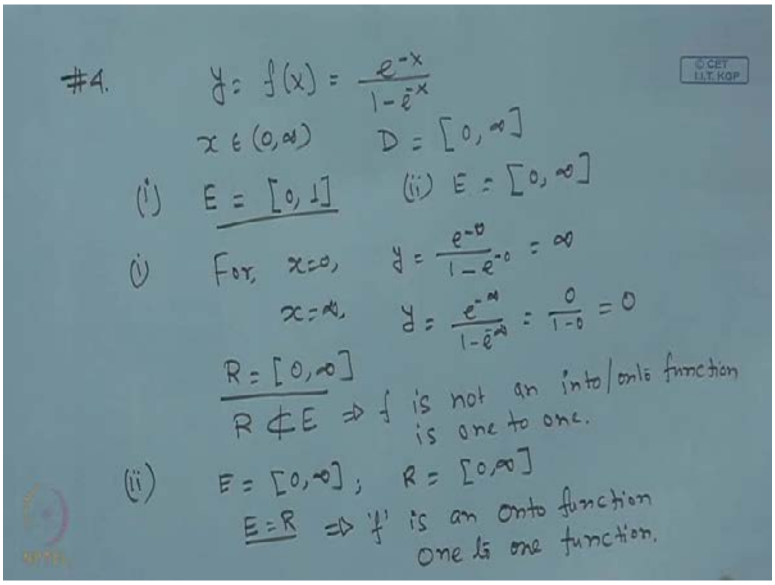


Now, depending on the value of a set y, we have to check the categorization of the function f. For case one E is a set 0 to 1; so, in this case, now let us find out for what is the range of y, so minimum value of x is 0 and the maximum value of x is infinity; so, therefore, for x is equal to 0 y is e to the power minus 0 is 1 divided by 0, so it is infinity; and for x is equal to infinity, y equal to e to the power minus infinity divided by infinity; so, it will be very small number, so it will be 0. So, range of y is a set lying between 0 to infinity.

Now, in this case, R is from, E is 0 to 1 and range is 0 to infinity; so, therefore, the set E is basically a subset of R not the other way around. Therefore, R is a bigger set R is not a subset of E; therefore, it is not so, function f is not an Into function neither it is an Onto function. But y is equal to e to the power minus x by x in the domain of x from 0 to infinity, it will be having a unique value for every value of x, y will be having a unique value; therefore, f is a One to One function.

Now, in case two, our E is 0 to infinity and we have already proved that our range is also from 0 to infinity; so, in this case R is equal to, exactly equal to E, therefore f is an Onto function, and of course, it is One to One.

(Refer Slide Time: 17:54)



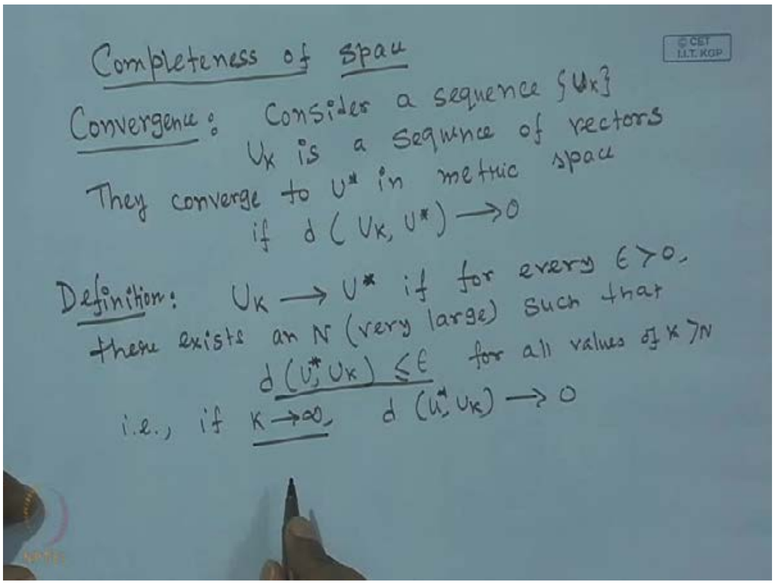
Now, let us go to the another problem; the problem number four may be, so in this case y is equal to f of x and it is e to the power minus x divided by 1 minus e to the power minus x. Now, x belongs to at the domain of x is from 0 to infinity; therefore, you can identify the domain from 0 to infinity. So, in this case, we have to find out, what is the nature of function f, whether it is Onto, Into or One to One, for different values of x and for different sets of. In case one, the set E is from 0 to 1; in case two, the set E is from 0 to infinity.

