Advanced Mathematical Techniques in Chemical Engineering

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Lecture No. # 26
Solution of Higher Dimensional PDEs

Good afternoon everyone. So, in the last class we have looked into several problems in parabolic partial differential equations, whatever can be solved using separation of variable. And we took up a relevant chemical engineering application - chemical engineering problems, namely the transient heat conduction problem in one dimension. And we looked into the basic problem, what is classified as the basic problem, which is a well-posed problem. Then, we talked about the ill posed problem where the initial condition is homogeneous, but the boundary conditions are now, one of them is homogeneous, another is non-homogeneous. Then, we have looked into the basic problem whichever the initial condition is non-homogeneous, but the boundary conditions are homogeneous. Then, we looked into the method of how to convert the ill posed problem into well-posed problem. And finally, we took up some of the real life chemical engineering heat conduction problem and we solve them using separation of variable.

Now, there are three kinds of two kinds of boundary conditions in the actual problem we have looked into: one is the boundary conditions at Dirichlet boundary condition in the next problem, we completely solved the problem; and the next problem is the boundary condition is Neumann boundary condition; the problem we are solving and you could not finish in the last class, where one of the boundary condition is non-homogeneous and it will be robin mixed boundary condition; so, that means whatever the heat that has been arrived at that boundary by conduction is taken away by convection.

So, we were solving that problem in an in between. So, let me complete that problem first; then, will move into multidimensional basic problem - parabolic in nature - which can be solved by using separation of variable.

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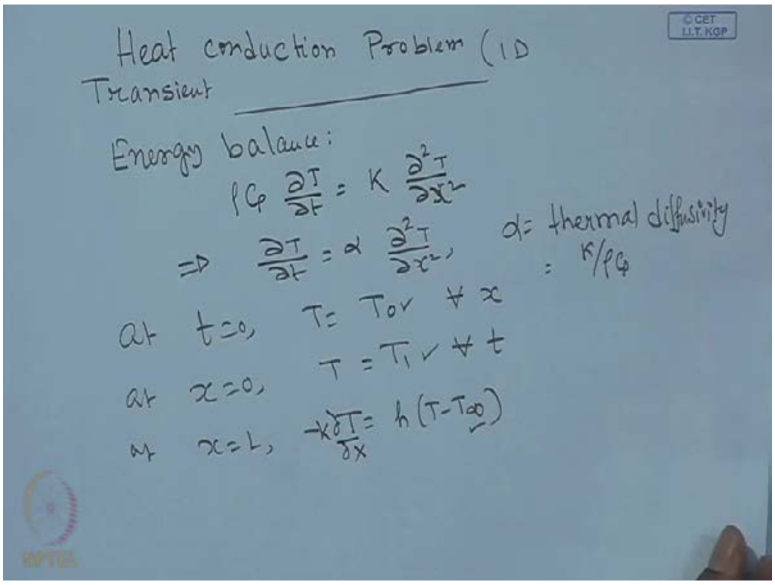

So, let me look into an actual chemical engineering problem - heat conduction problem, but one-dimensional transient heat conduction problem but one-dimensional. So, the governing equation is the energy balance equation. So, if you write down the energy balance equation, rho c p del t del t is equal to K del square T del x square; so, del T del t is nothing but alpha del square T del x square, where alpha is known as the thermal diffusivity, that will be having a value expression K by rho c p.

So, now, we fix up the general boundary conditions, at t is equal to 0, temperature of the problem was temperature of the plate - it was T naught and at for this valid for every x; for at x is equal to 0, the temperature was kept constant and maintained at a temperature T 1, that is valid for every t; and at x is equal to l, T is equal to minus k del t del x is equal to h T minus T infinity, where this term - left hand side represents minus K del T del X is equal to the energy, that has come at that boundary located at x equal to l by conduction. And that amount of energy has been taken away by the ambient by the convection present in the ambience. So, h is the heat transfer coefficient and T infinity is the ambient temperature.

Now, let us look into how many non-homogeneities are there in this particular problem. There are three sources of non-homogeneity: one is in the initial condition; another is the boundary condition and because the presence of this temperature T infinity in the boundary condition located at x is equal to l.

So, there are three sources of non-homogeneity in this problem. So, we make this problem non-dimensional, so that the number of sources of non-homogeneity will be reduced. And at least by defining and appropriate temperature - non-dimensional appropriate temperature, we can reduce the number of non-homogeneities from 3 to 2.

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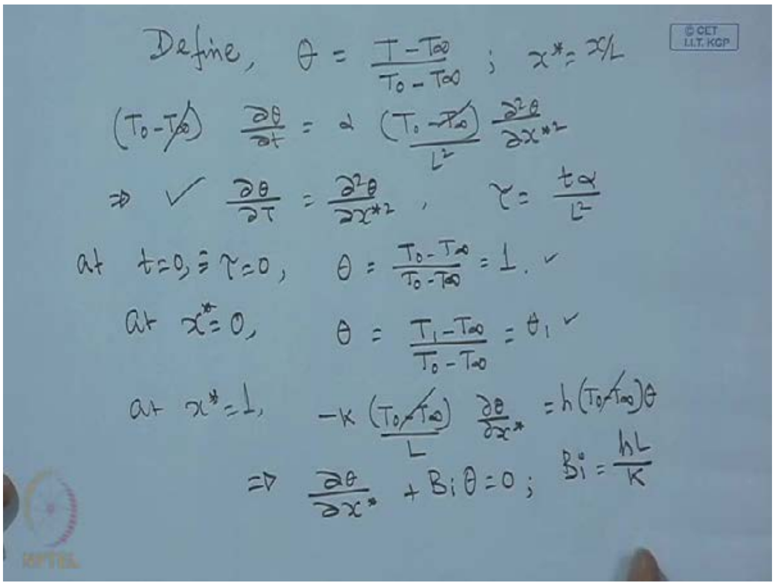

Now, let us look into that, by define a non-dimensional temperature theta as T minus T infinity divided by T 0 minus T infinity, let us define in that way. And it does not matter whether you write T 0 minus T infinity or T 1 minus T infinity, but the idea is, this has to be equal to 0 at one of the boundaries. So, let us define theta like this; and of course, x star is x upon l, where l is the length of the plate in the x direction.

So, if you put all these things in the non-dimensional form, so your basic equation will be T 0 minus T infinity del theta del t is nothing but alpha T 0 minus T infinity del square theta divided by l square del x star square. So, T 0 minus T infinity will be cancelled out; so, you can write, del theta del tau is equal to del square theta del x star square, where tau is non-dimensional temperature and it is t alpha over l square - that is the non-dimensional tau.

Let us look into the boundary condition initial condition, at t is equal to 0, that means at tau is equal to 0 - this is equivalent to tau is equal to 0 - theta was, t is equal to t naught, so, you just put T naught minus T infinity divided by T naught minus T infinity; so theta is equal to 1. At x is equal to… x star is equal to 0, T is equal to T 1, therefore theta is equal to T 1 minus T infinity divided by T 0 minus T infinity; this is a non-dimensional temperature and it is constant; so, let us put this value as theta 1.

And at x is equal to… x star is equal to 1, we have minus k T 0 minus T infinity del theta del x star, so there has to be an l here is equal to h T minus T infinity - you write T 0 minus T infinity times theta; so, you can, this will be cancelled out and you can bring it to the other side. So, del theta del x star plus B i theta is equal to 0; that is a mixed boundary condition at x star equal to 0, where B i is nothing but the Biot number and this is equal to h l over k.

