Advanced Mathematical Techniques in Chemical Engineering

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Module No. # 01

Lecture No. # 21

Adjoint Operator

Welcome to the third session of today's class. We were looking into the solution of a chemical engineering system in continuous domain.

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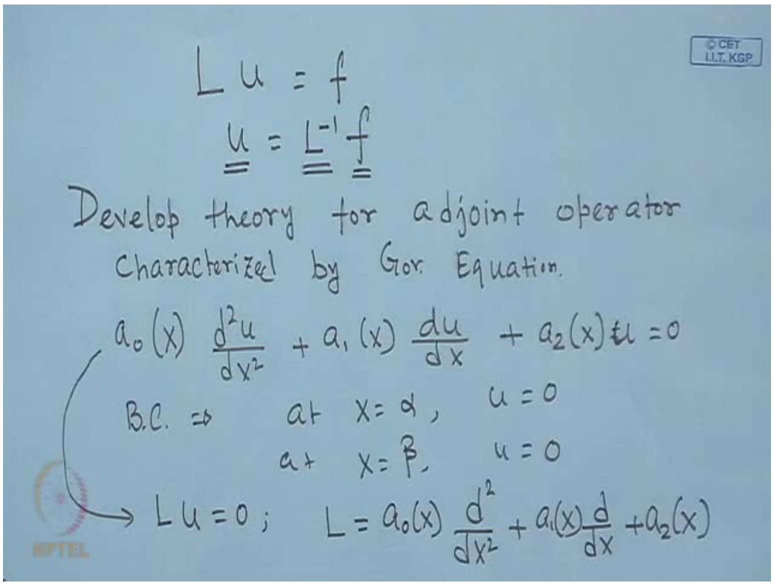

If you remember, whatever we are discussing is that, for a general system, we can mathematically characterize the system as L u is equal to f, where L is the operator, u is the solution function and f is the non-homogeneous term. The solution will be obtained exactly like the matrix operator u is equal to L inverse f. The inverse operators are sometimes it is met identical to adjoint operator. So, we need to in order to find out the solution u, you need to find out the adjoint operator and then f is of course known for, because it is system specific.

Now, our next aim is to get the solution in the continuous domain. Therefore, we would like to find out what is the adjoint operator, given an operator L. We develop the theory for adjoint operator, how to get it from a given an operator? Consider the chemical engineering system is characterized by the governing equation of second order a o, which is in general function of x d square u by dx square plus a 1 x du dx plus a 2 x multiplied by u is equal to 0.

If this is the equation and subject to boundary conditions that at x is equal to alpha, u is equal to 0; at x is equal to beta, u is equal to 0 - homogeneous boundary condition and homogeneous equation. So, this equation can be written in a compact form as Lu is equal to 0, where the operator L is given as a 0 x d square dx square plus a 1 x d dx plus a 2 x.

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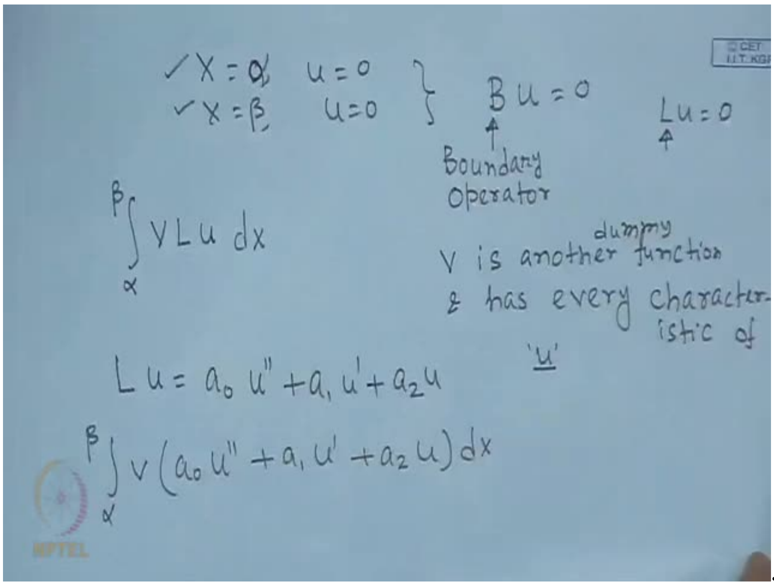

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So, this is the operator and we write the boundary condition x is equal to alpha, u is equal to 0; x is equal to beta, u is equal to 0. These are the homogeneous boundary conditions - we call them in general - Bu is equal to 0. This is the notation Bu is equal to 0, means homogenous boundary condition; Lu is equal to 0, means homogenous governing equation.

Now, BL is the Lu is equal to 0; this is the operator - main operator and B is known as the boundary operator. Now, consider the Lu is equal to f is given; consider the integration integral alpha to beta V Lu dx, where Lu is the operator that we have; L is the operator, u is the function - solution function - and V is again another function; it is a another dummy function and has every characteristic of u. That means, if u is differentiable 2 times and if u is continuous in domain, V is also differentiable and continues in the same domain.

So, if that is the case, we consider these integral over the domain of x from alpha to beta and we put the value of the expression of Lu; if you remember the expression of Lu is nothing but a 0 d square u dx square; we put it at double prime - plus a 1 du dx - we put a single prime - plus a 2 u. We put the expression of L there; so v times a 0 u double prime plus a 1 u prime plus a 2 u dx.

