**Process Control and Instrumentation**
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**Lecture -7**
**Dynamic Behavior of Chemical Processes (Contd.)**

In the last call, we discussed linearization of single variable systems, first we have linearize the non linear equation having single variable using Taylor series expansion and in the next, we represented that linearized model in terms of deviation variables. So, today, we will extend our discussion to linearization of multi variables systems.

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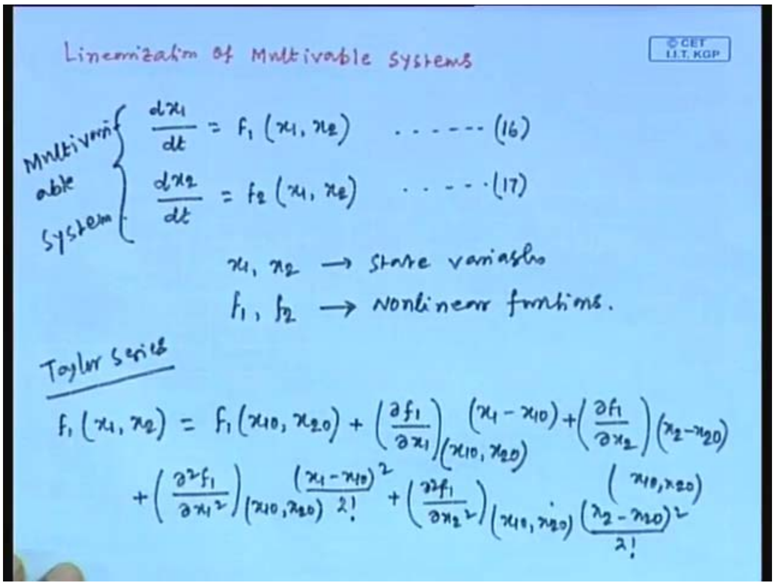


So, today's topic is linearization of multi variable system, the approach is quite similar what we did in the last class. So, today we will consider a multivariable system represented by these 2 equations, d x 1 d t equal to function of x 1 and x 2 and another equation is d x 2 d t, second function is f 2 and that is also the function of 2 state variables one is x 1, another one is x 2 and we will continue the equation numbers.

I mean yesterday, we have considered up to equation number 15. So, today we will start from equation number 16 and this is equation number 17. So, this is the representation of a multi variables system. So, by multivariable system, we mean a system having 2 state variables by multi variable system, we mean a system having 2 state variables, what are these variables, one is x 1, another one is x 2.

And so, these are 2 state variables one is x 1, another one is x 2, 2 functions are also involved one is f 1 and f 2, both these functions are non linear functions, these are non linear functions. So, what we need to. So, we need to linearize these 2 non-linear equations by the use of Taylor series.

Now, according to Taylor series what will be the linearized form of the first function that is f 1 x 1 and x 2, what will be the first term will be f 1 x 1 naught x 2 naught. What will be the next term will be the differentiation f 1 with respect to x 1 and we need to represent this at a particular operating point, that is suppose x 1 naught x 2 naught, it will be multiplied with x 1 minus x 1 naught.

In the next step, we need to differentiate it with respect to second state variable that is x 2. So, with respect to x 1 naught x 2 naught multiplied with x 2 minus x 2 naught, next we have to write the second derivative dou 2 f 1 dou x 1 square at the operating point x 1 naught x 2 naught then whole square of these divider by factorial 2. Similarly the second derivative with respect to dou x 2 x 1 naught x 2 naught multiplied by x 2 minus x 2 naught whole square divided by factorial 2.

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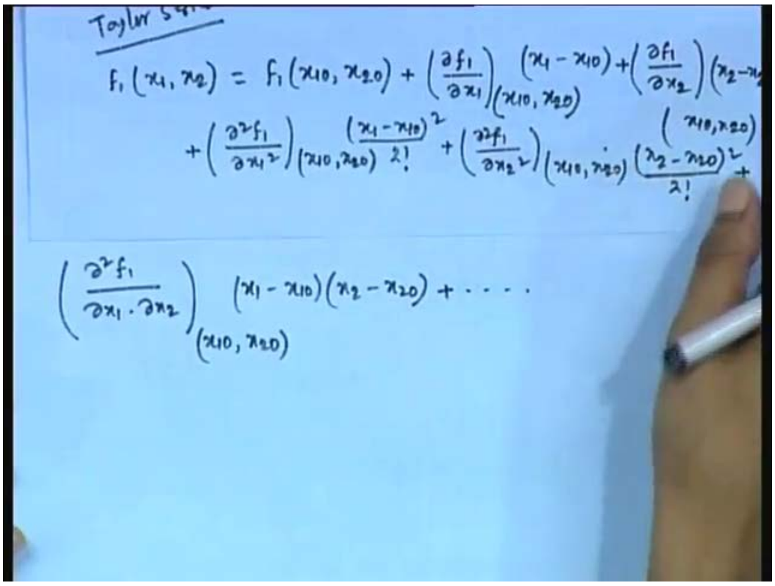

What will be the next term dou 2 f 1 dou x 1 dou x 2 with respect to I mean at the point x 1 naught x 2 naught multiplied by x 1 minus x 1 naught x 2 minus x 2 naught. So, this is the linearized form according to that Taylor series expansion of the non linear function f 1.

