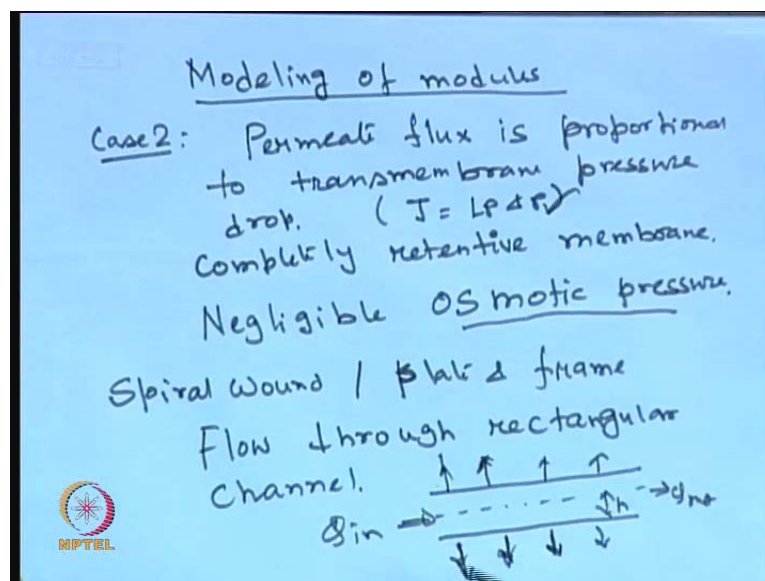


**Novel Separation Processes**  
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**Lecture No. 14**  
**Membrane Separation Processes (Contd.)**

Well now we will start with the module modeling and go to the next complication to be incorporated in the modeling of the modules.

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We have finished the simplified case number 1. Now, we will go to case number 2. In case number 2 we had basically talking about permeate flux. In the earlier case we can say permeate flux was constant. Now, we are talking about permeate flux is proportional to trans membrane pressure drop. Pressure drop and completely retentive membrane, that means  $c_P$  is equal to 0 and permeate flux is proportional to pressure drop means  $J$  is equal to  $L_P \Delta P$  that means you are neglecting the osmotic pressure. The osmotic pressure compared to the feed pressure which is negligible. Negligible osmotic pressure.

So, again we will start with the spiral module or plate and frame. That means you are talking about flow through, flow through rectangular channel. Now, the geometry remains the same. So, basically 2 channels and you will be these say  $h$  that is the middle of the channel, is  $q_{in}$  in the material going into the system,  $q_{out}$  the flow at the material that is going out of the system and will be having presence of membrane and these case

in fact this is not constant this  $J$   $W$  is not constant, it will be a function of  $x$  simply because the  $\Delta P$  is a function of  $x$  in this case as well.

But this is still not a realistic case because we are neglecting the osmotic pressure effects. So, in the next case, case number 3 will be dealing with that but before that, let us solve this case because we can have an analytical solution in these case and see how it know works out for this case.

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$J = L_p \Delta P.$

Governing Equation of trans-membrane Pressure drop.

$$\frac{d^2 \Delta P}{dx^2} = \frac{3\mu J}{h^3}$$

$$\frac{d^2 \Delta P}{dx^2} = \frac{3\mu L_p \Delta P}{h^3}$$

at  $x=0, \Delta P = \Delta P_{in}$

at  $x=0, \frac{d\Delta P}{dx} = -\frac{3\mu}{2h^3} \Delta P_{in}.$

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So,  $J$  becomes  $L P \Delta P$  and if you look into the expression of the you no trans membrane pressure drop governing equation. The governing equation becomes  $d^2 \Delta P / dx^2$  is equal to  $3 \mu h^3 \Delta P$ .

So, we have already derived that I am not I am taking up the expression. Now, we are putting  $J$  is equal to  $L P \Delta P$ . So, this becomes  $d^2 \Delta P / dx^2$  is equal to  $3 \mu / h^3 L P \Delta P$ ,  $L P$  is nothing but, the membrane **per** ability. So, the difference is that the are this time the right hand side is not constant. So, the earlier case it was this the since  $J$   $0$  itself was taken as constant but, this this time it will not be constant I was looking to the boundary conditions at  $x$  is equal to  $0$ , you had  $\Delta P$  is equal to  $\Delta P_{in}$ , at  $x$  is equal to  $0$  you have  $d \Delta P / dx$  will be nothing but  $3 \mu / 2 h^3 \Delta P_{in}$ .

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$$\lambda = \sqrt{\frac{3\mu L P}{h^3}}$$
$$\frac{d^2 \Delta P}{dx^2} - \lambda^2 \Delta P = 0$$
$$\Delta P = e^{\lambda x} e^{m x}$$
$$m^2 - \lambda^2 = 0$$
$$m = \pm \lambda$$
$$\Delta P(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$$

at  $x=0$ ,  $\Delta P = \Delta P_{in}$ .

$$C_1 + C_2 = \Delta P_{in}$$

So, let us see how this can be tackled. We considered we let us say lambda is equal to root of 3 mu L P over h 2. So, in that case this equation can be written down

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$$\Delta P(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$$
$$\frac{d\Delta P}{dx} = C_1 \lambda e^{\lambda x} - C_2 \lambda e^{-\lambda x}$$
$$\left. \frac{d\Delta P}{dx} \right|_{x=0} = (C_1 - C_2) \lambda$$

at  $x=0$ ,  $\frac{d\Delta P}{dx} = -\frac{3\mu}{2h^3} \frac{Q_{in}}{h}$

$$(C_2 - C_1) \lambda = \frac{3\mu}{2h^3} \frac{Q_{in}}{h}$$
$$\frac{C_2 + C_1}{C_1} = \frac{\Delta P_{in}}{\quad}$$
$$C_1 = \checkmark$$
$$C_2 = \checkmark$$

as  $d^2 \Delta P / dx^2$  is nothing but, minus lambda square  $\Delta P$  will be equal to 0. Now, this is a second order ordinary differential equation with the solution in the form of e to the power n x and you will be getting a characteristic equation in terms of m square minus lambda square. So, the solution will be in the form of e to the power m x, e to the **power** sorry e to the power lambda x the characteristic equation that

you will be getting **sorry** it will in the form of  $e$  to the power  $m x$  only. So, you will be getting  $m^2 - \lambda^2$  will be equal to 0,  $m$  will be  $\pm \lambda$

