Advanced Mathematical Techniques in Chemical Engineering
Prof. S. De
Department of Chemical Engineering
Indian Institute of Technology, Kharagpur

Module No. # 01
Lecture No. # 22
Theorems of Eigenvalues and Eigenfunction

Good afternoon everyone, so we will be starting this class with the view of whatever we have finished in the last class. And in the last class, what we did is that we covered, we looked into, examined, some of the standard equations in Cartesian coordinate, where the equation which is valid for cylindrical coordinate. The equation which is valid for the spherical polar coordinate and Euler's equation.

We have looked into different you know solution of different types of equations, under different set of boundary conditions, may be a Dirichlet boundary condition, may be a Neumann boundary condition or Robin mixed boundary condition. So, we have seen how depending on the square boundary conditions, the solution of such problems are changed and the eigenvalues and eigenfunctions become different for different boundary conditions.

And except the Cartesian coordinate, we have also looked into the Basel equation and Legendre function, Legendre equation. And we have seen how the eigenvalues and eigenfunction will be appearing in the form of Basel function or Legendre polynomial. And we have also looked into different properties this Basel functions or Legendre polynomial, they will obey… And also we have seen into the solution of Euler's equation, not only that, we looked into the, we have developed the theorem for to get the adjoint operator, given an operator L.

Now, what will be doing in this class, will be formulating, the we will carry forward the development of the eigenvalue problem and will be defining a standard eigenvalue problem or a Sturm-Liouville problem. Once we define a standard eigenvalue problem or Sturm-Liouville problem, will be looking into some of the theorems and axioms these eigenvalues and eigenfunctions will obey. And these properties we will be utilizing frequently when we will be solving the equations using a partial differential equations, using the separation of variable type of solution.

Once, we complete the relevant theorems and axioms of eigenvalues and eigenfunctions and then will get into the actual solution of partial differential equations. Let us start the formulation of standard eigenvalue problem or Sturm-Liouville problem.

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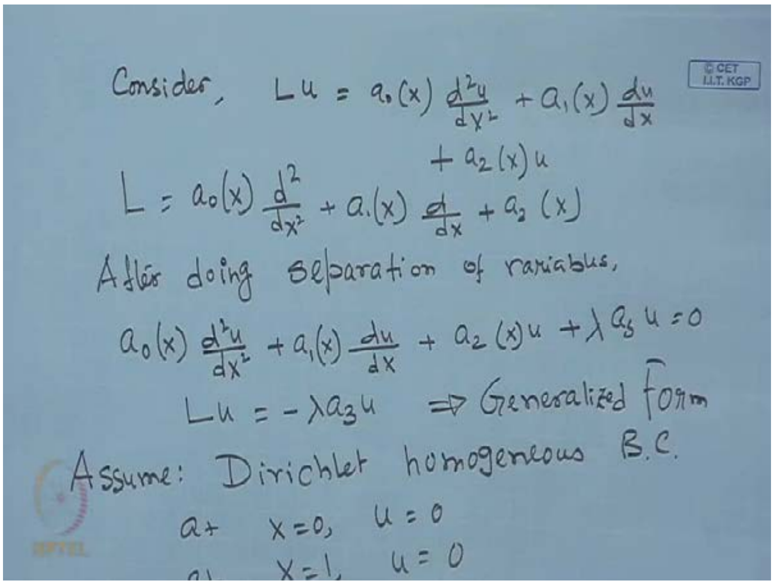

For that we consider the parameter or operator L L u is equal to a 0 x plus a 0 x d square u d x square plus a 1 x d u d x plus a 2 x u. Suppose we consider a function like this, where the operator L is nothing but a 0 x d square d x square plus a 1 x d d x plus a 2 x. Now, on separation, after doing separation of variables, we will be getting an equation something like this, a 0 x d square u d x square plus a 1 as a function of x d u d x plus a 2 as a function of x u plus lambda a 3 u is equal to 0.

Just consider this equation and will be doing the separation of, actual separation of variables later on. And for that time being, just take this equation as granted, it is basically in the form of L u is equal to minus lambda a 3 u, so this is a generalized form of the equation. In a particular equation, a 1 may be 0 or a 2 may be 0, but we are going to find out the different for different values of lambda, how this equation will be transformed into a standard eigenvalue problem.

Now, we assume a Dirichlet boundary condition homogenous. If you remember in the last class, we have looked into several types of boundary conditions, but all these boundary conditions are homogenous in nature. So, the generality of the solution does not change, the steps remains same and the formulation remains constant if we change a homogenous Dirichlet boundary condition to homogenous robin mixed or homogenous Neumann boundary condition. Let us consider that, we used a Dirichlet homogenous boundary condition and these are at x is equal to 0, u is equal to 0, at x is equal to 1, u is equal to 0.

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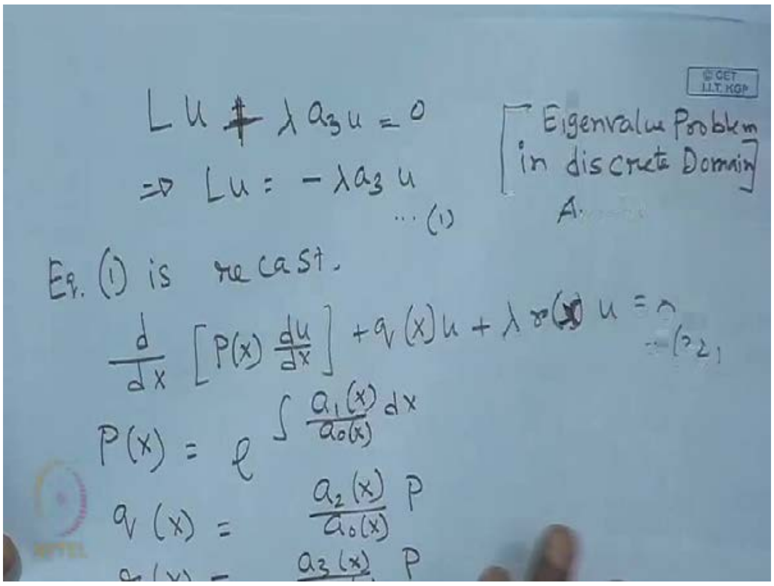

So, let us proceed with this, now the form of the equation is L u plus lambda a 3 u is equal to 0. So, you will be getting L u is equal to minus lambda a 3 u and if you look into the similarity of the eigenvalue problem in discrete domain, so you just remember, recall the eigenvalue problem in discrete domain that was a x is equal to lambda x.