So, let us look into the problem, we made this governing equation non-dimensional and made the independent variable x star well-behaved between 0 and 1. At the same time, we had the initial condition, theta is equal to 1, which was not equal to 0; this is non-homogeneous initial condition, the 1 - this term represents on non-homogeneity. Theta is equal to theta 1, at x star equal to 0, that is also non-homogeneous term, but on the other hand, the boundary condition at x star is equal to 1 has become homogeneous. So, in this way we have, by this way of defining the theta, we have reduced the number of non-homogeneities from 3 to 2.

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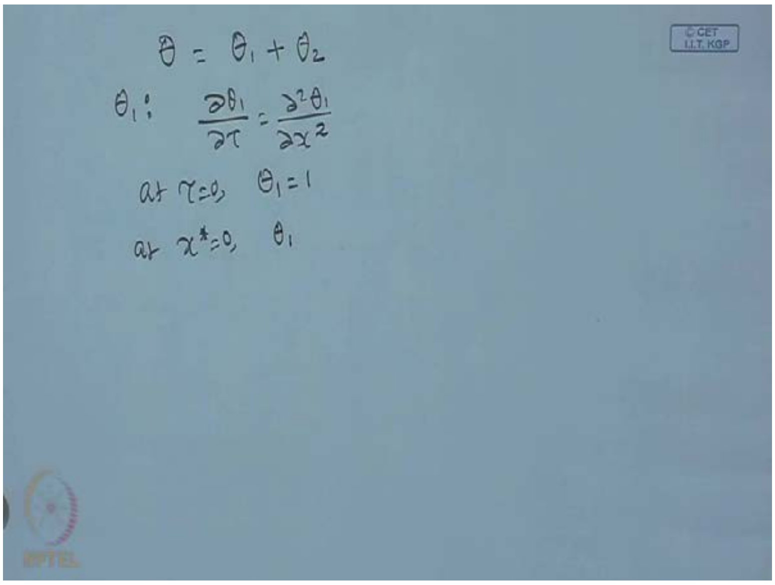

So, we define this problem into two sub problems, considering one non-homogeneity at a time. So, theta is equal to theta 1 plus theta 2. Let us define theta 1, theta 1 will be del theta 1 del tau is equal to del square theta 1 del x square; you just write x in terms in place of x star, so that means x is a non-dimensional length. At tau is equal to 0, you put theta 1 is equal to 1; at x star is equal to 0, we put theta 1.

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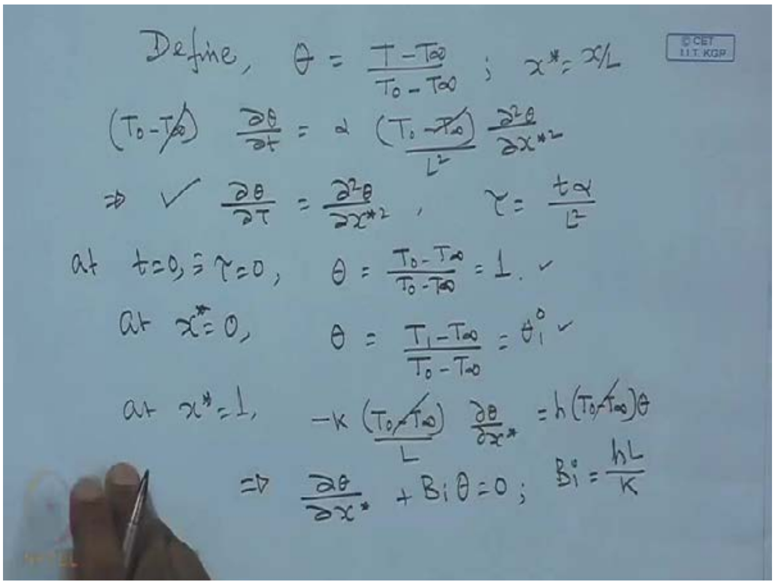

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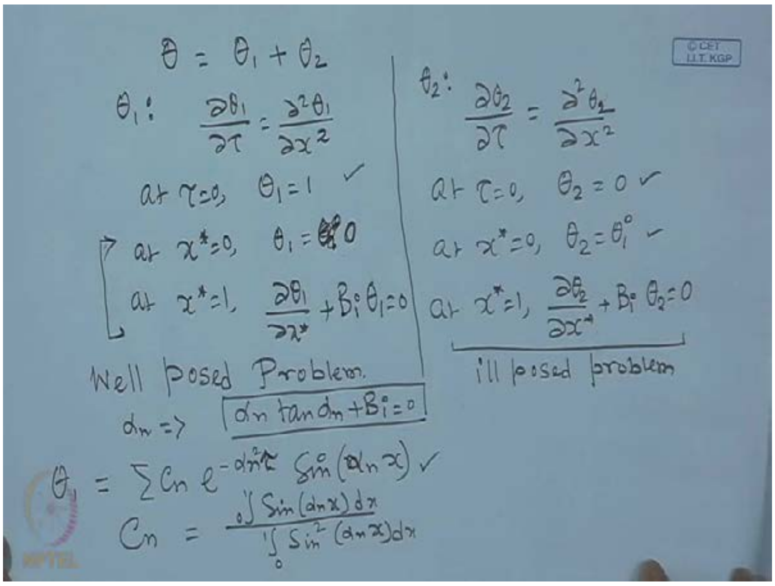

Since, there is a nomenclash and nomenclature, we call this as theta 1 superscript naught. So, at theta is equal to… at x star is equal to 1, theta 1 is equal to theta 1 naught; at x star is equal to 1… So, we put, we force this boundary condition, we force it to 0; so, we have, we consider only one non-homogeneity at a time. So, x star equal to 1, we have del theta one del x star plus B i theta is equal to 0; that is the definition of theta one.

Let us define theta 2, theta 2 is governing equation is del theta 2 del tau is equal to del square theta 1 del x square. At tau is equal to 0, theta 2 is equal to 0; at x star is equal to 0, theta 2 is equal to theta 1 superscript; and at x star is equal to 1, we have del theta 2 del x star plus B i theta 2 is equal to 0.

So, therefore, we have defined, we have divided this problem into two sub problems, considering one non-homogeneity at a time; we kept the non-homogeneity in the initial condition, as it is in the problem number theta 1. In problem theta 2… We in theta 1, we kept the non-homogeneity at time t equal to 0, but we forced the non-homogeneity present at x star equal to 0 to 0; so, we just force it to vanish. On the other hand, in case of theta 2, the initial condition was taken as homogeneous, but we kept the non-homogeneous term in the boundary condition as it is.

So, we divide this problem into two sub problems, so that we consider one non-homogeneity at a time. Now, if you look into the problem of theta 1, theta 1 is a well-posed problem, because it has a non-homogeneous initial condition and homogeneous boundary condition. So, this is definitely a well-posed problem

And if we look into the problem number theta 2 - second sub problem, then the second sub problem the initial condition is homogeneous, but one of the boundary condition is not homogeneous - it is in homogeneous and the other one is homogeneous; therefore, this is an ill posed problem.

