Novel Separation Processes

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Lecture No. # 11

Membrane Separation Processes
(Contd.)

Good morning every one. So, we are talking about the gel layer controlled filtration in the last class, and we keep on continuing doing the mathematical modeling of the same. But before going into mathematical details, we discussed how a gel layer can be formed on a membrane surface. And we found out there are two possible reasons to form gel layer an on membrane surface; number one, if I talking about the Trans filtration of a solute which will be having small lower molecular weight which will be having a significant osmotic pressure. It may happen that in a transient operation beyond the particular time of operation or in a steady state beyond the particular length of the module, the membrane surface concentration can exceed the solubility limit at the particular temperature. In that case, the solutes can be deposited over the membrane surface as a solid, there will there will be on set of phase separation.

Second case, if you are talking about filtration of some gelling agents; for example, pectin, polyvinyl alcohol; these polymer pectin is a polysaccharide available in a fruit juice which is harmful as for as the shelf life of the juice is concerned. Therefore, you have to remove it by some kind of filtration mechanism; for example, ultra filtration. The gel can be form from the very beginning of the operation. Now, again in this case we can have two scenarios for our gel layer controlled filtration. One is the steady state another is the transient that is the most realistic case.

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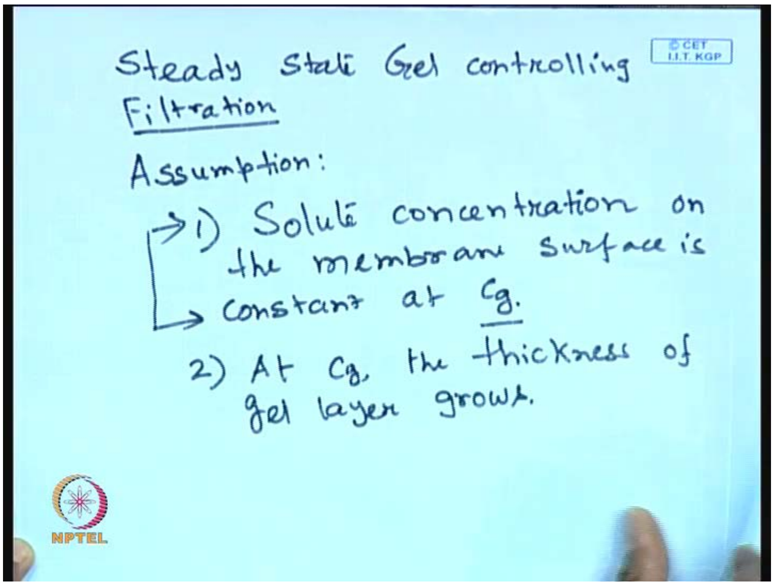

Steady State Gel controlling Filtration

Assumption:

1) Solute concentration on the membrane surface is constant at C_g.

2) At C_g, the thickness of gel layer grows.

So, first we will study the steady state gel layer controlled filtration. Then will talk about the most realistic case that is the transient gel layer controlling filtrations were the gel layer grows in kind. In case of gel layer filtration the first assumption is the solute concentration on the membrane surface is constant. Assumption number 1, solute concentration on the membrane surface is constant at a gel layer concentration C g. How to obtain this gel layer concentration C g? That will discuss later on. Solute concentration in the membrane surface is constant that is the C g, and at the particular concentration at C g more particles deposit over the membrane surface; and the thickness of gel layer grows. As the thickness grows it offers more resistance and that because is the decline in permeate flux and the throughput of the system.

So, we assume that the gel layer concentration is constant throughout the gel layer, and how to obtain the gel layer concentration that will see after sometimes. So, you see the contradiction of the (( )) difference between these model osmotic pressure control model, if you remember in the osmotic pressure control model; we have assumed that membrane surface concentration, solute concentration at the membrane surface is an increasing function of time. In case of a transient operation or it is an increasing function of X in case of steady state cross flow filtration system. On the other hand in gel layer controlling filtration the solute concentration is on the membrane surface is always constant and it is at the concentration C gel. Will see later on how to obtain the gel layer concentration.