So, this equation can be rewritten as this form, so equation 1 is recast as d d x of p of x d u d x plus q of x u plus lambda r as a as a function of x times u is equal to 0, let say this is equation number 2.

Now, these two equations will be identical, because we define p q r such that p of x is e to the power integral a 1 x divided by a 0 x d x, q of x is nothing but a 2 x divided by a 0 times p and r of x is nothing but a 3 x divided by a 0 x times p. Now, we can substitute p q and r in this equation and will be getting back this equation. So, let us try to do that and prove that the form of L u is equal to minus lambda a 3 u and this form, that is equation 1 and 2 are identical in nature.

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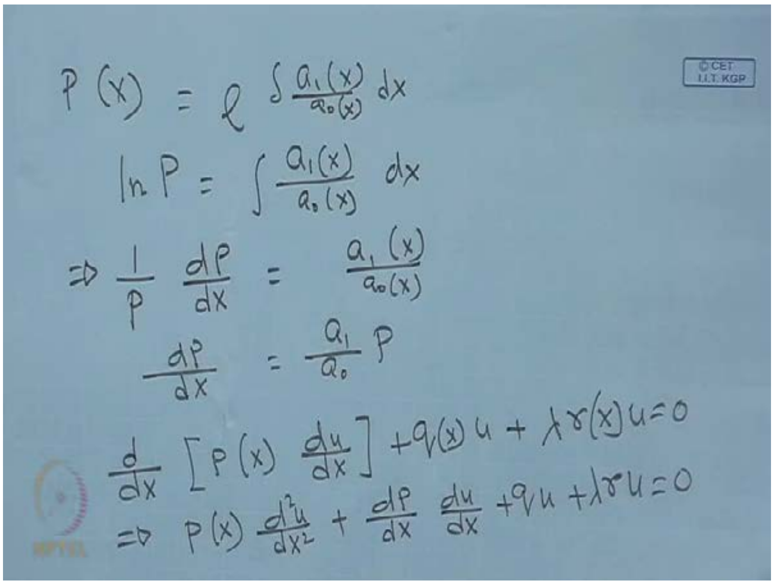

So, therefore, we start with this, so p of x is defined as e to the power integral a 1 x divided by a 0 x d x. If you take logarithm on both side, this becomes l n p, is nothing but a 1 x, integral a 1 x a 0 x d x. So, if we differentiate both sides, you will be getting with respect to x, you will be getting 1 over p d p d x, is nothing but a 1 divided by a 0 and you will be getting d p d x is a 1 divided by a 0 times p.

Now, if we put these values into the governing equation d d x of p of x d u d x plus q of x u plus lambda r of x u is equal to 0. Now, what we are going to do, we open up open this equation up, that means we differentiate this part, so this becomes p of x d square u d x square plus d p d x d u d x plus q u plus lambda r u is equal to 0. So, therefore, we write the equation as p of x, so this becomes, substitute the different values of p and q and r.

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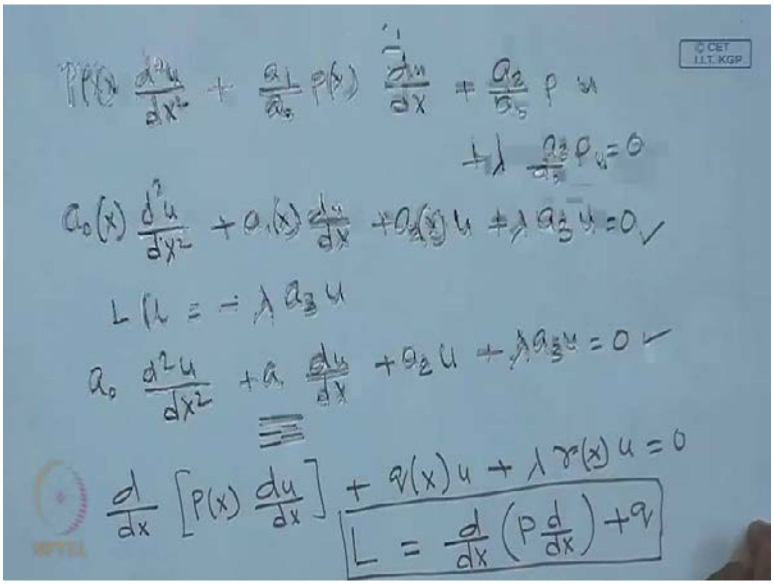

So, p of x d square u d x square plus d p d x, we have already found that found out that d p d x in the earlier one is a 1 over a 0 times p of x times d u d x plus q, q we have defined as a 2 by a 0 times p plus lambda r u and r we have defined as a 3 by a 0 times p is equal to 0. Now, if I have seen that in all the terms we have a constant quantity, that is p of x, we divide both sides by p of x and multiply both side by a 0. So, what you will be getting is a 0 as a function of x d square u d x square plus a 1 function of x d u d x plus a 2 function of x u plus, there is one u there, so u plus lambda a 3 u is equal to 0.

So, if we now compare this equation with the earlier one, that you had earlier as L u is equal to minus lambda a 3 u, so this becomes a 0 d square u d x square, write the operator L, so this becomes a 1 d u d x plus a 2 u plus, we take lambda a 3 on the other side, so lambda a 3 u is equal to 0. So, these two equations are identical, therefore this equation can be equivalently written as d d x of p of x d u d x plus q of x u plus lambda r, which is, which can be in general function of x times u. So, my operator, we can write it as d d x of p d d x plus q, so this becomes my operator and we can look into the adjoint operator to this particular operator.

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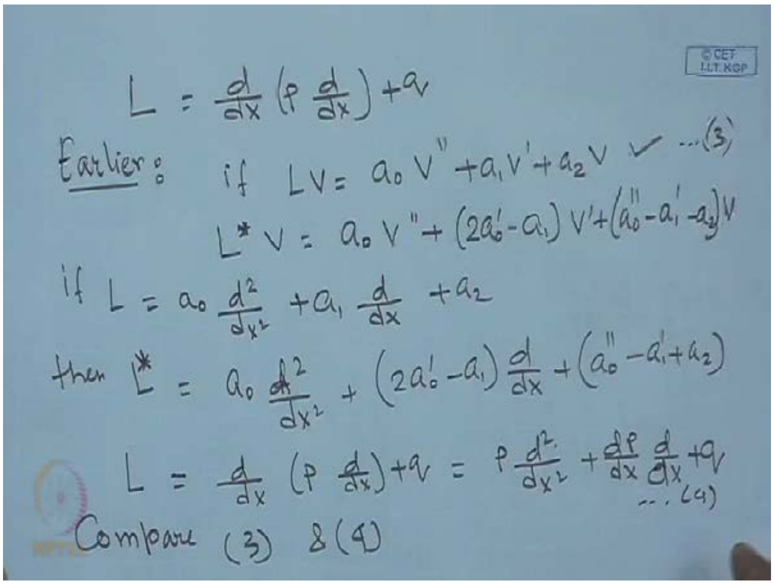

So, the operator becomes now L is equal to d d x of p d d x plus q, so we have already seen that earlier. We have in the last class, we have seen that if L of v is a 0 v double prime, that means d square v d d x square plus a 1 v prime plus a 2 v, then the adjoint operator L star v equal to a 0 v double prime plus 2 a 0 prime minus a 1 times v prime plus a 0 double prime minus a 1 prime plus a 2 times v. That means if my L is equal to a 0 d square d x square plus a 1 d d x plus a 2, then my L star is a 0 d square d x square plus 2 a 0 prime minus a 1 d d x plus a 0 double prime minus a 1 prime plus a 2.