We have already seen the solution of this kind of well-posed problem earlier and if you remember the solution is something like this, so the Eigenvalues for this problem are obtained by the roots of the transcendental equation; alpha n tan alpha n plus Biot equal to 0; roots of this transcendental equations are given by the are the Eigenvalues of this particular problem and sin functions are the Eigen functions. So, the complete solution of this problem is theta 1 is equal to summation of c n e to the power minus alpha n square tau sin alpha n x.

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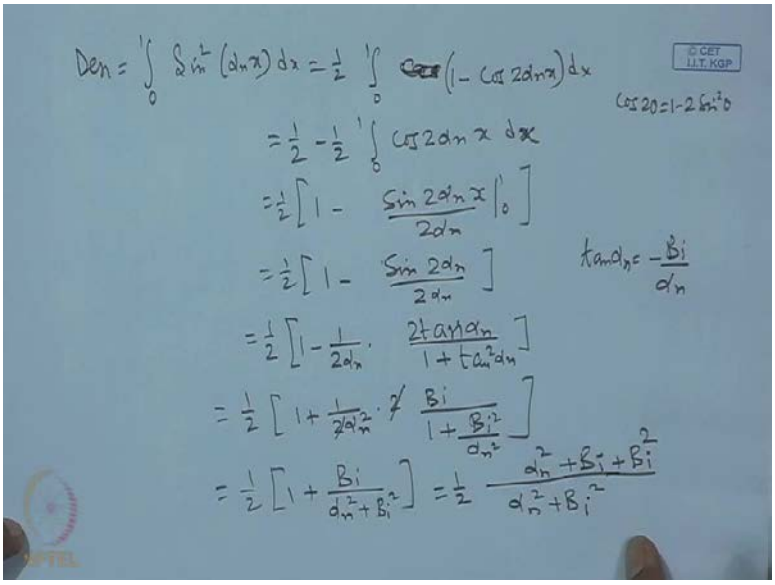

So, c n is obtained by carrying out this integral, sin square alpha n x d x from 0 to 1 and in the numerator, it is integral f x - f x is 1 in this case,- so, it will be integral of sin alpha n x d x from 0 to 1. So, this gives the complete solution of theta 1 and we know what is the solution of this; we have already proved this solution earlier.

So, if we want, we can look into the solution of c n once again. So, c n… the first we evaluate the denominator is 0 to 1 sin square alpha n x d x, so it will be half 0 to 1 2 sin square alpha n x; so, this will be nothing but cosine 1 minus… - cos 2 theta is equal to 1 minus 2 sin square theta, so 2 sin square - cosine 1 minus cosine 2 alpha n x d x. So, the first integral - first part is half minus half, this will be integral of cosine 2 alpha n x d x 0 to 1.

So, this will be half 1 minus sin 2 alpha n x divided by 2 alpha from 0 to 1; so, this will be half 1 minus sin 2 alpha n, we put x is equal to 1 minus sin 0 is 0, so divided by 2 alpha n; and we expressed, sin 2 alpha n in terms of cosine 2 tan alpha n; so this becomes 2 alpha n tan 2 tan alpha n divided by 1 plus tan square alpha n.

And if we look into the transcendental equation, transcendental must satisfy tan alpha n is equal to minus Biot - B i divided by alpha n. So, we are going to put it there, 1 minus 2 alpha n is 2 tan alpha n is - minus minus plus - B i alpha n square divided by 1 plus B i square divided by alpha n square. So this, so half 1 plus B i alpha n square plus B i square; so this becomes half alpha n square plus B i square and on the numerator, you have alpha n square plus B i plus B i square; so, that is, that goes the denominator

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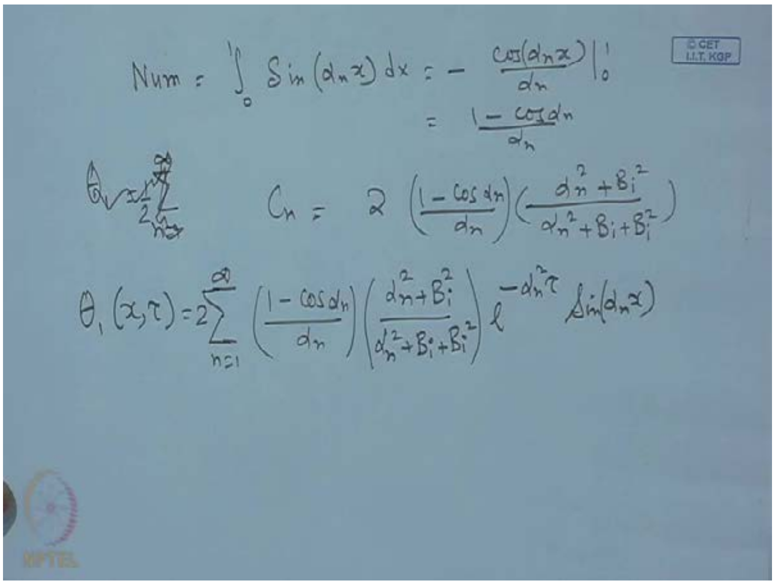

And if we look into the numerator, the numerator becomes integral 0 to 1 sin alpha n x d x; so, integration of this will be giving you, minus cos alpha n x divided by alpha n 0 to 1; so, it will be 1 minus cosine alpha n divided by alpha n.

So, we get the complete solution of theta 1. So, theta 1 will be summation n is equal to 1 to infinity c n, c n is given by this equation half c c n is… First, you find out c n, c n is numerator divided by denominator; so, it will be 2 1 minus cosine alpha n divided by alpha n and your denominator is alpha n square plus Beta i square divided by alpha n square plus Beta i plus Beta i square; so, that is the coefficient c n.

So, now you are in a position to write down theta 1 completely, summation 2 summation n is equal to 1 to infinity, we put the value of c n 1 minus cosine alpha n divided by alpha n multiplied by alpha n square plus Beta i square divided by alpha n square plus Beta i B i plus B i square Biot square e to the power minus alpha n square tau sin alpha n x; so, that gives the complete solution for the problem theta 1.

Now, let us look into the problem theta 2. Now, if you look into the problem theta 2, theta 2 is an ill posed problem and we just look into the solution of theta 2.

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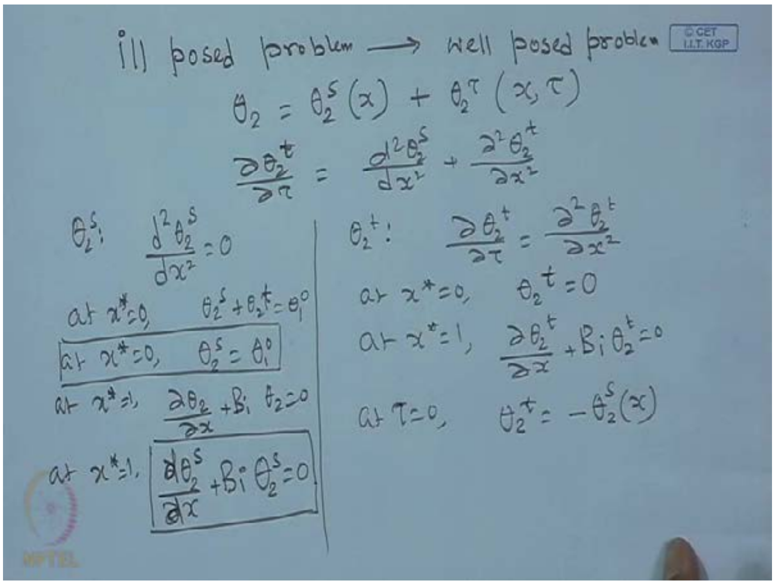

So, we make the problem ill posed we convert the ill posed problem to well-posed problem. Now if you do that, what will be getting is that, we since the time dependent part, the initial condition is homogeneous. We break down the problem into two sub problem: one is theta 2 s, that is a function of… that is a steady state solution - it is a function of x alone; another is the unsteady state solution transient problem, there is a function of x and tau both.

