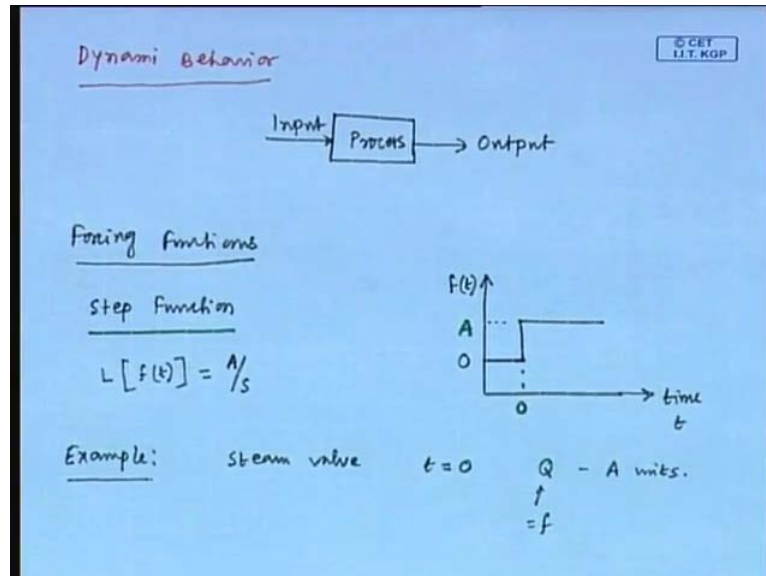


Process Control and Instrumentation
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Lecture - 6
Dynamic Behavior of Chemical Processes

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We will discuss the Dynamic Behavior of Chemical Processes. I have mentioned in the last class about this in brief, I mean what is transient behavior, how we can get the transient behavior. So, we will start from that only, we have a chemical process, this is input this is output, if we give a change in input variable, we will get some change in output variable. And that is the dynamic behavior and that we will observe in terms of the change in the output variable.

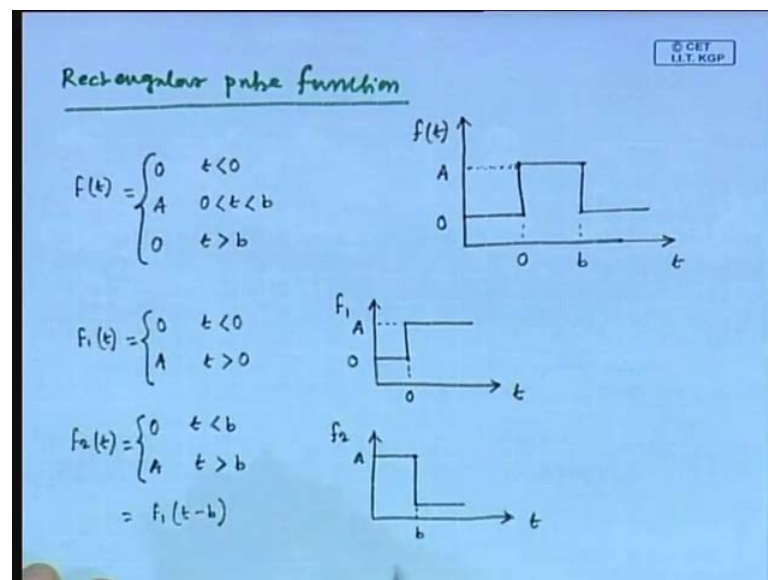
So, we will consider the different forcing functions, first we will start with step function. So, first I am drawing the step function, what it how it looks the function, we will represent by f . So, this is f t and this is time t , initially, if the process is at steady state condition, I mean there is no change. Now, we are changing the input variable in this way. So, initially it was 0 definitely, we are considering here f as A deviation variable, that is why that is 0. F we are considering as a deviation variable here, f is not function here, f is variable note this point

So, since f is the deviation variable, that is why at steady state, f equal to 0. Now, we are changing this, f value at suppose time t equals to 0 and the magnitude of this change is suppose A , this is a step function, which we have represented graphically here. Now, how we can represented mathematically, f t equals to A that is a step function. So, if we write the Laplace transform of this function f t , what would be the Laplace transform of this A by S .