So, therefore, we can compare these equation with the earlier one, the operator was written as d d x of p d d x plus q, therefore p d square x, d square d x square plus d p d x d d x plus q. So, therefore, we can compare this equation with this equation and if we can compare that a 0 becomes, if you compare, let say these, right, let us write it sees 3, this is 4, so you can compare 3 and 4 and can get different values.

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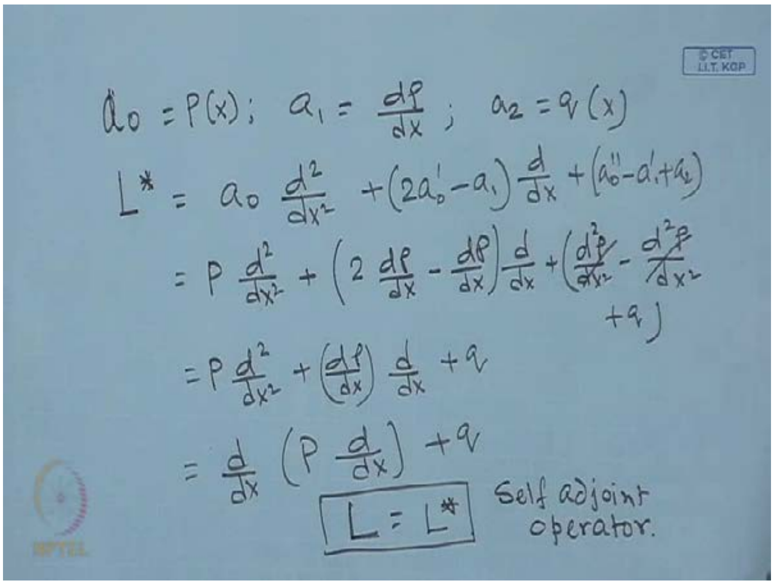

So, therefore, we can by comparing we can write p is a 0 is nothing but p of x, your a 1 is nothing but d p d x and a 2 is nothing but q of x. So, if we look into the L star, the L star is a 0 d square d x square plus 2 a 0 prime minus a 1 d d x plus a 0 double prime minus a 1 prime plus a 2. So, a 0 becomes p, so this becomes p d square d x square plus 2 a 0 prime, that means 2 d p d x minus a 1, a 1 is d p d x d d x plus a 0 double prime is d square p d x square minus a 1 prime is d square p d x square plus a 2 is q. So, this two will be cancelling out, so you will be getting p d square d x square plus, it will be one d p d x of d d x plus q, so this will be d d x multiplied by p d d x plus q.

Now, this will be if you look into the earlier slide, will be seeing that L is equal to L star in this particular problem. So, we are talking about a self-adjoint operator, this particular operator in general is known as this, in this particular operator in generally is known as the Sturm-Liouville operator. So, let us write down the Sturm-Liouville equation and we have to if we have to prove that if the Sturm-Liouville equation is a self-adjoint equation, we have already proved that L is equal to L star, but we have to prove that b is equal b star as well, the boundary operator should also match.

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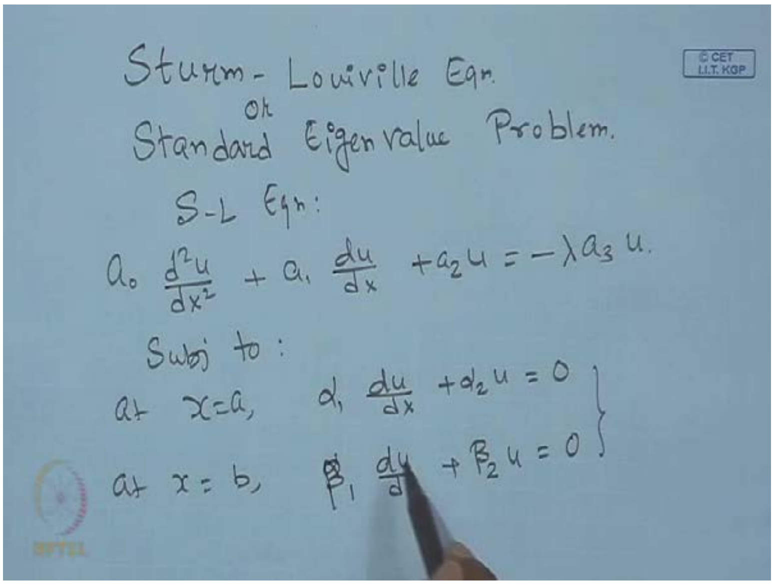

So, let us define now Sturm-Liouville operator or Sturm-Liouville equation. This Sturm-Liouville equation is also known as standard eigenvalue problem. So, Sturm-Liouville equation is defined as S L equation, it is defined as a 0 d square u d x square plus a 1 d u d x plus a 2 u is equal to minus lambda a 3 u.

So, subject to the boundary conditions, will use to general boundary condition, that is at x is equal to a, alpha 1 d u d x plus alpha 2 u is equal to 0 and at x is equal to b, we have beta 1 d u d x plus beta 2 u is equal to 0. So, we consider two most generalized boundary condition, that if alpha 1 and beta 1 are 0, then both of this boundaries are, they will be boiling down to Dirichlet boundary condition. If alpha 2 equal to 0 and beta 2 is equal to 0, will be getting a Neumann boundary condition, if both of them are non-zero, then will be getting a robin mixed boundary condition.