Now, let us first find out for, let us take up the problem number one; first let us find out, what is range of x for x is equal to zero, y is equal to e to the power minus zero divided by 1 minus e to the power 0; so, in the denominator, it will be 0, so it will be infinity. In another case, on x is equal to infinity, then y is equal to e to the power in minus infinity and 1 minus e to the power minus infinity; so, this will be 0, this will be 1 minus 0, this will be 0.

So, R is also from 0 to infinity. So, in case one, R is a bigger set compare to E; therefore, R is not a subset of E; therefore, f is not an Into function, it is neither an Onto function but it is One to One, because for every value of x, there exists a unique value of y.

Let us go through the second part, in this part E is equal to 0 to infinity; so, therefore an R is also 0 to infinity, so in this particular case, E is equal to R, whether R is equal to E. Therefore, f is an Onto function, and of course, it is an One to One function because whether it is Onto or Into, but for every value of x, there exists a unique value of y; therefore, it is an One to One function as well.

(Refer Slide Time: 21:28)



Next, we talk about a very important concept that is the completeness of space and convergence of a sequence and the basic definition of convergence and let us try to understand, what is meant by the convergence. So, that has something related to the completeness of space and it will be giving as an idea, what is the convergence, it gives an idea convergence criterion of a sequence.

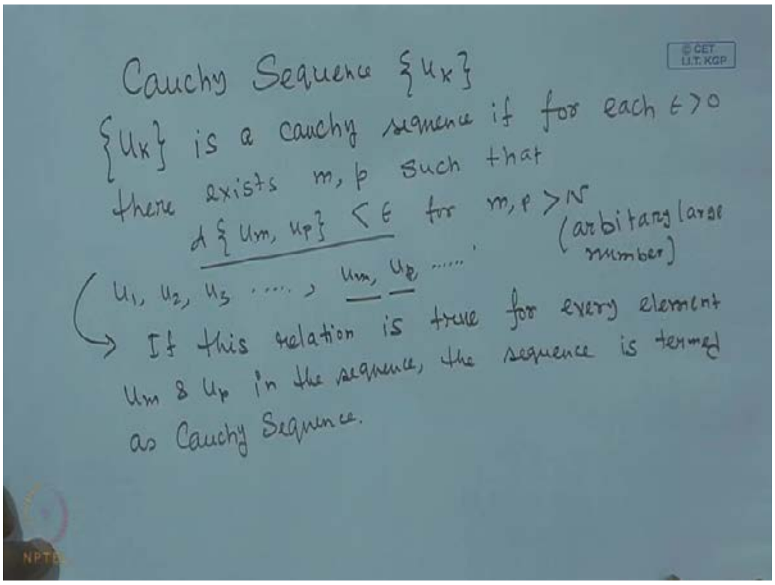
Let us first talk about what is the definition of convergence; let us talk about a sequence, consider a sequence u k, it is a sequence of vector; so, U k, it is a sequence of vector, so U k is a sequence of vectors and we talk these vectors. They converge to U star, that is in metric space, if the metric between U of k, k th element and U star tends to 0. If the metric between the k th element of the sequence and k is definitely a large number, k th element of the sequence and U star, then the metric between these two tends to 0, then we call this the sequence U k is converging to U star.

Now, what is the exact definition, to understand this concept, the exact definition is that U k, the sequence U k tends to U star, there is a element in the vector, if for every epsilon greater than 0 or epsilon is extremely small number, there exists an N which is very large, such that, metric between U and U k should be less than equal to epsilon for all values of k greater than N, that is, if k tends to infinity metric between u and U k should be tending to 0.