So, therefore, we put this equation into the governing equation. If you put this equation governing equation, then we will be collect the similar terms and will be getting the governing equation of theta 2 s and theta 2 T. So, this becomes del theta 2 T del tau is equal to d square theta 2 s d x square plus del square theta 2 del x square.

So, we define theta 2 s and we define theta 2 t. Now, if you look into the definition of theta 2 s, it will be simply d square theta 2 s d x square is equal to 0. Subject to the boundary condition that, at x star is equal to 0, we have theta 2 is equal to theta 1 0, so theta 2 s plus theta 2 tau is equal to 0 is equal to theta 1 0.

So, we associate this non-homogeneous term with the steady state solution to force the theta 2 T - the boundary condition theta 2 T to be homogeneous; that means, we put the boundary condition at x star as theta 2 s is equal to theta 1 0.

Now, let us put the governing equation of theta 2 t. del theta 2 t del tau should be is equal to del square theta 2 t del x square. Now, at x star is equal to 0, since we have associated the non-homogeneous part with the steady state solution, then theta 2 tau will be must be equal to, theta 2 t must be equal to 0, that simply makes the corresponding boundary condition of the transient solution to be homogeneous.

Now, let us look into the other boundary condition; that is, at x star is equal to 1, we have del theta 2 del x plus Biot theta 2 is equal to 0; put theta 2 is equal to theta 2 s plus theta 2 t and then separate, collect the similar order terms; so, we will be getting the boundary condition at half theta 2 s.

At x star is equal to 1, that will be del theta 2 s del x plus B i theta 2 s is equal to 0, so it becomes d theta 2 s d x - this becomes total derivative. So, this is the boundary condition on x star equal to 1; and this is the boundary condition on x star is equal to 0; and the other the boundary condition at x star is equal to 1, for so x star and x star basically same, for theta 2 will be del theta 2 t del x plus B i theta 2 t is equal to 0.

Now, look into the initial condition; at tau is equal to 0, your theta 2 was is equal to 0, therefore theta 2 t will be nothing but minus of theta 2 s x, which is a sole function of x; and in fact, that is the solution of the transient, this is a solution of the steady state part.

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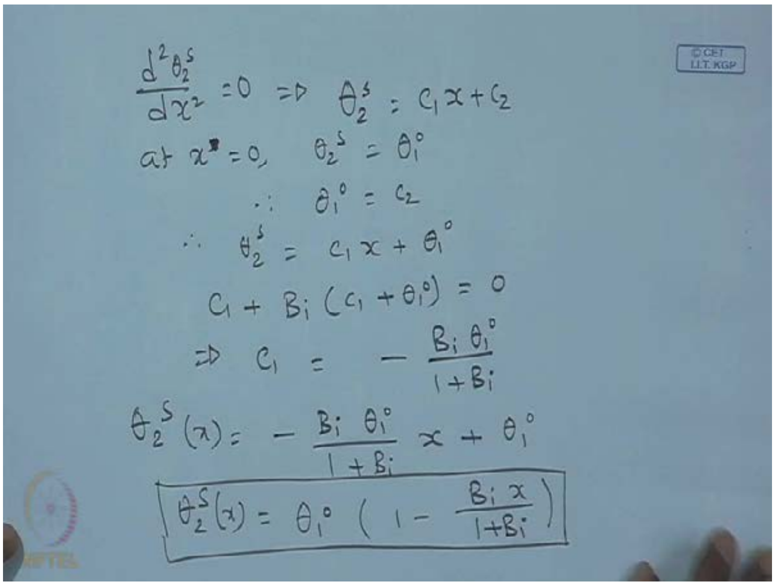

Now, if you look into this problem, this theta solution of the steady state part will be known; by solving, we will be getting this. So, if let us talk about the steady state solution, then will come back to the transient problem. So, d square theta 2 s d x square must be equal to 0, so this as a solution theta 2 s is nothing but c 1 x plus c 2. Now at x star is equal to 0, at x is equal to 0, theta 2 s is equal to theta 1 0; therefore, theta 1 0 is nothing but c 2. So, therefore, c theta 2 s is equal to c 1 x plus theta 1 0.

Now, we use the other boundary condition. At x star is equal to 1, d theta 2 s d x; d theta 2 s d x is nothing but c 1 plus B I, theta 2 s evaluated at x star is equal to 1, so this is equal to c 1 plus theta 1 0 is equal to 0. So, therefore, c 1 can be written as minus Biot theta 1 0 divided by 1 plus Biot number.

So, you will be having the solution of steady state part is nothing but minus B i theta 1 0 divided by 1 plus B i times x plus c 2 is nothing but theta 1 0; so, we have a solution theta 1 0 is equal to 1 minus B i x divided by 1 plus B i.

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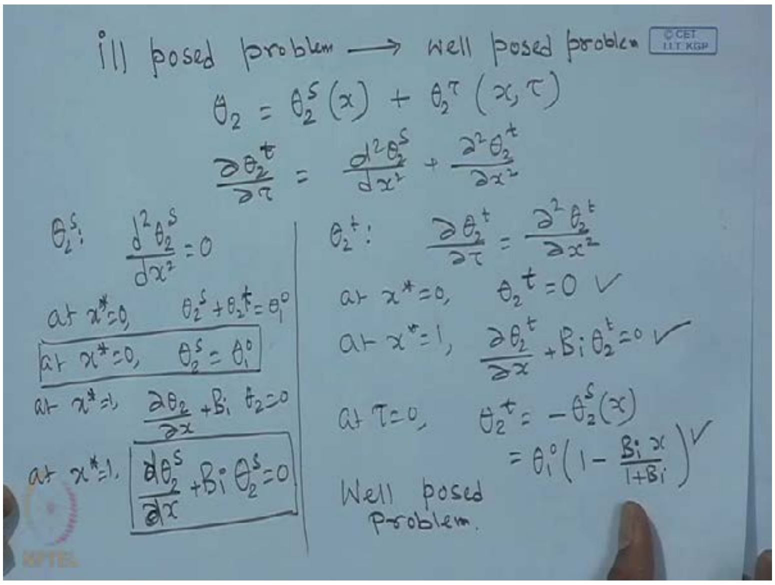

So, that is the profile of the solution of theta 2 s, as a function; theta 2 s, that is a steady state solution as a function of x. This profile I am going to put it over here, so theta 2 t at the transient part is nothing but theta 1 0 multiplied by 1 minus Biot times x divided by 1 plus B i.

So, if you look into this problem, the problem has the non-zero initial condition with a known function here. And homogeneous boundary condition at x star at x is equal to 0 and homogeneous boundary condition at x is equal to 1, so this is a well-posed problem. And we know how to solve this problem, we have seen it at least two times earlier.