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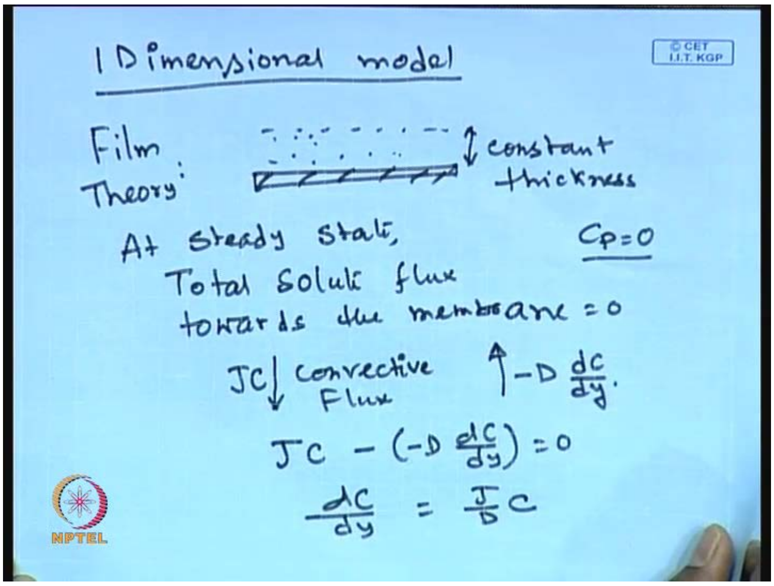

1 Dimensional model

Film Theory: constant thickness

At steady state, $C_p = 0$

Total solute flux towards the membrane = 0

$JC \downarrow$ convective Flux $\uparrow -D\frac{dc}{dy}$.

$$JC - \left(-D\frac{dc}{dy}\right) = 0$$

$$-\frac{dc}{dy} = \frac{J}{D}C$$

Now again we do the first simplistic model or one-dimensional model. The model remains the same as the film theory that we have talked earlier. It is basically a variation of film theory or variant of film theory; we assume that over the membrane surface. There is a film of solute is present of constant thickness, where solute concentration everywhere is at C gel and we do at the steady state. We do a solute balance equation and in a gel layer we have in gel layer filtration, we stated 2 assumptions; and the third assumption that has missed is that the permeate concentration in gel layer controlling filtration is always 0. In most of the cases it is 0 that is C p is equal to 0. Why C p is equal to 0? In most of the gel layer controlling filtration the solute particles are of larger in size, it has a very high molecular weight so that the effect of osmotic pressure is negligible.

If you remember pi is equal to nothing but R T C by M. If M is very high then the osmotic pressure becomes very small. In that case if the molecular weight is very high the size will be definitely very large, and the available membrane using even for the micro filtration membrane. In most of the cases these particles are completely returned by the membrane. So, it is most realistic to have a permeate concentration to be 0 in gel layer controlled filtration. At the steady state the total flux, total solute flux towards the membrane is equal to 0. What is total solute flux? One is J times C, J times C is nothing but the convective flux another is the diffusive flux minus D dc dy, because of the concentration gradient.

The concern the gel layer concentration is always greater than the bulk concentration so there will be backward diffusion from the surface to the bulk. So, at the steady state they should be equal so JC minus of minus D dc dy will be equal to 0. This is the governing equation of the concentration balance. So, these becomes dc dy will be nothing but J by D C, now you can have an integrate you can rearrange these things separate the variables and can integrate between the you know membrane surface on the bulk of the solution.

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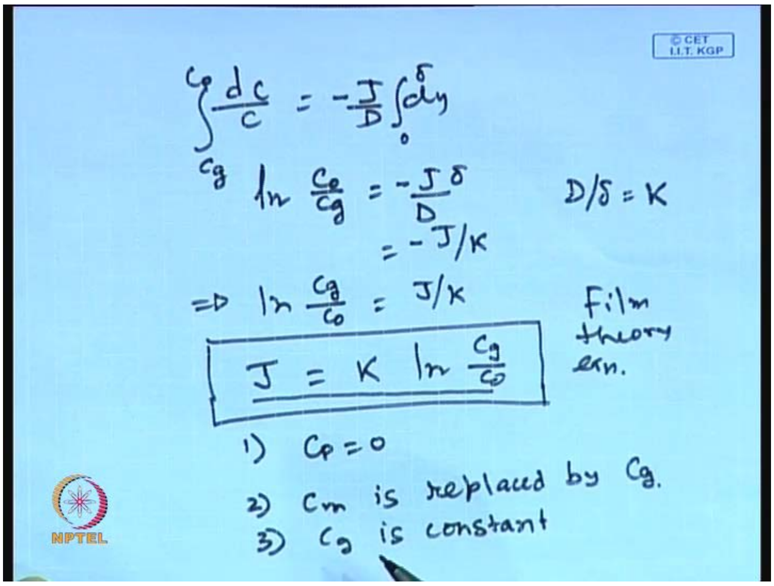