Similar expression, we will get for the case second case of f 2, so I am not writing that similar expression, we will get. Now neglecting the order, neglecting the terms of order two and higher what will get.

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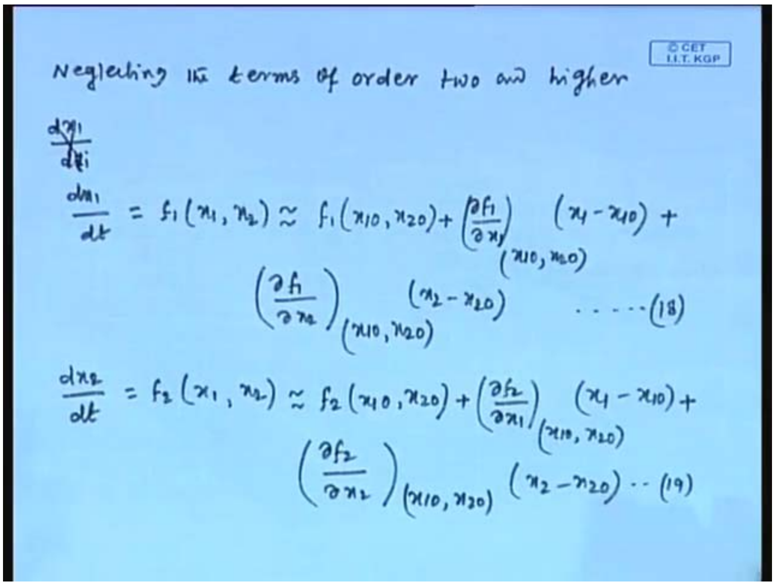


Neglecting the terms of order 2 and higher what will get, if we write from d x 1 d t d x 1 d t. So, it will be f 1 x 1 x 2. Since we are approximating this, so it will be f 1 x 1 naught x 2 naught plus this x 1 naught x 2 naught multiplied by this plus dou f 1 dou x 2 at x 1 naught x 2 naught multiplied by x 2 minus x 2 naught. This is the linearized form and by neglecting the terms of order true and higher.

So, we will give equation number 18 for this, we will use equation 18 for this equation similarly what will be the equation for the second equation. I mean second modeling equation is write, the second equation also d x 2 d t that will be f 2 x 1 x 2 x 1 naught x 2 naught plus dou f 2 dou x 1 at x 1 naught x 2 naught multiplied by x 1 minus x 1 naught plus dou f 2 dou x 2 x 1 naught x 2 naught multiplied by x 2 minus x 2 naught.

So, this is suppose equation number 19, what is the next step, we represent this linearized model in terms of deviation variables. So, what will be the next step, we need to write the equations considering steady state.

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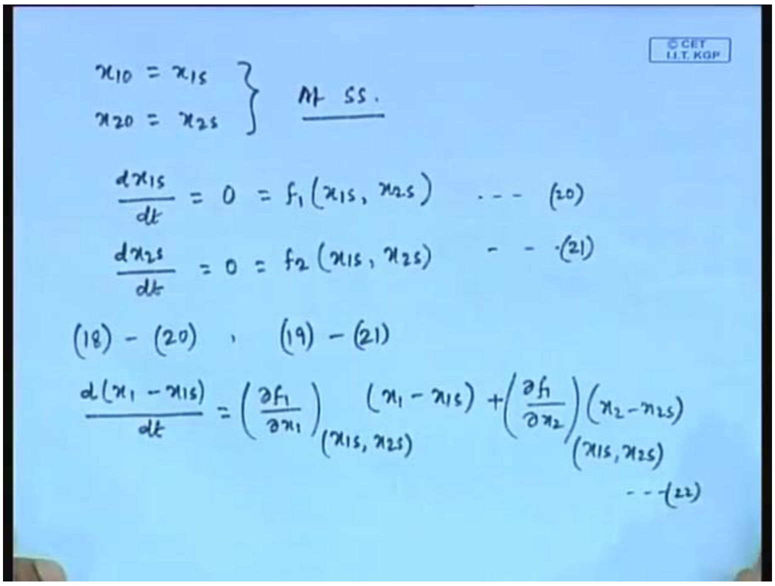


At steady state, we will assume that x 1 naught equals to x 1 s, basically we are replacing the suffix 0 by s. Similarly, we will write x 2 naught equals to x 2 s at steady state, we will consider x 1 s and s 2 s. So, how we can representing the modeling equation at steady state, first equation will be d x 1 s d t x 1 is s is if constant term. So, we can write it as 0, this equals to first function and that is the function of x 1 s and x 2 s. So, this is suppose equation number 20.

Similarly, second equation, we can write d x 2 s d t equal to 0 equal to f 2 x 1 s x 2 s, suppose this is equation number 21. So, what we need to do in the next step to represent in terms of deviation variables, we have to subtract the steady state models from linearized models.