So, you will be getting a solution  $\Delta P$  as a function of  $x$  as  $c_1 e^{\lambda x} + c_2 e^{-\lambda x}$ . Now, these constants  $c_1$  and  $c_2$  can be evaluated from the boundary conditions that we have already specified that is the first boundary condition is at  $x = 0$  you will be having  $\Delta P = 0$ . So, you will be getting  $c_1 + c_2 = 0$  and the second boundary condition in order to get the second boundary condition you have to take the derivative of these expressions and evaluate it at  $x = 0$  and see what you get.

$\Delta P(x)$  is equal to  $c_1 e^{\lambda x} + c_2 e^{-\lambda x}$ . So, derivative of that will be  $\frac{d \Delta P}{dx} = c_1 \lambda e^{\lambda x} - c_2 \lambda e^{-\lambda x}$ . Evaluate this at  $x = 0$ . So, you will be getting  $c_1 - c_2$  and the second if you remember the second boundary condition becomes  $x = 0$ ,  $\frac{d \Delta P}{dx} = -\frac{3 \mu}{2 h^3 W q}$  and you will be getting consequently  $c_2 - c_1 = \frac{3 \mu}{2 h^3 W q}$ . So, you can take  $\lambda$  to the denominator. So, this becomes  $\frac{c_1 - c_2}{\lambda} = \frac{3 \mu}{2 h^3 W q}$ . So, you can already  $c_1 - c_2$  plus  $c_1 = 0$  is equal to  $\Delta P = 0$ .

Now, you can have you can evaluate this you can just add them up and get the expression of  $c_1$ , can get the expression of  $c_2$ . Please do that I am not doing it and you please fill up the couple of steps and if you rearrange the solution the solution will become like will be constituted with the  $\cosh$  hyperbolic sign and hyperbolic  $\cos$  functions. So, I am just going to write down the final expression. So, you can really fill these gaps up.

(Refer Slide Time: 08:52)

$$\frac{\Delta P(x)}{\Delta P_i} = \cosh(\lambda x) - \frac{3}{2} \frac{\mu Q_i}{h^3 W \lambda \Delta P_i} \sinh(\lambda x)$$

Transmembrane Pressure profile  
 as a function of "x".  
 Profile of axial pressure drop.  

$$\Delta P_x = \Delta P(x) = \Delta P_i [1 - \cosh(\lambda x)] + \frac{3}{2} \frac{\mu Q_i}{h^3 W \lambda} \sinh(\lambda x)$$

$$\lambda = \sqrt{\frac{3 \mu L P}{h^3}}$$

And the expression that you will be getting is that delta P x divided by delta P i is nothing but cos h lambda x minus 3 by 2 mu q i divided by 2 h cube divided by h cube W lambda, W is the width divided by delta P I, delta P i means that the inlet sine h lambda x.

This is the this is a so this becomes, this is the pressure profile, transmembrane pressure profile. Transmembrane pressure profile as a function of x, as a function of actual distance. Now, this can be translated into the axial pressure drop. So, the profile of axial pressure drop becomes delta P i minus **sorry** minus delta P x this becomes delta P i 1 minus cos h lambda x plus 3 by 2 mu Q i divided by h cube W lambda sine h lambda x where the parameter lambda is given by under root 3 mu L P divided by h cube.

So, these expression gives the profiles of axial pressure drop even if you if you remember if you have already done the other courses that I am talking about the chemical engineering courses. In all the chemical engineering equipment whatever you are talking about the flow through a tube or for a any column design let us say packed pet or say furised pet or anything these pressure drop calculation becomes very important because everywhere you are you are doing the pressure drop calculation, the idea is you want has to calculate the pressure because in a flowing system pressure has to decrease because of the friction and other fittings and other losses. So, you have to design a path

accordingly so that you can overcome that much of pressure drop and can deliver at a particular flow rate at a desired flow rate.

So, all these calculation are basically aimed to design a particular equipment what is that equipment. Equipment is a either a pump in case of flow of the liquids or a compressor in case of flow of a gas. That is why pressure drop calculation becomes very important in all the chemical engineering equipments. So, this gives the profile of axial pressure drop and 1 can get the total pressure drop across the length of the module.

(Refer Slide Time: 12:08)

Total pressure drop across the module length.

$$\Delta P_i - \Delta P(L) = \Delta P_i [1 - \cosh(\lambda L)] + \frac{3\mu Q_i}{2h^3 W \lambda} \sinh(\lambda L)$$

Fractional recovery of feed:

$$f = \frac{Q_p}{Q_i} = \frac{\text{Flow rate in permeate}}{\text{Flow rate in}}$$

$$= \frac{2WL P \int_0^L \Delta P(x) dx}{Q_i}$$

So, this becomes  $\Delta P_i - \Delta P(L)$  is equal to  $\Delta P_i [1 - \cosh(\lambda L)] + \frac{3\mu Q_i}{2h^3 W \lambda} \sinh(\lambda L)$ .

Now, we can define the fractional recovery of the feed of the complete module the this, defined as  $Q_p$  divided by  $Q_i$  in that means flow rate you are getting out of the permeate, total flow rate in permeate divided by flow rate in that means material going out of in the permeate divided material going into system per unit time so this becomes  $2WL P$ , in this case since here  $\Delta P$  is a function of  $x$  and you will be you will be having 0 to  $L$   $\Delta P(x) dx$  divided by  $Q_i$ .