So, again we use the separation of variable type of solution. Since this is the mixed boundary condition, at x is equal to 1, we will be having a transcendental equation roots of which will be denoted by, the Eigenvalue will be obtained from the 0 is of roots of that equation and Eigen functions will be sin function. And by using the initial condition, we can finally evaluate the unknown coefficient of integration and finally, we can obtained the solution of theta 2 t.

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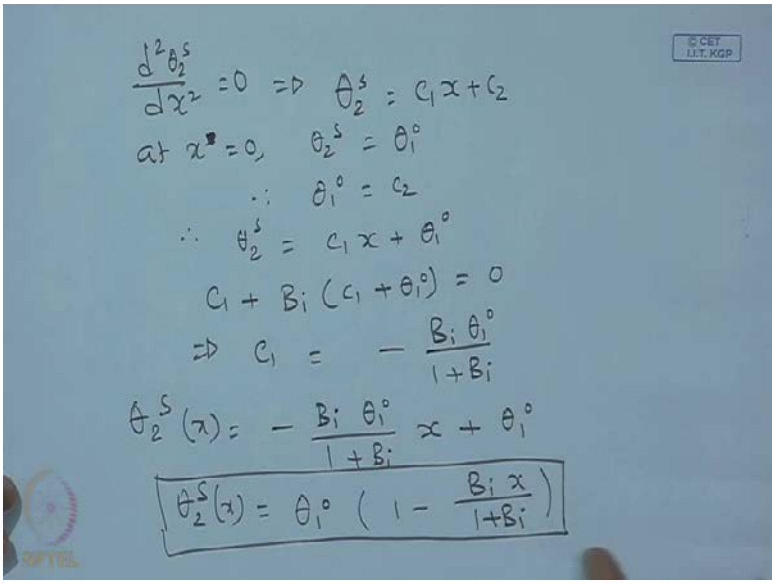

Since I we have already solve this problem earlier, I am not solving this problem here again; so, it will be a repetition only. So, we can get the complete solution of theta 2 t; we have already got the complete solution of theta 2 s; then, we have already the obtained the solution of theta 1.

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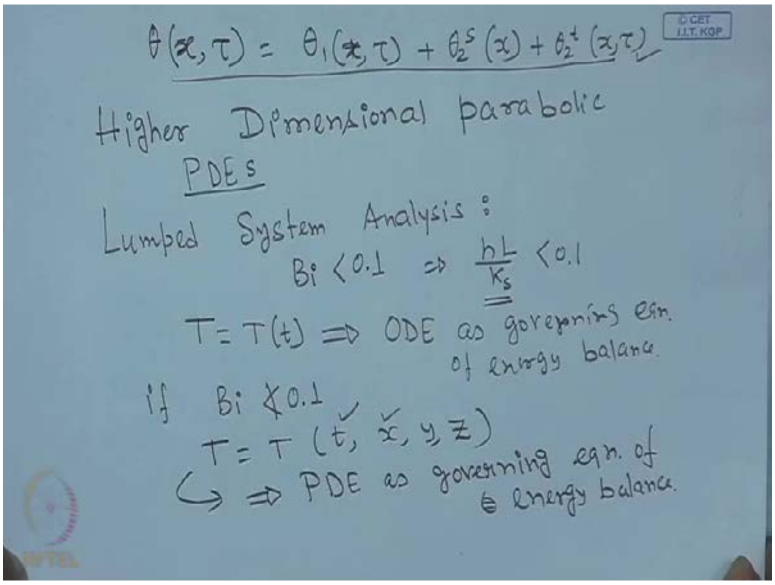

So, the overall theta profile as a function of x and tau can be obtained as theta 1 x and tau plus theta 2 s, which is a function of x, and theta 2 t which is a function of both x and tau. So, you got the complete overall solution of this particular problem, which is an actual heat conduction problem in chemical engineering application.

So, that covers more or less parabolic - two-dimensional parabolic partial differential equation: one dimension in space, one dimension in time. And we have looked into the problem will all source of, with all source of non-homogeneities present in the initial condition as well as in the boundary conditions. And what will be the effect of different boundaries and how different boundaries can be handled by considering, by breaking down the problem into sub problems considering one non-homogeneity at a time.

Now, we have also looked into the actual problem in chemical engineering system, where all the initial and boundary conditions are non-homogeneous. We have reduced the number of non-homogeneities by defining an appropriate non-dimensional quantity, so that the number of non-homogeneities reduced at least by 1 from 3 to 2.

And then, we bring down, we divided this problem into sub problems considering one non-homogeneity at a time and solve them completely. If the problems becomes a zero initial condition and homogeneous condition and non-homogeneous boundary condition, then we divided the problem again into two sub problems, considering one non-homogeneity, considering the problem is a linear combination of say - time d independent part, there is a steady part and time dependent part; that is the transient part and then, we formulate the respect, we define the corresponding problems of the steady state part and the transient part, and we solve them completely.

Now, that gives a white coverage of different types of parabolic equations in two dimensions one can encounter with. Now, we will be looking into higher dimensional parabolic partial differential equations. Now, we will be looking into this problem for higher order of… when the function, when the temperature may be a function of x and y, so actually if you look into the problem it becomes… If you remember the what is called lumped system analysis, in the lumped system analysis we have assumed it is again a heat conduction problem. If you remember what a lumped system analysis in heat transfer, then things becomes clear, whenever will be having a multidimensional problem or whenever will be and wherever will be having only a single dimensional problem in time only.

So, what is lumped system analysis? In lumped system analysis, if Biot number is less than point one, that means if what is Biot number? h l over k of thermal conductivity of solid is less than point one; h is the heat transfer coefficient, l is the characteristic length, k s is the thermal conductivity of solid.

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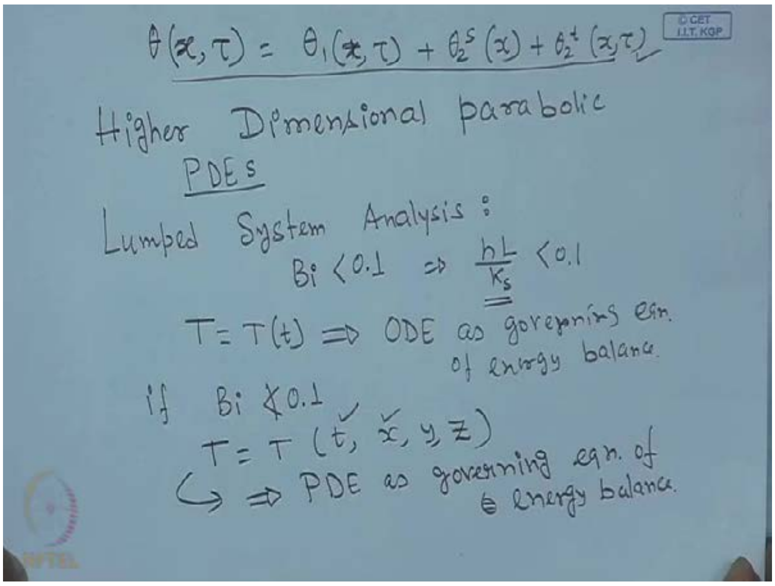

If that is less than point one, then the variation of temperature in space within the control volume can be neglected. So, that means conductivity is so high corresponding to the convective heat transfer coefficient. Then, this is less than point 1 means this thermal conductivity of solid is at least one order of magnitude higher compared to the heat transfer coefficient - convective heat transfer coefficient; that simply means, since the conductivity of the solid is pretty high, the temperature at different parts of the solid becomes almost instantaneously uniform.