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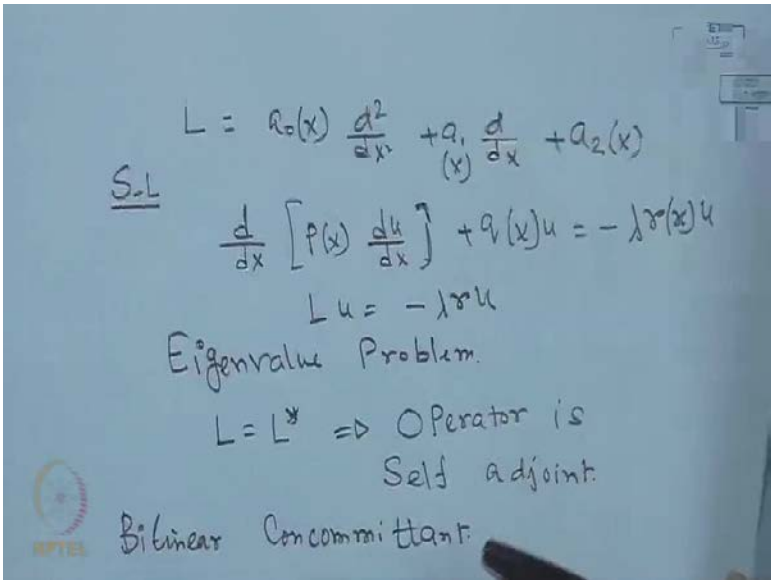

So, we can look into the operator, the operator in this problem is nothing but a 0 d square d x square plus a 1 d d x plus a 2, both a 1 a 0 a 1 a 2 they are function of x and this equation can be written in this form. Already we have seen that, that equation as Sturm-Liouville equation, can be written in this form d d x of p of x d u d x plus q of x u is equal to minus lambda r of x times u, so L u is equal to minus lambda r u.

So, this is a standard eigenvalue problem in continuous function, continuous domain. We have already seen that L is self-adjoint in the just earlier to this, we have already seen that L is equal to L star, so the operator is self-adjoint. Now, in the earlier class, we have seen this, if the system become self-adjoint, then the operator has to self-adjoint, as well as the boundary condition has to be self-adjoint. That means boundary operator must be is equal to, b is equal to b star, so for that you have to, what you have to do, we have to examine the bilinear concomitant J u v.

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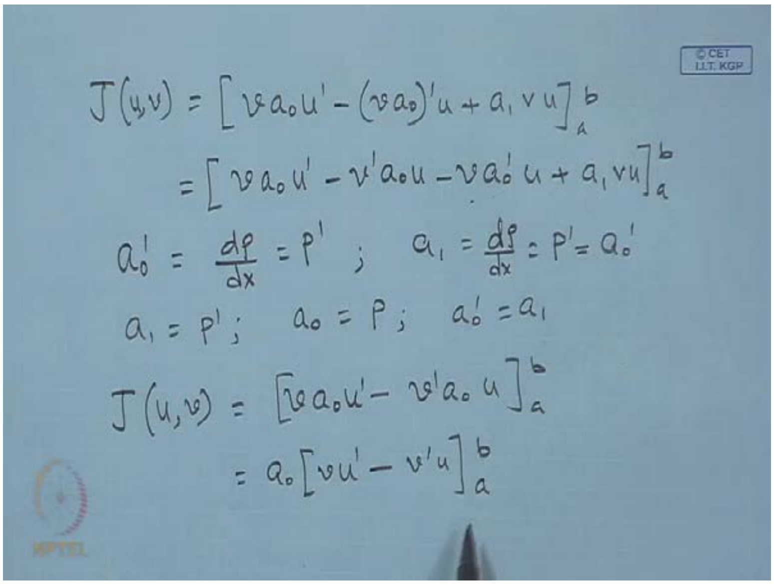

So, we have to look into the bilinear concomitant term. So, if you look into that will be getting, we will be writing J u v is equal to v a 0 u prime minus v a 0 prime of u plus a 1 v u evaluated from on the boundaries a to b. And this prime denotes differentiation with respect to x, so if you just open this up, this becomes v a u, v a 0 u prime minus v prime a 0 u minus v a 0 prime u plus a 1 v u from a to b.

We have already proved earlier that a 0 prime is d p d x and this is p prime and a 1 is nothing but d p d x is equal to p prime and that is equal to a 0 prime. So, a 1 is equal to p prime, a 0 is equal to p and a 0 prime is equal to a 1. So, if we since a 0 prime is equal to a 1, then last two terms of this bilinear concomitant they will vanish, so they will just out they will just out, they will be cancelled each other.

So, what is the form of bilinear concomitant? This will be v a 0 u prime minus v prime a 0 u evaluated between a to b. So, I can take a 0 common, so this becomes v u prime minus v prime u evaluated between a to b.

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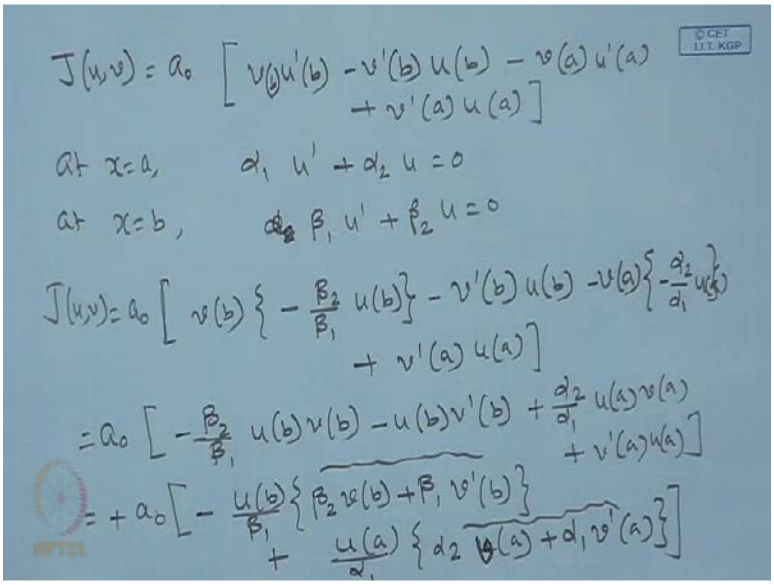

Now, will be simplifying it and check whether to make this bilinear concomitant to 0, what will be the conditions on v we have imposed. Now, let us put, let us evaluate bilinear concomitant term. So, this becomes a 0 v u prime at evaluated at b minus v prime evaluated at b u evaluated at b minus v evaluated at a u prime evaluated at a minus minus plus v prime evaluated at a u evaluated at a.