So, the Laplace transform of a step function is A by S where, A is the magnitude of that step function, what is the example of this step function. Suppose, we have a steam valve, we have one steam valve. Now open the steam valve at t equals to 0, such that Q changes by A units, we have a steam valve open the steam valve at time t equals to 0.

So, that Q changes by A units, here Q is the steam flow rate and that is basically, f fine this Q is basically f . So, if we change the Q by A units at t time equals to 0, that is the example of a step input. So, we can change the load variable, disturbance variable by this fashion to observe, the transient behavior of the process, this is the first type of forcing functions, which is commonly used.

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Second forcing function is rectangular pulse function, first we will draw the figure of this rectangular pulse function so, this is f t and this is time t , initially if f t is 0. Now it changes at time t equals to 0 by unit of suppose A , then it remains there up to time b then it returns to the old value, this is the graphical representation of a rectangular pulse

function. Initially $f(t)$ is 0 that means, the process is at steady state then at time t equals to 0, the input function changes by magnitude A , then it remains, there for a time duration of b and finally it returns to the old value.

Now, how you can represent it mathematically, $f(t)$ equals to 0 and time t less than 0, $f(t)$ equals to A , when time is in between 0 and b and again this is 0, when time is greater than b . So, this is the mathematical representation of this pulse function, this pulse function is basically the combination of 2 input functions, this is the combination of 2 input functions. So, what are these 2 input functions, if we consider one function is f_1 and another is f_2 then the graphical representation of f_1 is like this initially it is at 0 level, this is 0 and this quantity is A .

So, $f_1(t)$ equal to 0, when t is less than 0 and this is equal to A , when t is greater 0, another function is f_2 and this is suppose time t initially, it is at the value of A then it is coming back to this value and this time is b . So, $f_2(t)$ equals to 0, when t is less than b , $f_2(t)$ equals to A , when time is greater than b can, we write this for the function second function f_2 can, we write this, we are considering this is 0.

So, this is a initial value, we are considering 0 and this we are considering A that means, this is $f_1(t)$ minus b , here we are considering increasing, I mean it is increasing the step, here we are considering decreasing, initial value is 0 and this quantity is A by that way, we can write f_2 equals to $f_1(t)$ minus b .

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$$\begin{aligned}
 f(t) &= f_1(t) - f_1(t-b) \\
 \mathcal{L}[f(t)] &= \mathcal{L}[f_1(t) - f_1(t-b)] \\
 &= \bar{f}_1(s) - e^{-bs} \bar{f}_1(s) \\
 &= \frac{A}{s} (1 - e^{-bs}) \dots \text{Final form.}
 \end{aligned}$$

Now, $f(t)$ is a combination of these 2, $f_1(t) - f_2(t)$ minus sign is included here, because for f_1 function, we have considered that increase of step. But, for the case of second function, we have consider the decrease of step, that is why negative sign for f_2 and positive sign for m_2 sorry, this is f_1 .

Now, we will take Laplace transform of this, Laplace transform of $f(t)$. So, Laplace transform of $f_1(t) - f_2(t) - b$, this is equal to $f_1(s)$ and for this, we can write e^{-bs} to the power minus b $f_1(s)$. So, $f_1(s)$ is $\frac{1}{s}$ by A , $f_2(s)$ is $\frac{1}{s}$ by A , that is the step function only with the magnitude of A . So, $A \left(\frac{1}{s} - e^{-bs} \frac{1}{s} \right)$, this is the Laplace transform of a rectangular pulse function, this is a final form for the pulse function.

