Advanced Mathematical Techniques in Chemical Engineering

Prof. S. De
Department of Chemical Engineering
Indian Institute of Technology, Kharagpur

Module No. # 01
Lecture No. # 23
Solution PDE: Separation of Variables Method

Very good morning to everyone. In the last class what we did? We developed, we looked into the formulation of Sturm Liouville problem or standard Eigen value problem and various properties of Sturm Liouville problem. How will the Eigen values be evaluated? How will the Eigen functions be evaluated? What are the different properties that this Eigen functions will satisfy?

So, we have looked into all these aspects. Now, probably we have also proved that the Eigen functions are orthogonal to each other for distinct Eigen values and they are real. With this background, we will be able to start with the solution of any partial differential equation - linear and homogeneous - by the method of using separation of variable.

What I have done for going step-by-step solution of partial differential equations: first, I will take some examples of parabolic equations only. Initially, I will be talking about only a two-dimensional problem: one-dimensional in space and other one-dimensional in time; so transient parabolic problem.

We will be looking into basic problems first; the basic problem constitutes a problem that we have already discussed in the earlier class; it is a well-behaved problem. In that problem, you must be having a set of homogeneous boundary conditions and non-homogeneous initial conditions; then, that is a well-behaved problem. Well-posed problem will be called as a basic problem. So, we will be solving the basic problem for parabolic partial differential equations for various boundary conditions and we will look into the solution of this problem almost completely.

Next, what will I do? I will take up one problem that will be well-behaved parabolic partial differential equation; that means, the non-homogeneity appears in the boundary condition, but the initial condition is homogeneous. We have already discussed in the earlier classes that it is an ill-posed problem; we will be looking into the solution of the parabolic partial differential equations of such well-posed problems and then we will be looking into an actual heat conduction problem, which is basically parabolic in nature.

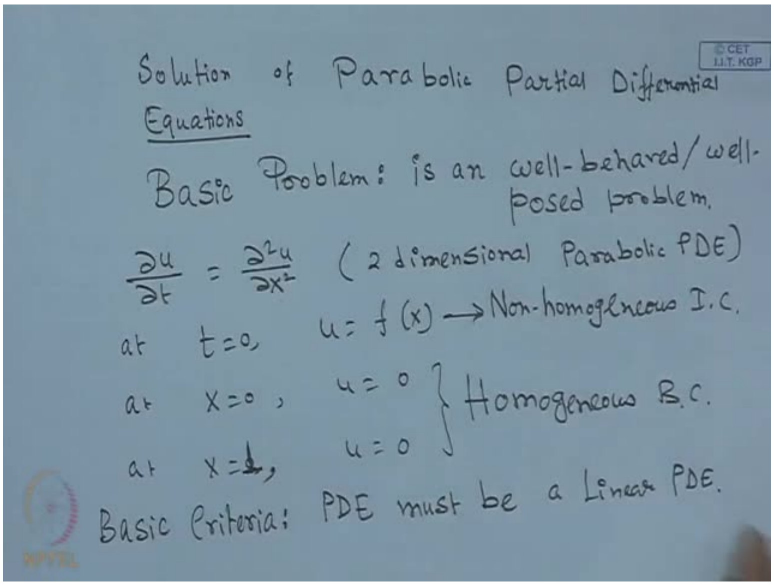
Let us start from the actual problem; we will make it non-dimensional and then we will go step by step. Follow the same procedure whatever you have done earlier, so that it will be giving the complete picture of how to solve the parabolic partial differential equation by using separation of variable type of method.

Next, I will talk about the multi-dimensional problem, higher-dimensional problem, three-dimensional problem and four-dimensional problem. Three-dimensional problem means two dimensions in space, one dimension in time and all these are parabolic. Similarly, the four-dimensional problem means one dimension in space, one dimension in time and three dimensions in space; it will be four-dimensional problem.

We will be looking into the basic problem as well as the complete solution of the problem, so that it completely covers the parabolic partial differential equation in rectangular coordinate or Cartesian coordinate. Then, we will take up some of the examples of elliptical partial differential equations and hyperbolic partial differential equations - all in Cartesian coordinate.

Then, we will move over to the cylindrical coordinate where you will be talking - dealing - with the (( )) functions. At the end, we will be moving over to the spherical polar coordinate system, where the Eigen functions are nothing but the Legendre polynomials. Once that will be over, then we will be going over to the non-homogeneous partial differential equations and the Green's function method; how to solve them by using separation of variable.

(Refer Slide Time: 04:37)



Therefore, for the time being, we will be looking into the basic problems only such as solution of parabolic partial differential equations. We will be looking into the solution of the basic problem; we will look into the solution of an well-posed problem and then we will be constructing the complete solution by taking up one actual example of an actual problem; so we will be looking into the basic problem. First of all, let me define, what do I mean by a basic problem? Basic problem is a well-behaved problem - well-behaved or well-posed problem.

So, del u del t is equal to del square u del x square - this is a typically a two-dimensional parabolic partial differential equation; it is also called one-dimensional transient problem; this one dimension in space and transient means it is time variant. Now, for this problem, if the initial condition is non-zero, non-homogeneous, at t is equal to 0, u is equal to f of x; at x is equal to 0, u is equal to 0; at x is equal to 0, u is equal to 0; that means, you have homogeneous and Dirichlet boundary conditions, and homogeneous boundary conditions, and non-homogeneous initial condition, then this problem is defined as well-posed problem.

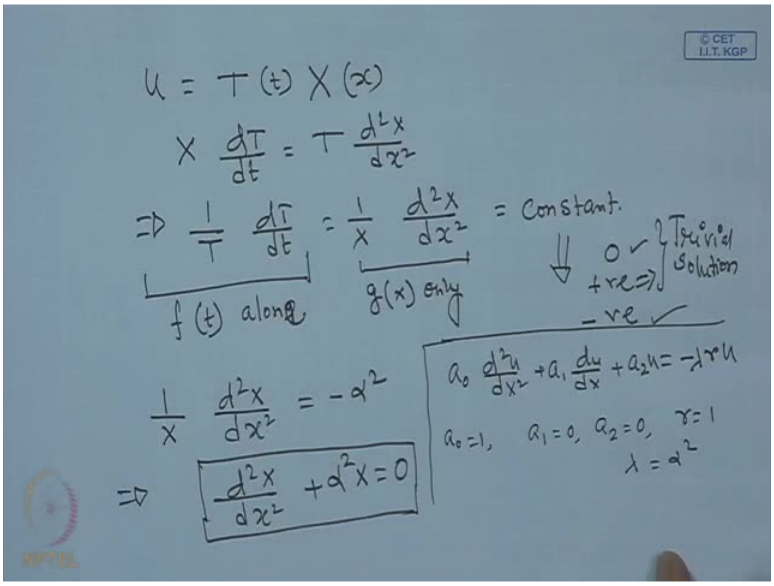
Similarly, we can have a Neumann boundary condition, when you are here, but it should be homogeneous. If any Robin mixed boundary condition may be present, then this x is equal to 1 at any boundary if that condition is homogeneous, then this is a well-posed problem. What is a well-posed problem? The well-posed problem is at time t is equal to

0, if you have a homogeneous initial condition but one of the boundary condition is non-homogeneous, then that will be a well-posed problem; we will be taking over the well-posed problem later on in this class itself.