So, that is the definition of actual convergence of a sequence U k; we call that this sequence is a converging sequence, for there exists a very large number of N for which the metric between the sequence U k and U star should be less than epsilon; epsilon is extremely small number, but it is positive, extremely small positive number.

This relationship, this metric holds for every value of k of k, where k is very large, very large k is greater than N, where N is very large. And in other words, if N is very large and k is greater than N, then k tends to infinity; the metric between u star and U k tends to zero, then we call this sequence U k as a converging sequence. This simply indicates that for very large number N, means, for the later stage of the sequence, the sequence will be tending to a particular element, in the vector space, in the metric space and that particular, if that is the case, then this sequence is called a converging sequence.

(Refer Slide Time: 25:56)

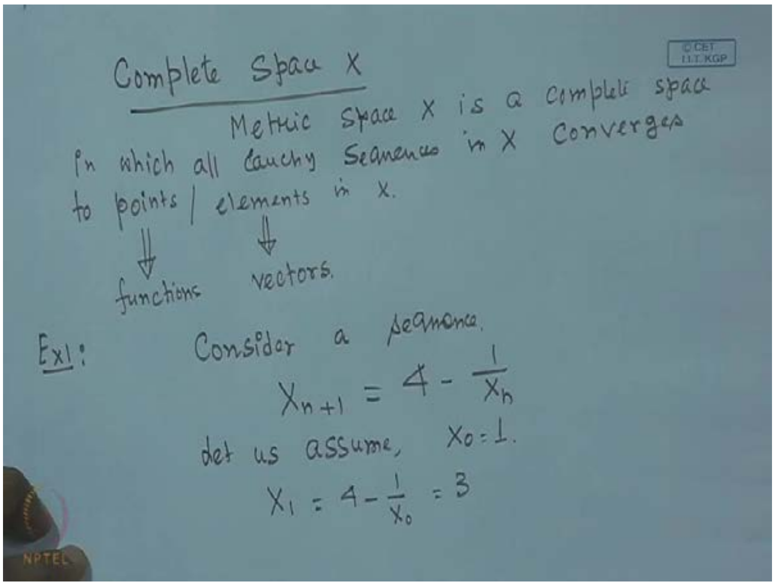


Next, we define a cauchy sequence; the cauchy sequence, let us say, u k, we call that sequence as a cauchy sequence, if U k is a cauchy sequence, if for each epsilon greater than 0, there exists m, p, such that, the metric between u m and u p should be less than epsilon for m and p are much, much greater than N, where N is an arbitrary large number. So, then this sequence is called a cauchy sequence, so what is the fundamentals behind this idea is that, in this case we are talking about the sequence u 1, u 2, u 3, like u m p; so, this is the sequence. So, we try to find out metric between the two sequence, m th sequence and p th sequence, where m and p values are extremely high. So, in that case for very large number, for if you have a sequence of functions, at a very later on stages the large value of N, the metric between the two sequences will be extremely small, that means, they will be less than epsilon; epsilon is a large, when epsilon is a small positive value and there will be converging to a particular value, there will be converging to a particular point, if it is, if the u is a function and there will be converging to a particular element in the space, if u is a vector.

So, therefore, if this relationship is true for every m and p, for every element u m and u p in the sequence, the sequence is termed as cauchy sequence; so, that means, if for particular value of m and particular value of p, where m and p is greater than N, if the metric between the two is very small, that will not help to term the sequence as cauchy sequence. For every value of m and p, both are large number, that means, for a very large number of N; that means, the elements in the later positioned sequence. They are, for every value of m and p, the metric between the two will be extremely small; that means, they will be converging at the point, for very high large number of N.

Then, this sequence is called a cauchy sequence; once we know the definition of cauchy sequence, then will be able to define the complete space at X.