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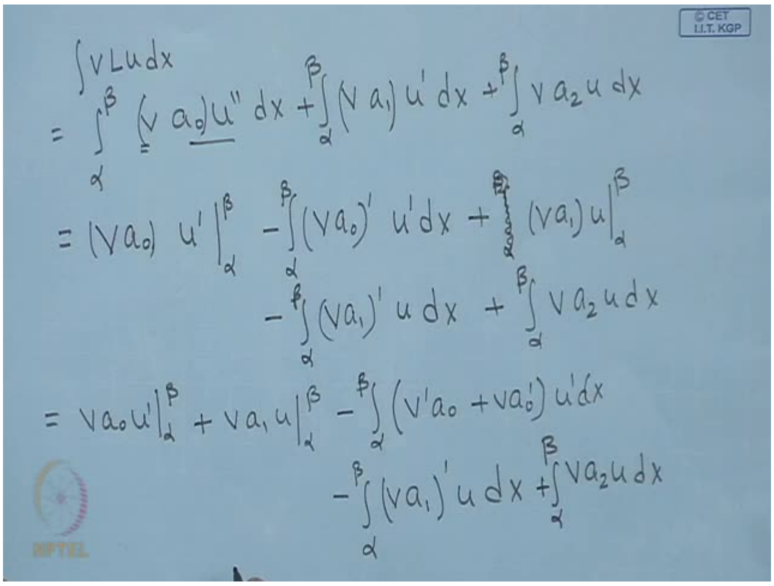

Next, what we do? We take up individual term and do the integration by parts and see what we get. If you do that, V Lu dx is equal to integration from alpha to beta V a 0 u double prime dx plus alpha to beta V a 1 u prime dx plus V a 2 u dx.

Therefore, we carry out this integration by parts; consider this as the first function and this as the second function (Refer Slide Time: 07:37). So you take V a 0 as the first function and u double prime as the second function. If you do that, what we will be getting is that first function integration of second function minus differential of the first function and integration of the second function.

We combine V times a 0 x as a first function and open up this integration by parts. So, V a 0 that is the first function, we put it in bracket; integration of the second function, so it becomes u prime evaluated from alpha to beta minus differential of first function; so V a

0. We denote a prime to indicate the differentiation multiplied by the integration of the second function plus, again, here we combine V a 1 as the first function; so this will be from alpha to beta. This will be the first function, integration of the second function V a 1; integration of the second function, the second function is u prime; so this becomes u from alpha to beta minus integration alpha to beta differential of the first function; so V a 1 prime; integration of the second function that is u dx plus alpha to beta V a 2 u dx.

We get V a 0 u prime evaluated from alpha to beta minus V a 1 u evaluated from alpha to beta. So, am just bringing this one here (Refer Slide Time: 58:07); that is, there is a plus sign there, so it will be plus. Now, am just writing it as alpha to beta, just open up this differentiation; it will be V prime a 0 plus V a 0 prime into u prime dx minus integration alpha to beta V a 1 prime of that u times dx.

If you look into these equations, we will be seeing that in the differential form inside the integration, this has u but this has u prime - u dash. So, we can break down or we can even carry out this integration once again and see what we get.

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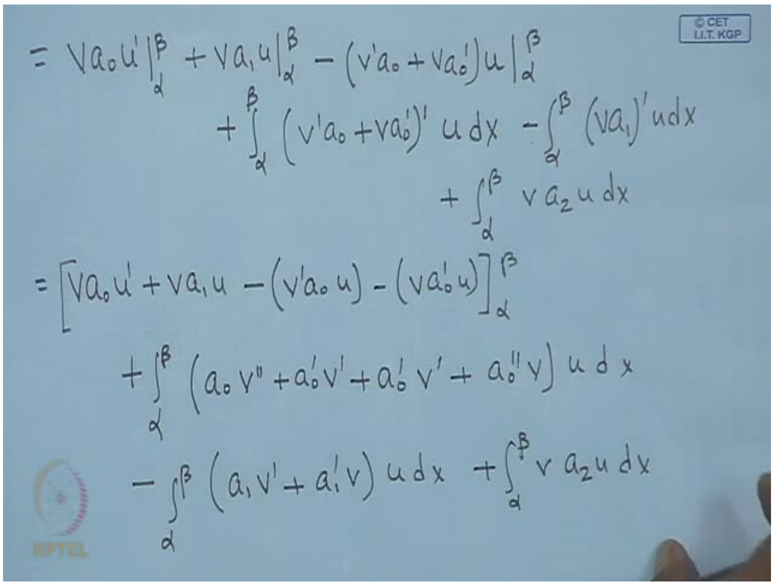

We have just one more term here; so we should write the last term alpha to beta V a 2 u dx. So, we carry a forward to next step, that is V a 0 u prime alpha to beta plus V a 1 u alpha to beta; then we integrate by parts minus first function that is V prime a 0 plus V a 0 prime; differential of the second function, that is, the u evaluate from alpha to beta minus; so it will be minus into minus plus from alpha to beta; differential of the first function, so V prime a 0 plus V a 0 prime. Differential of the first function multiplied by the integration of the second function; then, we keep the other two terms alpha to beta V a 1 prime of that u time dx plus integration alpha to beta V a 2 u dx.

We combine all these terms together; so what we will be getting is that V a 0 u prime plus V a 1 u and this one minus V prime a 0 u minus Va 0 prime u evaluated from alpha to beta. So, that takes care of these three boundary terms plus the governing equation plus alpha to beta V prime a 0 plus V a 0 prime of that; so this becomes a 0 V double prime plus a 0 prime V prime plus a 0 prime V prime plus a 0 double prime V times u d x; then we put other two terms V a 1 prime of that. So, it will be a 1 V prime plus a 1 prime V u dx and the last term is alpha to beta V a 2 u dx.

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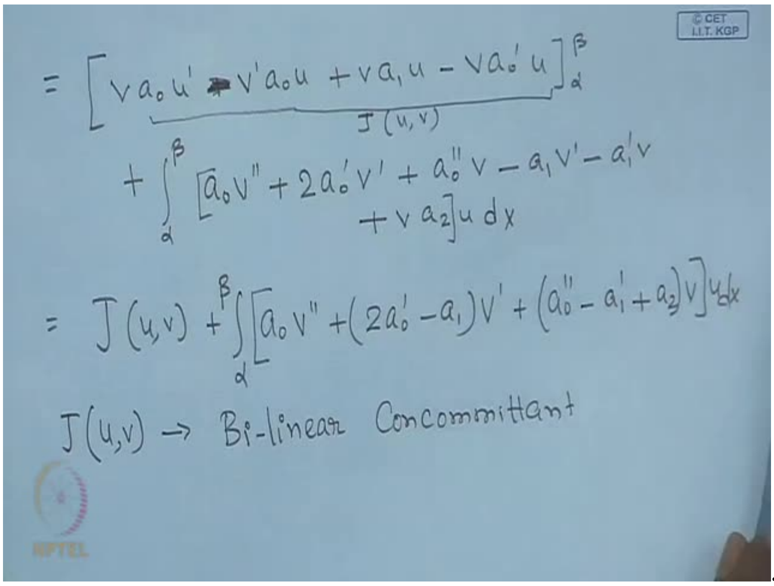