Let us look into the solution of the basic problem in parabolic domain, which is a well-posed problem. If you look into the solution of this by using separation of variable method, the solution is assumed to be composed of a product of two functions - one is a function of space alone, another is the function of time alone.

Before going into that, we must see under what conditions one can use separation of variable method to solve a partial differential equation? The basic criteria is that your governing equation has to be a linear one; PDE must be a linear partial differential equation and then only one can think of applying separation of variable method for the solution.

(Refer Slide Time: 09:08)



As I said that the separation of variable method, it assumes the solution u is composed of a product of two functions - one is function of time alone and another is the function of space alone. Therefore, u is supposed to be a product of function of time and multiplied by a function of space. So, the function capital T is a function of time only and X is a function of space only; once we do that, then we substitute these products.

The governing equation: if you put it in the governing equation this becomes dT dt and this becomes T d square x dx square; so, the partial differential becomes total differential, because T is a sole function of time and capital X is a sole function of space. Next, we separate the variables 1 over T dT dt is equal to 1 over x d square x dx square. If we separate the variables, if you look into this equation, the left hand side is completely a function of time alone and right hand side is entirely function of space.

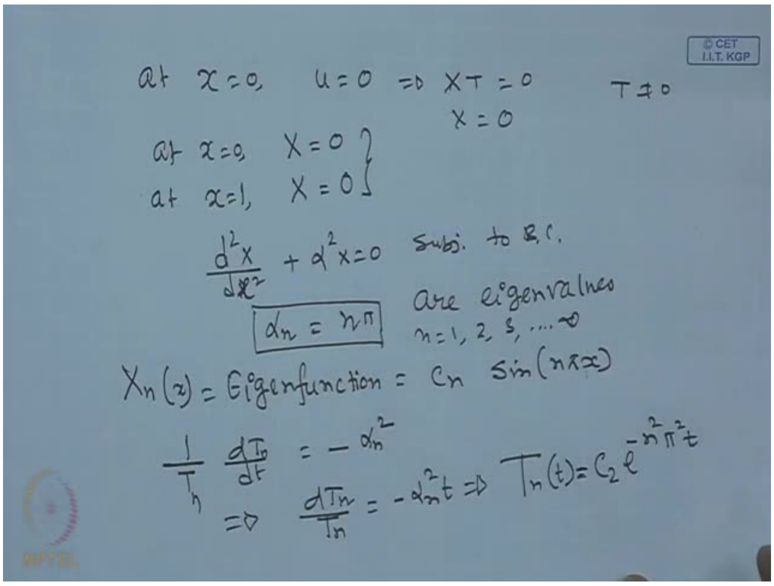
This is a function of time alone; this is a function of space alone; they are equal and they must be equal to some constant (Refer Slide Time: 10:27). We have seen earlier that this constant can have three options: it can be 0; it can be positive; it can be negative. We have already seen earlier that if this constant is 0, then we are going to get a trivial solution and we are not looking into that. If this function is positive, then also we will be getting a trivial solution that I have proved in the earlier classes. This constant has to be a negative constant in order to get a non-trivial solution.

Therefore, 1 over x d square x dx square is equal to minus alpha square; minus alpha square ensures that this constant is negative. We multiply both sides by x and bring it on the left hand side; so d square x dx square plus alpha square x is equal to 0. If you look into the general value with the general formulation of Sturm Liouville problem, this equation falls under standard Sturm Liouville problem under certain modifications.

I will just take a diversion here; if you remember what is the standard Sturm Liouville problem, it will be a 0 d square u dx square plus a 1 du dx plus a 2 which is equal to minus lambda r u a 2 u lambda r u; this is a generalized Sturm Liouville problem or Eigen value problem. This equation is a special case of Sturm Liouville problem, because in this case a 0 is equal to 1, a 1 is equal to 0, a 2 is equal to 0, r is equal to 1 and lambda is equal to alpha square. Under this condition the generalized Sturm Liouville problem falls down to this equation (Refer Slide Time: 12:58). This equation is also a Sturm Liouville problem.

Only thing is that in order to qualify this equation we have to just check whether the boundary conditions are homogeneous; we already known that the boundary conditions are homogeneous in the original problem; so, let us fix up the boundary conditions of this standard Eigen value problem or Sturm Liouville problem. These boundary conditions will be obtained from the boundary conditions of the original problem, if we do that.

(Refer Slide Time: 13:31)



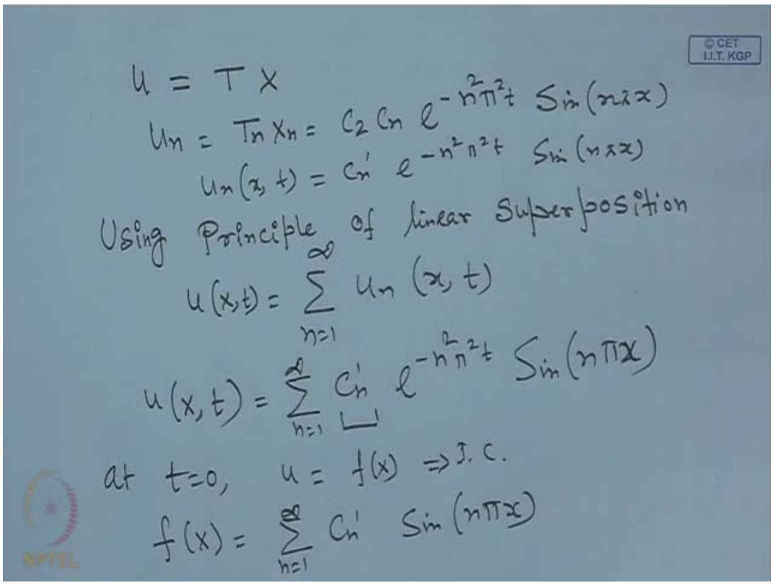
Let us look into the original problem; in the original problem, the boundary conditions is at x is equal to 0, u is equal to 0; the space-wearing part must satisfy this boundary conditions as well, because at x is equal to 0, u is equal to 0; you just put x multiplied by T is equal to 0 and T is not equal to 0; otherwise, it becomes a trivial solution and this simply indicates x is equal to 0; so, at small x is equal to 0, our capital X is equal to 0; similarly, at small x is equal to 1, you have our capital X is also equal to 0.