$$\int_{C_g}^{C_0} \frac{dc}{c} = -\frac{J}{D}\int_0^{\delta} dy$$

$$\ln \frac{C_0}{C_g} = -\frac{J\delta}{D}$$

$$= -J/K$$

$D/\delta = K$

$$\Rightarrow \ln \frac{C_g}{C_0} = J/K$$

$$\boxed{J = K \ln \frac{C_g}{C_0}}$$

Film theory eqn.

1) $C_p = 0$

2) C_m is replaced by C_g.

3) C_g is constant

So dc by C is nothing but J D by dy so from 0 to delta and C g to C naught so L n C naught by C g equal to J D delta. And D by delta is equal to always mass transfer coefficient, feed mass transfer coefficient K, so these become J by K. And I think I have with minus sign somewhere. I miss the minus sign here. Because this is the convective flux towards the membrane minus of diffusive flux hire from the membrane. So, it becomes minus, minus plus so only change the signs then becomes minus. Right so there is the minus here, so there is the minus there is the minus there. Therefore, l n minus can be observed in l n C g by C naught is equal to J by K so you can get an expression of membrane in the expression of permeate flux as K L n C g by C naught.

Now you see this is almost equal to the film theory equation of the osmotic pressure control filtration. But what is the difference? The difference is that is know C p, because C p equal to 0 in most of the gel layer control filtration. Number 1 let us write down C p is equal to 0. Number 2 the gel layer concentration is constant the assumption the gel layer concentration is constant at the concentration C g so C M is replaced by C g. Number 3 is that C g is constant and you can evaluate the value of C g from a separate set of experiment, how will do that? On the other hand, C M was a function of time or X whether the case was transient or steady state. These are the three differences.

So, this is the film theory equation for gel layer controlled filtration, and whenever we are talking about the calculation of the system performance; we are talking about the productivity are the permeate flux as well as the permeate quality. But the permeate quality is always the C p is always a gel layer concentration filtration gel layer control filtration therefore, there is no question of permeate quality. But these expression gives use the productivity of the permeate flux in the system. Now what are the short comings of this model? The short coming is that, if you remember film theory is always saying that there is a film or constant thickness of solute, that is depositing over the membrane surface which may not be correct, which is not correct, which is basically actually a growing you know thickness of the concentration boundary layer. That is depositing over the membrane surface.

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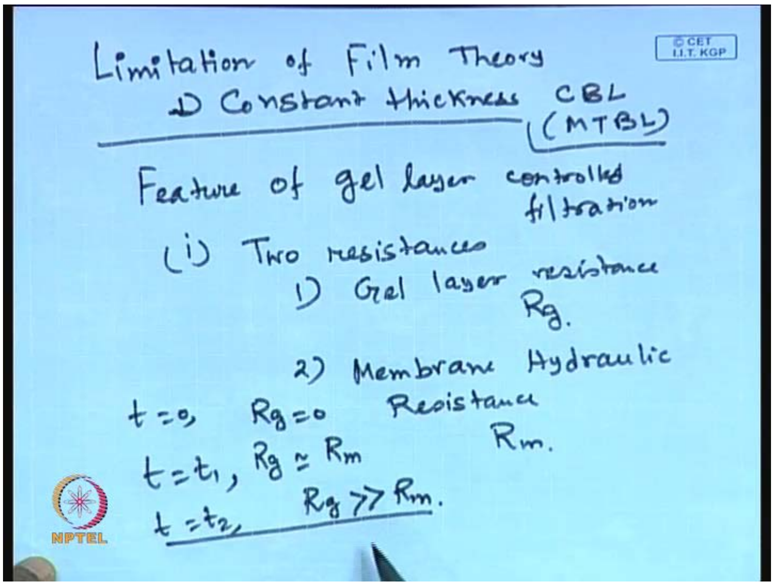

Limitation of Film Theory

1) Constant thickness CBL (MTBL)

Feature of gel layer controlled filtration

(i) Two resistances

1) Gel layer resistance R_g.

2) Membrane Hydraulic Resistance R_m.