We go forward and do further simplification and see what we get. So, just write this equation as V a 0 u prime plus V prime a 0 u minus sorry minus V prime a 0 u plus V a 1 u minus V a 0 prime u evaluated from alpha to beta and this becomes a 0 to collect all the terms in the integration a 0 V double prime plus a 0 prime V prime plus a 0 prime V prime. So, this becomes 2 a 0 prime V prime plus a 0 double prime V minus a 1 V prime minus a 1 prime V plus V a 2 u V a 2 and all of them will be multiplied by u times dx.

We write this as J u v (Refer Slide Time: 16:13). So this is J u v plus alpha to beta a 0 V double prime plus 2 a 0 prime minus a 1 V prime plus a 0 double prime minus a 1 prime plus V plus a 2 multiplied by V; then whole thing should be multiplied by u times dx.

So, let us examine all these terms one by one and see what we get. J u v is known as bi-linear concommittant. This bi-linear concommittant is basically the term which contains all the boundary conditions together. Now, let us look into the bi-linear concommittant term and how this bi-linear concommittant will be taking shape in this particular case. So, first we examine the different terms of bi-linear concommittant and see what we get.

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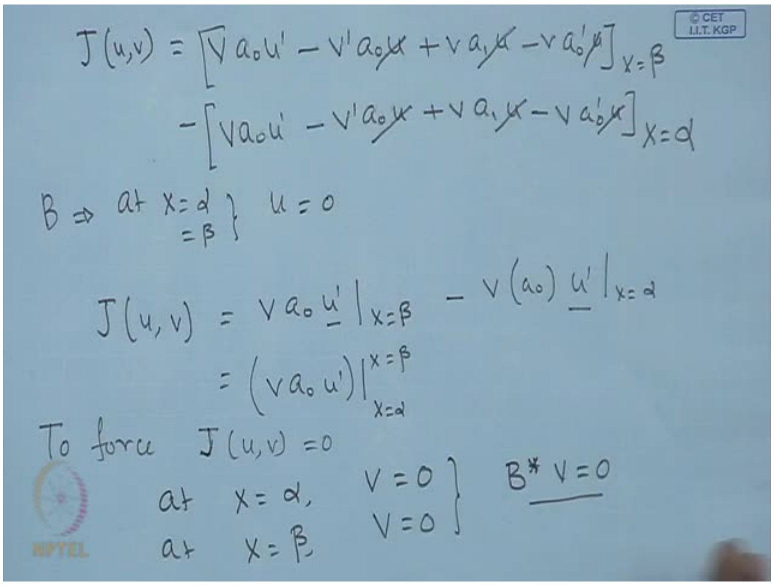

So, J u v is nothing but V a 0 u prime minus V prime a 0 u plus V a 1 u minus V a 0 prime u evaluated at x is equal to beta minus the same thing V a 0 u prime minus V prime a 0 u plus V a 1 u minus V a 0 prime u evaluated at x is equal to alpha.

Now, we have already seen, if you look into the boundary condition of B that at x is equal to alpha and x is equal to beta, we had u is equal to 0. Therefore, at x is equal to beta, u is equal to 0, so this term is gone; at x is equal to beta this term is also gone; so x is equal to beta this term is also gone.

Similarly, at x is equal to alpha, we had u is equal to 0. So, therefore, this term is gone; this term is gone; this term is gone irrespective of value of V and V prime and irrespective the value of a 0 a 1 are their primes.

So J u v becomes V a 0 u prime at x is equal to beta minus V a 0 u prime at x is equal to alpha; so we can write them in a notation V a 0 u prime evaluated at x is equal to alpha and x is equal to beta.

Now, in this case, in order to make the bi-linear concommittant 0, so you will be getting some condition on V (Refer Slide Time: 20:16). We do not know the value of u prime because only u is specified at the two boundaries and also a 0 is a function of x; so a 0 is evaluated at x is equal to beta and x is equal to alpha will be some non-zero constant.

So, in order to get bi-linear concommittant, J u v to be equal to 0 only option is left is that if we select V is equal to 0 at x is equal to beta and V is equal to 0 at x is equal to alpha, then only J u v will be equal to 0. So, to force J u v is equal to 0, the choice of parameters left with us is that at x is equal to alpha, we have to take V is equal to 0; at x is equal to beta, we have to take V is equal to 0.

Therefore, if we select these two boundary conditions on V, we can force the bi-linear concommittant to vanish. Therefore, these at the boundary conditions on the adjoint operator L star and these are known as B star V is equal to 0; that means, even for the adjoint operator, the boundary conditions are same Dirichlet and there are homogeneous as far the original problem.