So, next we will just subtract the equation 20 from 18, similarly we have to subtract equation 21 from 19. Basically we are subtracting the steady state modeling equations from linearized modeling equations. So, what equation, we will get for the first state equation x 1 minus x 1 s d t equal to with respect to I mean at x 1 s x 2 s x 1 minus x 1 s plus dou f 1 dou x 2 x 1 s x 2 s x 2 minus x 2 s. So, we will give some equation number say equation 22.

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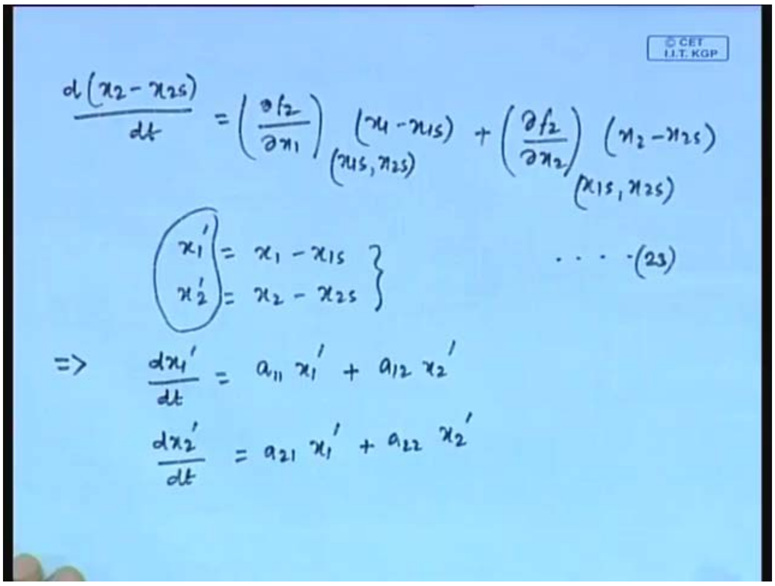


So, similarly we will get for the second state variable that will be d differentiation of x 2 minus x 2 s d t equal to dou x 1 at x 1 s x 2 s multiplied by x 1 minus x 1 s then partial differentiation with respect to second state variable x 2 minus x 2 s. Suppose, this is equation number 23.

Now, we assume the deviation variables one is x 1 prime another deviation variable, we assume x 2 prime, what is x 1 prime x 1 minus x 1 s, similarly x 2 prime is x 2 minus x 2 s, these are 2 deviation variables. Now, if we substitute these 2 deviation variable expressions in equation 22 and 23, what will be the expressions, I mean equations in terms of deviation variables, that we need to find in the next. We will just represent the equations 22 and 23 in terms of x 1 prime and x 2 prime. So, d x 1 prime d t equal to suppose, we using coefficients a 1 1 a 1 2 and for the next expression, we will use a 2 1 x 1 prime plus a 2 2 x 2 prime, if we write in this way.

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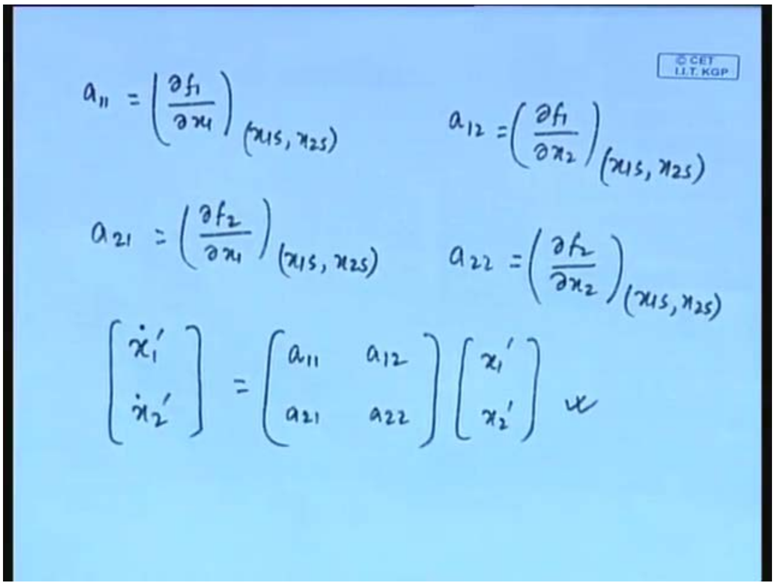


So, what would be the coefficient terms a 1 1 is x 1 s x 2 s a 1 2 is this 1, similarly a 2 1 will be this 1 and forth coefficient is a 2 2, that will be x 1 s x 2 s. So, we got the equations, now we can represent those linearized equations in terms of deviation variables in matrix form, we can represent the final expressions like this.

The dot symbol represents the time derivative, so coefficient matrix a 2 1 a 2 2 multiplied by x 1 prime x 2 prime is this equation. So, this is the linearized form for a multi variable system, in this multivariable system, we have considered only 2 state variables. So, similarly, we can derive the expression for more than 2 variables. Now, we will just take 1 example, which has 2 state variables. So, we have considered previously 1 C S T l example, that will continue to linearize using the Taylor series expansion.