Now, that simply indicates that there is no temperature gradient exist within the solid; that means, temperature is not a function of space - that means, temperature is not a function of x y z coordinate system. Temperature is almost instantaneously equal throughout the whole mass of the solid.

Therefore, in this case, in a whole the solid mass can be treated as a lumped mass and the temperature will be function of time alone. And you will be landing up an ordinary differential equation as governing equation of heat balance, of energy balance of the system.

On the other hand, if B i is not less than point 1, in that case, the thermal conductivity is not that high when compared to convective heat transfer coefficient. So, there exists a temperature gradient within the solid itself; that means, temperature profile in the solid is a function of space. In that case, temperature will be function x in one-dimensional problem temperature; function of x and y in two-dimensional problem; temperature function of x y and z in a three-dimensional problem. And temperature variation is most important and since it is varying as a function of x y z or in general - x and y; it becomes, it becomes no longer a one-dimensional problem, it becomes a two-dimensional problem or multi-dimensional problem, whatever the case may be.

If it becomes a multidimensional problem involving more than one dimension, that means it involves two more than one independent variable, therefore we are going to land up into an partial differential equation, instead of an ordinary differential equation in case of lumped system analysis.

So, in this case, Biot number is not less than point 1. Then, temperature is a function of time as well as space; in general, it is a function of x y z. And the governing equation and we will be landing up with a partial differential equation as governing equation of energy balance.

So, in the earlier examples, we have consider it is a function of temperature, function of time and space alone; so, we landed up with two independent variables. And therefore, we are landing up with a partial differential equation involving two dimensions: one in time, another in space x.

If we talk about a temperature function of x y and t will be landing up with a three-dimensional problem, if we consider that temperature function of x y z and t will be talking about a four-dimensional problem: three dimensions in space and one dimension in time, so these are the higher-dimensional parabolic partial differential equations.

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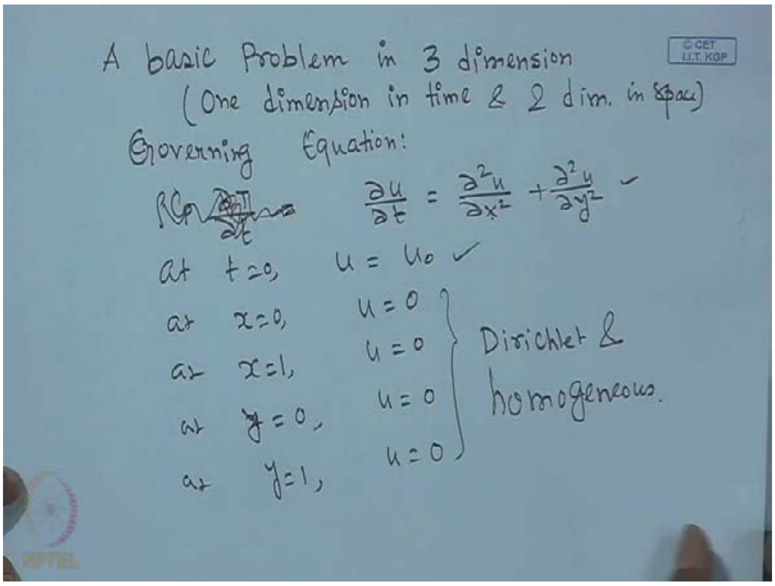

So, let us look into a well-behaved problem or a basic problem in three dimensions; that is, one dimension in time and two dimensions in space. So, a basic parabolic problem, so the governing equation of this problem will be rho c p del t del t; so, typical governing equation, we will be talking about the chemical engineering application later on.

Typical governing equation will be del u del t is equal to del square u del x square plus del square u del y square. So, therefore, we need to specify one initial condition on time, two boundary conditions on x, and two boundary conditions on y, because it is order two with respect to x as well as with respect to y.

Now, let us consider a basic problem; that means, the initial condition is non-homogeneous and all the four boundary conditions are homogeneous. Let us see what we get; that is a basic problem in three dimensions. At t is equal to 0, we have u is equal to u naught; and at x is equal to 0, we have u is equal to 0; at x is equal to 1, we have u is equal to 0; at x is equal to at y is equal to 0, we have u is equal to 0; at y is equal to 1, we have u equal to 0. That means, all the four boundary conditions are homogeneous and they are Dirichlet boundary conditions - Dirichlet and homogeneous boundary conditions. So, we talk about a basic problem, where the initial condition is non-zero and the four boundary conditions are homogeneous.

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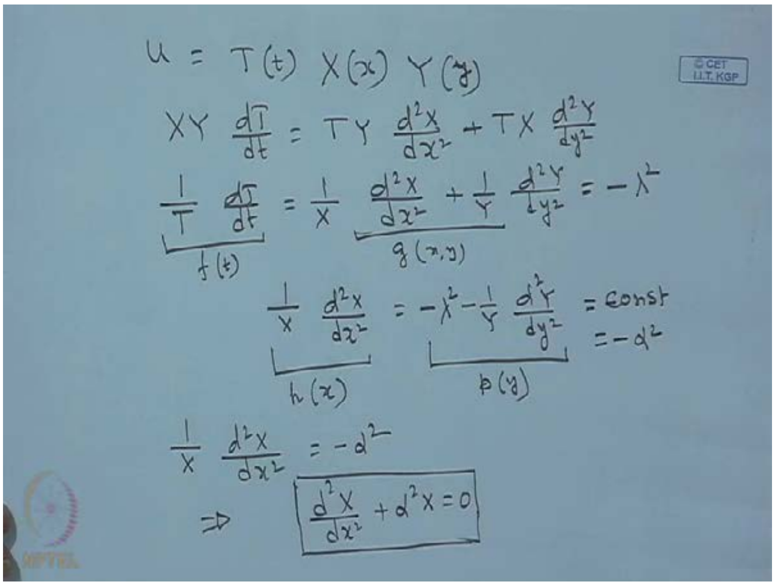

Now, since this operator is a linear operator, we again, we can use the separation of variable type of method. We assume that solution of u is composed of three parts; it is a multiplication of product of three quantities: one is function of time alone, another is a function of x alone, another is a function of y alone - that is a case. Then, we consider that, u is equal to function of time, x which is a function of x only, capital Y which is a function of small y only.

So, if that is the case, we put the equation in the governing equation - del u del t. So, this becomes X times Y d T d t is equal to del square u d del x square; that means, T into Y d square x d x square plus T into X d square y d y square. We divide the problem, we divide both side by x y and t.

So, we get 1 over t d T d t is equal to one over x d square x d x square plus 1 over Y d square Y d y square; so, the left hand side is a function of time alone, the right hand side is a function of space alone. So, therefore, they are equal and they will be equal to some constant; we call this constants as minus lambda square and this constant cannot be 0 or positive. If this is 0, they will be the giving a trivial solution.

So, we will be getting, so let us again consider this part, 1 over X d square x d x square is equal to minus lambda square minus 1 over Y d square y d y square. Now, the left hand side is a function of x alone, the right hand side is a function of y alone, and they are constant and they are equal and they will be equal to some constant; let us consider this constant is… let us say, so there will be some constant. Again, this constant can be 0, positive or negative. We have already seen if this constant is 0 and positive, we will be getting a trivial solution; so, this constant is negative, so this will be minus alpha square.