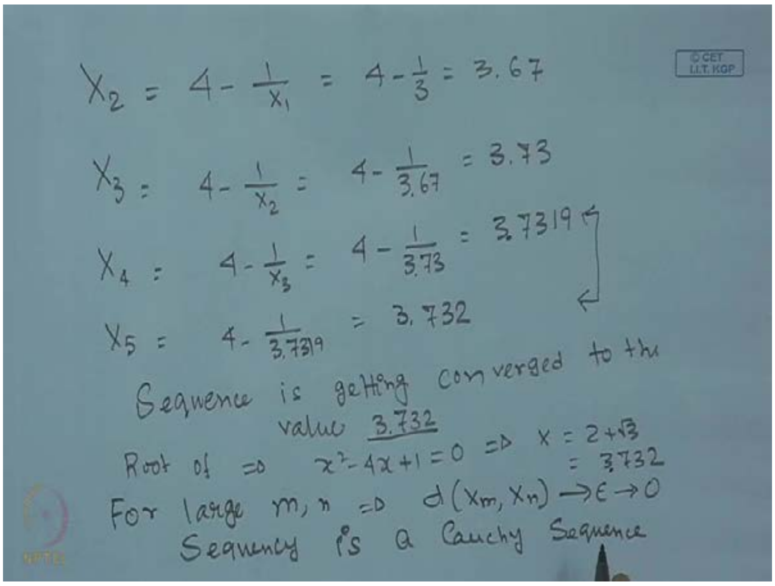
(Refer Slide Time: 29:59)



Let us define the complete space X, the metric space X is a complete space, in which all cauchy sequence in X converges to points or elements in X. Now, they will be converging to points, if we have talk about functions; they will be converging to elements, if we talk about vectors. So, we call that space a complete space, if all the cauchy sequence in that particular space, X converges to points or elements in X, then only that particular sequence is qualified to be termed as a cauchy sequence and the space is called a complete space.

Now, let us look into an example; in this example, we consider a sequence X n plus 1 is equal to 4 minus 1 over X n.

(Refer Slide Time: 32:28)



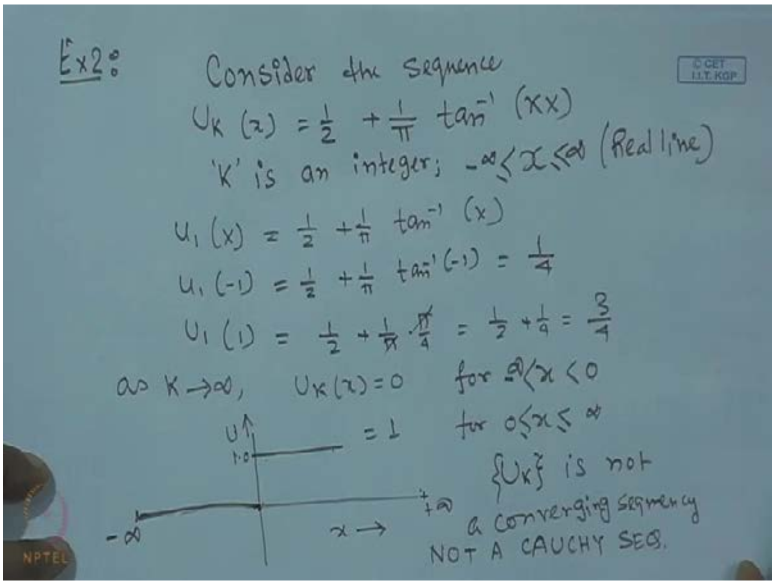
So, suppose, we have this sequence and let us try to generate this sequence and to start with it, let say let us assume, X 0 is equal to 1. So, we generate the sequence, so we put the value n equal to 0, 1, 2, like that. So, it will be X 1 will be nothing but 4 minus 1 over x zero so it will be 3. X 2 will be nothing but 4 minus 1 over x 1, x 1 is already we have found out it is 3, this is four minus 1 up on 3; so, it will be 3.67. Then your X 3 is nothing but 4 minus 1 over x 2, is equal to 4 minus 1 over 3.67; so, this will be 3.73. Now, X 4 is 4 minus 1 over x 3 and this will be 4 minus 1 over 3.73; so, this will be 3.7319. And X 5 is 4 minus 1 over 3.7319 and this will be 3.732.

