Advanced Mathematical Techniques in Chemical Engineering

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Lecture No. # 02
Introduction to Vector Space

Well, we will continue our discussion with the Linear Vector Spaces. So whatever we have done till now is that we have defined a real vector space in n-dimension; real as well as the complex space; and we have seen how you will define: the normed linear space, the metric linear space and the inner product linear space; the various properties these linear spaces will obey and satisfy; What is the inter relation between two is; and given a vector, how will you compute the normed metric or the inner product; and not only that, if we are moving from the discrete domain to the continuous domain, then how the inner product, the metric and norm will be defined for these continuous domain functions- whether it is a one-dimensional function, or two-dimensional function, or three- dimensional function.

Next, we are looking into some of the simplified examples and in the first example, we have seen, given two vector, how will you define the metric between the two, the inner product between the two and norm of each vector in discrete domain. The second example we have taken up is for a continuous two dimensional domain given by the functions f and g and how the various quantities like inner product, norm and metric are defined for these cases.

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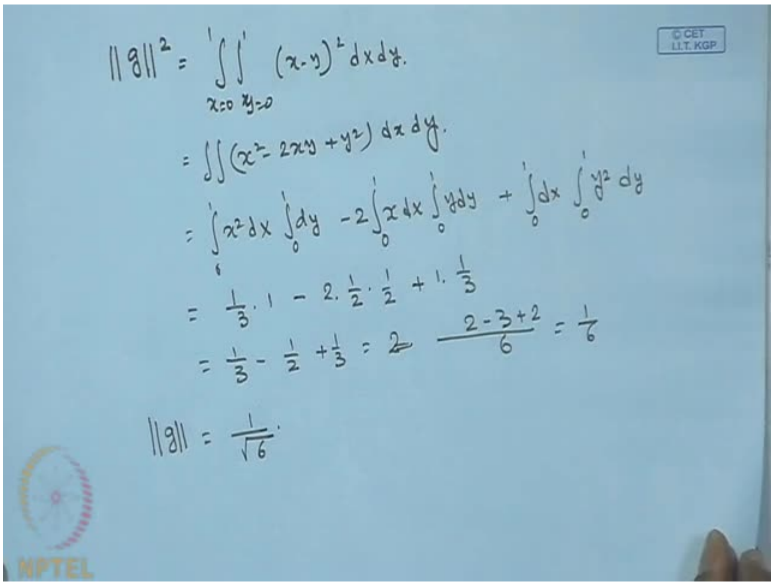

Now, we are looking into the second example, where we are talking about the function f and g. So, in the last lecture, we have looked into the norm of function f and now we will be looking into how to compute the norm of g. This is given as norm of g square, integral over x, integral over y, x minus y square dx dy; so, it will be double integral x square minus 2xy plus y square dx dy. So it will be from 0 to 1 x square dx into 0 to 1 dy minus 2x dx 0 to 1 into y dy 0 to 1 plus dx from 0 to 1 and y square dy from 0 to 1.

So, if you carry out this integration, this turns out to be 1 upon 3, this will be 1; minus 2 into 1 upon 2 into 1 upon 2; plus 1 into 1 upon 3. This will be 1 upon 3; minus 1 upon 2; plus 1 upon 3. So it will be 2 and if we simplify this thing it will be 2 minus 3 plus 2 . so 1 over 6; so norm of g for this particular case is 1 upon root over 6.

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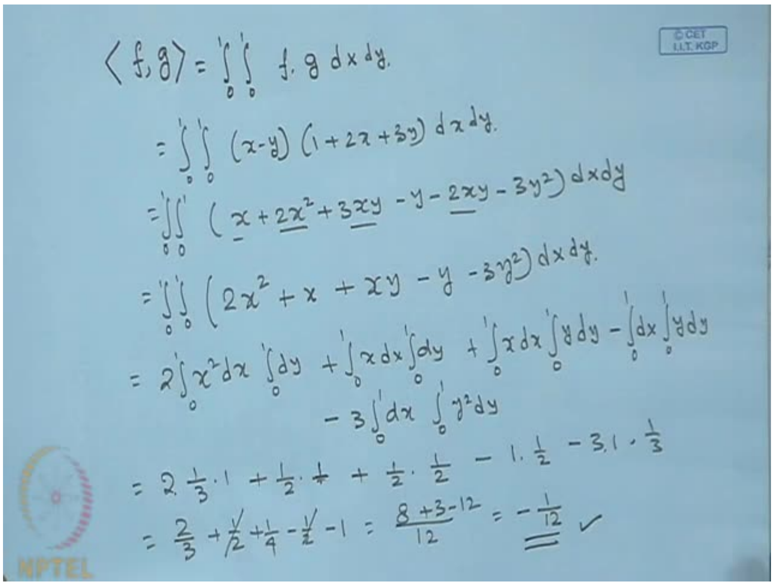

So, that is how we can develop the norm of f and norm of g. Next, we talk about norm, the inner product between f and g will be simply double integral 0 to 1; 0 to 1 f multiplied by g dx dy; that is the definition.

So, if we look into the expressions, we substitute the expression of g and f; so it will be x minus y, multiplied by 1 plus 2x plus 3y, dx dy.

So let us simplify this one, so x plus 2x square plus 3xy minus y minus 2xy minus 3y square dx dy; and that is a simplification will be 2x square plus x- and 3xy minus 2xy will be- plus xy minus y minus 3y square dx dy.

So, 2 integral x square dx integral dy 0 to 1 plus 0 to 1 x dx into integral dy from 0 to 1; third term will be integration x dx integration y dy; fourth term will be integration 0 to 1 x multiplied by y dy 0 to 1 and the last term will be 3 dx 0 to 1 y square dy 0 to 1. So this will be 2, after carrying out the integration- multiplied into 1 upon 3 into 1; plus 1 upon 2 multiplied by 1; plus 1 upon 2 into 1 upon 2; minus 1 into 1 by 2; minus 3 into 1 into 1 upon 3. So you will be getting 2 by 3 plus half plus 1 upon 4 minus half minus 1. So half and half will be cancelled out; so, it will be 1 upon 12; 3 into 4; and 4 into 2- 8; plus 3 minus 12. So it will be minus 1 by 12.