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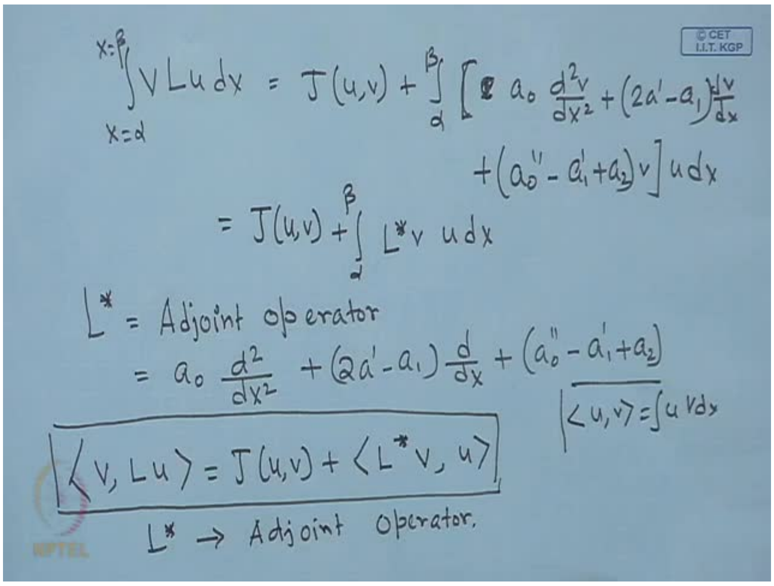

Now, let us look into the other part - the integration. We were doing integration V Lu dx from x is equal alpha to x is equal to beta and whatever we got is that J u v plus alpha to beta; we got a 0 d square v dx square plus 2 a prime minus a 1 dv dx plus a 0 double prime minus a 1 prime plus a 2 v multiplied by u dx.

Therefore, we can write this as J u v plus alpha to beta L star v u dx; so what is L star? L star is known as the adjoint operator and generally this adjoint operator can be written as a 0 d square dx square plus 2 a prime minus a 1 d dx plus a 0 double prime minus a 1 prime plus a 2 ; this is the adjoint operator (Refer Slide Time: 24:18).

We can write this integral, if you remember for the continuous function, inner product is nothing but given by integration. So, inner product of u and v is nothing but integration u v dx. We can write this equation as inner product of V and Lu should be is equal to J u v plus inner product of L star V comma u.

Now, these equations can be written in a compact form like this (Refer Slide Time: 25:08). So in this equation, L star is known as the adjoint operator.

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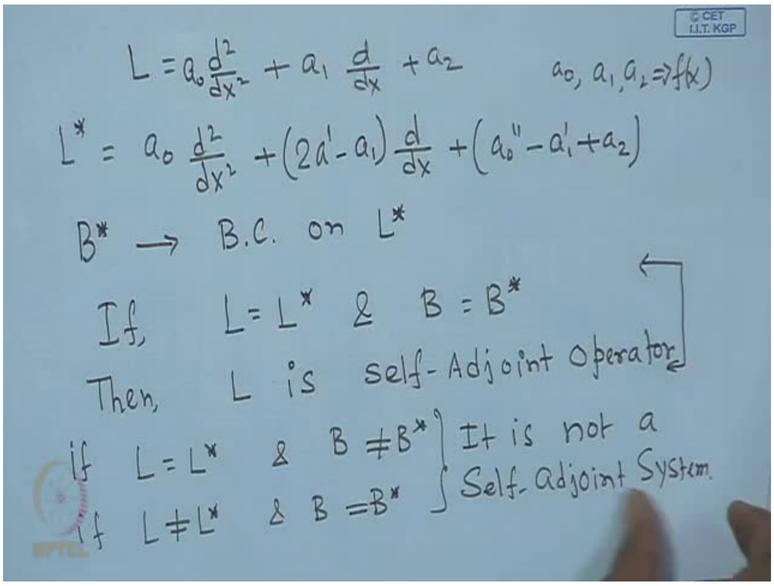

So, in the general case, whatever we are discussing in this class, if the operator is L is d square a 0 d square dx square is a 0 is a function of x in general plus a 1 d dx plus a 2; all a 0 a 1 a 2 are functions of x, then L star is given as a 0 d square dx square plus 2 a prime minus a 1 d dx plus a 0 double prime minus a 1 prime plus a 2.

So, given an operator, we will be able to obtain the adjoint operator. Similarly, by forcing bi-linear concommittant one can get the boundary condition of the adjoint operator. We call that boundary condition as B star; B star is the boundary operator and this gives the boundary conditions on L star or on V, so that is dummy variable, where the L star is defined.

So, this way, one can obtain the adjoint operator and adjoint boundary operator as well. If L is equal to L star and B is equal to B star, then we call L as the self-adjoint operator. In that case, L is self-adjoint operator. Now, if both these conditions have to be satisfied simultaneously in order to have a self-adjoint operator, if L is equal to L star and B is not equal to B star and the other way also - if L is not is equal to L star and B is equal to B star, then the problem is not a self-adjoint problem. If L is equal to L star, B is equal to B star, then only L is self-adjoint operator and the system is called a self-adjoint system. If any one of them is not equal, then it is not a self-adjoint system.

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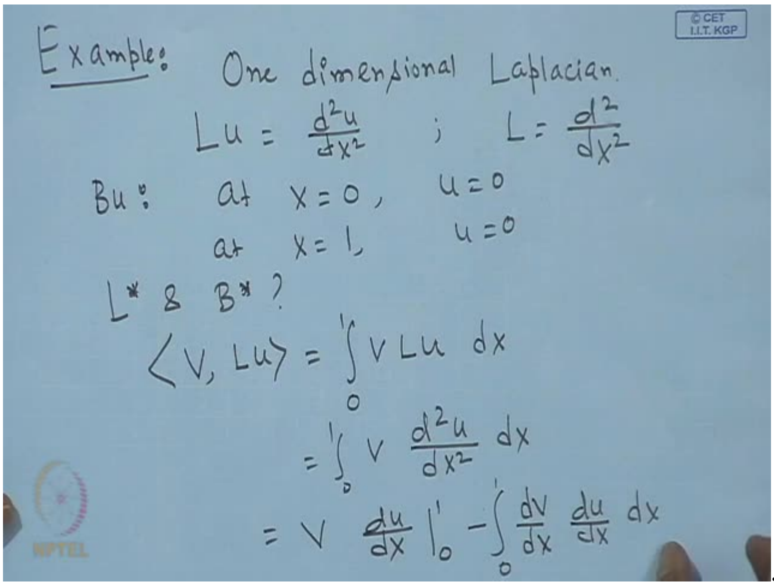

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Now, we will just take of couple of example, how to evaluate the self-adjoint operator and boundary operator for a given operator. The first example: we take a simple example for one dimensional Laplacian; so we take an example of one dimensional Laplacian. For example, Lu is equal to d square u dx square, so my operator L is one-dimensional Laplacian; so it is d square dx square.