If this is the case, then we will be, let us look into the x varying part, 1 over x d square x d x square is equal to minus alpha square. So, we will be getting d square x d x square plus alpha square x is equal to 0. So, this is again a standard Eigenvalue problem or a Sturm Liouville Problem.

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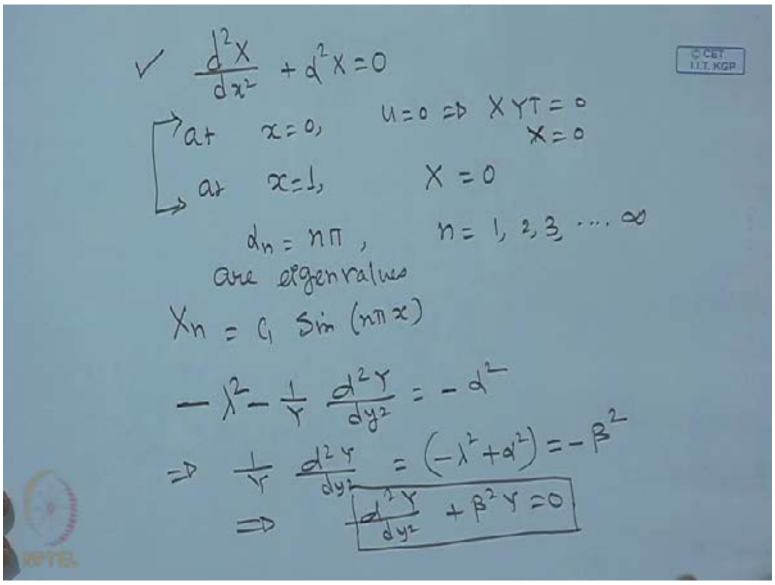

And let us fix up the boundary conditions of this standard Eigenvalue problem; so, d square x d x square plus alpha square x is equal to 0. Now, let us put the boundary conditions and the boundary condition of the original problem must be satisfied by the x varying part; boundary conditions on x must be satisfied that the x varying part. So, at x is equal to 0 in the original problem, your u was equal to 0, therefore X Y T is equal to 0;

y and t - we cannot comment on this, they will be not equal to 0; therefore, X is equal to 0 - at capital X must be equal to 0, at x is equal to 0.

Similarly, at x is equal to 1, u was equal to 0, therefore capital X is equal to 0. Now, if you remember, this is a standard Eigenvalue problem and these are the homogeneous Dirichlet boundary conditions, so we know the solution of this problem. The solution of this problem is that, we have the solution as alpha n is equal to n pi, where index n runs from 1 2 3 up to infinity; so, alpha n are Eigenvalues and Eigen functions are x n equal c 1 sin n pi x. So, we completely solved the x varying part.

Let us look into the y varying part. If you look into the y varying part, it becomes 1 minus lambda square minus 1 over Y d square Y d y square is equal to minus alpha square; so, we put it on the other side. So, 1 over Y d square y d y square should be is equal to minus lambda square plus alpha square.

So, this is a constant and again this constant has to be a negative constant; otherwise, this is a combination of two constant and this constant has to be a negative constant (Refer Slide Time: 42.17). Otherwise you will be landing up with a trivial solution, so this must be equal to minus Beta square. So, therefore, we will be getting the governing equation of y as d square Y d y square plus Beta square Y must be equal to 0. So, you obtain the governing equation of y varying part.

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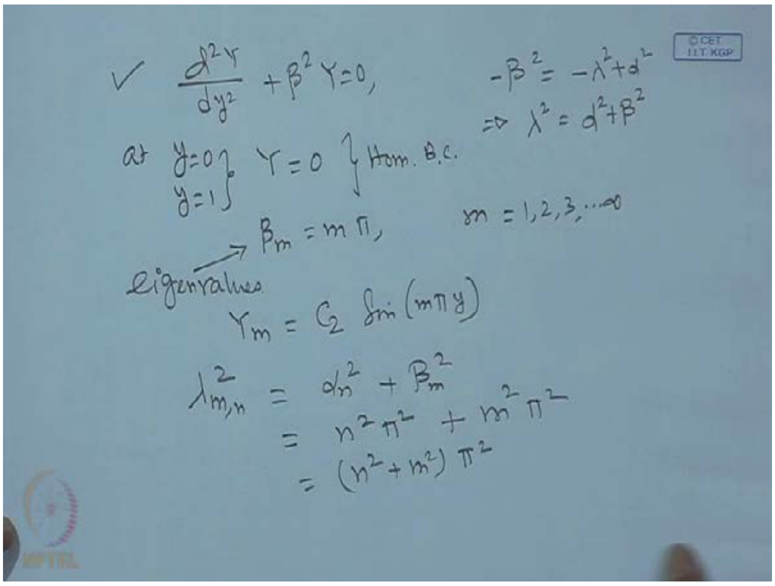

So, if we look into that, d square Y we get d square y d y square plus Beta square Y is equal to 0, where Beta square - minus Beta square is nothing but minus lambda square plus alpha square or lambda square is nothing but alpha square plus Beta square.

Now, let us look into the boundary condition of the y varying part. If in the original problem, at y is equal to 0 and y is equal to 1, we had u is equal to 0 - that means, y must be equal to 0 on the both boundaries. Now, again this is a standard Eigen value problem, as this is a special form of Sturm Liouville Problem and the boundary conditions are homogeneous boundary conditions. Again we know the solution of this, the solution is Beta m is equal to m pi, where the index m runs from 1 2 3 up to infinity; so, these are the Eigenvalues and the corresponding Eigen functions are Y m, that will be nothing but c 2 sin m pi y.

So, why we have put an independent index m? Because this is an independent Eigen value problem. Therefore, on the other hand, the x varying part was also independent Eigen value problem with the index n and this will be an independent Eigenvalue problem with index m. So, therefore, lambda - we can write it as lambda m n; so, lambda m n square is nothing but alpha n square plus Beta m square. So, alpha n is equal to nothing but n square pi square and Beta m square is m square pi square; so, therefore we can write it as n square m square pi square. So, this gives the complete specification of lambda m n.

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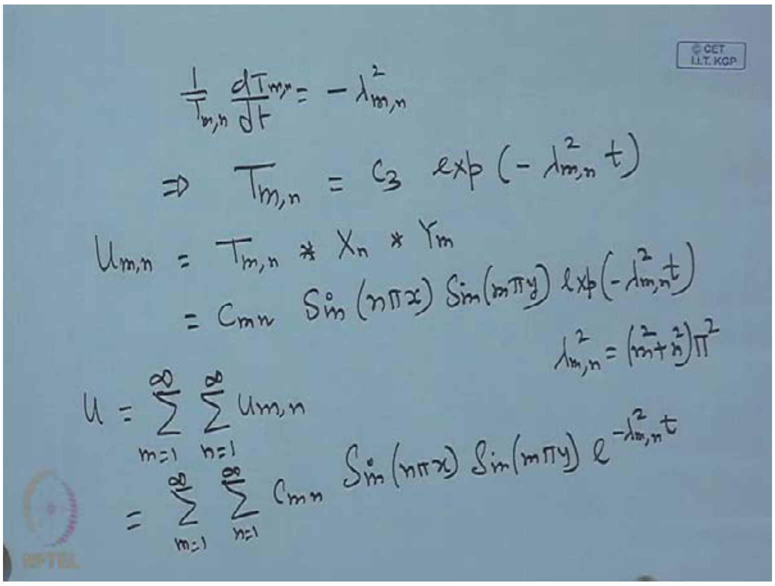

So, we have already almost solved the problem, only the time varying part has to be found out. If you looked into look into the time varying part, it will be 1 over T d T d t is equal to minus lambda m n square. So, we write it, correspondingly you write it T m n as a subscript; to correspond that, it will be corresponding to particular lambda m n value. So, this will be given a solution, T m n is nothing but another constant c 3 exponential minus lambda m n square t.