Now, what is the example of the pulse function, if we consider the steam valve, open the steam valve at t equals to 0. So, that Q changes by A units, hold at the new valve for the duration of b then return to the old valve, you just try to remember that configuration, I mean the graphical representation of pulse function again, I am telling open the steam valve at time t equals to 0. So, that Q changes by A units hold at the new valve for the time duration of b and return to the old valve, this is a example of a pulse function. Now, we will just modify that figure of pulse function, previously we have consider the time duration of b .

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$A \cdot \frac{1}{A} = 1$
 Unit pulse function, $\delta_A(t)$.

$\lim_{A \rightarrow 0} \left(A \cdot \frac{1}{A} \right) = 1$
 $\int_{-\infty}^{\infty} \delta(t) dt = 1$

Graph 1: Unit pulse function $\delta_A(t)$. The vertical axis is $F(t)$ and the horizontal axis is time t . The function is zero for $t < 0$ and $t > A$, and has a constant value of A for $0 < t < A$.

Graph 2: Unit impulse function (Dirac delta function), $\delta(t)$. The vertical axis is $f(t)$ and the horizontal axis is t . The function is zero for $t < 0$ and $t > 0$, and has an infinite value at $t = 0$.

Now, we will consider the time duration of A, this is $f(t) = 0$, initially then this is 0 and this time step is suppose A and we are considering here 1 by A, this is time t. So, what is the area is 1, A multiplied by 1 by A that equals to 1 and this is called unit pulse function, it is conventionally represented by $\delta_A(t)$. So, this is unit pulse and it is represented by $\delta_A(t)$

Now, we use to consider limit A tends to 0 A multiplied by 1 by A, how much is this 1, we are just approaching 0, we are approaching the time 0. So, how much it will be then 1, how we can represent it graphically, how we can represent this situation, graphically. The graphical representation of this is somewhat like this, this is time t equals to 0, this is 0, this function is called unit pulse function sorry, unit impulse function.

It is also called dirac function and it is conventionally, represented by $\delta(t)$, this is called dirac function or unit impulse function and it is represented by $\delta(t)$. So, what is the definition of this unit impulse function, it is defined as equal to 0 for all times except for time t equals to 0. So, we can represent this again by minus infinity to infinity $\delta(t) dt$ equals to 1, if we consider minus infinity to infinity then $\delta(t) dt$ equal to 1 within this range within this time range. Now, what will be the Laplace transform of this unit impulse function.

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$$\begin{aligned}
 L[\delta(t)] &= L\left[\lim_{A \rightarrow 0} \delta_A(t)\right] \\
 &= \lim_{A \rightarrow 0} \int_0^{\infty} \delta_A(t) e^{-st} dt = \lim_{A \rightarrow 0} \int_0^A e^{-st} dt \\
 &= \lim_{A \rightarrow 0} \left[\frac{1 - e^{-sA}}{s} \right] \\
 &= \frac{1}{s}
 \end{aligned}$$

Laplace transform of this $\delta(t)$ equal to Laplace transform of the unit pulse function in addition, I mean, if we consider limit a tends to then we can write here, unit pulse