So, that is the inner product between the two functions f and g; there is a difference between the two, incase the inner product of two vectors gives an angle; and in this case- incase of functions, the inner product is computed by this method.

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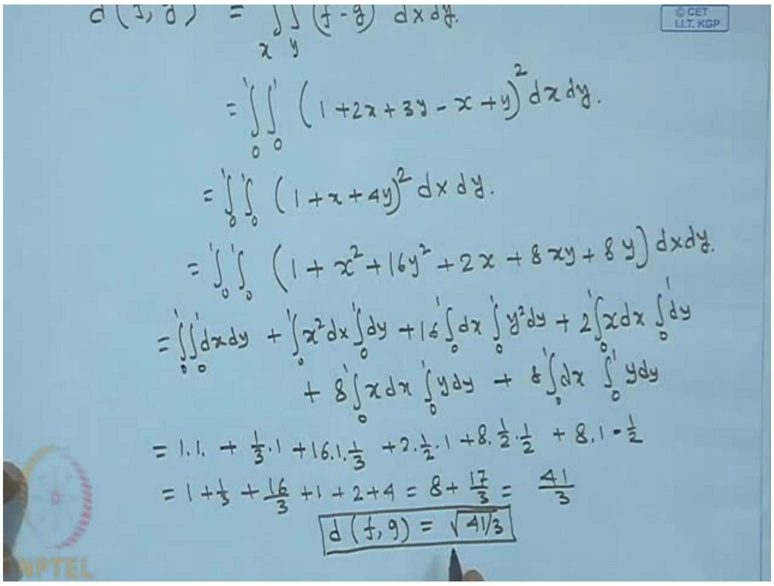

Next, we talk about the metric. Metric between the functions f and g; square of that is defined as double integral one over x and another over y, f minus g whole square dx dy.

So, it will be from 0 to 1, 0 to 1, f is 1 plus 2x plus 3y minus g is x minus 1x plus y dx dy. So it will be 0 to 1, 0 to 1, as simplify this one- 1 plus x plus 4y; square of that. So, just open up this square and if we open up this square this becomes a square plus b square plus c square plus 2ab plus 2bc plus 2ca dx dy.

So, your first one will be this one, second integral will be 0 to 1 x square dx dy from 0 to 1, third integral will be 16 dx 0 to 1 y square dy from 0 to 1, fourth integral will be 2x dx 0 to 1 dy 0 to 1, fifth integral will be x dx 0 to 1 y dy 0 to 1 and the last one will be 8 dx dy 0 to 1.

So, if you compute all these integrals, this becomes 1 multiplied by 1, plus- this will be- one-third multiplied by 1, plus 16 into 1 into one-third, plus 2 into half into 1, plus 8 into half into half, plus 8 into 1 into half; and this becomes 1, plus 1 upon 3, plus 16 upon 3, plus 1, plus 2, then this will be plus 4. So 4 plus 2- 6, 6 plus 1- 7 and 7 plus 1- 8; so, 8 plus 17 by 3. So, this will be 41 by 3. So, the metric between f and g will be root over 41 upon 3

So, that is how the inner product, metric and norm of the continuous two-dimensional functions can be computed. Next, we will look into some more examples of this concept.

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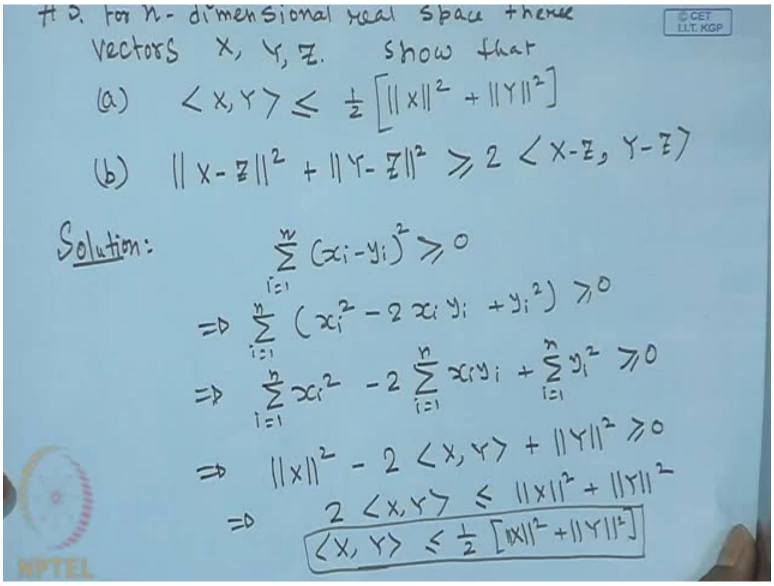

The next example is for n-dimensional real space; in real space, we have three vectors X, Y and Z and we have to show that first, the inner product between X and Y will be less than or equal to half, norm of X square, plus norm of Y square- this is the first relation we have to show. The second relation we have to show is that the norm of X minus Z, square of that; plus norm of Y minus Z, square of that; should be greater than or equal to 2 inner product between X minus Z and Y minus Z.

And these X, Y and Z are 3 general vectors having n number of real valued elements each. So, let us look into the solution.

We start with the identity that sum of the squares will be always greater than equal to 0.

So, summation of x i minus y i, square of that; will always be greater than or equal to 0. If x and y are identical, then this will be equal to 0; otherwise, it will be ever positive, because square is ever positive and sum of the squares will be always positive - so, i is equal to 1 to n. Just open up this square and if you open up this square, then what you will be getting is x i square, minus 2 x i y i, plus y i square; so, it will be greater than or equal to 0.