Next, the boundary conditions for this problem is given as at homogenous boundary conditions x is equal to 0, u is equal to 0; at x is equal to 1, u is equal to 0. We have to find out what is L star and what is B star - the adjoint operator and the boundary adjoint operator.

We proceed exactly the same way; we have developed the theory of the adjoint operator. So, we take the inner product of V Lu, where V is the some kind of dummy variable in the same domain; so it will be integration from alpha to beta; so it will from 0 to1 V Lu dx.

So, it will be 0 to 1, V L will be nothing but d square u dx square dx, then we integrated by parts. As you have done earlier, first function integration of - this is a first function; this is a second function. So, first function is integration of the second function evaluated within the limits 0 to 1 minus differential of the first function dv dx integration of the second function, that is du dx dx from 0 to 1. Then we do one more integration of that part, considering this is the first function, this is the second function. So, let us see, what we get.

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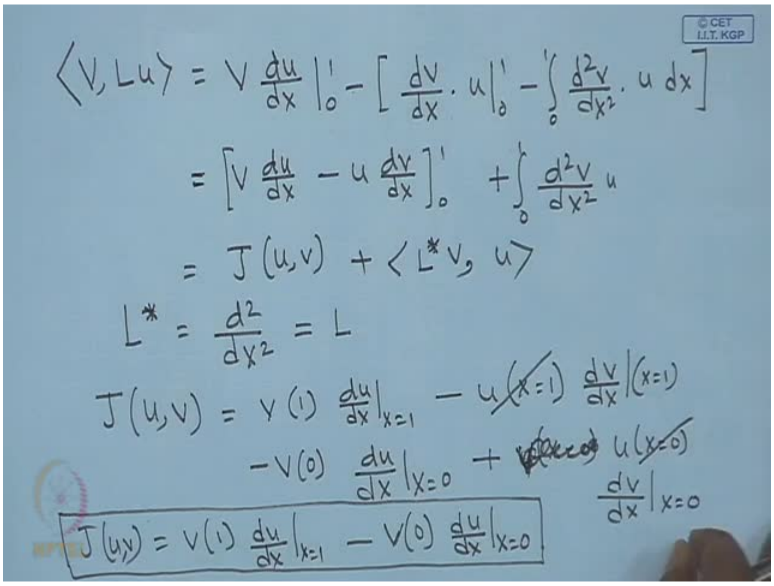

So, inner product of V times comma Lu will nothing but V du dx from 0 to 1 minus - take the dv dx as a first function; so dv dx integration of the second function is u from 0 to 1 minus differential of the first function; so d square v dx square multiplied by integration of the first function that u dx 0 to 1.

So, we collect the bi-linear concommittant term together V du dx minus u dv dx from 0 to 1 minus minus plus 0 to 1 d square v dx square u. This will be J u v plus inner product of L star v and u. Now what is L star? L star is equal to d square dx square and this is exactly same as L.

Now, we force bi-linear concommittant term to be 0 and see what we get as the d star. So, bi-linear concommittant term, J u v is nothing but v - we evaluate this one - v at x is equal to 1 du dx at x is equal to 1 minus u at x is equal to 1 times dv dx at x is equal to 1 minus V at x is equal to 0 du dx at x is equal to 0 minus minus plus u at V at x is equal to 0. So we that we have already done; so this will be u at x equal to 0 minus into minus plus, so u at x is equal to 0 times dv dx evaluated at x is equal to 0.

Now, we have already seen the original problem in u; it has a Dirichlet boundary condition that is u at x is equal to 0, and u at x is equal to 1, they are equal. So, basically, irrespective of the value of dv dx at x equal to 0 and x equal to 1 that will vanish. So, what is left is that V at x is equal to 1 du dx at x is equal to 1 minus V at x is equal to 0 du dx at x is equal to 0.

So, that is the fate of bi-linear concommittant after doing a little bit of simplification using the boundary condition du is equal to 0.

Now, if we would like to force this J u v to be 0, we do not have any idea about du dx at x is equal to 0 and du dx at x is equal to 1. We have to select, at x is equal to 0, V is equal to 0 and at x is equal to 1, V is equal to 1.

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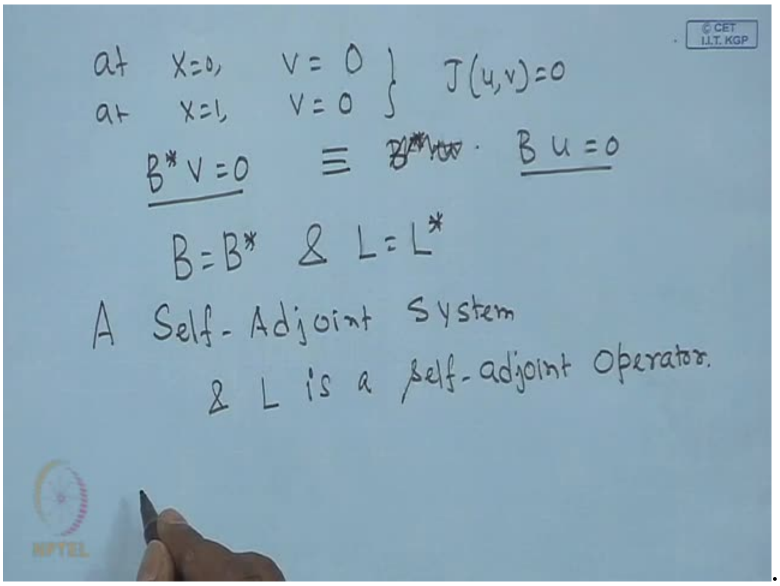

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So, if you do that, we will be getting that at x is equal to 0, V is equal to 0 and at x is equal to 1, V is equal to 0, under this condition only the bi-linear concommittant will vanish.

We will be getting the boundary operator; boundary operator is B square V is equal to 0. So, it is identical with the original boundary operator; that is, B square B star B that was B u this is identical equal to Bu is equal to 0. So, Bu is equal to 0 and you are getting B star is equal to 0; B is equal to B star in this particular case and L is equal to L star in this particular case. Therefore, this is a self-adjoint system and L is a self-adjoint operator.