Now, we can recall the boundary conditions on u, at x is equal to a, we have, if we recall the boundary conditions on u, it will be alpha 1 u prime plus alpha 2 u is equal to 0 and at x is equal to b, we have alpha beta 1 u prime plus beta 2 u is equal to 0. So, therefore, we substitute u prime a and u prime b from this equation. So, v of b and what is u prime b? u prime b is nothing but minus beta 2 by beta 1 u at b minus v prime b and u b minus v of a and what is u prime a? u prime a we substitute as minus alpha 2 by alpha 1 times u at a plus v prime a and u a remain as they are.

So, just simplify this equation, this becomes minus beta 2 by beta 1 u of b v of b, then we have minus u of b v prime of b, then minus minus plus alpha 2 by alpha 1 u of a v of a plus v prime a and u prime a multiplication on that. So, we have a 0, we take minus u of b as common, minus and also beta 1, to be divided by beta 1, minus u by b divided by beta 1 we take as common.

So, what will be getting is that b 2 v of b plus beta 1 v prime of b and from this two, we combine this two, will be getting plus u of a is common divided by alpha 1, so you will be getting alpha 2 u of a plus alpha 1 alpha 2 v of a plus alpha 1 v prime of a. Now, we do not have any idea about what is the value of u evaluated at a and u on the boundary x is equal to b. So, therefore, in order to make this bilinear concomitant to be vanished, the term in the second bracket, they should be put equal to 0, individually each of them.

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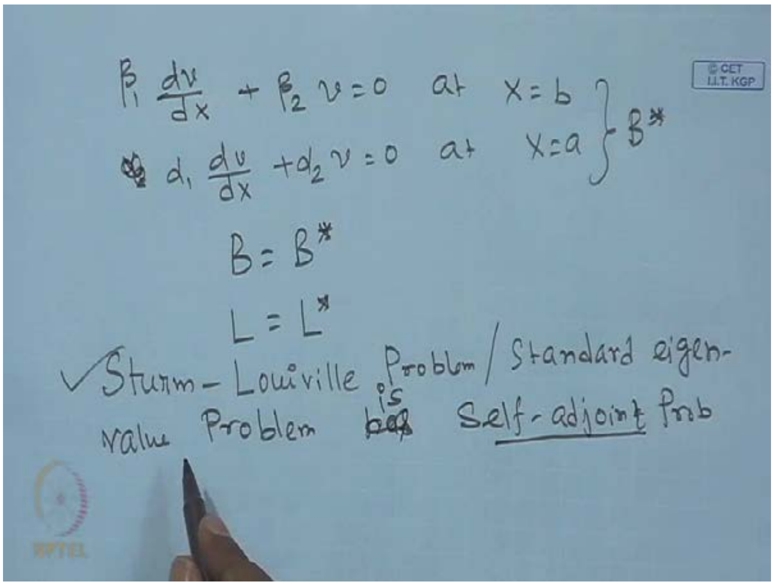

So, if you do that, then what will be getting is that beta 1 d v d x plus beta 2 v is equal to 0 at x is equal to b and from the other one, alpha 1 d v d x plus alpha 2 v is equal to 0 at x is equal to a. So, therefore, these two boundary conditions on v they emerge out from the by putting J bilinear concomitant equal to 0. And if we remember this is b star, this is the boundary operator of the adjoint problem, and if we remember the boundary conditions b on the original problem, that was at x is equal a alpha 1 d u d x plus alpha 2 u is equal to 0 and at x is equal to b alpha 1 d v d x plus alpha 2 u is equal to 0.

So, beta 1 d u d x plus beta 2 u is equal to 0 at x is equal to b. Therefore, both B and B star are identical and we had L is equal L star already proved earlier, so, therefore the we proved that Sturm-Liouville problem is problem or standard eigenvalue problem, has is self-adjoint problem.

So, you will be having L is equal to L star and B is equal to B star and therefore this proves that Sturm-Liouville problem or standard eigenvalue problem is a self-adjoint problem. And standard eigenvalue operator or Sturm-Liouville operator is a self-adjoint operator, so it does not matter what kind of boundary conditions it have, we have consider the most general robin mixed boundary condition from which the Dirichlet and Neumann conditions are specially derivable under special conditions.

So, in the most general conditions of boundary condition and the governing equation, the Sturm-Liouville problem is a self-adjoint problem.

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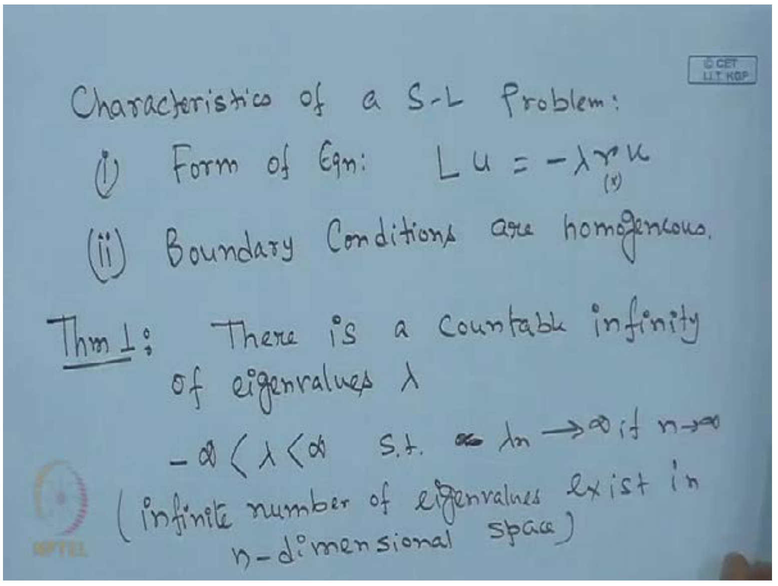

So, therefore, if we have a so if we can now identify what are characteristics of a Sturm-Liouville problem, there are two characteristics of Sturm-Liouville problem. The first one is that, if the form of the equation must be in this form L u is equal to minus lambda r u or may be a function of x in general. So, that is the form of the equation, governing equations and the boundary conditions are homogeneous. If these two conditions are satisfied, these two characteristics are satisfied, then we can have a Sturm-Liouville problem.

Next, we will be looking into some of the theorems of eigenvalues and eigenfunctions. The first theorem goes like this, there is a countable infinity of eigenvalues lambda that means lambda must be lying between minus infinity to plus infinity, such that lambda n tends to infinity if n tends to infinity. So, we should this statement is equivalent that there are infinite number of eigenvalues exists in n dimensional space. And in case of function, continues function this n dimension becomes too large, it becomes very large, it tends to infinity. So, in case of continuous functions, there are countable, but infinite number of eigenvalues present.