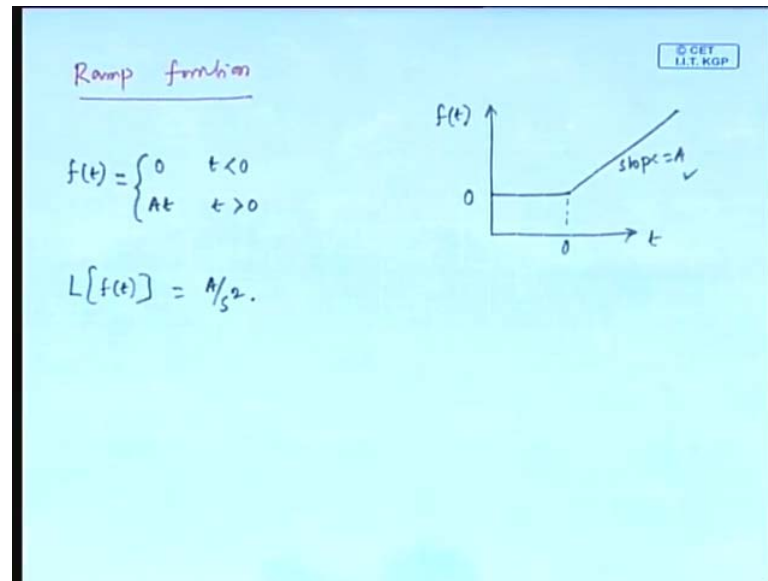
function. So, this is unit pulse function that means, integration 0 to infinity limit A tends to 0, $\int_0^A e^{-st} dt$, we are just considering the definition of Laplace transform. Integration of 0 to infinity limit A tends to 0 and this is the Laplace transform of unit pulse function.

We will just take out the limit A tends to 0, integration 0 to infinity $\int_0^A e^{-st} dt$. This is equal to limit a tends to 0, this part, we have already calculated how much is that $1 - e^{-sA}$ divided by sA , we have basically, consideration in the previous example the time duration b, but in this example, we are considering A.

So, you just replace b by A and in the unit pulse in the discussion of pulse function, we have consider the magnitude that is A, here we are considering 1 by A. So, we have derived the Laplace transform of pulse function, that is $A(1 - e^{-sA})/s$, we need to replace b by A and A by 1 by A then, we will get this equation, so how much it will be 1, how by the use of el hospital's rule.

So, this is the Laplace transform of the unit impulse function can you give any example of this unit impulse function, I mean we can continue the same example say, we have 1 stream valve can you give the example of this impulse function. It is impossible to realize perfectly, but we can give the example like this open the stream valve and instantaneously return to the old value open the stream valve at t equal to 0 and as soon as physically possible. There after return to old value, this is the example of impulse function.

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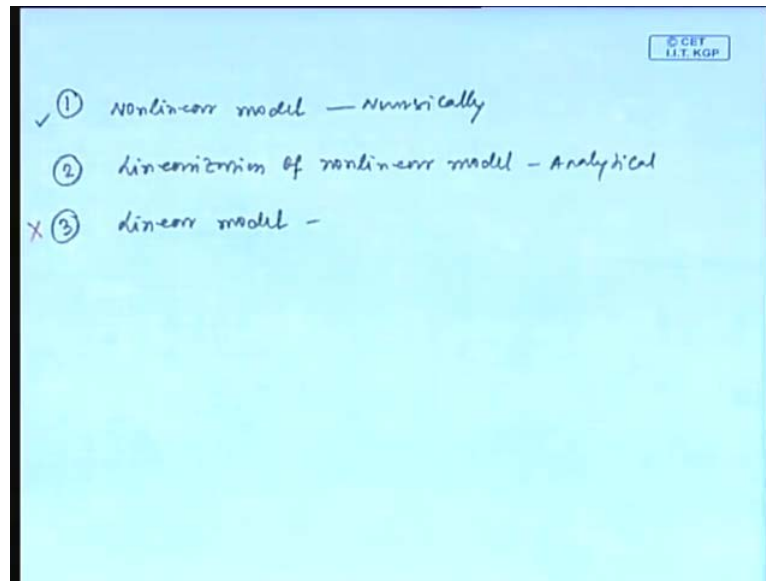


Another input function is also used in process control that is ramp function that is ramp function, we can represent this function in this way time t initially, it is 0 then it changes like this, suppose the slope is A . Initially, it is 0 and at time t equals to 0, it changes linearly with slope A . So, if $f(t)$ is ramp function then, it is 0, when t less than 0, it is $A t$ when t greater than 0 and the Laplace transform of this is A by S square. So, what will be the example gradually open the stream valve.