$t = 0$, $R_g = 0$

$t = t_1$, $R_g \simeq R_m$

$t = t_2$, $R_g >> R_m$.

So, let us look into the formulation when we can how we can overcome this limitation. Constant thickness, concentration boundary layer or mass transfer boundary layer; and in this case that is not a limitation will see how to overcome these limitation shortly write now, another think that we have to mention about the feature of gel layer control filtration. There are 2 resistances in series. Two resistances 1 is Gel layer resistance R g we called as R g, another is membrane hydraulic resistance. In most of the cases R g starts from 0, because the gel layer starts depositing. So, then it increases and after sometime of operation it becomes almost in the same order magnitude R m. And then after sometime R g becomes very large.

So, when we are talking about the steady state operation when you are talking about the steady state operation. So, you see the (( )) at time t is equal to 0 R g is equal to 0 sometime t is equal to t 1 R g becomes almost in same order magnitude of R m for t is equal to t 2 R g is becomes much greater than R m. So, when you almost reach the steady state that mean when you reach at the almost at the end of the operation, then there is no existence of membrane. So, basically gel layer called resistance control becomes the controlling in one not the membrane resistance. So, whenever we talking about the and how will control the gel layer resistance when you are reaching the steady state.

The gel layer thickness is growing with time, and at the steady state near the steady state its growth is arrested by the by what by external turbulence. How external turbulence is created? If you are talking about a stirred cell, it is the external turbulence is create by the stirred speed, if you are talking about the cross flow cell, the external turbulence is created by the cross flow of the feed over the membrane surface. So, when you are talking about the steady state operation under the gel layer controlled filtration; the role of R m the membrane resistance becomes negligible. And it is totally gel layer controlled resistance, and the flux will be dominate by the dominated by the mass transfer.

Mass transfer that will be hindering are trying to control the gel layer thickness of the gel layer resistance outside. And what is the mass transfer? It is relate to the turbulence in the system Reynolds number. Reynolds number if the system, and you know how to define the Reynolds number in case of a stirred operators or in case of a cross flow system so that is very important to understand.

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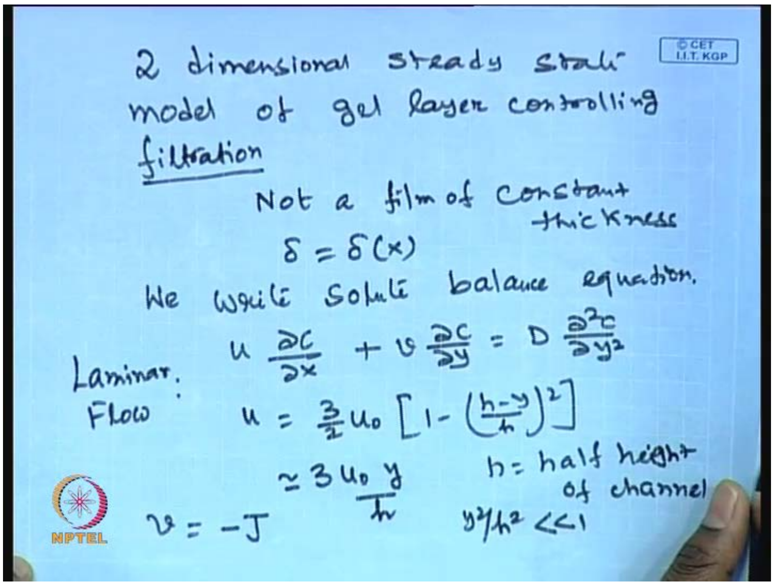

2 dimensional steady state model of gel layer controlling filtration

Not a film of constant thickness

$\delta = \delta(x)$

We write solute balance equation.

$$u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} = D \frac{\partial^2 c}{\partial y^2}$$

Laminar Flow:

$$u = \frac{3}{2} u_0 \left[1 - \left(\frac{h-y}{h}\right)^2\right]$$

$$\simeq 3 u_0 \frac{y}{h}$$

h = half height of channel

$v = -J$

$y^2/h^2 << 1$

Now let us look into the 2 dimensional models or improve the version of 2 dimensional steady state model of gel layer controlling filtration. So, what is the feature of this gel layer grows and the thickness of the and the concentration boundary layer will be a function of X so it is growing it is not a constant film. So, not a film of constant thickness that is present on the membrane surface the delta is a function of X; so we write down the solute balance equation u del C delX plus v delC dely is equal to D del square C del y square. The same equation that we have obtain earlier now here, then we insert the velocity profile. The velocity profile for the laminar flow we consider the laminar flow in fact we can extent this analysis to turbulence flow as well; u is equal to 3 by 2 u naught 1 minus h minus y divided by h square, were h is the half height of channel that means flow is occurring through rectangular channel.