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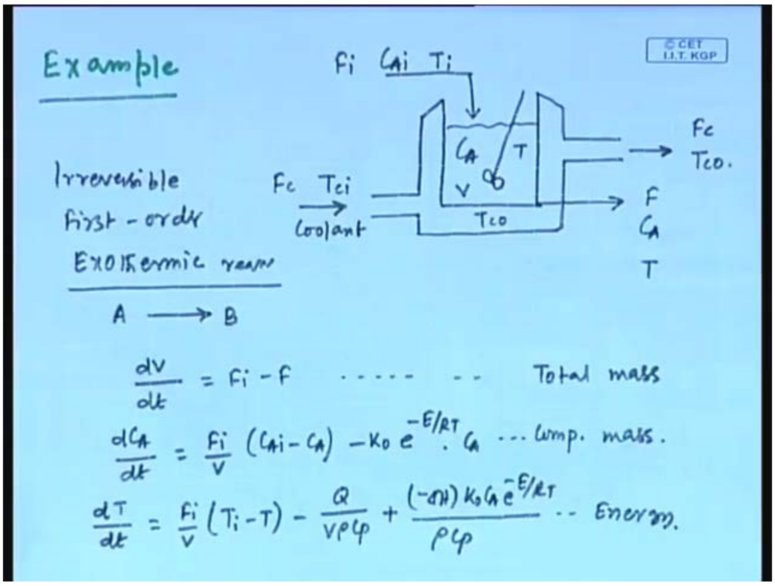


So, next we will consider one example, which we have discussed earlier, this is a jacketed C S T l, the jacket fluid is entering here and it is coming out here. Feed is introduced with a flow rate of f, I concentration of C A i and temperature of T i, suffix i indicates input and product is coming out with a flow rate of F concentration of component A that C A and temperature T. So, this composition of C A temperature is T volume of the reacting mixture, volume of the mixture is suppose B.

Previously, we have considered 1 A reversible first order exothermic reaction. So, irreversible first order exothermic reaction, that is represented by this A is the reactant and B is the desired product. So, remove the exothermic heat from reactor, we need to introduce coolant steam in the jacket. So, this is basically A coolant, which has the flow rate of suppose F c and temperature is T c i, the coolant is coming out at the same flow rate F c and temperature T c naught.

And we are assuming the temperature is T C naught. So, based on the standard assumptions, which we have considered earlier, we got 3 modeling equations. So, we recall these modeling equations one is d v d t equals to F i minus F, F is basically the volumetric flow rate, this equation we got based on total mass balance.

Similarly we got another equation for component mass balance considering component mass balance, we got d C A d t equal F i divided by V C A i minus C A minus K naught exponential of minus e divided by R T into C A. This equation we got considering component mass balance. Third equation, we got considering energy balance.

So, that equation is d t, d t equal F i divided by V T i minus T minus Q divided by V rho C P plus minus of dell h K naught C A exponential of minus e divided by R T, whole divided by rho C P, this equation we got performing energy balance.

You see all these 3 equations are state equations, because these equations have been derived by the application of the conservation principle on fundamental quantities. Now, we have considered the formulation of linearization for 2 variable systems. So, in this particular example, we will consider constant volume. So, that we get 2 variable system. So, in the we will discuss basically the linearization technique step wise.

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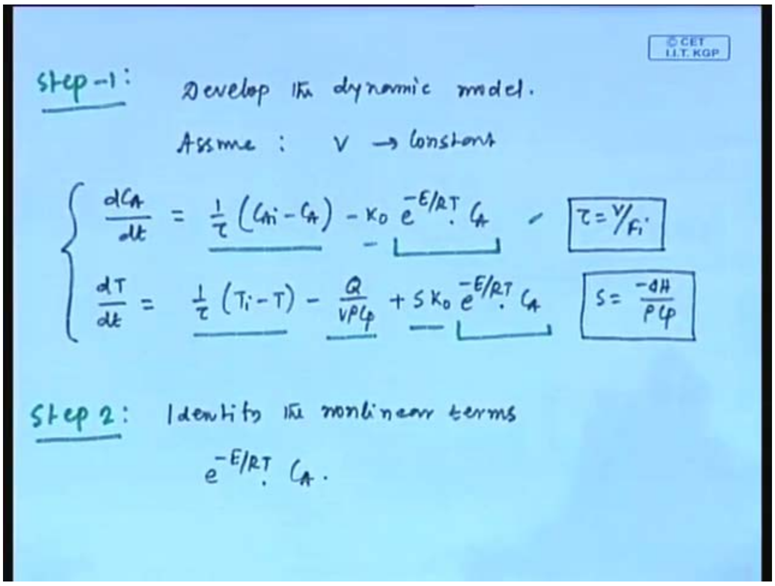


So, in the first step in the first step say this suppose step 1 develop the dynamic model in the first step, we need to develop the dynamic model, we will continue this example c s t r, further assuming constant volume system. So, we are basically assuming here, V is a constant quantity, if that is the case then there will not be in existence of total mass balance. So, the modeling structure will reduce to a structure having 2 equations.