So, the sequence, so from this calculation, we can observe that the sequence, the metric between the two sequences - conjugative sequences; let us say 4 and 5, 3 and 4, 4 and 5, 5 and 6, the metric between them two, means, the distance between that in two of them will be extremely small and it will be extremely small epsilon which is greater than 0.

So, sequence is getting converged to the value 3.732. So, this is basically if you look into this equation, this particular value is nothing but the root of the quadratic equation x square minus 4 x plus 1, for this particular problem. So if you look into the root, one of the root is, 2 plus root 3; so, it is 3.732 root 3 is 1.732 so plus 2; so, it will be 3.732.

So, therefore, this particular sequence that, whatever we have developed for starting point x naught is equal to 1, it will be converging to a value 3.732. Therefore, this sequence is a converging sequence for large value of m and n. For large value, large m and n, the metric between x m and x n tends to epsilon, that is very small number, that means, they tend to 0; therefore, the sequence we are talking about is a cauchy sequence.

(Refer Slide Time: 35:32)



Now, let us look into the problem number, example number two, where in this example, we are talking about a sequence which is not a cauchy sequence. So, we consider the sequence U k x is equal to half plus 1 over pi tan inverse k x. Let us talk about this sequence, where k is an integer and x is lying between minus infinity to plus infinity, its domain is basically the real line.

So, let us find out, what is u 1 x? u 1 x is nothing but half plus 1 over pi tan inverse x and u 3 x is just put k is equal to 3.

So, what is u 1 minus 1 is half plus 1 over pi tan inverse minus 1; so, this will be 1 up on 4. And u 1 for the value x is equal to 1, this will be half plus pi by 4, so 1 by pi 1 by pi will be cancelling out; so, this will be pi, so 1 by 2 plus 1 by 4 is equal to, so in this case it will be minus 1 by 4, so 1 by 2 minus 1 by 4, it will be 1 by 4. And in this case 1 by 2 plus 1 by 4, it will be 3 by 4.

Now, as k tends to infinity for large value of k, as k tends to infinity, U k x is equal to 0; k tends to infinity, means, it will become 0 for x lying in between minus infinity to 0 and this value will be 1 for x lying between 0 to infinity.

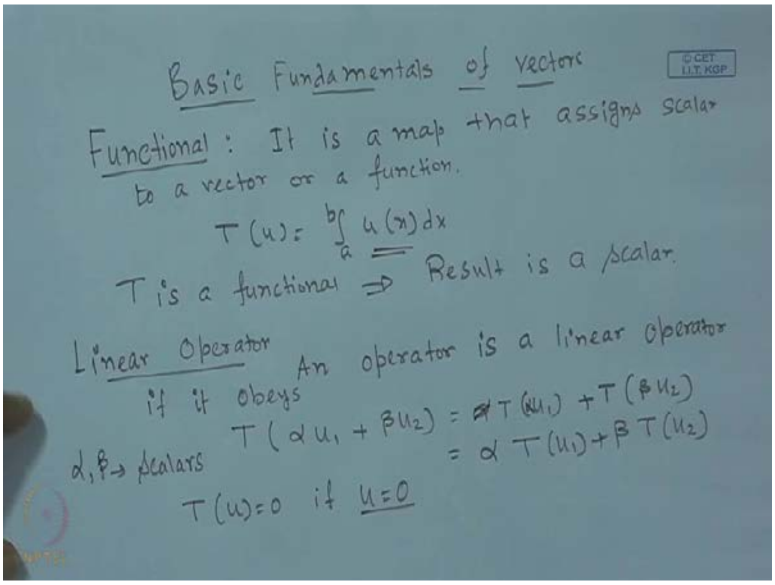
So, therefore, if we plot this sequence, if U is in this direction and x is in this direction, then we have minus infinity here, we have plus infinity here; so, from minus infinity to 0 the U k value is basically the x axis; so, from minus infinity to this plus infinity, this minus infinity from minus infinity to 0, U is equal to zero; so, you will be having a value here just superposing on the x axis. On the other hand, for x is equal to 0 to infinity, we have the value of U as 1 here.