Next, you open up this summation; so it will be summation x i square, i is equal to 1 to n; minus 2 summation, i is equal to 1 to n, x i y i; plus summation i is equal to 1 to n, y i square.

So by definition, summation of x i square is norm of X, square of that; so this will be norm of X square; minus and if you look into the definition of the inner product, this will be the inner product between X and Y vector; section Y will norm of Y square.

Therefore, if we rearrange this equation, we will be getting 2 of norm the inner product between X and Y, should be less than equal to norm of X, square of that; plus norm of Y, square of that; and after that it follows the relationship that we are going to prove, that is one will be inner product between X and Y should be less than half of norm of X square, plus norm of Y square- so, that proves the relationship we wanted to prove.

The next one will be the norm of X minus Z square, plus norm of Y minus Z square should be greater than 2 times the inner product between X minus Z and Y minus Z.

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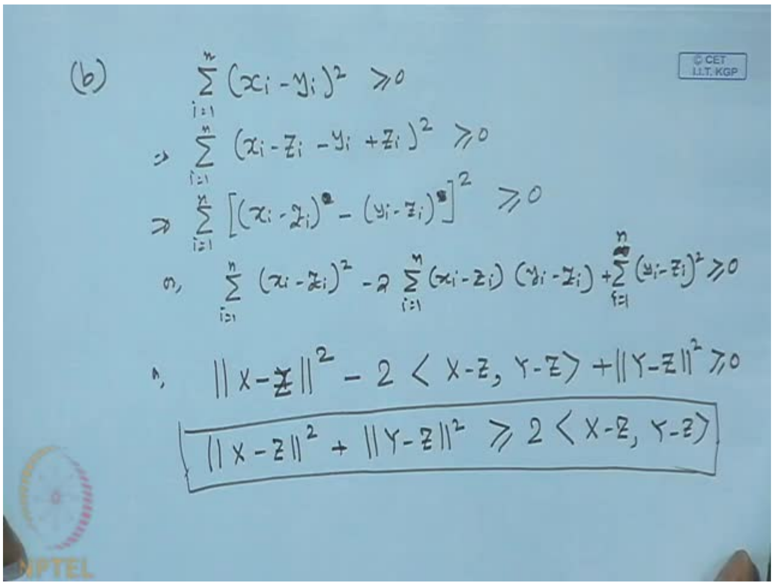

In this case also, we will be starting with the identity, but with the next identity- that means summation of same identity will start with x i y i square, should be greater than or equal to 0 and we will subtract and add z i; so, this will be summation x i, minus z i, minus y i, plus z i; square and again this is an identity therefore, we rearrange this one as xi minus z i square minus yi minus z i square sorry square greater than or equal to 0.

Now, we will open up this square; so, it will be given as summation i is equal to 1 to n, x i minus z i square; minus 2 summation i is equal to 1 to n, x i minus z i into y i minus z i; plus summation i is equal to 1 to n, y i minus z i square; greater than or equal to 0.

So by definition, summation x i z i, whole square is the norm of the vector X minus Z, square of that; and this will be the inner product between the vectors X minus Z and Y minus Z and this will be the norm of vector Y minus Z, square of that; so, the whole equation is obtained and it has to be rearranged and we will be getting our desired result.

So, this will be norm of X minus Z, whole square; plus norm of the vector Y minus Z, square of that; should be greater than or equal to 2 times the inner product between the vector X minus Z and Y minus Z and that completes the proof of the third problem that we have looked into.

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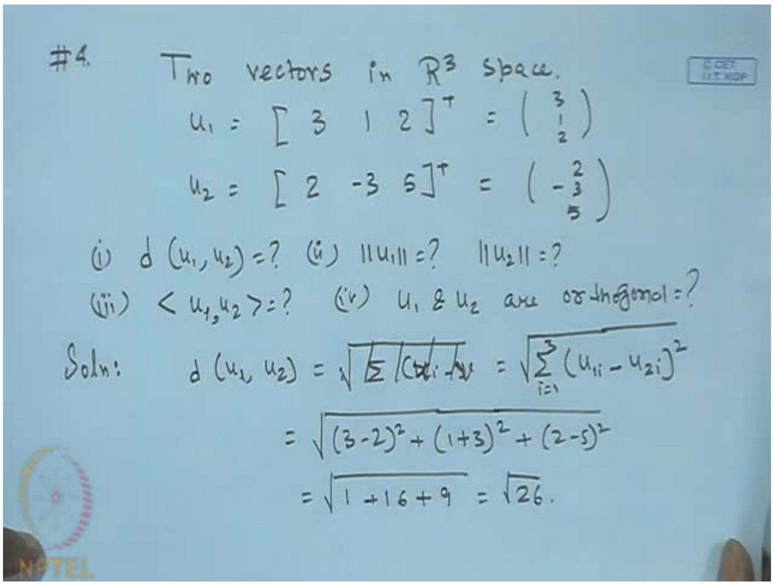

Let us go to the next problem, let us look into some more problems so that the method and our intension are absolutely clear and the computation procedure will also be clear.

Now, consider the 2 vectors in R to the power 3 space; that means each vector will be containing 3 elements; let us say u 1 equal to 3, 1, 2 transpose and this notation is the same as 3, 1, 2.

And u 2 is 2, minus 3, 5 transpose and this is 2, minus 3 and 5.

Now what we have to do, we have to compute the metric between u 1 and u 2; the second part is to compute norm of u 1 and norm of u 2; third part is to compute the inner product between u 1 and u 2 and the last part is, we have to check whether u 1 and u 2 are orthogonal or not.

Orthogonal means they are perpendicular to each other or not. If they are orthogonal means, the inner product between the two will be 90 degrees- that means they are oriented at 90 degree to each other.