Now, we have the expression of  $\Delta P$  as a function of  $x$  just going to substitute there and 1 can get this expression of fractional recovery of feed over the complete module. Again, I am just putting the I am giving to the final expression what you have to just

simply do, we have already derived the you know expression of delta P x substitute there carry out this integration and get the value of fractional recovery, expression of fractional recovery.

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$$f = \frac{2WL\Delta P_i}{\lambda Q_i} \left[ \sinh(\lambda L) - \frac{3\mu Q_i}{2h^3\kappa\lambda\Delta P_i} \{ \cosh(\lambda L) - 1 \} \right]$$

Profile of Permeate flux:

$$J(x) = L_P \Delta P(x)$$

$$J(x) = L_P \Delta P_i \left[ \cosh(\lambda x) - \frac{3\mu Q_i}{2h^3\kappa\lambda\Delta P_i} \sinh(\lambda x) \right]$$

Governing eqn. of Q.

$$Q = 2xh\mu$$

If you do that what you will be getting is f is equal to 2 W L P delta P i divided by lambda times Q i bracket sine hyperbolic lambda L minus 3 mu q y divided by 2 h cube W lambda delta P I cos h lambda L minus 1. It becomes a complicated expression.

So, I can put the various values of width of the channel permeate of the membrane in a in the delta P transmembrane pressure drop at the inlet that is known to you because you have a you have pressure gas there, Q i is the flow rate that is also known to because you have the rotor meter there length is known all the other parameters are known, lambda is basically parameter it is a combined parameter of viscosity permeability channel geometry that is h. So, that will be known so I can estimate the total recovery of the feed in the permeate.

And the profile of permeate flux I can obtain because profile of permeate flux is very simple to obtain in this case the permeate flux profile becomes because you have J as a function of L P times delta P but, delta P is a function of x, you can get a profile of permeate flux L P delta P i cos h lambda x minus 3 mu Q i divided by 2 h cube W lambda L lambda delta P i times sine h lambda x. So, I can get the expression of how the permeate flux varies as a function of x in this case and I can integrate it out over the full



length of the module you can get the how much is the length average permeate flux you going to get.

Now, using 1 can, 1 can get the expression of how velocity the cross per velocity is varying as a functions of x, if you put if you if you look into the governing equation of Q, just just get the governing equation of Q, Q varies the flow rate and substitute Q is equal to 2 W, 2 x h u 2 W becomes x for that differential element like and this will be the area times you know so this is the cross sectional area. It should be W times h width times height not x. So, if that becomes a cross section area of the flow multiplied by the u that will be Q.

So, if you substitute in the governing equation of Q if you remember we have to govern a equation of Q d q d x is equal to minus 2 W J something like that. We have substitute in that expression and see what you get.

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$$\boxed{\frac{du}{dx} = -\frac{J}{h}} \rightarrow u \text{ is varying as a fnctn. of } x$$

$$\int_{u_i}^u du = \frac{L \delta P_i}{h} \int_0^x \left[ \frac{3 \lambda \delta P_i}{2 h^3 \lambda \delta P_i} \sinh(\lambda x) - \cosh(\lambda x) \right] dx$$

Profile of cross flow velocity:

$$\frac{u(x)}{u_i} = 1 - \frac{L \delta P_i}{h \lambda u_i} \left[ \sinh(\lambda x) - \frac{3 \lambda \delta P_i}{2 h^3 \lambda \delta P_i} \{ \cosh(\lambda x) - 1 \} \right]$$

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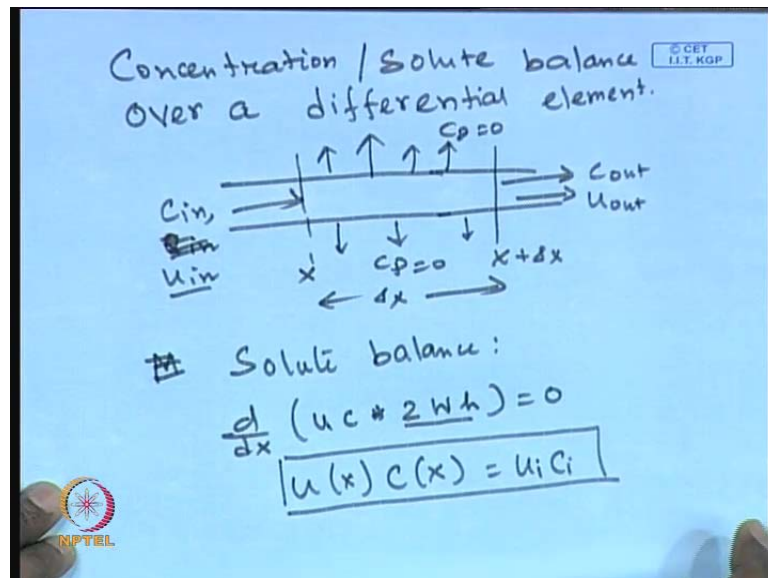
If you really do that you will be getting d u d x is equal to minus J over h. So, this is the governing equation how u is varying as a function of x. How the velocity the axial velocity is varying as a functional x in fact it is decreasing because of the because you are taking symmetrical out of it and you know the expression of J as a function of x which has derived it and substitute it there and integrate over the over the x and can get the value of u at any x location or at the end of the channel if you want.



So, if you really carry out the integration this becomes  $u$  and they will be formed inlet  $u_i$  to any location  $u$   $L P \Delta P_i$  divided by  $h_0$  to  $x$  velocity is  $u$  at any location  $x$ ,  $3 \mu Q_i$  divided by  $2 h^3 W \lambda \Delta P_i \sin h \lambda x - \cos h \lambda x dx$ . We carry out this integration and rearrange and you will be getting the profile of cos flow velocity,  $u$  as  $u x$  divided by  $u_i$  is nothing but,  $1 - L P \Delta P_i$  divided by  $h \lambda u_i \sin h \lambda x - 3 \mu Q_i$  divided by  $2 h^3 W \lambda \Delta P_i \cos h \lambda x - 1$ .