So, we will write the final model structure d C A d t equal to 1 by tau, C A i minus C A minus K naught exponential of minus e divided by R T multiplied by C A, here tau is V divided by F i , tau is V divided by F i. Similarly, we will write another equation that is d

T, d t equal to 1 divided by tau T i minus T minus Q divided by V rho C P plus S K naught exponential of minus E R T C A. So, what is S is minus of dell H divide by rho C P, this is S and you see this S is a constant quantity, similarly tau is also constant. So, this is the final model structure, which we need to develop in step 1.

Next is step 2 next 1 is step 2 in step 2, we need to identify the non linear terms present in the modeling equations. You see in the first equation I mean, which we got based on component balance, the first right hand term, this is linear term k naught is pre exponential factors. So, that is constant

So, only non-linear term is here, this one C A multiplied by exponential of minus E divided by R T, that is only non linear term present in the first equation. Similarly, in the second equation first right hand term that is linear term, second right hand term V rho C P, they are all constant only variable is Q. So, this is also linear term.

In the third right hand term S is constant K naught constant. So, here also the non linear term is exponential of minus E divided by R T multiplied by C A. So, in this model structure, there is only a single non linear term that is exponential of minus E divided by R T multiplied by C A. So, we will basically, linearized this term using the Taylor series expansion. So, that is the third step.

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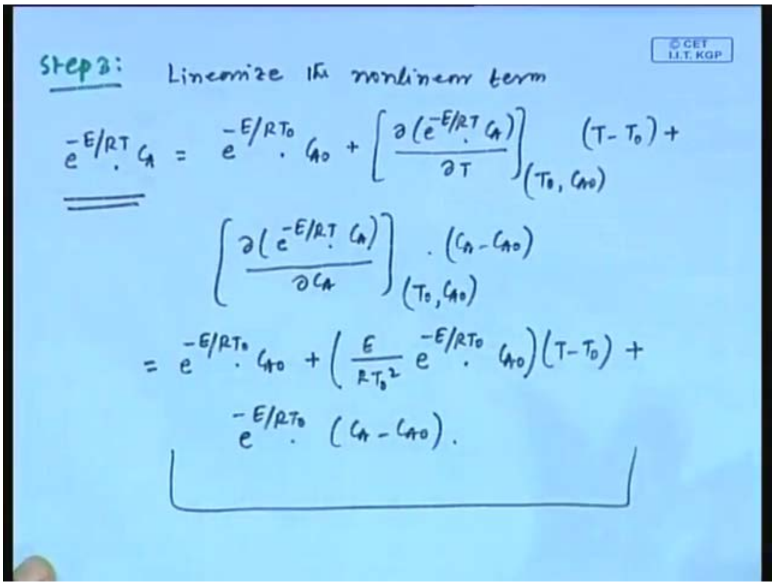

So, step 3 linearize the non-linear term, using Taylor series expansion you linearize it the non-linear term. So, that is exponential of minus E divided by R T multiplied by C A, this is equals to exponential of minus E divided by R T naught multiplied by C A naught then, we will derive this E divide by R T C A with respect to temperature at T naught C A naught multiplied by T minus T naught. Similarly, we will derive we will differentiate with respect to C A, at T naught C A naught and we will multiplied with C A minus C A naught

Here also, we are neglecting the terms of order true and higher. So, what will get finally, we will get exponential minus E divide by R T naught C A naught plus E divided by R T naught square exponential of E divide by R T naught into C A naught T minus T naught. Third term will be exponential E divided by R T naught minus E divided by R T naught into C A minus C A naught. So, in this third step, we have just linearize the non linear tau.

In the next step, I mean step 4, we will substitute this linearized expression in the non linear modeling equations, we will substitute the linearized term of E to the power minus E divided R T C A, in the non linear modeling equation substitute this. In place of this substitute this term, I mean this part.

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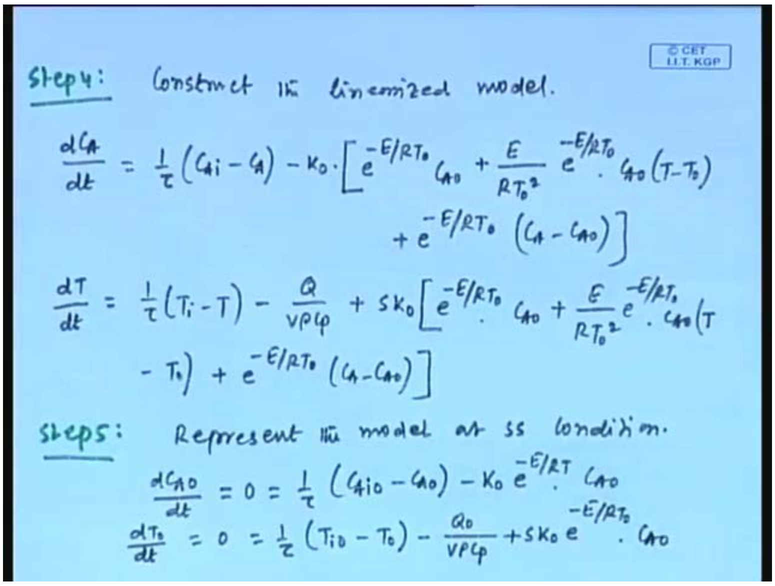


So, in step 4, we need to construct the linearized model. So, for the case of concentration, we will get this equation d C A d t first right hand term is 1 by tau C A i minus C A.