So, therefore, this sequence is not at all a converging sequence, it does not converge to a point in the space, rather than, it assumes two different values in a one domain of the space, it is assuming a value 0; in another domain, it assumes a value 1; so, it is not a converging sequence, those are the conclusion is that the sequence U k is not a converging sequence and the space comprising of this sequence is not a cauchy sequence.

Converging sequence and the space comprising of this is not also a complete space; U k is not a converging sequence, it is not a cauchy sequence and the space where this sequence belongs to that space is not a complete space.

So, we have more or less covered the different aspects of Onto function, Into function, One to One function, as well as, the define the cauchy sequence, we defined the convergence of a sequence, we defined the completeness of space and how the cauchy sequence will be connected to the completeness of a space and how to calculate the how to determine, whether the sequence is a converging whether it is a not converging and how it is related to the completeness of the space.

(Refer Slide Time: 40:35)

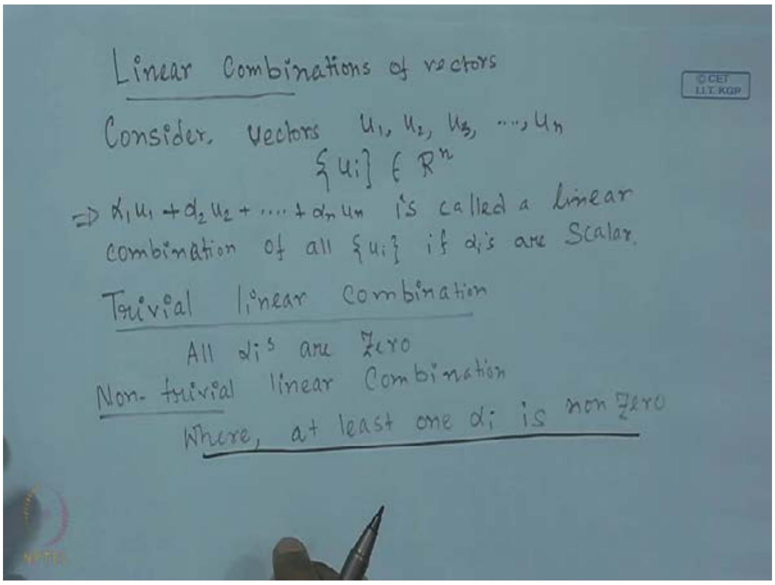


Next, we talk about some of the basic fundamentals of vectors, this will be very important. As we go along with this course, the first definition, you just refreshing our idea, the first definition is called a Functional. What is a functional? Functional is a map that is assigns scalar value to a vector; that means, it is a map that assigns scalar to a vector or a function; so, that is called a functional. For example, T of u is equal to a to b u x d x; so, u x is a function between the limit x is equal to a and b and T is basically a functional, it, this functional operates on u x within the limit a to b in this particular manner, it is in the form of integration; therefore, what is the result? The result is a scalar, the result is basically a point and that point is nothing but a scalar.

So, a functional is a map that operates on a vector or a function and assigns it to a scalar and what is that scalar? That scalar may be an a element in the vector space or it can be a point in a function space.

Then, we talk about an operator and a linear operator; we term that operator as a linear operator, if it satisfies the following criteria; operator is a linear operator, if it obeys T alpha u 1 plus beta u 2 is equal to alpha T u 1, basically T alpha u 1 or alpha being a scalar, it will be coming out of the functional. So, in more generalize form, it should be T alpha u 1 plus T beta u 2 and both alpha and beta are arbitrary scalars; therefore, since they are scalar, they will not take part in the operation or the mapping of the functional, then they will be coming out of the functional; so, it will be alpha T of u 1 plus beta T of u 2. If this relationship is obeyed then the operator T is called a linear operator and T of u is equal to 0, if the function or vector is itself is a zero vector.