And if you replace  $x$  by  $L$  then you will be getting the velocity at the end of the module and how it vary so these expression gives how  $u$  varies as a function of  $x$  and if you put  $x$  is equal to  $L$  that will give you the velocity at the end of channel. Now, next you can do the what we have we have done till now we have used the **nevestes equation** to get the  $x$  governing equation of  $\Delta P$ . We did a overall material balance over differential elements we got the in governing equation of  $Q$  or  $u$ . Now, if you do a resist balance equation or solute balance equation over the differential element you will be getting an expression of concentration.

(Refer Slide Time: 21:23)



Let us do that concentration or solute balance equation, solute balance over a differential element. Please, look it at  $x$  is look it  $x$  plus  $\Delta x$  so these thickness is basically nothing but,  $\Delta x$  and  $c_{in}$  and  $Q_{in}$  or it is a  $u_{in}$  is a velocity of the inlet,  $c_{in}$  is the concentration under inlet,  $c_{out}$  and  $u_{out}$  are basically the concentration of the solute and

the velocity that is going out of the system and you are going to get the permeate but, remember if since we are talking about a completely retentive membrane the  $c_P$  will be equal to 0 on both sides. We will be going to basically recover pure water out of this.

So, if you really do a material balance or you know solute balance what you will be you are going to get is that you are going to get  $d \cdot dx$  of  $u$  times  $c$  times  $2$  times  $W$  times  $h$  becomes equal to 0 because nothing is and from the from the permeate side nothing is going out because no concentrations of the solute that is no solute is going out therefore,  $u$  times  $2 W h$ ,  $2 W h$  is the surface area. So,  $u$  times  $2 W$ ,  $2 W h$  is basically the cross sectional area. Why it becomes 2 why it is 2? Because  $h$  is the half height so there is a total height multiplied by the  $W$  that gives the cross sectional area multiplied by  $u$  that will be the total flow rate, flow rate multiplied the concentration that will be giving you the  $kg$  permeate square second.

So, that will be equal to 0 and what you are going to get is that  $u$  at  $x$ ,  $c$  at  $x$  must be equal to  $u$  at  $i$  and  $c$  at  $i$  that means  $u$  times  $c$  is constant  $d \cdot dx$  of  $u c$  equal to 0 means  $u$  times  $c$  is constant that means at any location  $u$  times  $c$  is nothing but, the product at the inlet. Now, we know the variation or the expression of  $u$  at  $x$  and divided by  $u_i$ . So, from this you can get what is the you know profile of concentration as a function of  $x$ .

(Refer Slide Time: 24:15)

$$\frac{c(x)}{c_i} = \frac{u_i}{u(x)}$$

$$= \frac{1}{1 - \frac{L \Delta P_i}{h \lambda u_i} \left[ \sinh(\lambda x) - \frac{3 \lambda h_i}{2 h^2 \lambda \Delta P_i} \{ \cosh(\lambda x) - 1 \} \right]}$$

In the downstream of the module concentration increases.

J

$\uparrow c$

$\uparrow u$

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So,  $c$  at  $x$  divided by  $c$  in is nothing but,  $u_i$  divided by  $u$  of  $x$  that simply means you and you have already obtained the expression of  $u$   $x$  there, just substitute their so.

So, you will be getting I think these itself I think we have derived  $u_y$  so it will be inverse of that. So, it becomes  $\frac{1}{1 - \frac{L P \Delta P}{h \lambda u}} \sin \frac{h \lambda x}{2} - \frac{3 \mu Q_i}{2 h^3 W \lambda \Delta P} \cos \frac{h \lambda x}{2}$ . This simply means that as since  $u$  is varying as a function of  $x$  and it is a decreasing function of  $x$ , as you go down the channel or the module the velocity will be decreasing because you are taking a extracting the material out of it. As the velocity decreases you know these numerator they the whole  $c_x$  by  $c_i$  will be increasing.

So, basically in the downstream of the module concentration is increasing where the that means the solution becomes more concentrated. Why it will be become more concentrated? Because you are not it is a total retentive membrane and you are you are extracting the water out of it so it becomes more concentrated. Concentration increases. So, if you now plot a profile your pressure drop will increase as you go down, your flux will decrease something like this your per you your velocity will be decreased, it will be decreasing, this will be  $u$ , the cross flow velocity it concentration will be increasing, this is the profile of concentration. So, you can get various profiles of all the parameters velocity, concentration, permeate flux as well as the transmembrane pressure drop and I can do the appropriate selection of the form accordingly.

Now let us so, this gives a more realistic system on your on your  $\Delta P$  transmembrane pressure drop or operating pressure drop becomes very high compared to the osmotic pressure drop.

(Refer Slide Time: 27:20)

Tubular Module

Eqn. of motion.

$$u = \frac{R^2}{8\mu} \left(-\frac{dP}{dx}\right) = \frac{R^2}{8\mu} \left(-\frac{d\Delta P}{dx}\right)$$

$$\frac{d\Delta P}{dx} = -\frac{8\mu}{R^2} u$$

Material balance over differential element:  $J = LP\Delta P$

$$\frac{du}{dx} = -\frac{2J}{R} = -\frac{2LP\Delta P}{R}$$

Diff. w.r.t  $x$

$$\frac{d^2\Delta P}{dx^2} = -\frac{8\mu}{R^2} \frac{du}{dx}$$

Now, let us look into the tubular module because we should as we have discussed the modules earlier there are 2 coordinate system that that we can think of in all the modules either tubular radial polar coordinate or in the rectangular coordinate. Let so let us we have done the rectangular polar coordinate let us look into the tubular coordinate or the radial coordinate, I will use full for the tubular modules. How the expression changes and how they look like.