Next term is K naught multiplied by the non linear term. So, K naught exponential of minus E divided by R T naught C A naught plus E divided by R T naught square E to the power, I mean exponential of minus E divided by R T naught C A naught T minus T naught and the last term is exponential of minus E divided by R T naught multiplied by C A minus C A naught, am I correct.

So, for the temperature, I mean d T d t equation will be the first right hand term is 1 by tau in terms of temperature, 1 by tau multiplied by T i minus T and Q divided by V rho C P is there that is also linear term then the third term was S K naught multiplied by the non linear term. So, you will just substitute here, exponential minus E divided by R T naught C A naught then E divided by R T naught square exponential minus E divided by R T naught multiplied by C A naught multiplied by T minus T naught.

Last term is exponential minus E divided by R T naught C A minus C A naught. So, after substituting the linearized expression in the non linear model, we get these 2 equations what is the next step, in the we need to write the modeling equations at steady state, because ultimately, we have to write the equations linearized equations in terms of deviation variables.

So, step 5 is we have to write the modeling equations at steady state. So, represent the model at steady state condition. So, if we represent at steady state then we get this equations, the it was d C A d t. So, we are just adding the suffix 0 to represent the steady state, we are adding suffix 0 to represent the steady state 1 by tau C A i naught minus naught minus K naught exponential minus E divided by R T C A naught, then this is the steady state representation of the component mass balance equation.

Similarly, d t naught d t equal to 0 equal to 1 by tau T i naught minus T naught minus Q naught divided by V rho C P plus S K naught exponential minus E divided by RT naught C A naught, this is the steady state representation of the second modeling equation. So, sometimes, we are using a suffix S, sometimes we are using suffix 0 to represent the steady state.

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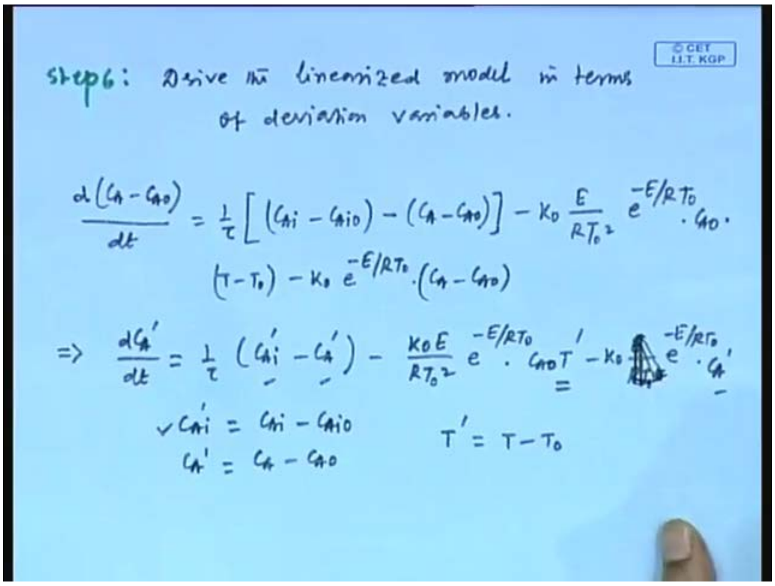


In the final step that is step 6, we have to derive, the linearized model in terms of deviation variables. So, we will write the final expression by subtracting steady state model, steady state modeling equation from linearize modeling equations. So, step 6 is derive the linearized model in terms of deviation variables, if we subtract steady state equations from linearized modeling equations, we get these 2 equations, I am writing those 2 equations.

So, 1 is C A minus C A naught d t equal to 1 by tau C A i minus C A i naught, next 1 is C A minus C A naught. Then K naught E divided by R T naught square E to the power minus exponential of minus E divided by R T naught multiplied by C A naught multiplied by T minus T naught the last term is K naught E to the power minus E divided exponential of minus E by R T naught C A minus C A naught.

If, we subtract the steady state modeling equations from linearized equation, we get this component mass balance equation. So, we will represent this in terms of deviation variables. So, what will be that d C A prime d t equal to 1 by tau C A i prime minus C A prime minus K naught E divide by R T naught square exponential of minus E divided by R T naught C A naught T prime. Last term will be K naught E divide by R T naught square exponential minus E divided by R T naught multiplied by C A prime.

Here, we are considering last term this will not be there no here. So, this will be K naught exponential of minus E divided by R T naught C A prime. So, here we are considering basically C A i prime, that is C A i minus C A i naught, this is 1 deviation variables, we have consider in this equation, another one is C A prime that is C A minus C A naught, another deviation variable is here T prime, that is T minus T naught, so this one and this one. So, 3 deviation variables, we have basically considered. So, similarly we need to write the equation in terms of temperature.