So, that Q increases linearly, gradually increase the stream valve. So, that Q increases or Q changes linearly and ramp ends, when the valve is fully open. So, these are the forcing functions, which are used to observe the transient behavior, definitely in presence of controlled scheme, but presently, we are not discussing the control part, we are just discussing the simulation results, I mean the model output.

Next, we will discuss the different how to solve the different model structures, practically most of the chemical processes are non linear in nature, almost all chemical processes, so non linear dynamics. Now analytical solution of non linear equations are not available.

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So, how we can proceed, one option is we can solve the modeling equations numerically, if we have the non linear equations, non linear model, we can solve it numerically. So, keeping the numerical simulation of the modeling equations, we can proceed for the development of non linear model equations. Another option is we can linearize the non linear model. So, we can linearize the non linear model. So, linearization of non linear equations or non linear model and we can go for analytical solution, if we can linearize the non linear equations, we can go for analytical solution.

Third option is we can develop the linear model for the non linear processes and we can solve the linear equations again analytically. So, there are 3 options, one is development of non linear model and go for solution by the use of numerically techniques, another one is linearize, the non linear equations and follow the analytical solutions and third one is develop the linear model structure and simulate analytically. But, this third option was used in the past. Now a days it is not used, due to the advancement of different mathematical techniques.

So, this option is quite old. So, we will not discuss any more this in the first option, we have option to model the process by the development of non linear equations and the solution of those equations numerically, I hope you have studied the numerical simulations in your mathematics courses. So, we will not discuss about that in more details, I mean, we do not want discuss in this course the numerical simulation of non-

linear equations. Second option is linearization of non linear model structure, so that we will discuss in brief.

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Linearization of Single-variable systems

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$$\frac{dx}{dt} = f(x, t) \quad \text{----- (1)}$$

$$\frac{dx}{dt} = f(x) \quad \text{----- (2)}$$

Taylor Series

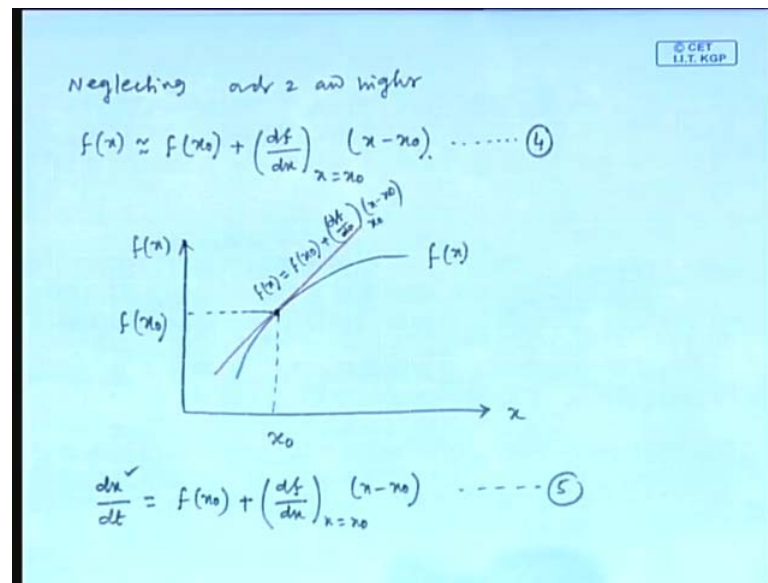
$$f(x) = f(x_0) + \left(\frac{df}{dx}\right)_{x=x_0} (x-x_0) + \left(\frac{d^2f}{dx^2}\right)_{x=x_0} \frac{(x-x_0)^2}{2!} + \dots +$$

$$\dots + \left(\frac{d^n f}{dx^n}\right)_{x=x_0} \frac{(x-x_0)^n}{n!} + \dots + \dots \quad \text{----- (3)}$$

So, first we will start the linearization of single variable system, we will consider a simple ordinary differential equation represented by $\frac{dx}{dt} = f(x, t)$ equal to function of x and t , suppose this is equation number 1, here x is the only variable, I mean dependent variable and t is the independent variable that means, x is also varies with time. So, we can omit the time t within this function, we can write in this form also, this is suppose equation number 2.