Equation of motion or Navier Stokes equation will give you the expression of Q or u, what is that? The u becomes R square by 8 mu divided minus d P d x and we have already seen d P d x is nothing but, d delta P d x the transmembrane pressure drop so we write it safely as R square divided by 8 mu minus d delta P d x. So, you can get an expression of d delta P d x as minus 8 mu over R square times mu. Now, again you can write down a material balance over differential element. As we have done earlier we will be going to get and I am assuming that density of the feed is equal to density of the permeate, we are going to get d u d x is equal to minus 2 J over R and since the osmotic pressure is negligible J is nothing but, L P times delta P, so this will be nothing but, minus 2 L P delta P over R and just so that that means you can differentiate this expression with respect to x once more.

So, differentiate with respect to x so what you get is d square delta P by d x square is nothing but, minus 8 mu over R square d u d x and substitute d u d x here. So, you will

be going you are going to get the governing equation of transmembrane pressure drop. So, if you really do that I am not going just put it there you will be getting  $16 \mu L P y R q$  into  $\Delta P$

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The image shows handwritten notes on a blue background. At the top, the governing equation is boxed: 
$$\frac{d^2 \Delta P}{dx^2} = \frac{16 \mu L P}{R^3} \Delta P$$
 Below this, it says "2 B.C. for  $\Delta P$ ." and lists two conditions at  $x=0$ :  $\Delta P = \Delta P_{in}$  and  $\frac{d\Delta P}{dx} = -\frac{8 \mu}{R^2} u_{in}$ . A large arrow points to the text "Cauchy B.C." followed by "Both boundary conditions are specified at the same location." There are logos for "CET I.I.T. KGP" in the top right and "NPTEL" in the bottom left.

So, we are going write down the expression of transmembrane pressure drop in case of the tubular module  $d^2 \Delta P dx^2$  becomes  $16 \mu L P$  over  $R^3$  times  $\Delta P$ .

So and we require 2 boundary conditions for  $\Delta P$  and you have already seen how they will be set at  $x$  equal to 0 both the boundary conditions are to be specified,  $\Delta P$  is equal  $\Delta P_{in}$  and at  $x$  equal to 0 you have  $d \Delta P / dx$  is nothing but  $8 \mu / R^2$  times  $u_{in}$ ,  $u_{in}$  times in. Now, using these 2 boundary conditions now when both the boundaries are specified when the both the boundary conditions as specified at the same boundary raises specific name to the system do you know that is known as the Cauchy boundary conditions. Cauchy boundary conditions is basically both boundaries are specified at the same location, the same location.

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$$\Delta P(x) = \Delta P_i \cosh(mx) - \beta \sinh(mx)$$

$$m = \sqrt{\frac{16\mu L p}{R^2}} \quad \& \quad \beta = \frac{8\mu u_i}{m R^2}$$

Axial pressure drop profile.

$$\Delta P_i - \Delta P(x) = \Delta P_i [1 - \cosh(mx)] + \beta \sinh(mx)$$

Total axial pressure drop

$$\Delta P_i - \Delta P(L) = \Delta P_i [1 - \cosh(mL)] + \beta \sinh(mL)$$

Now, if you really solve these set of equation with this boundary conditions I am just going to write down the final expression, this becomes solution becomes delta P as a function of x is nothing but, delta P i again it will firm by the cos h hyperbolic cos in sine functions cos h m x minus beta times sine h m x by m is equal to under root 16 mu L P divided by R cube and beta equal to 8 mu u inlet divided by m R square and the axial pressure drop profile can be obtained, profile and module becomes delta P i minus delta P x is equal to delta P i 1 minus cos h m x plus beta sine h m x and the total axial pressure drop among the module across the module when you replace x by L.

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Permeate flux profile:

$$J(x) = Lp [\Delta P_i \cosh(mx) - \beta \sinh(mx)]$$

$$\bar{J}_L = \frac{1}{L} \int_0^L J(x) dx.$$

Profile of cross flow velocity:

$$\frac{u(x)}{u_{in}} = 1 - \frac{2Lp}{mR} [\Delta P_i \sinh(mx) + \beta \{1 - \cosh(mx)\}]$$

Concentration Profile:

$$\frac{C(x)}{C_i} = \frac{u_{in}}{u(x)} = 1 - \frac{2Lp}{mR} [ \dots ]$$

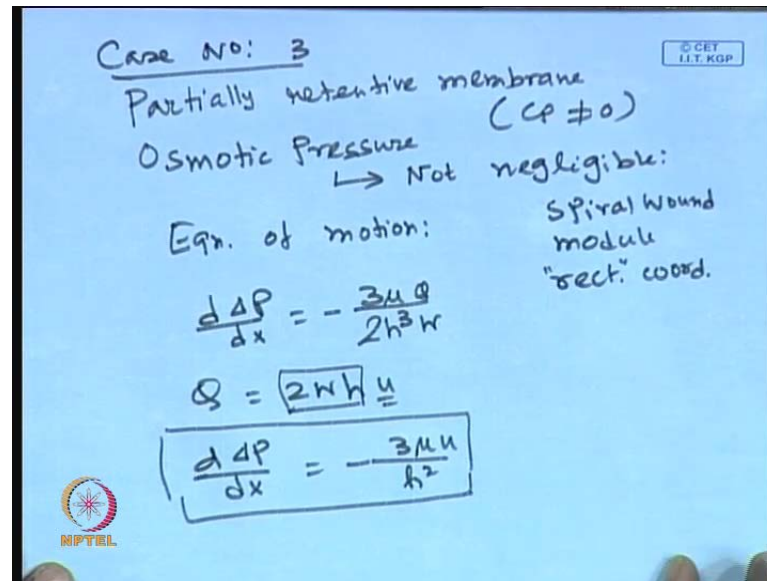


So, since you know the pressure drop profile  $\Delta P$  you can go to the expression of permeate flux which is  $J$  is equal to  $L P \Delta P$ . Since you know the  $\Delta P$  as a function of  $x$  you can get the permeate flux as a function of  $x$ . So, if you really do that just substituting the  $\Delta P$  profile into the expression of  $J(x)$ , you can get the profile of the permeate flux, permeate flux profile becomes  $J(x) = L P \Delta P \left[ \cosh\left(\frac{h}{m}x\right) - \beta \sinh\left(\frac{h}{m}x\right) \right]$  you can go to the integration or length averaging over the full length and can get the value of the length averaged permeate flux. So, length average permeate flux over the length becomes nothing but,  $\frac{1}{L} \int_0^L J(x) dx$  just put the expression and integrate the out will be getting.