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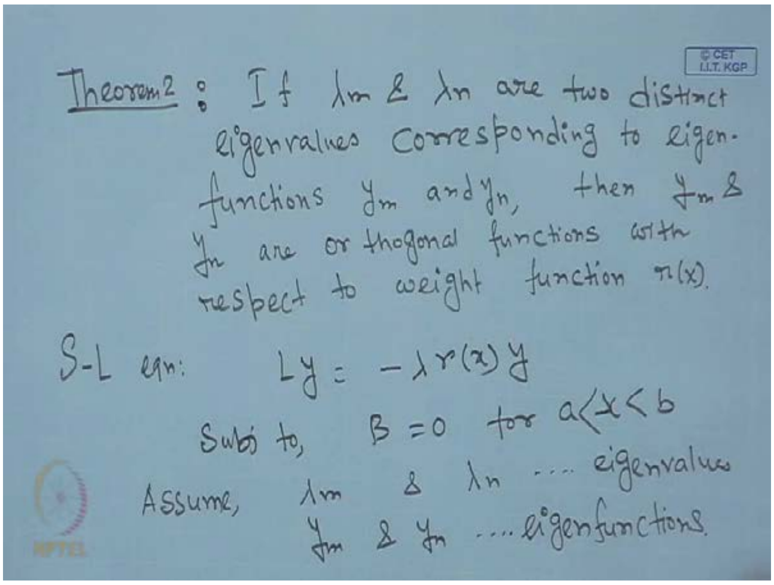

Next, we look into theorem 2. This theorem says that if lambda m and lambda n are two distinct eigenvalues corresponding to eigenfunctions y m and y n, then the eigenfunctions y m and y n are orthogonal functions with respect to weight function r. So, these are for this lambda m and lambda n are the eigenvalues corresponding to Sturm-Liouville equation. So, let us write down the Sturm-Liouville equation, this is L y is equal to minus lambda r, which is in general function of x and multiplied by y.

Now, subject to the boundary operator B is equal to 0, for x lying between small a and small b, so let us assume lambda m and lambda n are distinct eigenvalues there and the corresponding eigenfunctions are y m and y n, are corresponding eigenfunctions. So, this y m and y n must satisfy this equation, when y becomes y m then lambda becomes lambda m, when y become y n lambda becomes lambda n.

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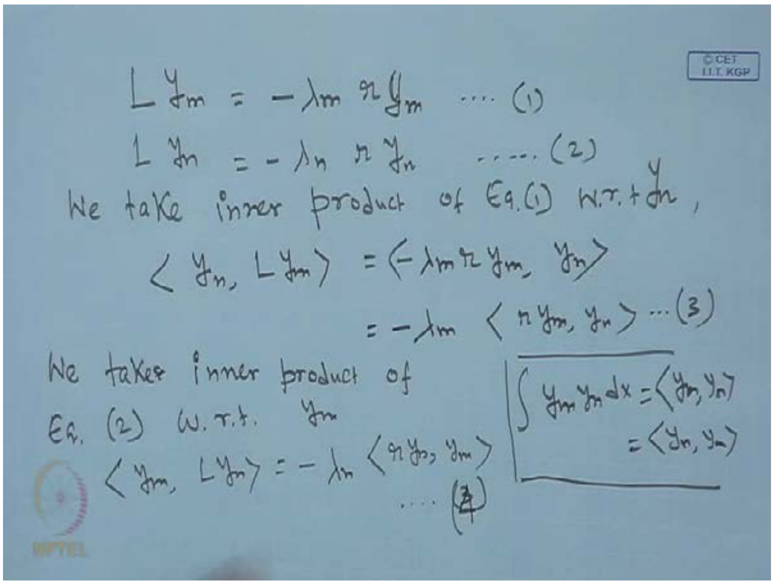

So, therefore, we should write these two equations as L y m is equal to minus lambda m r y m and L y n is equal to minus lambda n r y n. So, this is equation number 1, this is equation number 2, we take inner product of equation number 1 with respect to n y n and see what we get. We get y n, inner product of y n and L y m is equal to minus lambda m r y m, inner product between these two.

Now, lambda m being a constant, it will be coming out of the equilibrium sign, inner product sign with the minus, so this becomes inner product of r y m and y n. So, if you remember in case of continuous function y m y n d x integration, y m y n d x is nothing but the inner product of y m and y n and this is identical to inner product of y n and y m.

So, what will be getting out of this, so you will be again take the, so this is number 1 equation number 3. Then we take inner product of equation 2 with respect to y m, so if you do that you will be getting inner product of y m L y n should be is equal to minus lambda n inner product of r y n coma y m, this is equation number 4.

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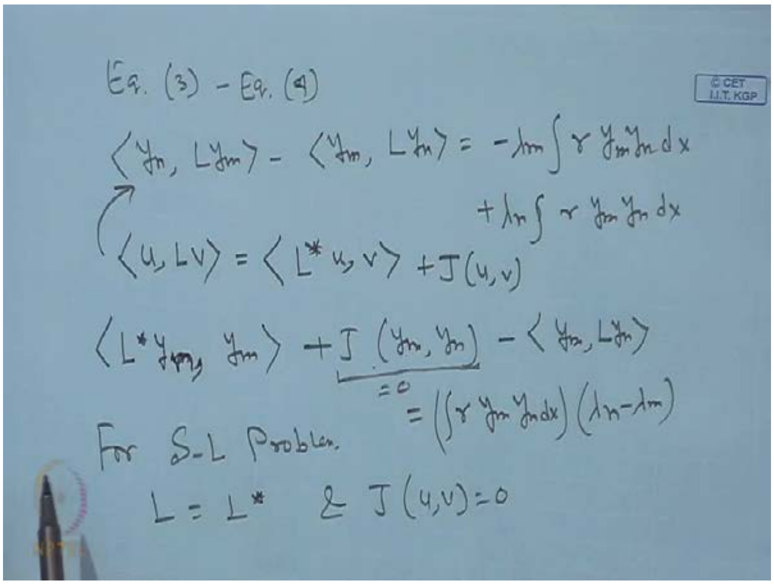

Now, what we do, we subtract equation number 4 from 3 and see what we get. So, if we subtract equation 4 from equation 3, will be getting inner product of y n, L y m minus inner product of y m and L y n. And this will be minus lambda m inner product of r y m y n, can be written as this integral r y m y n d x minus minus plus lambda n r y m y n d x.

So, if we now utilize the relationship that we have already derived in the last class, that is u inner product between u and L v must be equal to inner product of L star u comma v plus J u v. So, we write that here, so what you get is that get inner product of L star y m and v it was basically y m, so L star y n, y m plus J inner product of y m and y n minus inner product of y m, L y n is equal to, we take integral r y m y n d x common and this becomes lambda n minus lambda m.