Now, we will linearize this function f according to Taylor series expansion. So, we will use the Taylor series to linearize $f(x)$ accordingly, if $f(x)$ equal to $f(x_0)$ here, x_0 is the reference x , I mean reference value of x plus $\left(\frac{df}{dx}\right)_{x=x_0} (x-x_0)$ plus $\left(\frac{d^2f}{dx^2}\right)_{x=x_0} \frac{(x-x_0)^2}{2!}$ plus \dots plus $\left(\frac{d^n f}{dx^n}\right)_{x=x_0} \frac{(x-x_0)^n}{n!}$ plus \dots like this.

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This is the Taylor series expansion of $f(x)$ neglecting the term of order 2 and higher, we get $f(x)$ is approximately equal to $f(x_0) + \left(\frac{df}{dx}\right)_{x=x_0} (x - x_0)$, you give this Taylor series equation, as equation number 3, the Taylor series expansion of $f(x)$, you put equation 3 for that and this approximate linearization, we are representing by equation 4.

Now, this linear approximation is satisfactory, when x is very close to x_0 , this is $f(x)$, this is x . Suppose this is the function $f(x)$, this point is x_0 and corresponding $f(x_0)$, now if, we linearize with respect to this point. So, this is basically, if this linear function is represented by $f(x) = f(x_0) + \left(\frac{df}{dx}\right)_{x=x_0} (x - x_0)$.

So, from this graphical representation, it is very clear that the linear approximation is satisfactory, when this x is very close to x_0 , if we move away from this x_0 this part is not linear at all fine. So, the deviation increases as we move far from x_0 to x_0 .

Now, we can write equation 4 like this way also $\frac{dx}{dt} = f(x_0) + \left(\frac{df}{dx}\right)_{x=x_0} (x - x_0)$. Because, $f(x) = \frac{dx}{dt}$ that is our equation number 2, we have just replaced $f(x)$ equals to $\frac{dx}{dt}$ in equation 4 and then we get equation 5, this is a linearization of $f(x)$ according to Taylor series expansion.

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Liquid Tank System

$A \frac{dh}{dt} = F_i - f_o \dots$ Total mass bal. ----- (6)

✓ Case 1: $f_o \propto h \Rightarrow f_o = \beta h$

Eq. (6): $A \frac{dh}{dt} + \beta h = F_i \dots$ linear ODE.

Case 2: $f_o \propto \sqrt{h} \quad f_o = \alpha \sqrt{h}$

Eq. (6): $A \frac{dh}{dt} + \alpha \sqrt{h} = F_i \dots$ (7) ✓

Now, we will take 1 example to linearize the model of that particular example. So, we will consider the example of liquid tank system, this is the schematic of the liquid tank system, liquid height and cross sectional of the tank input flow rate is f_i and outlet flow rate is f_o . We have derived earlier the modeling equation for this system. The modeling equation is $A \frac{dh}{dt} = F_i - F_o$, this equation, we can derive based on total mass balance.

Now, this outlet flow rate is basically the function of liquid height the outlet flow rate F_o depends on the liquid height in the tank, we will consider 2 cases. In first case, we will consider F_o is proportional to h anyway, you give this as equation number 6. In the first case, we will consider F_o is proportional to h that means, F_o equals to βh , if we substitute this in equation 6, I mean if we substitute F_o equals to βh in equation 6 then, we get $A \frac{dh}{dt} + \beta h = F_i$. If we substitute F_o equals to βh in equation 6, we get this form and this is a linear ordinary differential equation. So, there is no need of any linearization, we can solve it analytically.