Let us look into the solution to this problem. So inner product between u 1 and u 2 is- if you look into the definition, it is under root summation x i- that means it is under root summation u 1i minus u 2i, square of that; i equal to 1 to 3 in this particular case. So it will be 3 minus 2, square of that; plus 1 plus 3, square of that; plus 2 minus 5, square of that; so, it will be root over 1, plus 16, plus, minus 3, square and so, it will be 9. So, it should be root over 26- that is the inner product between u 1 and u 2.

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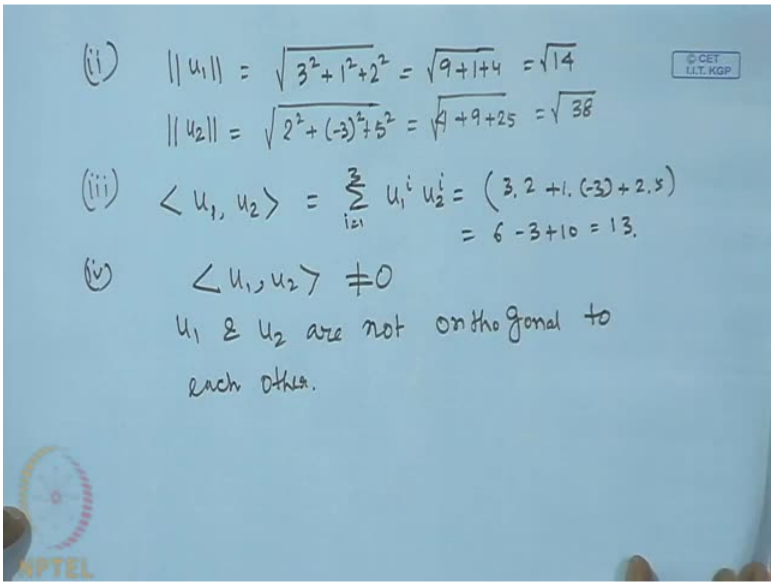

Next, we will compute the norm of u 1 and u 2- part two. So in part two, we will compute norm of u 1 as root over square of each element and add them up.

So it will be 3 square, plus 1 square, plus 2 square; so, it will be root over 9, plus 1, plus 4. So, it will be root over 14 units.

Similarly, you can compute the norm of u 2, it will be 2 square, plus minus 3 square, plus 5 square. So it will be 4, plus 9, plus 25; so, it will be root over 38.

Next, we take up that the third part of the problem that computes the inner product between u 1 and u 2. So, if you look into the definition, it will be summation over i equal to 1 to 3; u 1i multiplied by u 2i. So it will be 3 into 2, plus 1 into minus 3, plus 2 into 5; so, it will be 6, minus 3, plus 10 and so it will be 13.

So, that is the inner product between u 1 and u 2. Therefore, the last bit of this problem is that we have to check whether u 1 and u 2 are orthogonal to each other or not. They are not orthogonal, because inner product between u 1 and u 2 is not equal to 0.

Therefore, the conclusion is that u 1 and u 2 are not orthogonal to each other. So, that goes for the problem number four.

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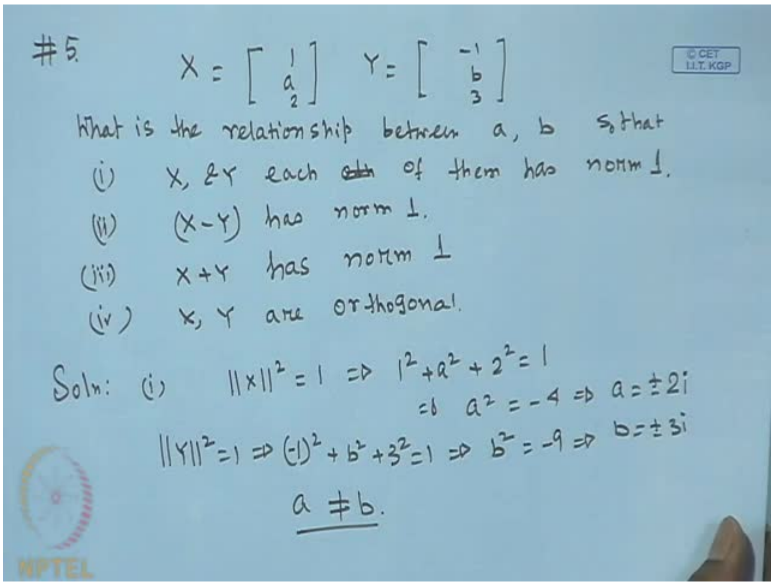

And let us look into the next problem. The next problem says that we have 2 vectors, 3-dimensional vectors: 1, a and 2; so that belong to n-dimensional space.

The elements can be really complex or whatever; so, Y will be minus 1, b and 3. Then we have to find out the relationship between a and b for different cases.

Let us say case one: X and Y each of them has norm 1.

Second case: X minus Y has norm 1 and third case: X plus Y has norm one.

Last one is, X to Y are orthogonal. In each of these cases what is the relationship between a and b, is the question.

So, let us look into the solution, in the first part, the X has norm 1- that means norm of X equal to 1and therefore, it will be 1 square, plus a square, plus two square equal to 1. So, a square equal to minus 4; so, a equal to plus or minus 2i.

Y has norm 1- that means minus 1 square, plus b square, plus 3 square equal to 1. Therefore, b square will be equal to minus 9 and so b equal to plus or minus 3i.

Now in this case, the relationship between a and b is that a is not equal to b.