Therefore, d square x dx square plus alpha square x is equal to 0 subject to boundary conditions of this (Refer Slide Time: 14:22). So, these are the homogeneous boundary condition and this is a standard Eigen value problem. We have solved this problem in detail in the earlier class; we already know that there are n numbers of Eigen values present in this case, when these are equal to n pi are the index n runs from 1 to infinity - the Eigen functions and the sine functions.

So, we can write alpha n is equal to n pi are Eigen values, where n is equal to 1, 2, 3 upto infinity. The corresponding Eigen functions are x n is equal to C n sin lambda n x or n pi x, these are the Eigen functions. Now, we can look into the time-wearing part; if you look into the time-wearing part, it is 1 over T dT dt is equal to minus alpha n square - just put corresponding to alpha subscript n; the corresponding solution of time-wearing part you just put a subscript n there.

If we really solve this problem, this becomes dT n T n is equal to minus alpha n square t or will be getting the solution as T n, which is a function of time; it is nothing but one more constant. Let say as C 2 e to the power minus n square pi square t, where alpha n is nothing but n pi in this particular case (Refer Slide Time: 15:54).

(Refer Slide Time: 16:29)



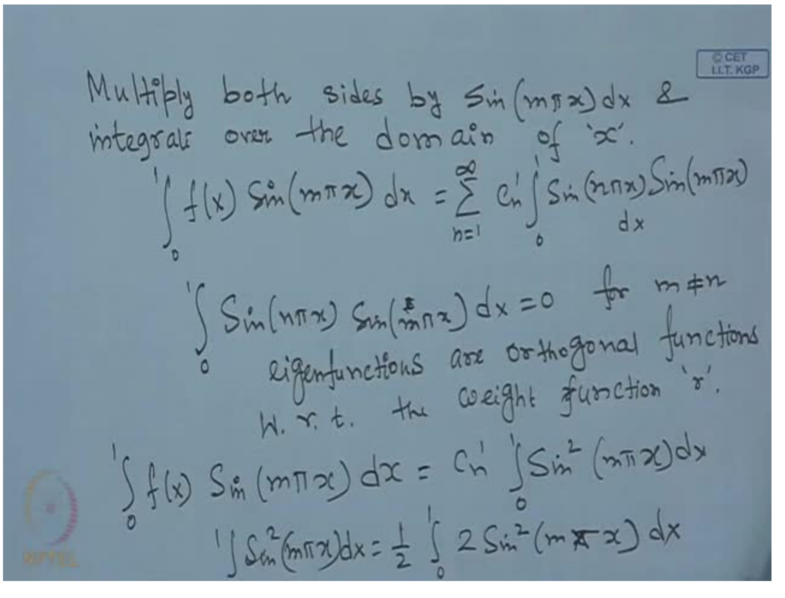
Now, we construct the complete solution, if you look into the solution, the solution is; if you remember u was product of T and X, corresponding value for nth Eigen value will be having u n is equal to T n X n.. This will be nothing but C 2 multiplied by C n e to the power minus n square pi square t sin n pi x. Now, C 2 multiplied by C n, it will be a new constant; we called that constant as C n prime e to the power minus n square pi square t sin n pi x; so, that is nth solution corresponding to the Eigen function Eigen value, nth Eigen value and nth Eigen function.

As you remember that Eigen functions are orthogonal functions, the complete solution will be constituted by summation of all the Eigen functions; I will be getting the overall solution by using principle of linear superposition. Using principle of linear superposition, we can construct the complete solution and the complete solution will be linear superposition of all these solutions - n solutions, infinite number solutions.

So, u as a function of x and t should be summation of u n as a function of x and t, n is equal to 1 to infinity; so you construct the solution summation n is equal to 1 to infinity C n prime e to the power minus n square pi square t sin n pi x. Now, what is left is we have got the complete solution almost, but only one constant is not evaluated; we have to evaluate this constant - C n prime (Refer Slide Time: 18:43).

If you remember, there were three conditions to define this parabolic problem: two boundary conditions and one initial condition; if you remember that we have already utilized the two boundary conditions, so we can utilize the unused initial condition and can evaluate the constant C n prime. If you remember that at x is equal to at t is equal to 0 for any value of x, u was equal to f of x that was the initial condition we had and we have not utilized it till now; now, we are going to utilize it; so, at t is equal to 0, we put this equation f of x is equal to summation n is equal to 1 to infinity C n prime sin n pi x.

(Refer Slide Time: 19:54)



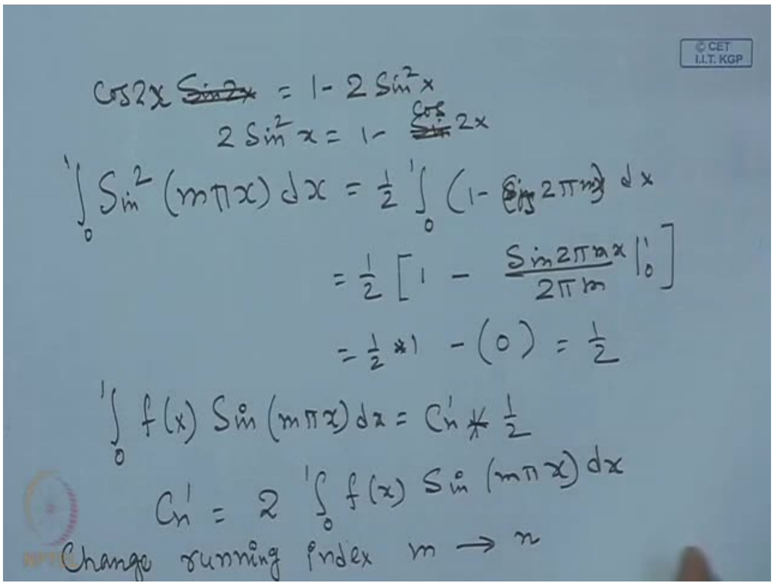
From this equation, one has to evaluate C n prime. Let us see, how will we evaluate that? So what we do? We multiply both sides of this equation by sin m pi x dx and integrate over the domain of x; so, multiply both sides by sin m pi x dx and then integrate over the domain of x; if you do that it will be f of x sin m pi x dx 0 to 1 which is equal to integration n that is equal to 1 to infinity C n prime sin n pi x sin m pi x dx and integration over 0 to 1.

Let us open up this summation and we can utilize the property of orthogonal functions that we have already proved - the sine functions and cosine functions are orthogonal function; that means, since the Eigen functions are orthogonal functions, sin n pi x sin m pi x dx from 0 to 1 is equal to 0 for m is not equal to n; this is the property that we have already looked into - the property of the Eigen functions - that the Eigen functions are orthogonal functions with respect to the weight function r.