(Refer Slide Time: 44:34)

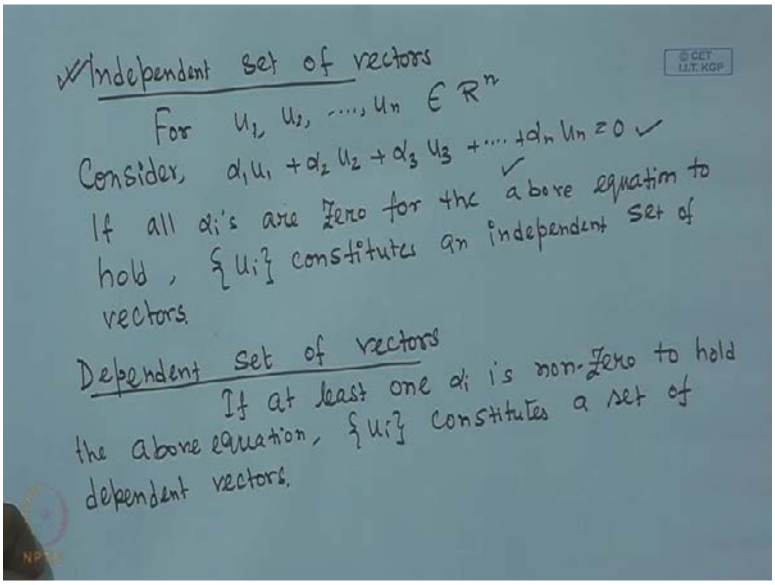


Once, we define the linear operator, then will be in a position to define the linear combination of vectors. This concept is very important as for as solution of set of algebraic equations or set of ordinary differential equations are solved those will be appearing quite often in any chemical engineering application.

So, let us consider n number of vectors in n-dimensional real space u 1, u 2, u 3, up to u n, where each of u i belongs to n-dimensional real space, then we called this combination alpha 1 u 1 plus alpha 2 u 2 plus alpha 3 u 3 plus alpha n u n is called a linear combination of all u i‘s, if alpha i‘s are scalar. So, this expression gives a linear combination of all the vectors u 1 to u n and each vector will be multiplied by a scalar and if it is presented in a linear functional linear form, then this linear form is called a linear combination of all the vectors u i.

Now, there is something called trivial linear combination, we call that combination as a trivial linear combination; if all the alpha i’s are zero, then we call that linear combination as a trivial combination. And what is non-trivial linear combination? Non-trivial linear combination is that combination, where at least one alpha i is non-zero and will be getting into a non-trivial linear combination at least one alpha i is non-zero.

(Refer Slide Time: 48:14)