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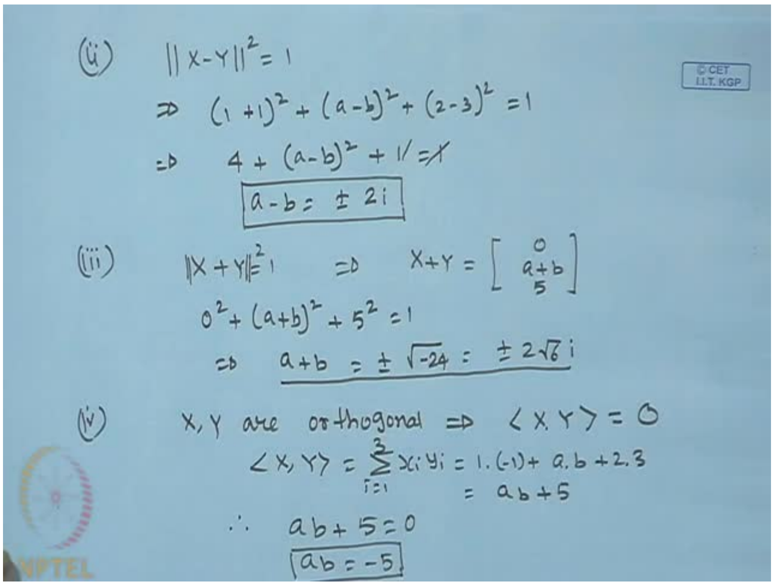

Now let us look into the second part, where X minus Y has norm 1. So norm of X minus Y should be 1- that simply means 1, minus minus 1, square, plus a, minus b square, plus 2, minus 3 square; equal to 1.

So we will be having 4 plus, a minus, b whole square; plus 1 equal to 1. So a minus b equal to plus or minus 2i. So that is the relationship between a minus b.

And the third part is that X plus Y has a norm 1, so let us look into the vector X plus Y. X plus Y will be having the elements 0, a plus b and 2 plus 3- that are 5. So, norm of them is equal to 0 square, plus a, plus b square, plus 5 square; equal to 1.

So, a plus b equal to plus or minus root over minus 24- that will be plus or minus 2 root 6 times i and that is the relationship between a plus b.

And the last part is that, X and Y are orthogonal. This relationship is indicated by this mathematical expression.

The inner product between X and Y should be equal to 0. So, let us compute the inner product between X and Y.

So it will be inner product between X and Y should be summation of x i y i; i equal to 1 to 3.

So, it will be 1 into minus 1, plus a into b, plus 2 into 3. So, it will be ab plus 5, so this should be equal to 0 because X and Y are orthogonal.

Therefore ab plus 5 must be equal to 0 and so ab equal to minus 5. So, this is the relationship between a and b- that we will be getting for the various conditions satisfied are given by this problem.

Next, we just look into another problem that will be a very interesting problem.

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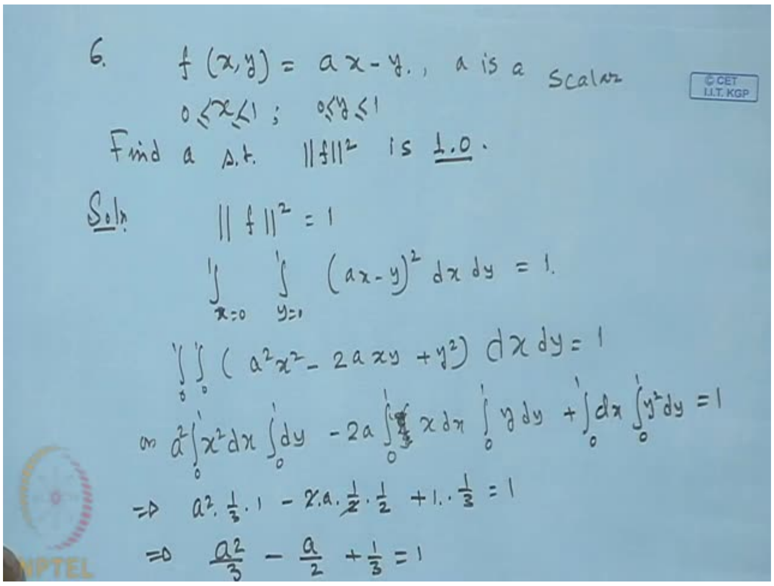

The sixth problem, it is in the case of a continuous function in a continuous domain and it is a two-dimensional domain. So, we define a continuous function f as a function of x and y, where x and y are the two independent domain that we know in coordinate system.

So this is given by ax minus y and both x and y belongs to the same domain- that means x varying between 0 and 1 and y is also varying between 0 and 1.

Now, we have to find a such that norm of f is equal to 1. The solution is that, norm of f square is equal to 1. So, the norm of f will be integral 0 to 1 over x and another integral is over y. So it will be double integral ax minus y square dx dy; should be is equal to 1. So. we just carry out this integration and it will be 0 to 1 and 0 to 1; open up this square- a square x square, minus 2axy, plus y square, dx dy; should be equal to 1. Then term wise you carry out the integration: a square x square, dx dy; this is a first term. Second term will be x dx from 0 to 1; y dy from 0 to 1 and third term will be plus dx, 0 to 1; y square dy from 0 to 1; is equal to 1. So, a square into 1 upon 3 into 1, minus 2a into 1 upon 2 into 1 upon 2, plus 1 into 1 upon 3 is equal to 1.

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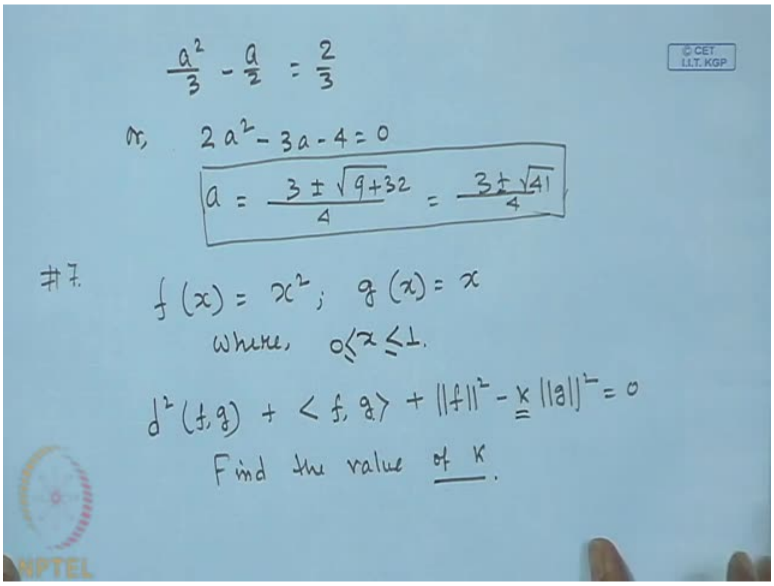