If you remember that just few minutes back we have identified that r is equal to 1 for a Cartesian coordinate in this particular problem. So, if you open up this summation series, then all the terms will be equal to 0, where m is not equal to n and only one term will survive on the right hand side, that will be for m is equal to n.

So, if you do that, you will be getting 0 to 1 f x sin m pi x dx is equal to C n prime integral sin square m pi x; that is, m is equal to n dx from 0 to 1. If you would like to evaluate sin square m pi x dx, let us see what you get integration 0 to 1 sin square m pi x dx, it will be you just multiply and divided by 2 and it will be half 0 to 1 2 sin square m pi x dx.

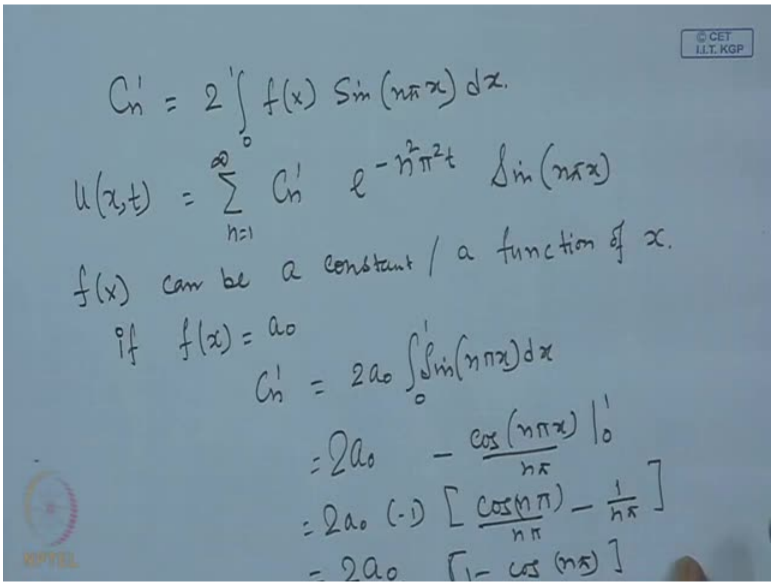
(Refer Slide Time: 23:32)



If you look into the relationship of sin 2 theta and sin (( )), it will become sin 2 x which is nothing but 1 minus 2 sin square x. So, 2 sin square x will be 1 minus sin 2 x; substitute there, 0 to 1 sin square m pi x dx, it will be half 0 to 1 in place of 2 sin square m pi x. You just substitute 1 minus sin 2 x, so it will be 2 m pi dx. This is the form of cosine 2 x; so cosine 2 x will be 1 minus cosine, there is mistake here (Refer Slide Time: 24:25). So cosine 2 x is 1 minus 2 sin square x; so, 2 sin square is nothing but 1 minus cosine 2 x; it will be cosine 2 x cosine 2 pi m x dx.

Carry out this integration - so half the first one will be 0 to 1, it will be 1 minus integration of cosine 2 pi m x; it is nothing but sin 2 pi m x divided by 2 pi m from 0 to 1. So, half into 1 multiplied by sin 2 pi into 2 pi m divided by 2 pi m; so sin 2 pi m will be equal to 0, sin 2 pi will be always 0 and sin 0 is 0. You will be getting a 0 contribution from there; so it will become half; integration of sin square m pi x dx is nothing but half. Therefore, what we will be getting is from 0 to 1 f of x sin m pi x dx is equal to C n prime multiplied by half, C n prime is nothing but 2 0 to 1 f of x sin m pi x dx.

(Refer Slide Time: 26:12)

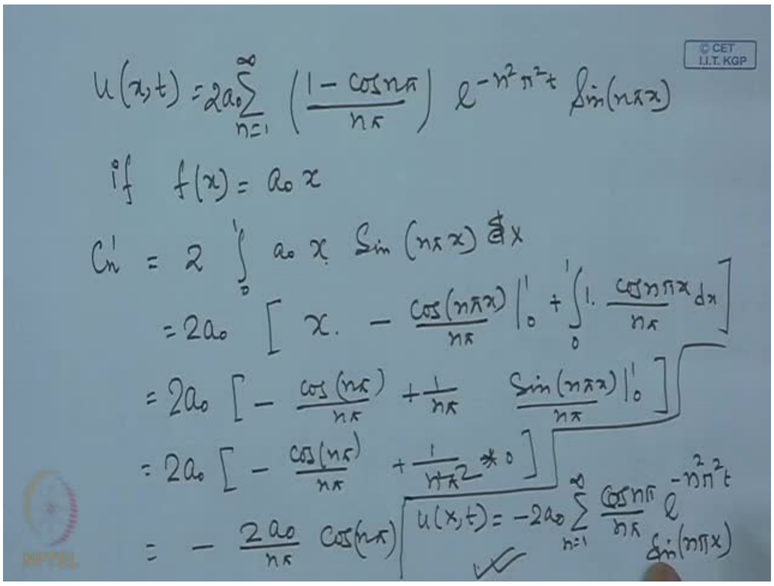


Now what we do? We change the running variable m from m to n, change running index m to n, we will be getting C n prime in terms of n. C n prime is nothing but 2 integral 0 to 1 f of x sin n pi x dx; so, that gives the complete solution that u as a function of x and t, it should be equal to n; this is equal to 1 to infinity C n prime e to the power minus n square pi square t sin n pi x, where the constant C n prime is given by this expression. If the value of f of x is known to you, then you can integrate this out analytically or if effects is a complicated function of x, then this integration has to be evaluated numerically using a Trapezoidal Rule or Simpson’s Rule.

These effects can be a constant or in general it can be a function of x, if f x is a simple function of x, write x or x square or a to the power x or cosine x k x. something like that; then, this integration can be solved analytically. Otherwise, one has to take request to numerical techniques; for example, I will just solve this problem for f x is constant; if f x is constant, let's say a 0, let see what we get out of this equation; C n prime is nothing but 2 a 0 integral sin n pi x dx from 0 to 1.

Therefore, you will be getting 2 a 0 integral of sin n pi x is nothing but minus cosine n pi x divide by n pi; it will be from 0 to 1; so 2 a 0 with a minus sign, it will be cosine n pi divided by n pi minus 0, it will be cosine 0; so it will be 1 over n pi you will be getting; you can observe the minus sign inside, it will become 2 a 0 over n pi 1 minus cosine n pi.