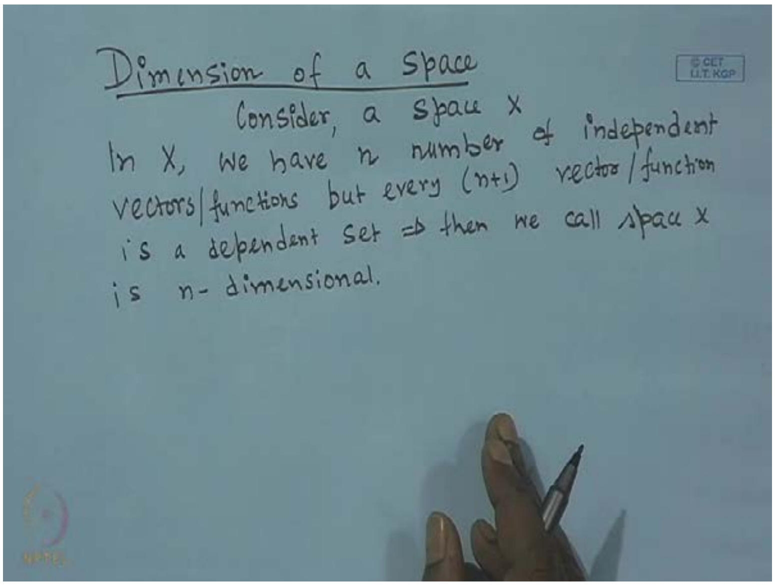


Once, we get this linear combination and concept very clear, then we will be in a position to define independent set of vectors, dependent set of vectors and dimension of space. What is independent set of vectors? For example, if we consider the class of sequence of n number of vectors u 1, u 2, up to u n, so that they belong to n-dimensional real space and consider the linear combination alpha 1 u 1 plus alpha 2 u 2 plus alpha 3 u 3 plus alpha n u n is equal to 0. If this relationship holds for all values of if all alpha i's are zero for the above equation to hold, then u i constitutes an independent set of vectors; that means, if we have an equation, if you have a linear combination of all n number of vectors as alpha 1 u 1 plus alpha 2 u 2 plus up to alpha 1 u 1; if that linear combination is equal to 0, to hold this relation good, you have to, have all the alpha you should be individually equal to 0; then this set of vectors is called independent set of vectors. In fact this concept is absolutely important, when you are solving the set of algebraic equations in any chemical engineering process, probably in the chemical kinetics is one of the example.

So, in that case, if you have three unknown and if you have three equations or three vectors then any fourth vector or fourth equation will be redundant. So, if the three unknowns of there, then there will be three equations should be minimum required to solve the system. So, independence of vector or dependence set of vectors will be directly related to the redundancy of the solution, whether the solution is a unique solution, whether there is a multiple solution that is existing in your system. So dependent vectors and independent vectors is definitely a very extremely important concept for the chemical engineers.

Next, we talk about the dependent set of vectors; in this case, if at least one alpha i is non-zero to hold the above equation, this is the equation, then u i constitutes a set of dependent vectors. So, again the dependent vector, independent vector, there it is, they are related to the redundancy of the system and uniqueness of the problem.

(Refer Slide Time: 52:23)



Next, we talk about the dimension of space, we will look into actual chemical engineering applications after defining all these concepts clearly. It called a dimension of space X, supposed we consider a space X, and in this space, in X, we have n number of independent vectors, they can be functions as well, so either vectors or functions, but every n plus 1 vector or function is a dependent set, then we call space X is n-dimensional.

So, therefore, if we have a three dimensional space, then we have to have a three independent vectors, every other vector present in the space can be expressed has a linear combination of these three independent vectors. This simply means, therefore the dimension of the space is n, so fourth vector n plus 1, the fourth vector will be since for a three dimensional space; the fourth vector can be or expressed as a linear combination of the term independent vectors.

So, if we have a 10 dimensional space, then so 11th vector will be basically a linear combination of the 10 independent vectors. Suppose, these will be having a typically application in case of in chemical engineering application, in case of chemical reaction system; suppose, we have a chemical reaction system having there are around 10 number of spaces involved. So, therefore, if you like to solve the composition of the all spaces after the reaction, then probably will be writing 10 number of equations, then 10 independent equations we must be having in order to solve this set of equations of ten unset of system of ten unknowns.

Any 11th equations that we make a write for whatever over all must balance of or whatever balance, the 11th equation will be a redundant one. So, you must be having total number of, so it, it has a direct relations to the degree of freedom of the system; so whatever the degree of freedom in your system, you must be maintaining that.

So, if you have a three unknown system, you must be having three equations; so, correspondingly the dimension of the space is very important that directly dictates the number of equations you are going to handle. And any if you would like to form an any equation that will be characterizing the system, beyond the dimension n, then you will be talk you will be over specifying the system or the reduction the equation becomes a redundant one.

So, will be in the next class we will be looking into how to test whether a set of equations, set of vectors are dependent vectors or independent vectors. And how they are related to the chemical engineering systems and what are the different characteristics of these vectors, whether they are orthogonal vectors, orthonormal vectors, if they are. And how they will simplify the whole root of the calculations procedure and we will look into all these aspects in the next class. Thank you very much.