So, we will be getting: a square by 3- 2 and 2 cancels out; so, a by 2, plus 1 upon 3 is equal to 1. We further simplify this equation and this becomes a square by 3, minus a by 2 is equal to 2 by 3. Now, we land up in a quadratic of a and this will be simplified into two: a square, minus 3a, minus 4 is equal to 0 and this quadratic has to be solved. So, we will be getting the value minus b- so it is plus 3, plus or minus under root b square, minus into minus plus 4ac- 32; divided by 2a- that is 2 into 2 is 4. So, a is equal to 3, plus or minus root over 41; divided by 4. So, that this function f, which is a function of dimension x and y and so it is a 2-dimensional function that will be having a norm 1.

Next, we look into the next example and again it is a problem of continuous function and this function, in this case, is a 1-dimensional function. Let us say f of x is equal to x square and g of x is equal to x, where the domain of x is basically from 0 to 1.

Now, in this problem, we have a relationship - that is the metric between f and g, square of that; plus inner product between f and g; plus norm of f square, minus k, norm of g square; is equal to 0. If this algebraic relationship is valid, then find the value of k.

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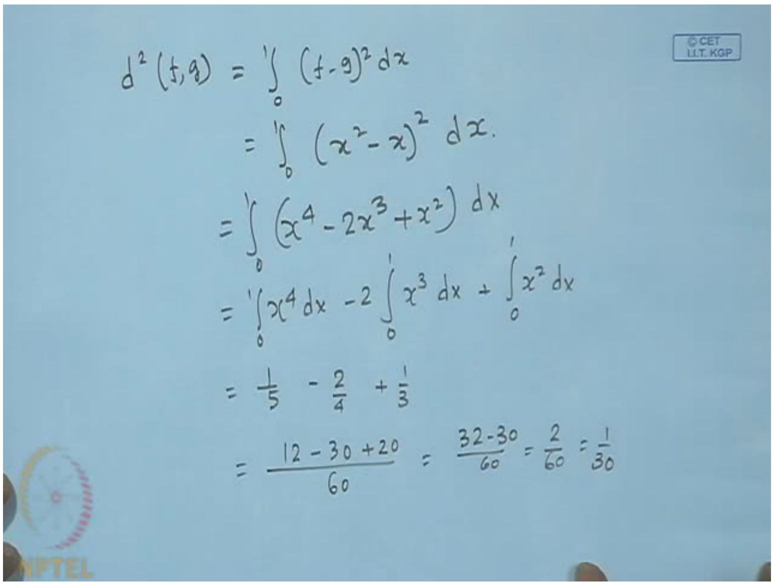

So again, this problem is a one-dimensional function and we have this algebraic relationship is satisfied, so that evaluates the value of this scalar quantity k. So, the procedure remains almost same - we have to evaluate the metric between f and g, the inner product between f and g, the norm of f and norm of g and then substitute those values here and solve this one equation one unknown problem; that will give you the value of k. Let us look into the solution; first, the metric between f and g - so, it will be integral 0 to 1, f minus g square dx; since, it is one-dimensional problem, it will be only one integration and this variable is dx and so it will be 0 to 1, f minus g. So, x square, minus x, square of that, dx; just open up this square, it will be x to the power 4, minus 2 x cubed, plus x square dx; and we carry out this integration - x to the power 4, dx; minus 2 x cubed, dx; plus x square, dx; so, this becomes 1 upon 5, minus 2 over 4, plus 1 upon

3. So, just carry out this and simplify this one; the LCM is 60 - so, it will be 12, 15 into 2 - 30, plus 20. So, it will be 32, minus 30; divided by 60; and it will be 2 by 60. Therefore, it will equal to 1 upon 30; that is the value of metric square of that.

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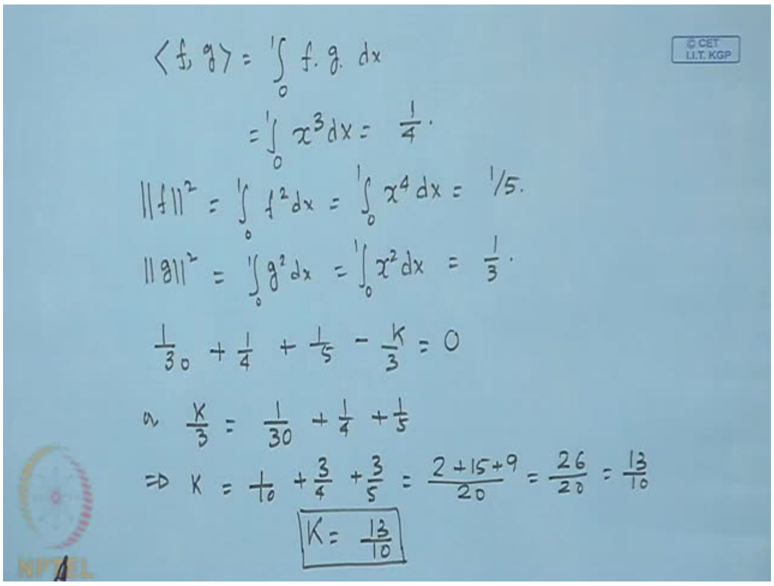

Then, we evaluate the inner product between f and g; that will be simply integral 0 to 1, f multiplied by g dx; So, it will be 0 to 1, f is x square; and g is x and so, it will be x cubed dx; so, it will be 1 upon 4. So, that is the value of inner product, then we find out the norm of f, square of that; so, it will be 0 to 1, f square dx; and it will be 0 to 1, x to the power 4 dx; this value is 1 upon 5.