So, both the self-adjoint - in this particular example - we have seen that for a Laplacian operator in one-dimensional that is the self-adjoint operator because L is equal to L star as well as B is equal to B star. Now, the next point and automatically the question arises - that if we change the boundary condition from Dirichlet to Neumann, what happens to the analysis? That system remains the self-adjoint system or not?

In order to examine that, we change the boundary condition - the first boundary condition; may be, replace the Dirichlet boundary condition by Neumann boundary condition and check whether the system remains self-adjoint or not.

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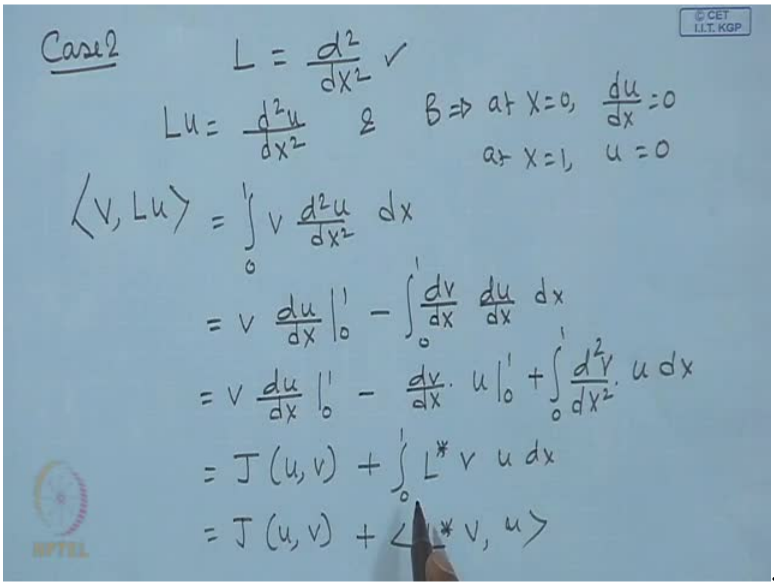

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Let us formulate the problem once again; let us put it under case number 2. In this case, the operator is same one-dimensional Dirichlet; so, Lu is equal to d square u dx square and let us say, what is B? B is at x is equal to 0 du dx is equal to 0 and at x is equal to 1, u is equal to 0. If you just identify, we have changed only the boundary condition at x is equal to 0 of the earlier problem by a Neumann boundary condition and the other boundary condition remains a Dirichlet boundary condition.

Now, the question is whether L constitutes a self-adjoint system, whether L is equal to L star or B is equal to B star or not. We start with the same formulation inner product of V and Lu; so it will be integration 0 to 1 V d square u dx square dx. Again, we proceed the same way; we take request to the integration by parts; so this will be first function, integration of the second function 0 to 1 minus differential first function that is dv dx integration of the second one du dx dx from 0 to 1; then we take this as first function, take this as second function, carry out the next integration step; so it will be V du dx 0 to 1 minus first function dv dx integration of the second one will be u from 0 to 1 minus minus into minus plus. So, it will be 0 to 1 differential of the first function will be d square V dx square integration of a second 1 u dx.

The first two terms will be constituted by the bi-linear concommittant term, so we write it as J u v plus 0 to 1 L star V times u dx. So, write it as J u v plus inner product of L star V and u.

So, if you look into the value of expression of L star; L star becomes d square dx square this is same as L. So, therefore, L is equal to L star for this problem as well.

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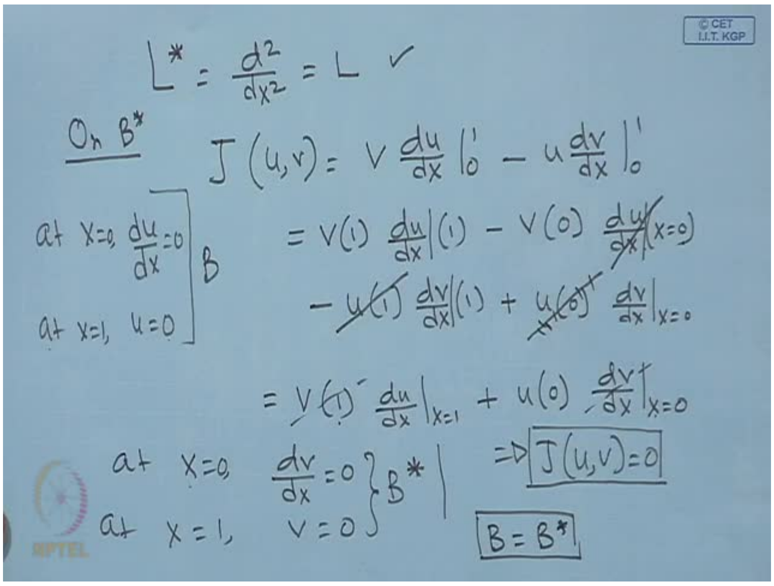

Next, we have to look into the d star. So, if you look into the L star, L star becomes d square dx square and this is same as L and on B star; if you check on B star, so for that we have to very carefully look into the bi-linear concommittant; so J u v will be nothing but v du dx from 0 to 1 minus u dv dx from 0 to 1.

We open up this bi-linear concommittant term and evaluate at both the terms at both the boundaries. So, this will be v at 1 du dx at 1 minus v at 0 multiplied by du dx evaluated at x is equal to 0 minus u at 1 dv dx at 1 minus minus plus u at 0 dv dx evaluated at x is equal to 0, and if you remember what are the boundary conditions on u? At x is equal to 0 du dx was equal to 0; at x is equal to 1, u was equal to 0.

These were the boundary conditions on u; so we immediately put this equal to 0 because u itself is 0 at the boundary x is equal to 0 and u at and du dx is equal to 0.So this is not equal to 0, u equal to 0 at x is equal to 1; so this will be equal to 0 because at x is equal to 1, u is equal to 0 and at x is equal to 0 du dx is equal to 0; that means, at x is equal to 0 du dx will be equal to 0, we cut this off. So, what is remaining of bi-linear concommittant term is V at 1 du dx evaluated at x is equal to 1 plus u at 0 dv dx evaluated at x is equal to 0 (Refer Slide Time: 43:42).