In another case, we will consider this F_o is proportional to root h , F_o is the square root of liquid height in the tank that means, F_o equals to say $\alpha \sqrt{h}$. Now, if we substitute again F_o equals to $\alpha \sqrt{h}$ in equation 6, we have $A \frac{dh}{dt} + \alpha \sqrt{h} = F_i$, if we substitute F_o equal to $\alpha \sqrt{h}$ in

equation 6, we get equation 7, this is not a linear ordinary differential equation. So, which non linear term is present in this equation, the non linear term is alpha root h. So, we will linearize this alpha root h term using the Taylor series expansion, we will linearize this alpha root h term, using the Taylor series expansion and then we substitute that in equation 7.

(Refer Slide Time: 40:08)

The image shows a handwritten slide titled "Taylor series" with a logo in the top right corner that says "GET I.T.KGP". The main content is a Taylor series expansion of the function $\alpha \sqrt{h}$ around a point $h = h_0$. The expansion is given as:

$$\alpha \sqrt{h} = \alpha \sqrt{h_0} + \left[\frac{d}{dh} (\alpha \sqrt{h}) \right]_{h=h_0} (h-h_0) + \left[\frac{d^2}{dh^2} (\alpha \sqrt{h}) \right]_{h=h_0} \frac{(h-h_0)^2}{2!} + \dots$$

Below this, it says "Neglecting terms of order 2 and higher," and then shows the linearized form:

$$\alpha \sqrt{h} = \alpha \sqrt{h_0} + \frac{\alpha}{2\sqrt{h_0}} (h-h_0). \checkmark$$

Then, it presents "Eq. (7):" which is a differential equation:

$$A \frac{dh}{dt} + \frac{\alpha}{2\sqrt{h_0}} h = F_i - \frac{\alpha}{2} \sqrt{h_0} \quad \dots \text{--- (8)}$$

At the bottom, it notes that $h = h_0$ and that "Eq. (7) - nonlinear" and "Eq. (8) - linear" are "Same".

So, according to Taylor series alpha root h equal to alpha root h naught plus differentiation of alpha root h at point h naught, here h naught is the reference height, h minus h naught. Similarly, if the second term, the second derivative will be like this at h equals to h naught h minus h naught whole square divided by factorial 2. We neglect the terms of order 2 and higher, neglecting terms of order 2 and higher, we get alpha root h equals to alpha root h naught plus alpha divided by 2 root h naught multiplied by h minus h naught. This is the final form of alpha root h after linearization.

Now, we will substitute this alpha root h term in equation 7 substitute it, if we substitute in equation 7, then we get A d h d t plus alpha divided by 2 root h naught multiplied by h equal to F i minus alpha by 2 root over of h naught. This is suppose equation number 8. So, this is the linearized form of the non linear model or non linear modeling equation.

Now, if we substitute h equals to h naught, if we substitute h equals to h naught equation 7 and 8, they are identical, you see if we substitute h equals to h naught equation 7 and equation 8. They are same, equation 7 represents the non linear form and equation 8

represents the linear form exactly at x_s , they are identical. So, that is why, we have mentioned earlier that the linear approximation is satisfactory, when x is very close to x_s . So, this is the linearization. Now, we will try to represent the model in terms of deviation variables.

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The image shows a handwritten derivation on a blue background. At the top right, there is a small logo for '© CEF I.I.T. KGP'. The text is as follows:

Deviation variables

$$\frac{dx_s}{dt} = 0 = f(x_s) \quad \dots \dots (9)$$

Taylor series

$$f(x) = \frac{dx}{dt} = f(x_s) + \left(\frac{df}{dx} \right)_{x=x_s} (x - x_s) \quad \dots (10)$$

$$(10) - (9) \quad \frac{d(x - x_s)}{dt} = \left(\frac{df}{dx} \right)_{x=x_s} (x - x_s) \quad \dots (11)$$

$$x' = x - x_s.$$

$$\text{Eq. (11):} \quad \frac{dx'}{dt} = \left(\frac{df}{dx} \right)_{x=x_s} x' \quad \dots (12)$$