Next one is, norm of g, square of that; it will be 0 to 1, g square dx; so it will be 0 to 1, x square dx; it will be 1 upon 3. Then we substitute all this quantities in the given expression. So, if you do that you will be getting 1 by 30; minus plus inner product between f and g - that is 1 by 4; then norm of f square will be 1 upon 5, minus k times norm of g, square; so, it will be minus k by 3 and should be equal to 0.

So, substitute this k by 3 and should be equal to 1 by 30, plus 1 by 4, plus 1 by 5; so, k should be equal to 1 by 10, plus 3 by 4, plus 3 by 5 and the LCM is 20. 10 in it is 2 5 5 3 15 plus 5 3 3 3 9, 26 by 20. So it will be 13 by 10 and the value of k is 13 by 10 for this particular problem.

So, in this way you have seen how the inner product metric and the norm of different vectors and functions can be calculated from either in the discrete domain or in the continuous domain and once we know the procedure to calculate them, we will be able to utilize them for solving different mathematical problems in Chemical Engineering applications.

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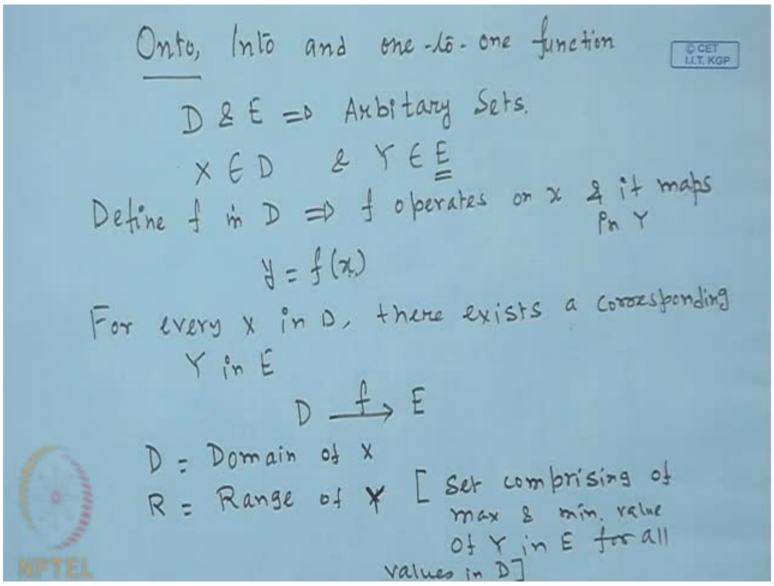

Next, we will look in to another topic that will be quite important and again we will just go through the introductory stuff- that is the definition of Onto, Into and One-to-One function and they will be quite critical in various Chemical Engineering applications. Now Onto, Into and One-to-One function, we consider there exist two sets- D and E sets and they are arbitrary sets.

The element x belongs to the D set and the element y they belongs to the E set. Now we define a function f in the D domain and it assumes it operates on x, so we define f in D set and f operates on x and it maps in y. So we get and after operation it maps in y so basically we are talking about something like y is equal to f of x.

So f and x are defined on D and after operation, the x is defined on D and after operation y after f is operated on x; we get the value of y and y is defined in the set E. Therefore, we can say that for every x in D there exists a corresponding y in E and so it is a map. So, f is a map and when D is operated by f it gives a map in the domain E. So, D is domain of x.

And next we define the R; R is range of y; so it gives set comprising of maximum and minimum value of y in E, for all values in D. So, that gives the range of y and next we look into the relationship of this set R and E, because both are in y domain now.

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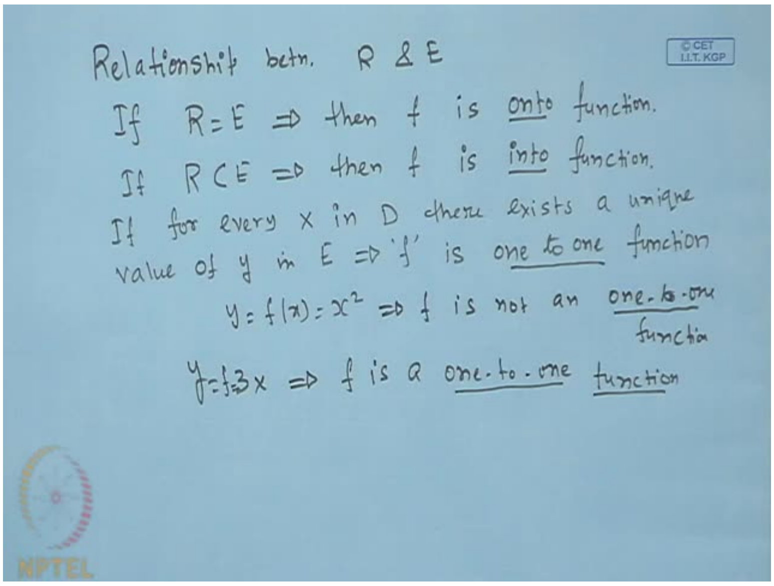

So next, we will have a relationship between R and E. Now, depending on this relationship between R and E, we define the function as Into or Onto; and the definition is if R is equal to E, then f is Onto function and if R is a subset of E, then f is Into function.

If, for every x in D, there exists a unique value of y in E, then f is called one-to-one function; that is very important; so, that means if y is equal to f of x is equal to x square and both x and y lie in the real domain from minus infinity to plus infinity; then we can see that for two values of x, y will be assuming the same value that means for x is equal to plus 1; y is equal to plus 1, for x is equal to minus 1; y is equal to plus 1, for x is equal to minus 5; y is equal to 25 and for x is equal to plus 5; y is equal to 25.

So for different values of x in the D, there does not exist a unique value of y in E and therefore, this function f is not a one-to-one function.