Now, we have a specification of value of u at x is equal to 0; that was du dx is equal to 0 but you do not know any idea - what the value of u at x is equal to 0 is. Therefore, in order to vanish the bi-linear concommittant term, to force the value of it is to be 0. V at x is equal to 0 must be equal to 0, and dv dx at x is equal to 0 must be equal to 0; then only J u v to be equal to 0.

So, if we select at x is equal to 0, dv dx is equal to 0 and at x is equal to 1, v is equal to 0; then both these terms of bi-linear concommittant will vanish and that will give you a term J u v is equal to 0. If we force bi-linear concommittant to vanish we will be getting the boundary conditions on V; so this will constitute B star (Refer Slide Time: 44:51). These are the boundary operator on B star.

Now, if you look into B and B star - so this is B - if you look into B and B star just see the similarity, at x is equal to 0, du dx was equal to 0; at x is equal to 0, dv dx is equal to 0; at x is equal to 1, u is equal to 0 and at x is equal to 1, V is equal to 0. So, we have B is equal to B star in this particular case as well and also L is equal to L star and we have B is equal to B star.

So, therefore, this system is represented by a self-adjoint system. The operator is a self-adjoint operator; the boundary operator is also a self-adjoint operator. Therefore, the system is completely self-adjoint system.

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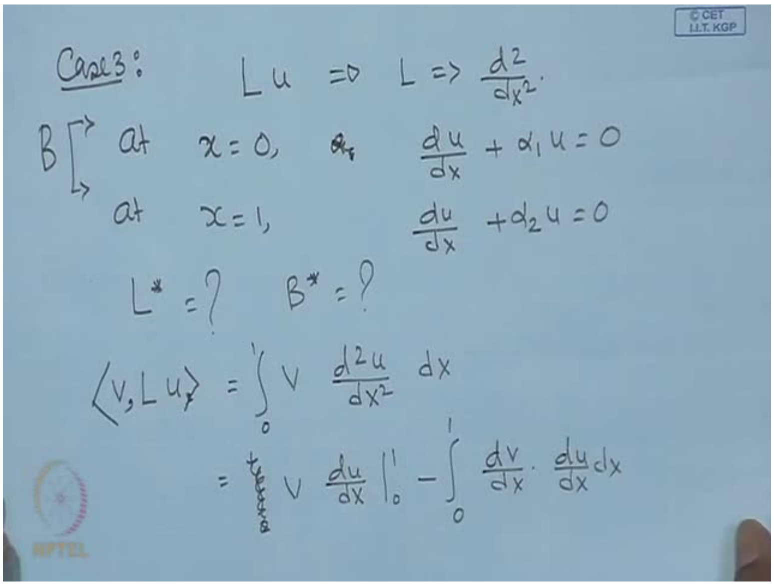

We will take up one more example, where one of the boundary is both the bound - let us say both the boundaries are replaced by the Robin mixed boundary condition and see what we get, whether the system remains a self-adjoint system or not.

Case 3: we take up the same problem - same operator - the one dimensional Laplacian. So, this becomes L u, where L is equal to d square dx square. The boundary conditions on this problem at this at x is equal to - so this at the B; so the boundary operator B is that at x is equal to 0, we have in general du dx plus alpha 1 u is equal to 0 and at x is equal to 1, du dx plus alpha 2 u is equal to 0. Both alpha 1, alpha 2 are some constants.

In this case, if you observe that we have kept the operator same, but we have replaced the boundary conditions by a Robin mixed boundary conditions and these boundary conditions are more generalized, because if alpha 1 is equal to 0 we will be getting a Neumann boundary condition. If alpha 1 is infinitely large, then you will be getting the Dirichlet boundary condition. Therefore, let us find out what is L star and what is B star in this particular problem.

Now, again, we proceed the same way as earlier; so we evaluate inner product of V and Lu; so, if we evaluate inner product of V and Lu, it will be integration 0 to 1 V d square u dx square dx. Again we do integration by parts; so this will be integration 0 to 1, first function integral of second function from 0 to 1 minus integration from 0 to 1, differential of first function dv dx, integral of the second function du dx into dx.

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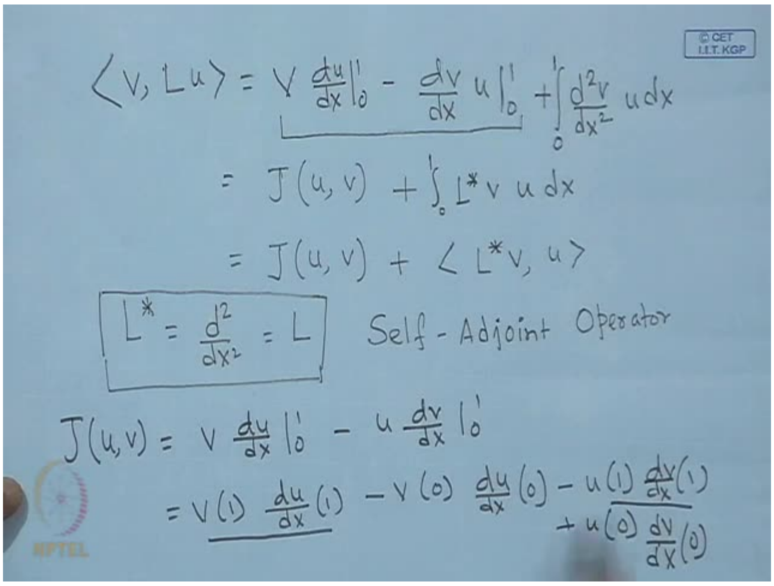

Then, we proceed for the next step. The next step will be simply V du dx from 0 to 1 minus, take dv dx as the first function, du dx as a second function. So, first function dv dx integral of the second function and that is u from 0 to 1 minus - minus minus - plus differential of the first function and integration of the second function d square by dx square u dx from 0 to 1.