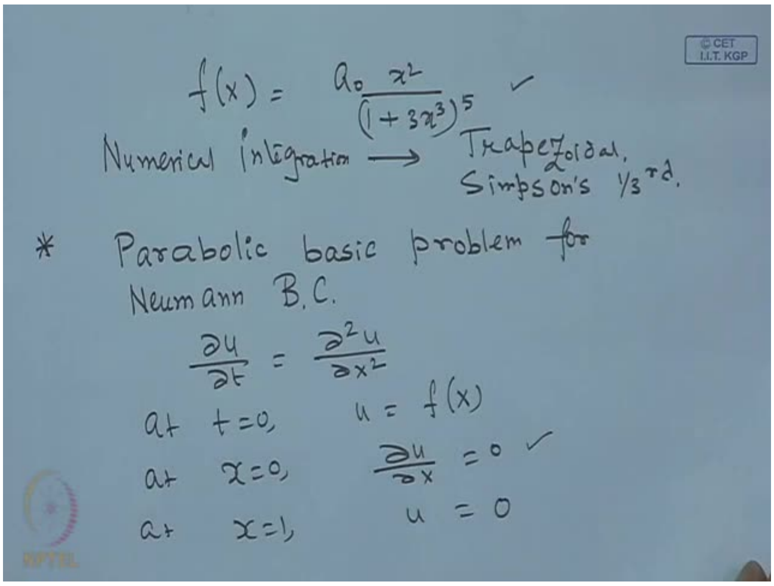
(Refer Slide Time: 28:49)



In this case, the complete solution becomes u x t is nothing but summation of n is equal to 1 to infinity 2a 0 1 minus cosine n pi over n pi e to the power minus n square pi square t sin n pi x. If f x is a known function of x, let's say f x is nothing but a linear function a 0 x and how will we evaluating this C n prime? The C n prime is 2 0 to 1 f of x sin x, so it will be a 0 x sin n pi x dx. Therefore, 2 a 0 will be taken out and this will be x sin n pi x dx; we carry out this integration by parts; take this as the first function, this is the second function (Refer Slide Time: 29:46). So, first function integral of second function will be minus cosine n pi x divided by n pi from 0 to 1 minus differential of the first function is 1; integration of second function minus minus plus cosine n pi x divided by n pi dx from 0 to 1 (Refer Slide Time: 30:10); so, it becomes 2 a 0 minus cosine n pi divided by n pi and when you put it 0, the whole term becomes 0. We just evaluate this one; so integral of cosine n pi x is nothing but sin n pi x divided by n pi from 0 to 1; this becomes 2 a 0 minus cosine n pi over n pi plus 1 over n square pi square sin n pi is 0 minus sin 0 is 0. So, the whole contribution becomes 0, you will be having minus 2 a 0 over n pi cosine n pi in this particular case.

If you look into the complete solution - I am just writing it over here - the complete solution is given as u as a function of x and t; this becomes minus 2 a 0 summation n is equal to 1 to infinity cosine n pi divided by n pi e to the power minus n square pi square t multiplied by sin n pi x; that gives the complete solution for this particular case. It will be 0, there it is fine, this is a perfect solution and this is a correct solution.

(Refer Slide Time: 32:31)



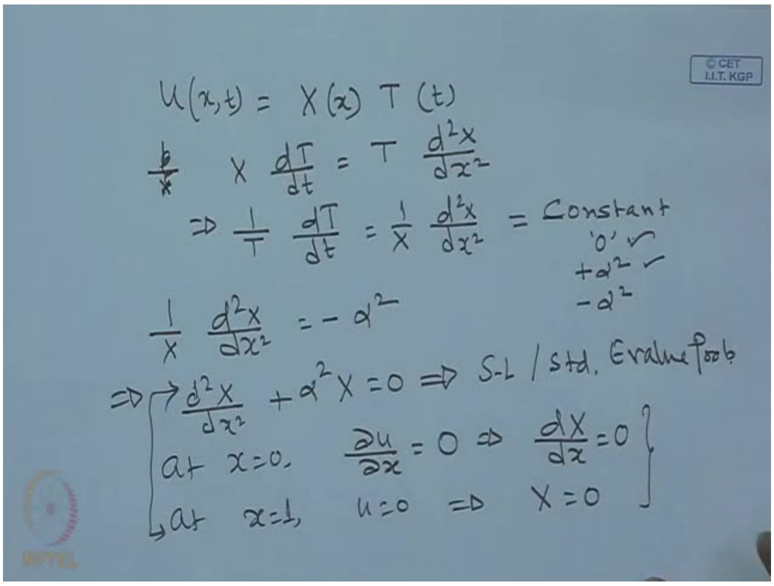
Similarly, if we have another function of f x is a known function, let's say a 0 x square or 2 x square or x cube or let's say e to the power k x something like this, the C n prime can be evaluated by integration by parts and you can construct the complete solution.

On the other hand, if f x is a complicated function like where the numerical analytical solution by using the integration by parts, it is may not be possible; for example, a case like f x is equal to a 0 x square divided by 1 plus 3 x to the power 3 x hole to the power 5. If it is a complicate function like this, one cannot do the integration by parts; one has to do numerical integration to evaluate this integral integration of f x integration 0 to 1 f x sin n pi x d x, that integration has to be evaluated numerically in one step.

A numerical integration is required using Trapezoidal Rule or Simpson's one-third. So, whatever the method, one can take recourse to solve this problem completely; so, that completes the solution of parabolic basic problem with a Dirichlet boundary condition.

Next, we will be taking up a parabolic basic problem in case of Neumann boundary condition; parabolic basic problem for Neumann boundary condition. This problem will be something like this - del u del t is equal to del square u del x square at t is equal to 0. We have u is equal to f of x and at x is equal to 0 will be having del u del x is equal to 0; at x is equal to 1, you have u is equal to 0. Now, only non-homogeneity is present in the governing equation. Therefore, this boundary condition is a Neumann boundary condition; so, again we defined this problem by using separation of variable.

(Refer Slide Time: 34:40)



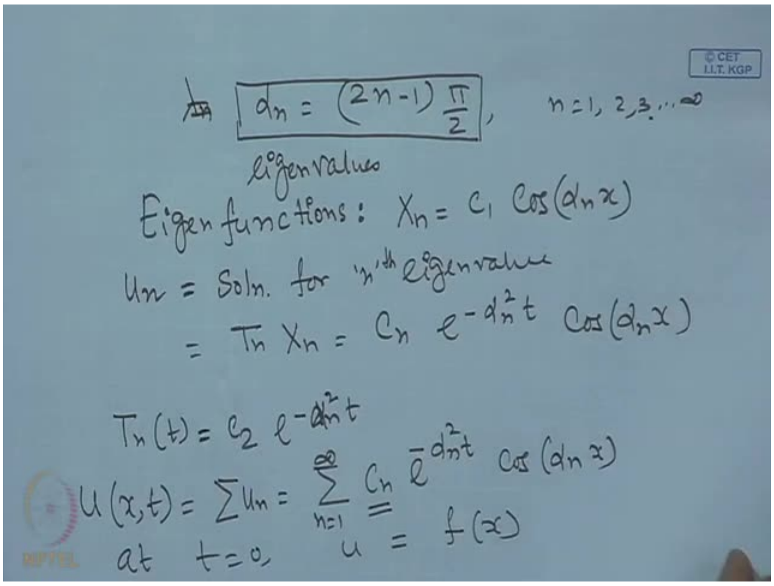
So, u has to be a product of two terms: one is assumed to be a function of x alone and another is a function of time alone. If you substitute in the governing equation this becomes 1 over x, this becomes x dT dt is equal to T d square x dx square. So, this become 1 over T dT dt is equal to 1 over x d square x dx square; so, again the left hand side is a function of time only and the right hand side is a function of space only; they are equal and they must be equal to some constant - let this be some constant; again this constant can be 0, can be positive, can be negative.