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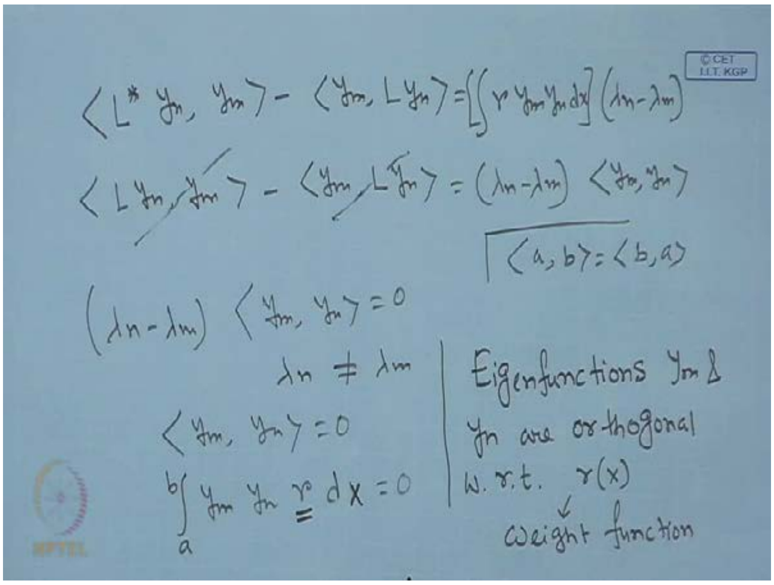

So, we have already proved earlier that for Sturm-Liouville problem L is equal to L star and J u v is equal to 0, so, therefore J y m y n will be equal to 0, so this will be equal to 0 in the case of Sturm-Liouville problem. And what we have now is that inner product of L star y n and y m minus inner product of y m comma L y n is equal to integral r y m y n d x multiplied by lambda n minus lambda m.

Now, since, L is equal to L star, we can write as L y inner product of L y n and y m minus inner product of y m L y n is equal to lambda n minus lambda m inner product of y m and y n with respect to weight function r. Therefore, we have already proved the relationship of inner product, that is inner product of a and b should be is equal to inner product of b. And therefore inner product of L y n and y m should be is equal to inner product of y m and L y n.

So, this two will be equal and identical, they will be cancelling out, so what will be getting is lambda n minus lambda m inner product of y m and y n should be is equal to 0. Now, lambda n and lambda m are two distinct eigenvalues, therefore lambda n is not equal to lambda m, therefore to satisfy these equation only option that is left is inner product of y m and y n should be is equal to 0, that means integral a to b y m into y n r d x should be is equal to 0.

So, therefore, this proves that the eigenfunctions y n and y m are orthogonal to each other with respect to the weight function r, so this proves that eigenfunctions y m and y n are orthogonal with respect to weight function r x, this is known as the weight function. So, this completes the proof that for the Sturm-Liouville problem, the eigenfunctions are orthogonal functions with respect to the weight function r.

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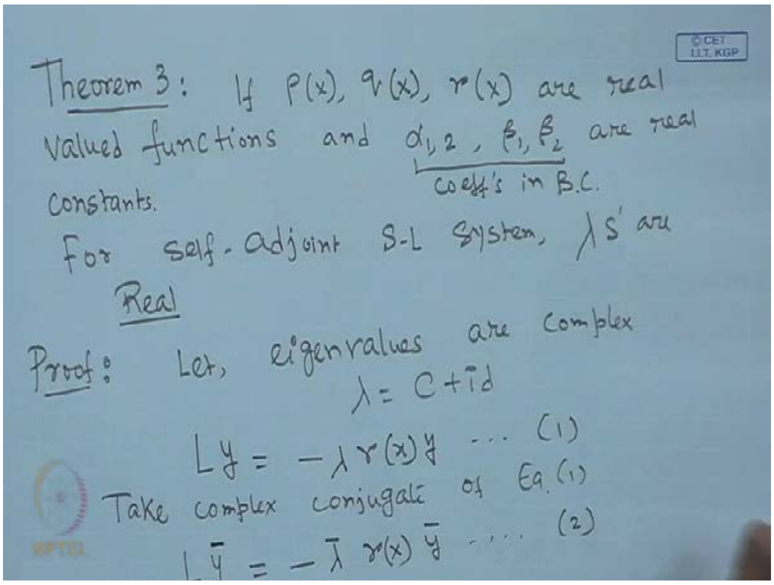

Next, we go to theorem number 3. If p of x, q of x, r of x are real valued functions and alpha 1, alpha 2, beta 1, beta 2 are real constants, these are the coefficients in boundary conditions, these are real and these coefficients are 0, then for self-adjoint S L system, the eigenvalues are real. If the functions the coefficients functions are real, if the coefficients in the boundary conditions are real, then there is no reason that eigenvalues becomes unreal or imaginary, they will be also real.

So, let us proof this and proof goes like this, let us assume that eigenvalues are complex, let eigenvalues are complex, therefore lambda is equal to C plus i d. So, it has a real part and it has a complex part, so lambda is equal to C plus i d, so we have to we write the eigenvalue problem L y is minus lambda r y. So, we take so this is equation number 1, so that is the eigenvalue, so that is a standard Sturm-Liouville problem, we take the complex conjugate of this equation.

We have already stated that r x is real, so L y bar, let us say bar is the complex conjugate of y, so complex conjugate of y is replaced by the, is denoted by the bar on the top of it. So, L y bar is equal to minus lambda bar r, r remains r, because it is a real, that is our assumption times y bar, so this is equation number 2, so this is the complex conjugate.