So, this is the bi-linear concommittant part; so we write it as J times, J of function of u and v plus, this is 0 to 1 L star V times u dx; so we can write it in this form, J u v plus inner product of L star v and u (Refer Slide Time: 50:50). If you look at that - what is L star? L star is d square dx square and it is same as L. So, we can identify that the operator is the self-adjoint operator.

Now, the next check is whether the system is a self-adjoint system or not. For that we have to find out what B star is. We follow the same procedure for evaluation of B star; we evaluate J u v and force the bi-linear concommittant to be 0. So, J u V is given as V at V du dx from 0 to 1 minus u dv dx from 0 to 1. We evaluate this one; so V at 1 du dx at 1 minus V at 0 du dx at 0 minus u at 1 dv dx at 1 minus minus plus u at 0 dv dx at 0. We take the boundary conditions at 1 and boundary conditions at 0 together and see what we get.

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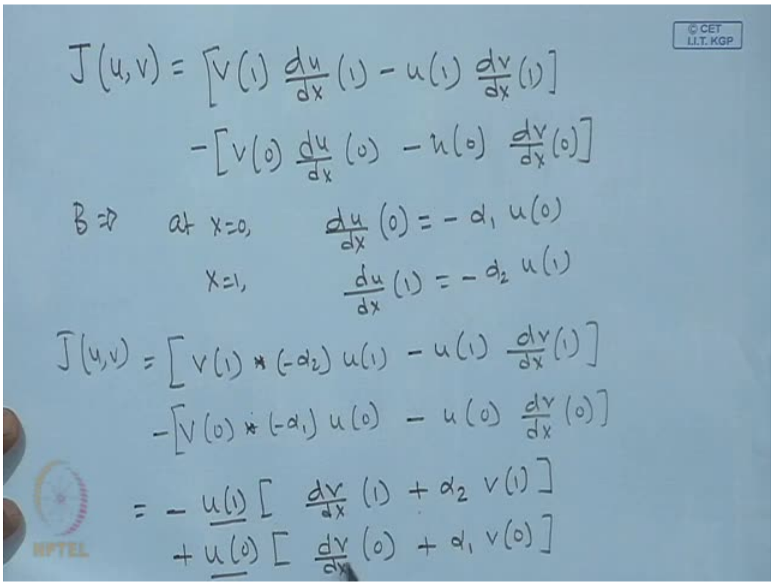

So, J u V will be V at 1 du dx at 1 minus u at 1 dv dx at 1; so this is one minus V at 0 du dx at 0 minus u at 0 dv dx at 0. We have already seen that if you look into the boundary conditions B, that is, at x is equal to 0, if you remember the boundary conditions that du dx at x is equal to 0 should be i equal to minus alpha 1 u at 0 and x equal to 1 du dx at 1 should be equal to minus alpha 2 u at 1.

So, we replace, the du dx at 1 and du dx at 0 by this terms; so it will be V at 1, du dx at 1 should be replaced by minus alpha 2; so minus, this should be multiplied by minus alpha 2 u at 1 minus u at 1 dv dx at 1 minus V at 0 - replace du dx at 0 by this one - minus alpha 1 u at 0 minus u at 0 dv dx at 0.

Next, we take minus common from this and u at 1; so if you take minus u at 1 common, so what is left behind is dv dx at x is equal to 1 plus alpha 2 V at x is equal to 1 minus into minus plus take u at minus u at 0 common; so it will be u at 0; so this becomes dv dx at 0 plus alpha 1 V at 0. Now, in order to get this bi-linear concommittant to vanish, we do not have any conditions at u 1 and u at 0. Therefore, dv dx plus alpha 2 at x is equal to 1 must be equal to 0 and dv dx plus alpha 1 at x equal to 0 must be equal to 0.

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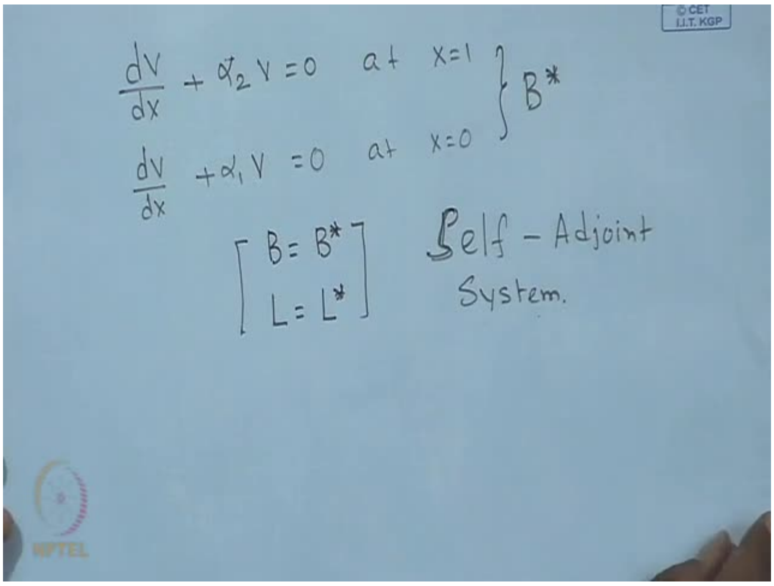

Therefore, we will be getting dv dx plus alpha 2 V is equal to 0, at x is equal to 1 and dv dx plus alpha 1 V must be equal to 0, at x is equal to 0. Therefore if you compare, these are the boundary operator, we get B star; if you compare B star with B you will find out that B is equal to B star and we have already observed that L is equal to L star, so the system is that self-adjoint system.

This is a self-adjoint system and one can get the B star the way you have defined. So, this is the theory of adjoint operator and how to get the boundary operator and we can identify whether the system is self-adjoint or not.

So, we stop here in this class. In the next class, we will take up various important theorems of Eigen values and Eigen functions. So that we can proceed for the solution of partial differential equations safely using those theorems and properties of Eigen values and Eigen functions.