In the earlier classes, we have proved that if this constant is 0, then it will be giving a trivial solution; for positive also it will be giving as trivial solution. But, if it is minus alpha square negative, then it will be giving a non-trivial solution. Let us do that; so 1 over x d square x dx square is equal to minus alpha square; you will be having 1 over d square x dx square plus alpha square x is equal to 0.

Now, let us construct the boundary condition of the x wearing part, it must be satisfying the boundary condition in the space of the original problem. If you look into that equation at x is equal to 0, you will have del u del x is equal to 0; therefore, you just put x u is equal to x into T, T is independent of x; so you will be getting dx dx is equal to 0. It will be a total derivative, because capital X is a sole function of small x.

Similarly, at x is equal to 1, we have u is equal to 0; therefore, your capital x must be equal to 0. Again, we will be having a homogeneous boundary condition - a standard form and a special form of generalized Sturm Liouville problem. So, this is a standard Sturm Liouville problem or standard Eigen value problem with the homogenous boundary conditions (Refer Slide Time: 37:15). We have already looked into the non-trivial solution for this particular problem in all earlier class.

(Refer Slide Time: 37:33)



We have seen that the Eigen values will be constituted by alpha n are the Eigen values they are 2n minus 1 pi by 2, where n is equal to 1, 2, 3 up to infinity. So, these are the

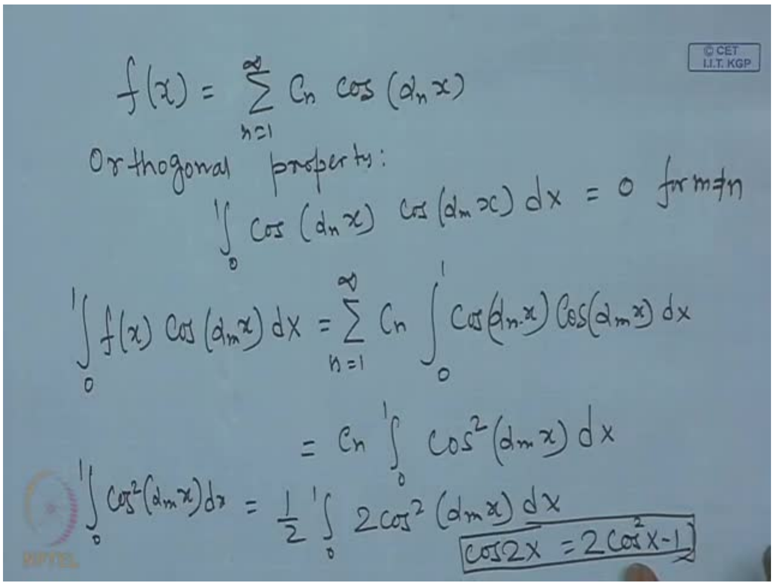
Eigen values to this problem and the corresponding Eigen functions are given by the cosine functions.

We have already seen in this case the Neumann boundary condition in the Eigen functions are cosine functions, and x n is corresponding to nth Eigen value, x n is corresponding to nth Eigen function. This will be C 1 cosine alpha n x, where alpha n is nothing but 2n minus 1 pi by 2.

We construct the u n solution for nth Eigen value and Eigen function. So, this will be nothing but T n multiplied by X n; therefore, it will be, if you look into the solution of T n, T n will be nothing but exactly what we have done earlier. This will be C 2 e to the power minus lambda n alpha n square times T. Therefore, if you multiplied T n and X n, it will be C_1 into C 2; so it will be a new constant. Let us say new constant is C $n_{,}$ it will be e to the power minus alpha n square t cosine alpha n x, where alpha n is given by this expression (Refer Slide Time: 38:43).This is the nth solution corresponding to nth Eigen value; so, we can linearly superposed for all the Eigen solution and we can construct the complete solution. Let us construct the complete solution; the complete solution will be summation of u n, this will be nothing but n is equal to 1 to infinity C n e to the power minus alpha n square t cosine alpha n x.

We have utilized the two boundary conditions, only one constant is left to determine and completely specify this problem; therefore, we have to use the unused initial condition of the original problem to solve this problem. Therefore, if you remember the initial condition at T is equal to 0, u is equal to some known function of x.

(Refer Slide Time: 40:27)

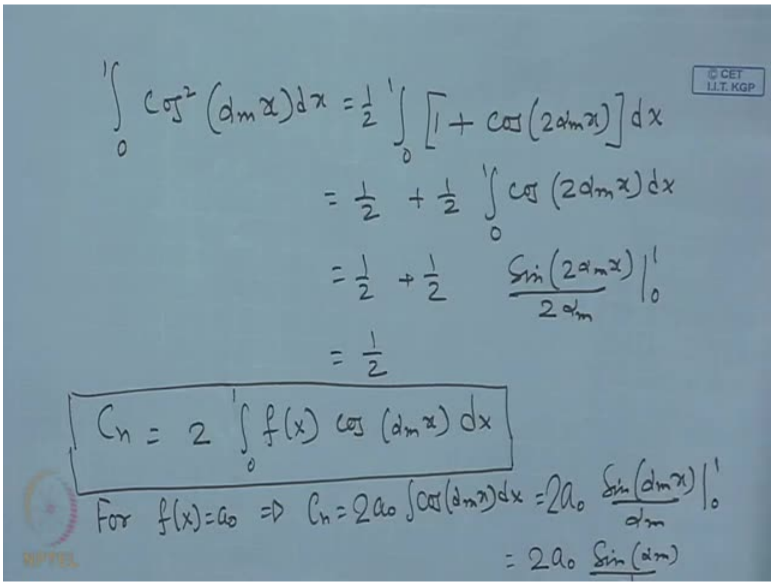


So, we put it there, this becomes f of x; it should be equal to summation C n cosine alpha n x, n is equal to 1 to infinity. Again, the Eigen functions are orthogonal to each other for a Sturm Liouville problem, we have already proved it earlier. Therefore, in the orthogonal properties of cosine functions, we can use these as integrations of cosine alpha n x cosine alpha m x dx is equal to 0, for m is not equal to n.