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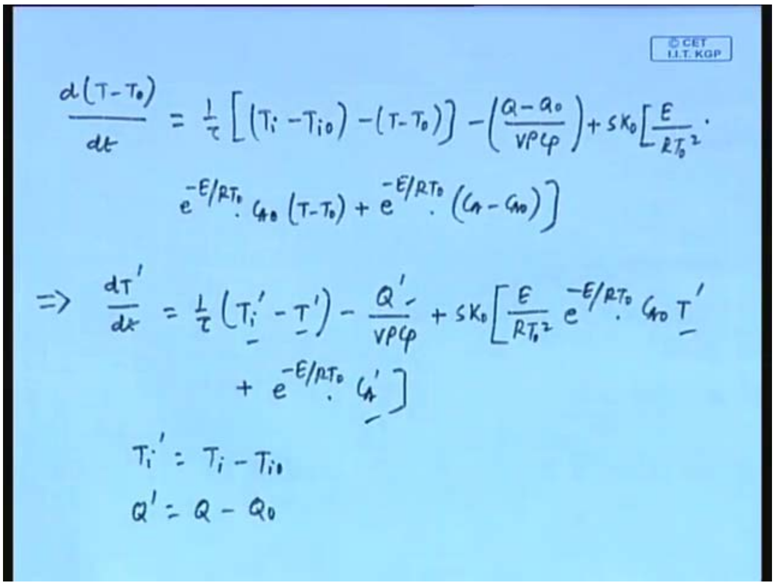


That will be T minus T naught d t equal to 1 by tau T i minus T i naught minus T minus T naught then Q minus Q naught divided by V rho C P plus S K naught multiplied by E divided by R T naught square exponential of minus E divided by R T naught C A naught T minus T naught plus exponential of minus E divided by R T naught into C A minus C A naught.

After subtracting the steady state equation, we get to this. So, if we again substitute the expression of deviation variables, we get d T prime d t equal to 1 by tau T i prime minus T prime minus Q prime divided by V rho C P then S K naught E divided by R T naught square exponential of minus E divided by R T naught C naught C A naught T prime plus exponential of minus E divided by R T naught C A prime.

So, here one deviation variable is T i prime, that is T i minus T i naught, this already we have defined T prime, another one is Q prime that is Q minus Q naught this is define, this is define. So, these are the 2 modeling equations for the example C S T S system in linearized form and in terms of deviation variables. So, next we will discuss the derivation of transfer functions, which is used in process control.

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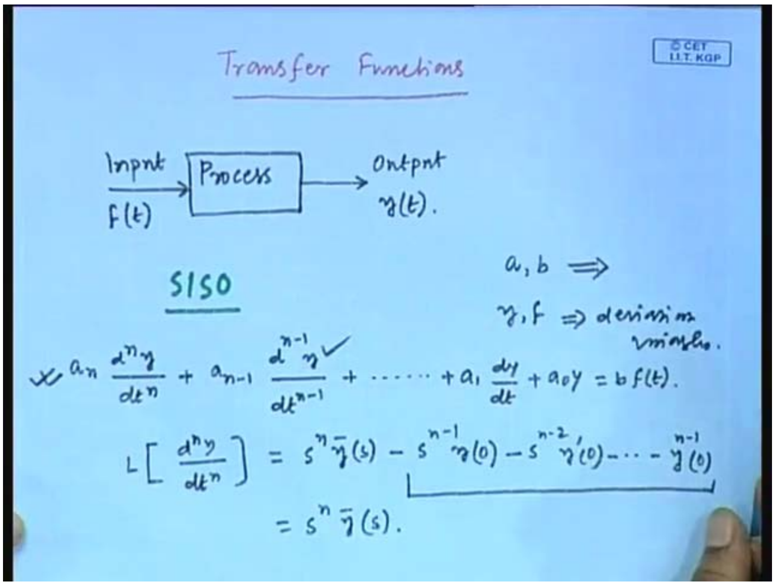


So, next topic is transfer functions, usually the process is I mean the modeling equations are represented in time domain, but the transfer function is represented in S domain Laplace domain. So, that job we need to do to represent the model in terms of deviation variables in terms of in Laplace domain.

So, first we will consider one process having inputs and outputs in time domain, suppose this is the block of a process then, this is the input, which will represent by f and time will use as independent variable. This is the output of the process, which will represent by y t, now here by process, we mean the model structure of the process, we will use the modeling equations of a process to derive the transfer functions.

So, we will consider one example first, which is a single input, single output system first, we will consider, we will first, we derive the transfer function for a single input single output system that is S I S O. So, a single input single output process, we can represent by this form, which is definitely linear or linearized version, a single input single output system, we are representing by nth order linear or linearized equation n minus 1 y d t n minus 1, we are trying to formulate in general. That is why, we are we have consider nth order equation, differential equation a 1 d y d t plus a naught y equal to b f t.