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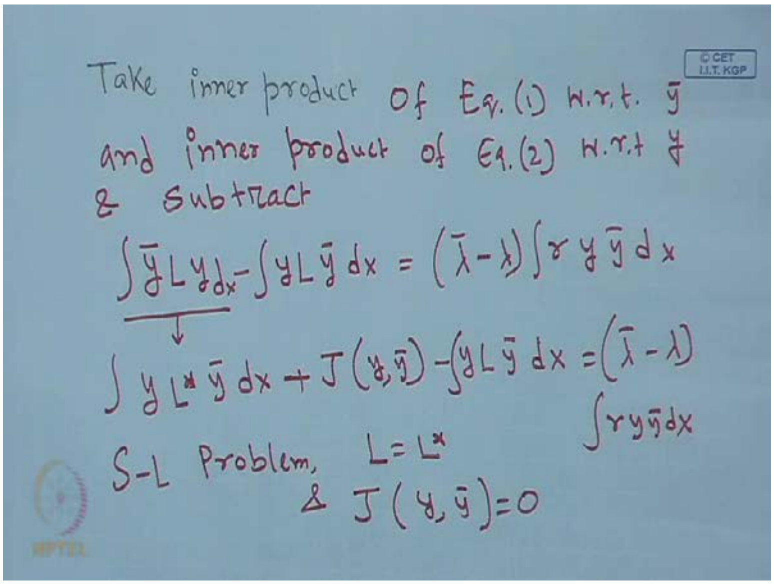

Now, what we will do, we next we take the inner product of equation 1 with respect to y bar and we take inner product of equation 2 with respect to y and subtract one from another. Let us see what we get, if we really do the subtraction, you will be getting inner product of y L y bar y bar L y minus y L y bar may be d x here, is equal to lambda bar minus lambda integral r y y bar d x.

Now, we write this equation, we substitute is as by y L star y bar d x plus bilinear concomitant between y and y bar minus y integral y L y dash y bar d x is equal to lambda bar minus lambda integral r y y bar d x. Now, since it is a Sturm-Liouville problem, we have L is equal to L star and J between y and y bar should be is equal to 0, bilinear concomitant vanish, as well as the operator is self-adjoint operator.

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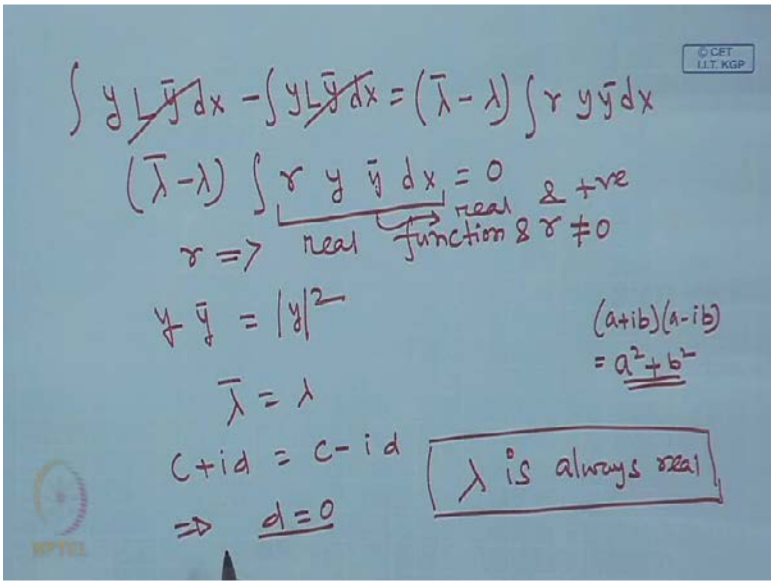

So, once we know this facts, then we can simplify the equation as y L y bar d x plus J will be equal to 0 minus y L y bar d x is equal to lambda bar minus lambda integral r y y bar d x. So, these two quantities on the left hand side, they are identical to each other and opposite in sign, so they will subtract, so lambda bar minus lambda becomes in lambda bar minus lambda and will be will be taken out and r y and y bar d x is equal to 0.

So, we have already seen that r is a real quantity, real function and it is a non-zero function, so r cannot be equal to 0. So, what is the product of y and y bar? Product of y and y bar, this is a complex number you have said and this is a complex conjugate of that, if you just do the product, if we just product two quantities, which is complex and its conjugate, so this becomes a square minus b square, so this become a square, i square is minus 1 plus b square,

So, multiplication of a complex and its conjugate will be always giving raise to a real part. Therefore, y multiplied by y bar is nothing but mode of y square of that, so this is a real part. What I mean is that the part in the integral r y y bar d x will be always real and positive, so this is a real and positive. So, therefore, in order to satisfy this equation, only option is left is lambda bar is equal to lambda. That means complex conjugate equal to real part, that means the complex part does not exists, this simply means C plus i d is equal to C minus i d, so this simply means that d is equal to 0 that means lambda is always real.

So, if you have a function p q r, which are all real functions and coefficients in the eigen on the boundary conditions beta 1, beta 2, alpha 1, alpha 2 all real, then eigenvalues of the system or the equation will be always real, you would not be having a complex eigenvalue. That means for real system, the eigenvalues are real and it will be a self-adjoint system.

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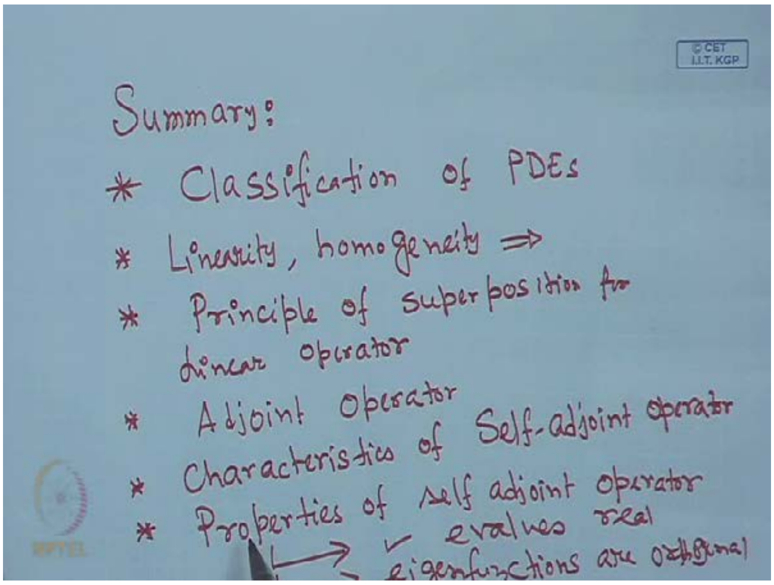

Now, let us take a stock of whatever we have done, the (( )) summarize. First is, we define the various classifications of differential equations of PDEs. Their linearity homogeneity And we have already we checked and defined. Then, we define the principle of superposition for linear operator, we define the adjoint operator, we define the characteristics of self-adjoint problem, adjoint operator and then we looked into the properties of self-adjoint operator.

So, the important properties are number one, the eigenvalues are real and eigenfunctions are orthogonal to each other, eigenfunctions are orthogonal. So, with this background, we will be in a position to solve the partial differential equation, linear partial differential equation by using separation of variable method. So, in the next class onwards will be taking up the solutions of partial differential equation. And up to this class, we are equip with all the weapons in order to attack, to solve the partial differential equations by using separation of variable.

Thank you very much.