So, we construct the corresponding solution in U m n, that will be T m n multiplied by the X n multiplied by the Y m. So, therefore, T m n will be c 1 c 2 c 3, it will be new constant. Let us say, C m n sin n pi x sin m pi y exponential minus lambda m n square t, where lambda m n square is nothing but m square plus n square pi square. And we can construct the complete solution u as a double summation of U m n, 1 over index m is equal to 1 to infinity, this is form index n equal to 1 to infinity. So, this will be double integral m is equal to 1 to infinity, n is equal to 1 to infinity C m n sin n pi x sin m pi y e to the power minus lambda m n square t.

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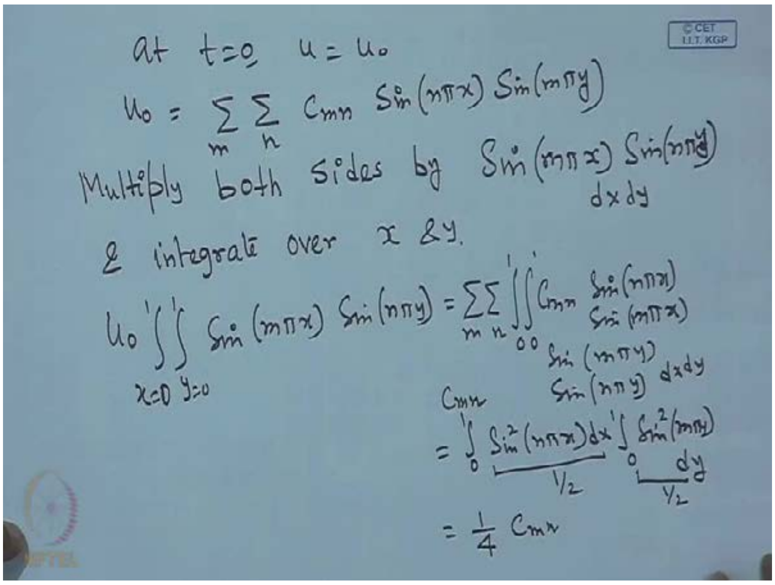

So, we have got almost the complete solution. So, what is left is, we have to evaluate the constant C m n; for that, we have to use the initial condition that has been unutilized till now. So, we have already utilized two boundary conditions in x, we have already utilized two boundary conditions in y; so, one initial condition at t is equal to 0 is unutilized. We use that, that is, at t is equal to 0, u was equal to u naught; so, you put it that u naught is equal to double summation 1 over m, another is over n, C m n sin n pi x sin m pi y.

Now, what we do? We multiply both side by sin m pi x and sin n pi y d x d y and integrate over the domain of x and y; and utilize the property, property that the Eigen functions are orthogonal functions. So, Multiply both sides by sin m pi x sin n pi y d x d y and integrate over x and y. If you do that, since u 0 is a known function, so it will be u 0 is constant; so, u 0 is taken out, integral x is equal to 0 to 1, y is equal to 0 to 1, sin m pi x sin n pi y is equal to summation double integral c m n sin n pi x sin m pi x sin m pi y sin n pi y d x d y.

Now, if we open up this summation series, this integral 1 is over x, another is over y. So, if we open up this integral, all the terms will vanish, because of the orthogonal property of the Eigen functions - sin function; only the terms where m is equal to n, they will be remaining and they will survive; so, it will be 0 to 1. So, only one term will survive, that will be sin square n pi x d x and 0 to 1 sin square m pi y d y.

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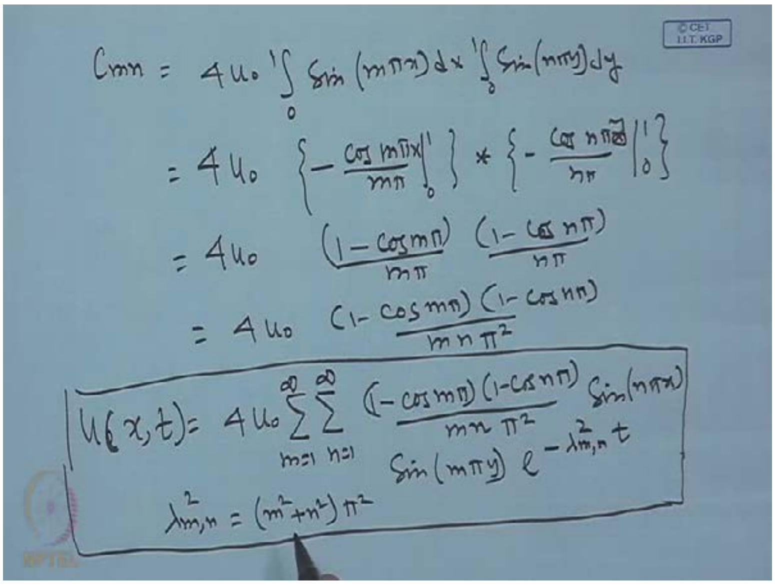

So, the answer of this is half, this will be half, so you will be getting one fourth. So, there will be a C m n here; so, we can 1 by fourth times C m n. So, we will be getting C m n as 4 times u naught integral 0 to 1 sin m pi x d x 0 to 1 sin n pi y d y. So, this becomes 4 u naught, it will be minus cosine m pi divided by m pi m pi x divided by 0 to 1 multiplied by minus cosine n pi x n pi y divided by n pi divided from 0 to 1.

So, this will be 4 u naught 1 minus - cos 0 is 1 - so, it will be 1 minus cosine m pi divided by m pi and it will be one minus cosine n pi divided by n pi. So, ultimately you will be getting 4 u naught 1 minus cos m pi into 1 minus cos n pi divided by m n pi square.

So, we will be in a position to get the complete solution, u as a function of x and time is equal to 4 u naught double summation, one summation is over m 1 to infinity, another summation is over n 1 to infinity, 1 minus cosine m pi 1 minus cosine n pi m n pi square sin n pi x sin m pi y e to the power minus lambda m n square times t, where lambda m n square is nothing but m square plus n square into pi square. So, this gives the complete solution in three-dimensional problem.

So, this gives the complete solution in a three-dimensional problem, where all the boundary conditions are Dirichlet boundary condition. Next, we will be talking about some of the conditions are may be robin mixed or some condition may be a Neumann boundary condition, then, what will be the form of this solution and then we will take up couple of standard chemical engineering problem to solve by using separation of variable. Once you do that, next we will move over to a four-dimensional problem; that is, the most general problem in chemical engineering application is one dimension in time and three dimensions in space.

So, we stop here. We will take up this, we will solve those problem, whatever I just mentioned in the next class.

Thank you very much.