Similarly, when you are talking about we can get the as we have discussed earlier you can get the expression of no profile of cross flow velocity, we can do the same thing I am just writing the final expression  $u(x)$  divided by  $u$  at inlet is nothing but,  $1 - \frac{2 L P}{m R \Delta P} \left[ \sinh\left(\frac{h}{m}x\right) + \beta \left(1 - \cosh\left(\frac{h}{m}x\right)\right) \right]$  and you get the concentration at any location which is  $u(x)$  at any location  $x$  and you can get the concentration profile at any location of  $x$  is inverse of  $u(x)$  divided by  $u$  at  $x$  and is own over this whole expression  $1 - \frac{2 L P}{m R \Delta P} \left[ \sinh\left(\frac{h}{m}x\right) + \beta \left(1 - \cosh\left(\frac{h}{m}x\right)\right) \right]$ .

So, as velocity decreases the concentration increases because you are extracting the material out of it. Now, in the so, that goes the derivation for the for the case where your permeate flux is proportional to  $\Delta P$  with the negligible osmotic pressure in the final realistic more realistic case we will be getting we will be doing the osmotic pressure is not really negligible and it will be some substantial value of significance.

(Refer Slide Time: 37:10)



So, case number 3 basically partially retentive membrane that means  $c_P$  is not equal to 0 that is a very realistic some amount of solute will be retained by the membrane, some amount is not. Partially retentive membrane and the osmotic pressure is not negligible.

Fine so, we do the again we do the similar type of calculations we start with the Navier Stokes equation and book it up with the velocity and you will be getting the governing equation with respect to or governing equation of transmembrane pressure drop with respect to  $x$  then you do the overall material balance that will give you the governing equation of the velocity cross flow velocity, you do an overall you do a you do a salute balance equation within the differential element that will give you the profile of the concentration salute concentration.

So, we go step by step so Navier Stokes equation and that will give the equation of motion, let us do a in a spiral module. The first thing we are we are going to for the spiral and module, rectangular coordinate that means rectangular coordinate,  $d\Delta P/dx$  it becomes minus  $3\mu Q$  divided by  $2h^3 W$  that that you get from the equation of motion and substitute in the flow rate  $Q$  is equal  $2Whu$  in the above equation because this is the cross sectional area and this is the velocity  $2$  comes because  $2$  times  $h$  is the half height  $2h$  is the full height.

(Refer Slide Time: 40:02)

Darcy's law:

$$J = L_p (\Delta P - \Delta \pi)$$

$$J = L_p \left[ \Delta P - B_1 C_m R_r - B_2 C_m^2 \{1 - (1 - R_r)^2\} - B_3 C_m^3 \{1 - (1 - R_r)^3\} \right]$$

Solute balance over the differential element:

$$\frac{d(u c)}{d x} = - \frac{J C_p}{h}$$

Diagram illustrating a differential element of length  $\Delta x$  between  $x$  and  $x + \Delta x$ . The element is bounded by two vertical lines. Above the element, three upward-pointing arrows represent flow. Below the element, two downward-pointing arrows represent flow, labeled  $J$  and  $C_p$ . The concentration variables are  $u, c$  at  $x$  and  $u, c$  at  $x + \Delta x$ . The MPTEL logo is visible in the bottom left corner.

So, this becomes  $d \Delta P$ ,  $d x$  becomes  $-3 \mu u$  divided by  $h^2$ . So,  $J$  becomes. Now, in this case since the osmotic pressure is not negligible we can use the Darcy's law as  $J$  is equal to  $L P \Delta P$  minus  $\Delta \pi$  and  $J$  becomes  $L P \Delta P$  minus  $B_1 C_m R_r$ . I think we already derived it number of times  $B_2 C_m^2 \{1 - (1 - R_r)^2\}$  minus  $B_3 C_m^3 \{1 - (1 - R_r)^3\}$ .

So, we already know the governing equation of  $\Delta P$ , we and here we will be getting the so it this becomes  $J$  becomes the functions of membrane surface concentration and salute balance over the differential element, elements gives you  $d u c$   $d x$  is equal to  $- J C_p$  divided by  $h$ , you can really derive this thing I am not doing it. So, if you want I you can do on a differential element, this is at  $x$ , this is at  $x$  plus  $\Delta x$ ,  $u$  at location  $x$ ,  $c$  at location  $x$  and is  $u$  and  $c$  at  $x$  plus  $\Delta x$ .

So and what is going out is nothing but,  $J$  and  $C_p$  is the permeate concentrations. So, total salute coming into the system is  $u$  times cross sectional area multiplied by  $c$  that is the total salute going into the system.