For m is not equal to n, cosine alpha n x multiplied by cosine alpha m x dx is equal to 0, where alpha n and alpha m are two distinct Eigen values. If we multiply both sides by cosine alpha n x dx and carry out the integration from 0 to 1 and see what you get? You multiply both sides by cosine alpha m x dx and carry out the integration over 0 to 1 domain of x; so n is equal to 1 to infinity C n integration cosine alpha n x cosine alpha m x dx from 0 to 1.

If you open up the summation series, all the terms will vanish; when alpha m for m is not equal to n, only one term will survive, which one is m is equal to n. So this becomes C n 0 to 1 cosine square alpha m x dx; if you evaluate this integral 0 to 1 cosine square alpha m x dx, if you remember the expression of cosine 2 alpha m x cosine 2 alpha m, you just divided by half from both side. So, it will be half 0 to 1 2 cosine square alpha m x dx - utilize the identity cosine 2x is nothing but 2 cosine square x minus 1.

(Refer Slide Time: 43:16)

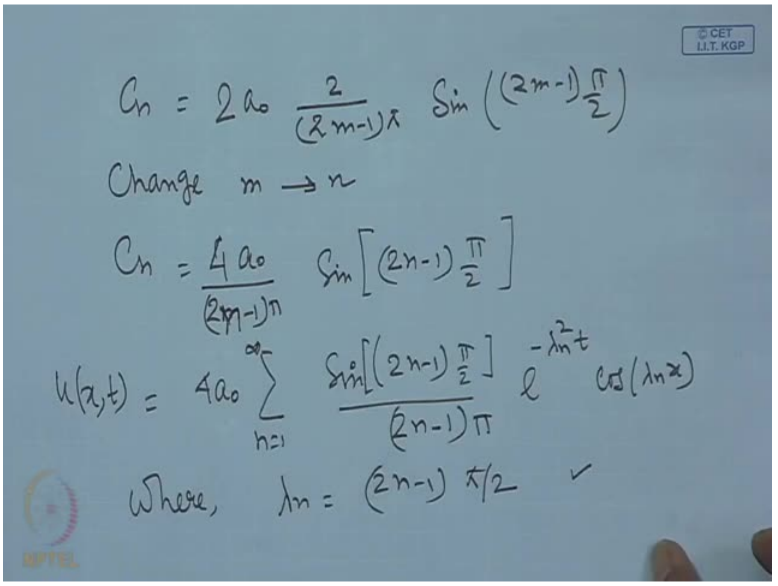


So, utilizing this identity, we can substitute this 2 cosine square alpha m x as 1 plus cosine 2 alpha m x. If you do that, what we will be getting is that integral 0 to 1 cosine square alpha m x d x is nothing but half 0 to 1 1 plus cosine 2 alpha m x dx; so, carry out this integration; this will be half plus half integral of cosine 2 alpha m x dx from 0 to 1; this becomes half plus half, it will be sin 2 alpha m x divided by 2 alpha m from 0 to 1; it becomes sin 2 alpha m is nothing but 0 and this will be sin 0 is also 0; we will be getting half. So, integral cosine square alpha m x dx is also half.

So, what we will be getting is that the final form is f of x. So, C n will be nothing but two times 0 to 1 f of x cosine alpha m x times dx. Again, like in the earlier case, what we have done in the earlier case in case of Dirichlet boundary condition? For a known function of f x, we can evaluate this expression of C n completely and you will be getting an analytical solution; if it is a constant, you can evaluate analytically; if it is a simple function of x, you can evaluate it analytically by using the integration by parts; if it is a complicated function, there we have to take recourse to numerical technique and we have to carry out this integration probably numerically.

So, I will just solve this problem for constant value of x, for f x is equal to a 0. This C n will be 2a 0 integration of cosine alpha m x dx and this becomes 2a 0 sin alpha m x divided by alpha m from 0 to 1; so this becomes 2a 0 sin alpha m divided by alpha m.

(Refer Slide Time: 46:10)

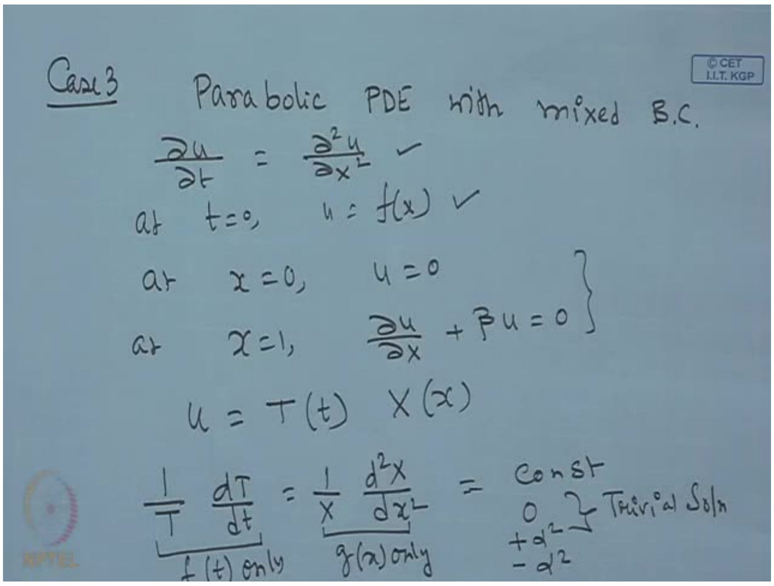


So, what you get is that C n becomes 2a 0 and we put the value of alpha m as 2m minus 1 pi by 2 and this becomes 2m minus 1 pi by 2; so, pi is there, 2 is there, this will be sin 2m minus 1 pi by 2; change the running index m to n, so you will be getting a C n is nothing but 4a 0 divided by 2m minus 1 pi sin 2n minus 1 pi by 2.

In this particular case, that is the expression of sine, you will be getting a complete solution as u as a function of x and t is 4a 0 in summation n is equal to 1 to infinity sin 2n minus 1 pi by 2 divided by, this is 2n, 2n minus 1 times pi e to the power minus lambda n square t cosine lambda n x, where lambda n is 2n minus 1 pi by 2.

So, that gives the complete solution if the function f x is equal to the constant a 0; if it is a known function like - x, x square e to the power kx, sin kx, cosine kx or whatever, then that integral has to be evaluated by parts; if it is a complicated function as we mentioned earlier, one can use the numerical technique to evaluate that. So that gives the complete solution in case of Neumann boundary condition of parabolic partial differential equation.