So, y is a output, f is a input a and b, they are the coefficients and what else, t is a independent variable. So, as I mention that the transfer function should be represented in S domain. So, we have to take Laplace transform of this equation. What is the Laplace transform I mean, if we consider the Laplace transform of d n y d t n, this is equals to S to the power n y bas S minus S n minus 1 y naught minus S n minus 2 y prime naught like this last term is y n minus 1 naught. This is the Laplace transform of nth order derivative.

Now, if we consider in this equation, in this process model y and f both are the deviation variables, these 2 are deviation variables, if that is the case, I mean, if y is the deviation variable then definitely all these terms become 0. So, we can write this as S n y bar S similarly, we can consider another derivative. So, if we take the Laplace transform of this linear or linearized nth order differential equation.

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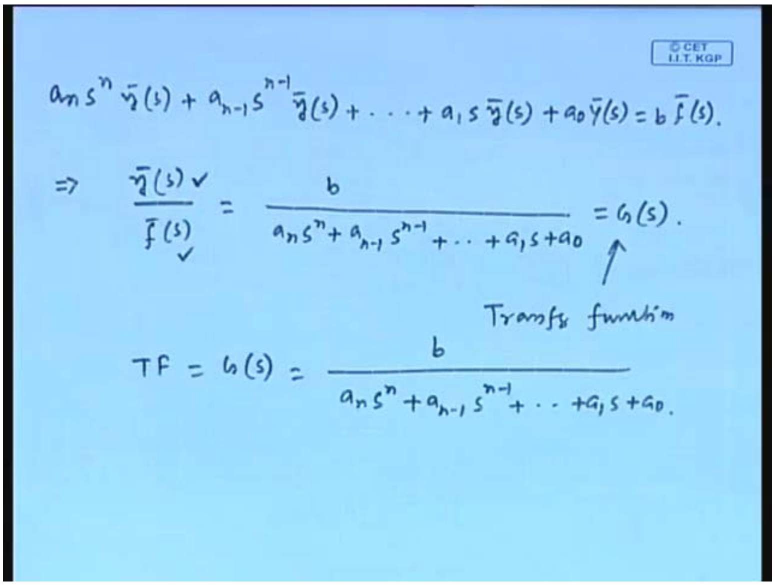


We get a n S to the power n y bar S plus a n minus 1 S to the power n minus 1 y bar S like this a 1 S y bar S plus a naught y bar S equal to b f bar S can, we write this. If we take the Laplace transform of the linear or linearized nth order differential equation, we get this.

Now, we can rearrange this equation like this equal to b divided by a n S to the power n a n minus 1 S to the power n minus 1 a 1 S plus a naught. This is the output in S domain, this the input in S domain, this is the right hand side, this right hand side is basically the transfer function, which is represented by G S, this is the transfer function, I mean the transfer function is represented by G S and that is b divided by a n S to the power n a n minus 1 S to the power n minus 1 a 1 S plus a naught.

So, this transfer function basically represents the linear or linearized input output model in terms of deviation variables, you see this is the input output model. So, transfer function represents the linear or linearized input output model in S domain Laplace domain. So, we have represented using the block the processing time domain, now we need to represent in terms of Laplace domain the whole process. How we can represent it, it is very simple.

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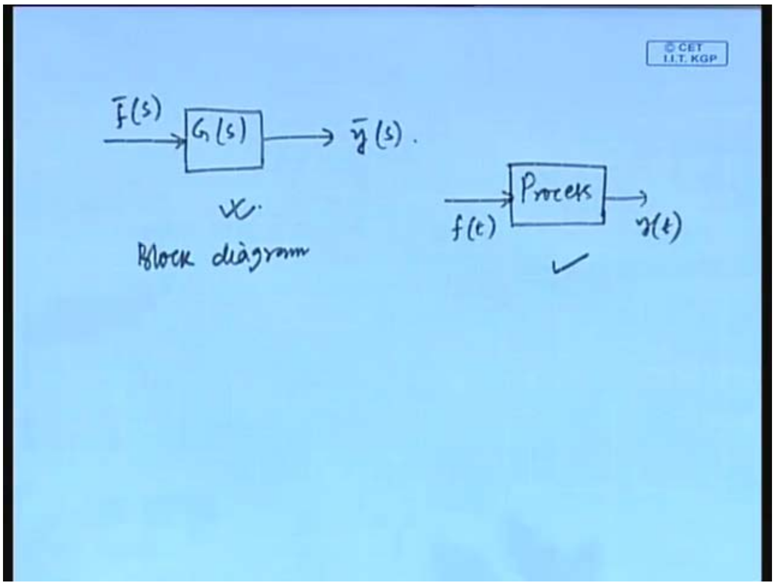


We will represent the process by G S here, our input is we will write in s domain our output is in S domain, you recall the representation in time domain. Now, will Laplace domain, I mean S domain, we will represent in this way. So, this is the block diagram, this is the block diagram of this process. So, this is the formulation of transfer function for a single output single input, S I S O system and in the next day, we will discuss multi input multi output, multi input single output transfer functions.

Thank you.