(Refer Slide Time: 42:23)

$w * 2h = C. \& \text{ area.}$   
 $2uc wh|_x = 2ucwh|x+dx + J * (W*2dx) C_p$   
 $2wh (c|_{x+dx} - c|x) = -2J C_p W dx$   
 $\frac{uc|x+dx - uc|x}{dx} = -\frac{J C_p}{h}$   
 $\boxed{\frac{d}{dx} (uc) = -\frac{J C_p}{h}}$   
 $\boxed{\frac{dc}{dx} = \frac{J}{u*h} (c - C_p)}$

So, if you do that what is the cross section area?  $W$  times  $2h$  that is the cross sectional area. So,  $uc$   $2Wh$  that is at  $x$  in is equal to out and what is out  $uc$   $2Wh$  and  $x$  plus  $\Delta x$  plus the material that is going out with the permeate.  $J$  times meter cube per meter square second what is the surface area? Surface area is nothing but,  $W$  times  $\Delta x$  that is the surface area so meter cube per meter square second, we multiplied by meter square so and then multiplied by  $c_p$ . This becomes the solvent multiplied by the concentration from meter cube  $k$  it becomes  $k$  g per second.

So, what you get is that  $2Wh$  common  $c$  at  $x$  plus  $\Delta x$  minus  $c$  at  $x$  is equal to minus  $J$  times  $c_p$   $W$  times  $\Delta x$ . There are 2 surfaces so there will be a 2 here  $W$  times  $\Delta x$  my multiplied by 2 so there will be 2 here. So, your 2 will be really will be cancelling so this becomes  $c$  at  $x$  plus  $\Delta x$  minus  $c$  at  $x$  divided by  $\Delta x$  is equal to minus  $J c_p$  divided by  $h$ . You see here no instead of  $W$  it will be  $u$ ,  $uc$  at  $x$  plus  $\Delta x$ , this one no its correct, correct  $uc$  at  $x$  plus  $\Delta x$  minus  $uc$  it will be differentiation of  $d/dx$  of  $uc$  that is correct because  $uc$  is also function of  $x$ .

So, it is  $d/dx$  of  $uc$  is minus  $J c_p$  divided by  $h$  and we have already derived the expression of  $d/dx$  of  $uc$  **sorry**  $d/dx$  earlier and sub so we will just open it up that means  $u$   $d/dx$  plus  $c$   $d/dx$  and substitute  $d/dx$  from the earlier expression and finally, you will be getting  $u$   $d/dx$  is equal to  $J$  times  $h$   $c$  minus  $c_p$ . So, expression of  $d/dx$  will be nothing but,  $J$  divided by  $u$  times  $h$ ,  $c$  minus  $c_p$ . Now,  $c_p$  is the permeate

concentration. Now, you will you got an ordinary differential equation of delta P, you got an ordinary deferential equation of delta of u, you got an ordinary definitional so from the Navier Stokes equation and combination of overall material balance you got the o d or the governing equation of delta P from the overall material balance over the differential element you will be getting the governing equation of u and from solute balance equation over the differential element, we will be getting the and combining with the expression of u you will be getting the governing equation of c.

Now, what is now left is how J is related to c m and how c m is related to c P that interface. So that will be we will be taking request to the definition of mass transfer coefficient.

(Refer Slide Time: 46:20)

$\checkmark k(c_m - c) = -D \left( \frac{\partial c}{\partial y} \right)_{y=0}$   
 At the S.S. the boundary condn. on membrane surface.  
 $\checkmark J(c_m - c_P) = -D \left( \frac{\partial c}{\partial y} \right)_{y=0}$   
 $k(c_m - c) = \frac{J}{c_m} R_{Tz}$   
 $\boxed{\frac{k(c_m - c)}{c_m R_{Tz} L_P} = \left[ \Delta P - (A_1 c_m + A_2 c_m^2 + A_3 c_m^3) \right]}$   
 $A_1 = B_1 R_{Tz}; A_2 = B_2 [1 - (1 - R_{Tz})^2]$   
 $A_3 = B_3 [1 - (1 - R_{Tz})^3]$

And we will we will we will calculate that k times c m c minus c is nothing but, d del c del y at y is equal to 0. If you remember that the value of c that we are putting here that is nothing but, the cross sectional average bulk concentration. It is basically nothing but the cross section average concentration so this is basically the bulk concentration we are talking about earlier.

So therefore, if you remember that definitions of mass transfer coefficient we had k times concentration the wall minus concentration of the bulk. So, here in this case it is not c not it is basically c that is appearing in the governing equation so my bulk conditions are now a function of x as far as the module is concerned. Initially, previously we had it with

the tedious constant. Now there will be function of the  $x$  there is only the difference. Now, let us see at the steady state the boundary condition on the membrane surface membrane surface becomes  $J$  times if you remember  $c_m - c_p$  is equal to  $-\frac{d}{dy} c$  at  $y$  is equal to 0.

So, that we have already used so many times. Now, I will equate these 2 equation because the right hand sides are both same so you will be getting  $k$  times  $c_m - c_p$  is nothing but,  $J$  times  $c_m$  times real retention  $R_r$ , I eliminate  $c_m$ , I eliminate  $c_p$  in favor of  $c_m$  multiplied with the real retention  $R_r$  which is constant. Now, we have already got an expression of  $J$  probably the  $J$  is equal to  $L P$  into  $\Delta P - \Delta \pi$  and we have expressed how  $\Delta \pi$  can be expressed in terms  $c_m$  and real retention etcetera.

So, just inside there here inside that equation here so you will be getting  $k$  times  $c_m - c_p$  divided by  $c_m R_r L P$  is equal to  $\Delta P - \Delta \pi$  plus  $a_1 c_m$  plus  $a_2 c_m^2$  plus  $a_3 c_m^3$  where  $a_1$  is nothing but,  $b_1 R_r$ ,  $a_2$  is nothing but  $b_2 (1 - R_r)^2$  and  $a_3$  is nothing but  $b_3 (1 - R_r)^3$ .

These expression, these expression gives you an algebraic relationship between the bulk concentration and membrane surface concentration. So, when I am writing  $J$  is equal to  $L P$  into  $\Delta \pi$  and  $\Delta \pi$  is a function of  $c_m$  now I should have a relation between the membrane surface concentration  $c_m$  and the bulk concentration  $c$  because all my governing equations are now in  $c$ . So this is the algebraic relation that gives you an interface or you know the relationship between the surface concentration on membrane and the bulk concentration  $c$ .