(Refer Slide Time: 48:21)

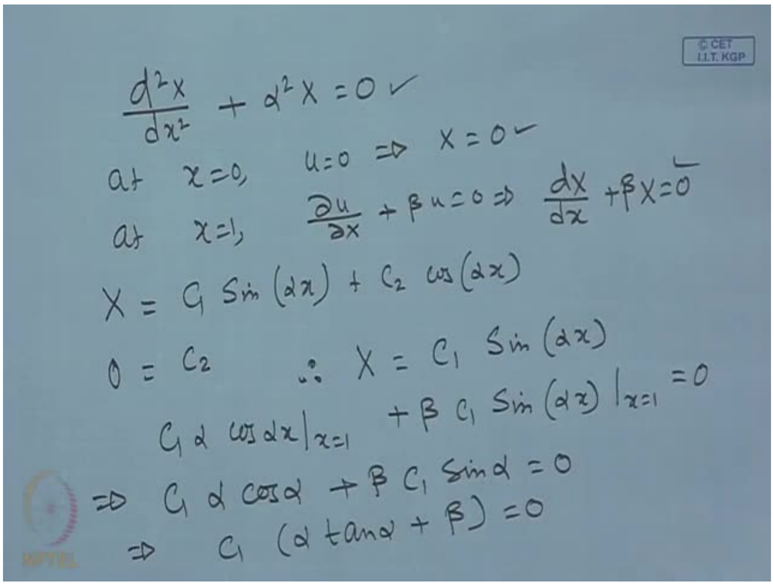


Then, we look into case number 3 where we deal with a parabolic PDE - parabolic PDE of a well-posed problem with mixed boundary condition. If you do that, the problem is something like this, del u del t is equal to del square u del x square subject to at t is equal to 0, u is equal to f of x. At x is equal to 0, you have u is equal to 0; at x is equal to 1, you have del u del x plus beta u is equal to 0.

Again, this problem has the homogeneous boundary condition and non-homogeneous initial condition. So, it is a well-behaved problem and it is a linear partial differential equation; so you can use the separation of variable type of method. So, u is assumed to be a product of two functions: one is completely a function of time, another is completely a function of space.

If you substitute this there and separate the variable, what you will be getting is 1 over T dT dt is equal to 1 over x d square x dx square; the left hand side u is a function of t only, the right hand side is a function of space only and they are equal; they must be equal to some constant; again, this constant can be 0, can be positive, can be negative. We have already seen that for 0 and positive value, we will be getting a trivial solution; therefore, in order to get a non-trivial solution, this constant must be equal to negative.

(Refer Slide Time: 50:30)

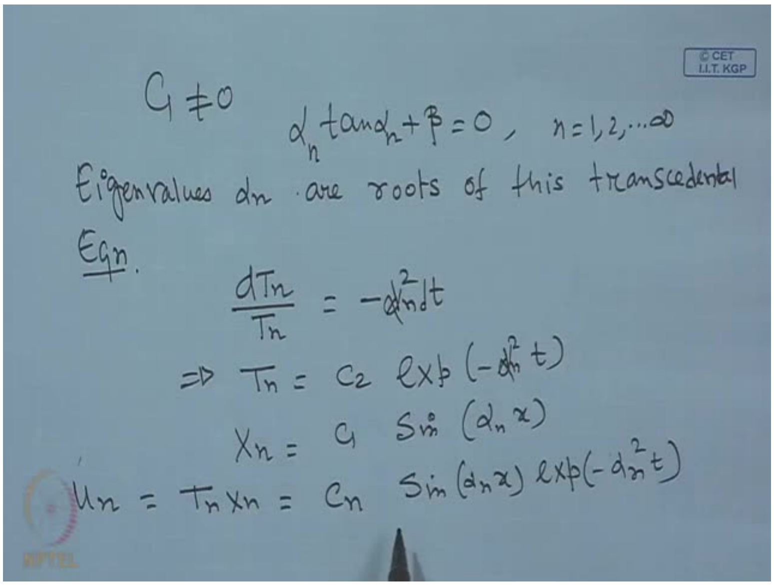


If you look into the solution, this becomes d square x dx square plus alpha square x is equal to 0. So, this is again a standard Eigen value problem or Sturm Liouville problem. Now, we have to see whether the boundary conditions are homogeneous; at x is equal to 0, your original boundary condition was u equal to 0 and therefore capital X must be equal to 0. They are the space-wearing part must be satisfying the boundary condition of the original problem in space.

At x is equal to 1, you had del u del x plus beta u is equal to 0 - put u is equal to t into x - this becomes at x is equal to 0, dX dx plus beta x is equal to 0; so, you have the standard Eigen value problem with homogenous boundary conditions and it can be solved. If we remember, the Eigen functions are given by the sine functions and Eigen values are given by the groups of a transcendental equation, but I do not remember the form of the transcendental equation; let us derive that.

Solution of this will be x is equal to C 1 sin alpha x plus C 2 cosine alpha x; solution is composed of the combination of the sine and cosine function. If you put the first one x is equal to 0, C 1 sin 0 is 0; this becomes C 2 cosine 0 is 1; so, C 2 equal to 0 (Refer Slide Time: 52:10). Therefore, the Eigen functions are nothing but the sine functions C 1 sin alpha x.

(Refer Slide Time: 53:14)



Now, we put the other boundary condition dx dx, it will be C 1 alpha cosine alpha x evaluated at x is equal to 1 plus beta C 1 sin alpha pi x; so, C 1 sin alpha x evaluated at x is equal to 1 should be equal to 0. Therefore, C 1 alpha cosine alpha plus beta C 1 sin alpha is equal to 0. So, divided by sin alpha, what we will be getting? We will get C 1 alpha tan alpha plus beta must be equal to 0. In order to get a non-trivial solution, C 1 must not be equal to 0; so, C 1 cannot be equal to 0. The solution is alpha tan alpha plus beta must be equal to 0; this equation has infinite number of the roots - each and every root is the Eigen value of the particular problem.

We write the root by n - the n index runs from 1, 2 up to infinity. Eigen values - alpha n - are roots of this transcendental equation; these roots can be evaluated numerically and one can get those values. So, corresponding Eigen functions are sine functions, we can get the time-wearing part. As earlier, we have seen that the dT n T n is minus lambda n square dt; so integrated out, you will be getting T n is equal to some constant times exponential minus lambda n square t. If you look into the space-wearing part, the corresponding to nth Eigen value X n is nothing but the C 1 sin alpha n times x.

So, we can construct the nth solution corresponding to nth Eigen value is T n X n; you multiply this 2 C 1 and C 2, it will be giving a new constant; this will be C n sin alpha n x exponentially minus, this is alpha n. This is not lambda; this is alpha; so it will be having alpha n square t (Refer Slide Time: 55:10).

So, I stop here at this class. In the next class, I will be completing this problem. I will be taking up this problem from this point onwards and complete this problem completely. Then, we will be moving forward how to solve the well-behaved problem, thank you.