On the other hand y is equal to 3x- that is y is equal to f of x is equal to 3x. For every value of x there exists a unique value of y; so in this case, f is a one-to-one function. Now this concept will be absolutely clear if we look into some of the examples.

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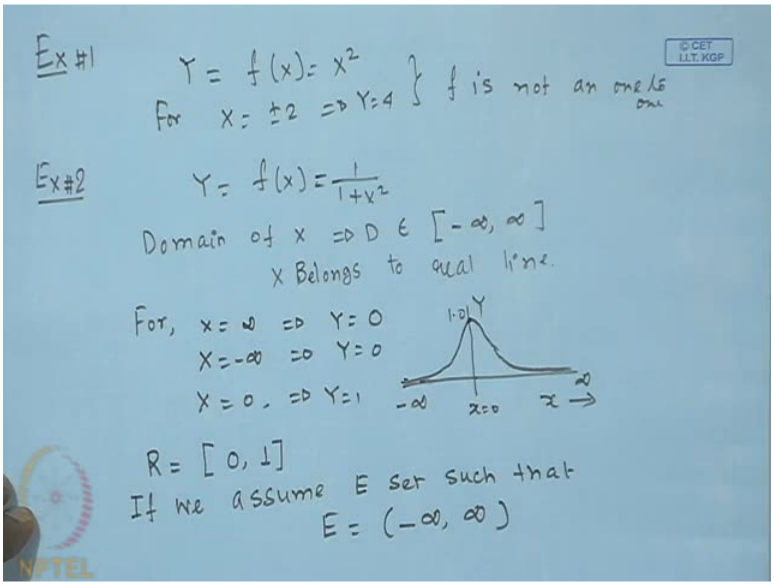

Now, consider the function Y is equal to f of x is equal to X square. The first one you have just talked about for X is equal to plus or minus 2 y is equal to 4. Therefore, f is not a one-to-one function.

Let us look into the second example, Y is equal to f of x is equal to 1 over 1, plus X square. Now, in this case let us say the domain of X- that is D is minus infinity to plus infinity- that means X belongs to real line and domain of X is D is minus infinity to plus infinity.

Now for X is equal to plus infinity; Y is equal to 0 and for X is equal to minus infinity; Y equal to 0, but for X is equal to 0; y equal to 1.

So if you plot Y as a function of x, so in one case it is minus infinity and in another case it is plus infinity and for x is equal to 0, its value is 1. Let us say this value is 1 so the curve looks something like this and this will be having a maximum at x is equal to zero and its value will be asymptotically goes down to 0, when x is approaching plus infinity to minus infinity.

Therefore the range of Y is from 0 to 1 and if we assume the E set; such that E is comprising of minus infinity to plus infinity, then we can see that the range of set R is a subset of E, because E is having minus infinity to plus infinity and on the other hand range is only from 0 to 1.

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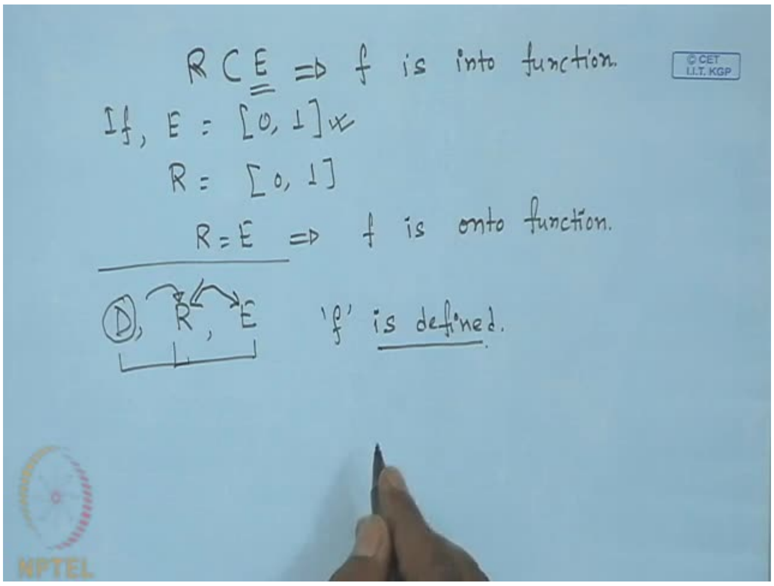

Therefore, R set is a subset of E and therefore on this particular case, f is Into function. If we define the E set as from 0 to 1 and we have already seen if E set is from 0 to 1 and we have already seen the R set is also from 0 to 1; so, in that case R is equal to E. Therefore for this definition of E set, f is an Onto function.

So depending on the value of E set the domain x are basically everything depends on the domain D, range R and the E set whatever we are talking about. So based on these three values- based on the domain of x R will be decided, based on the domain of R the relationship between R and E will be decided and depending on that the function definition, f is defined.

So let us try to conclude this session that we have already seen, whatever we have done till now are we have seen how the metrics, we have first define the linear metric, linear vector, linear space and we define the metric linear space, the norm linear space, the inner product space and we have found what are the various definitions this metric spaces will be obeying and where the assumes will be obeying and given for a discrete domain. For example, in vectors how these three quantities will be computed for the continuous domain like functions, how this three quantities will be computed in case of functions. If it is one-dimensional space, it is two-dimensional space, it is three-dimensional space how these three functions will be computed and what are the physical significance of metric norm and the inner product and how they are calculated for the functions with the continuous domain, for the vectors in the discrete domain and then we moved over to the nature of functions: Into functions, Onto functions and one-to-one functions and we have seen that how we can define the range of the function for it is basically a map of the we talking about a function f, which is a map that takes of a value in x that is in D domain and maps into the y domain that is in the E domain.

And depending on the range of y under domain space of E whether it is a subset of E, whether it is equal or whether it is not a subset depending on that we can define the function one-to-one, Into or Onto

We look into some of the examples of Onto function, Into functions and we will move over to the definition of completeness of space and a cautious sequence which will be very important in Chemical Engineering applications.

Thank you very much.