Now, what is the expression of mass transfer coefficient we can take now we can we can we can insert the generalized mass transfer coefficient that we have derived for the (( )) so the these module the third case deals with all the complexities involved in this system.



(Refer Slide Time: 49:56)

$k(x) = \frac{1}{I} \left( \frac{u D^2}{h x} \right)^{1/3}$   
 $I = \int_0^{\infty} \exp\left(-\frac{\eta^3}{3} - 0.42 \lambda \eta\right) d\eta$   
 $\lambda = \bar{J} \left( \frac{d L}{u D^2} \right)^{1/3}$   
 $\bar{J} = \frac{1}{L} \int_0^L J(x) dx.$   
 ODE 1  $\rightarrow$  Gov. eq.  $dP = v \frac{d^2 P}{dx^2} = v$   
 ODE 2  $\rightarrow$  Math. bal  $\rightarrow \frac{du}{dx} = f_1$   
 ODE 3  $\rightarrow$  Solute  $\rightarrow \frac{dc}{dx} = f_2$   
 System of DAE - IVP

So, the  $k$  if you remember the value the expression of mass transfer coefficient was  $1$  over our  $i u d$  square by  $h x$  rise to the power  $1$  up on  $3$  from the cyanide solution and  $i$  can be expressed the definite integral can be calculated from  $0$  to impunity exponential minus  $\eta$  cube by  $3$  minus  $0.42 \lambda \eta$  times  $d\eta$  where the section parameter  $\lambda$  is nothing but,  $\bar{J}$  that is the length  $R$  bit permeate flux divided by  $d d$  times  $L$  multiplied by  $u d$  square rise to the power  $1$  upon  $3$  while  $\bar{J}$  is nothing but,  $1$  bar  $1$  bar own our  $L$   $0$   $2$   $L$   $J$   $x$   $d x$ .

So the, what is the algorithm? The algorithm goes something like this to  $J$   $c$  value of  $\bar{J}$  bar guess value of  $\bar{J}$  bar and once you guess a value of  $\bar{J}$  bar the length of its permeate flux you will be getting the expression of  $\lambda$ , you will be in a position to evaluate the expression of  $i$ , the integration in definite integral  $i$ , once we able to integrate  $i$ , you can aim be able to calculate  $k$  mass transfer coefficient as a function  $x$ . Now, in this expression what is this value of  $u$ ?  $u$  is a basically bulk velocity we are talking about this is this was replace by  $u$  naught earlier is bulk velocity is now a function of  $x$  and we have a governing equation of  $x$  of  $u$  as  $d u d x$  equal to something.

So, once you can calculate the value of  $x$  then you can calculate the value the value of  $i$  no in a so  $k$  as a function of  $x$  will be known. So, these gives you  $c_m$  as a the relationship between  $c_m$  and  $c$ . Now all  $3$  governing equation if you remember all  $3$  o  $d$  is o  $d$  number  $1$  will be the governing equation of  $\Delta P$ , you have  $d$  square  $\Delta P d x$

square equal to something  $R$  or  $d \Delta P / dx$  equal as a function of  $u$  so you have the order 2 that gives you the overall material balance that gives you  $du/dx$  equal to some functional form  $f_1(u)$  order 3 that gives you solute balance equation over differential element that gives  $dc/dx$  is equal to some functional form  $f_2$ .

Now, this order 3 is and you have you there any basically initial value problem you know the initial condition all of them. So, these order 3 is as to solved by using Runge-Kutta method and every step of Runge-Kutta you have to solve this algebraic equation you have to solve this algebraic equation to update the value of  $c_m$  or  $c$ . There after you will be generating after integrations so the basically it is a system of differential algebraic equation.

You have 3 differential equation, 1 algebra equation coupled you have to solve this system of differential equations there initial value problem and will be generating the profile of  $\Delta P$  as a functions of  $x$ ,  $u$  as a function of  $x$ ,  $c$  as a function of  $x$ ,  $c_m$  as a function of  $x$  and  $J$  as a function of  $x$ .

(Refer Slide Time: 53:41)

$$\left. \begin{aligned} \Delta P &= \Delta P(x) \\ u &= u(x) \\ c &= c(x) \\ c_m &= c_m(x) \end{aligned} \right\}$$

$$c_p = c_p(x) = c_m(1-R_T)$$

$$J = J(x)$$

$$\bar{J} = \frac{1}{L} \int_0^L J(x) dx$$

So, what is the output? Output will be  $\Delta P$  as a function of  $x$ ,  $u$  as a function of  $x$ ,  $c$  as a function of  $x$ ,  $c_m$  as a function of  $x$  because  $c_n$  or  $c_m$  are related by the algebraic equation, mass transfer coefficient as a function of  $x$ ,  $c_p$  as a function of  $x$  because  $c_p$  is nothing but,  $c_m$  into 1 minus real it tension and of course  $J$  as a function of  $x$ . After doing this Runge-Kutta you calculate you do a length of average for you use a Simpson's rule in a trapezoidal rule as in Simpson's rule between in this case 3 fourth or 3 8 Simpson's rule so over

our  $0.2 L J \times d \times$ . Calculate the value of length have rest permeate flux and check by whether this calculated length our permeate flux is matching with the guess value of the length average permeate flux or not. That way 1 can generate the full profile of all the depending valuable and can get the expire at the value of no trans membrane trans module pressure drop as well as the concentration of the permeate of the outlet of the module.

So, you have to guess value of length of a permeate flux, re do this calculation and can and once giving profile you have to do the shim song averaging over the length and calculate the length average permeate flux and check whether they are equal or not, if not we have redo once again. So, that is how we have to do a act actual module on a module calculation so module modeling. So, in the next class so I stop here next class what I will do while extend this method for the tubular module and for the turbulent floor regime. Thank you.