We will represent this model in terms of deviation variables, how we can do that at steady state x is basically represented by x_s , x at steady state, we will represent by x_s , if x equals to x_s at steady state then we can write dx/dt equal to 0, equal to function of x_s see the steady state value of x is x_s x_s is basically constant. That is why the derivative of that constant term is 0 and this is the function of x , now we are replacing x by x_s that is why that is $f(x_s)$, this is say equation number 9.

Now, according to Taylor series expansion, we can write $f(x)$ just by replacing x by x_s dx/dt equals to $f(x_s)$ plus df/dx at x equals to x_s x minus x_s the linearization of $f(x)$, we got earlier here, we have just substituted x equals to x_s and we get equation 10. Now, we will just subtract, we will subtract equation 9 from equation 10 then we can write differentiation of x minus x_s dx/dt equal to df/dx at x equals to x_s multiplied by x minus x_s , this is equation number 11.

In the next, we will consider one deviation variable that is represented by x' equal to x minus x_s x is the deviation variable and that is equal to x minus x_s then equation 11 becomes dx'/dt equal to df/dx at x equals to x_s x' . Simply, we

have substituted in place x minus x prime sorry, we have substituted in place of x minus x s x prime then, we get equation 12. Now, we will continue that example again that liquid tank system.

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linearized form (liquid tank).

$$A \frac{dh}{dt} + \frac{\alpha}{2\sqrt{h_s}} h = F_i - \frac{\alpha}{2} \sqrt{h_s} \quad \dots (13) \checkmark$$

$$SS: A \frac{dh_s}{dt} + \alpha \sqrt{h_s} = F_{iS} \quad \dots (14) \checkmark$$

(13) - (14)

$$A \frac{d(h-h_s)}{dt} + \frac{\alpha}{2\sqrt{h_s}} (h-h_s) = F_i - F_{iS}$$

$$A \frac{dh'}{dt} + \frac{\alpha}{2\sqrt{h_s}} h' = F_i' \quad \dots (15)$$

$$\left. \begin{array}{l} F_i' = F_i - F_{iS} \\ h' = h - h_s \end{array} \right\}$$

So, previously we got the linearized form for the liquid tank system, we got the linearized form in equation 8. So, the linearized form is $A \frac{dh}{dt} + \frac{\alpha}{2\sqrt{h_s}} h = F_i - \frac{\alpha}{2} \sqrt{h_s}$ say this is equation number 13, we get equation 13 from equation 8 directly.

So, for the liquid tank system, this is the linearized form at steady state, this equation 13 yields $A \frac{dh_s}{dt} + \alpha \sqrt{h_s} = F_{iS}$ to represent the steady state, we have included suffix S with all variables. Then we get $A \frac{dh_s}{dt} + \alpha \sqrt{h_s} = F_{iS}$ now, we will just subtract equation 14 from equation 13, if we subtract equation 14 from equation 13, we get a differentiation of $h - h_s$ plus $\frac{\alpha}{2\sqrt{h_s}} (h - h_s) = F_i - F_{iS}$. Now, we will define 2 deviation variables, one is F_i' that is $F_i - F_{iS}$, another deviation variable, we will define that is h' equal to $h - h_s$, these are the 2 deviation variables.

Now, if we write the above equation in terms of deviation variables, we finally get $A \frac{dh'}{dt} + \frac{\alpha}{2\sqrt{h_s}} h' = F_i'$, this is suppose equation number 15. So, for the liquid tank system after linearization, we got one equation and in terms of deviation variables, we got equation 15. So, this is basically, the

linearization of non linear equations for a single variable system, I mean only one non linear equation is involved in this example. In the next, we will discuss the linearization of multivariable systems.

Thank you.