And we know that the thickness of the boundary layer will be extremely small compare to the half channel height and we can neglect y square by h square so these becomes linear profile. So, we have already seen earlier 3 u 0 y by h by neglecting y square by h square; the y will be the thickness of the boundary layer will be extremely small compare, to the half channel height which will be in the order of millimeter. That will be in the order of micron ten to the power minus six. So, it will be extremely small and v will be nothing but minus J because of thickness will be extremely thickness of the boundary layer will be extremely small, and whatever the value of permeate velocity at the boundary that will be almost prevailing throughout that small thickness, but this is a very valid assumption. So, let us write down the lets let us inside the velocity profile in the governing equation.

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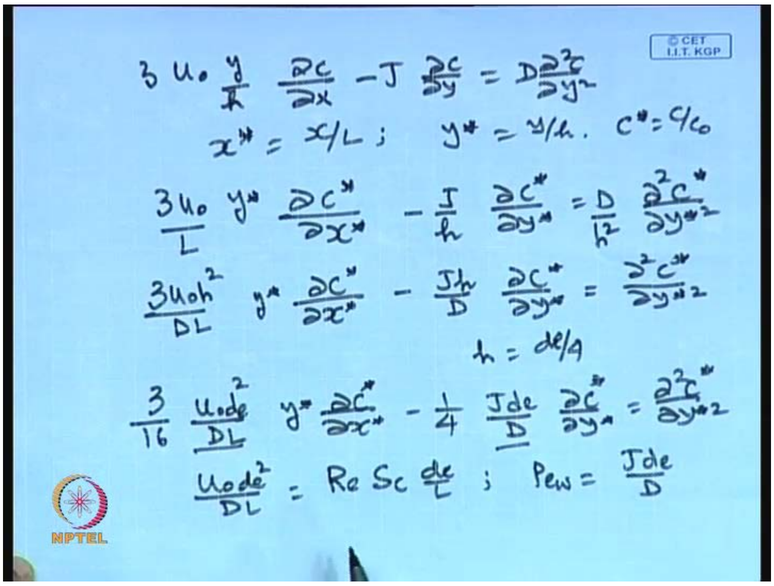

$$3 u_0 \frac{y}{h} \frac{\partial c}{\partial x} - J \frac{\partial c}{\partial y} = D \frac{\partial^2 c}{\partial y^2}$$

$$x^* = x/L \; ; \quad y^* = y/h . \quad C^* = C/C_0$$

$$\frac{3 u_0}{L} y^* \frac{\partial c^*}{\partial x^*} - \frac{J}{h} \frac{\partial c^*}{\partial y^*} = \frac{D}{h^2} \frac{\partial^2 c^*}{\partial y^{*2}}$$

$$\frac{3 u_0 h^2}{DL} y^* \frac{\partial c^*}{\partial x^*} - \frac{Jh}{D} \frac{\partial c^*}{\partial y^*} = \frac{\partial^2 c^*}{\partial y^{*2}}$$

$$h = de/4$$

$$\frac{3}{16} \underline{\frac{u_0 de^2}{DL}} y^* \frac{\partial c^*}{\partial x^*} - \frac{1}{4} \underline{\frac{J de}{D}} \frac{\partial c^*}{\partial y^*} = \frac{\partial^2 c^*}{\partial y^{*2}}$$

$$\frac{u_0 de^2}{DL} = Re\, Sc \frac{de}{L} \; ; \quad Pe_w = \frac{J de}{D}$$

And see what you get? 3 u 0 y by h that is u del C del X minus J del C dely is equal to D del square C del y square. Now we make this equation non dimensional will be easier for us. So, we write X star is equal to X by L y star is equal to y by h L is the characteristics length in the X direction, h is the characteristics length in the y direction. If you do that this becomes 3 u 0 y by h itself is y star and in C star is equal to C by C naught with the bulk concentration. This is becomes del C star will be L comes out so this comes del X star minus J over h del C star del y star D by h square del square C star del y star square, multiplied both side by h square by D. So, what you get 3 u 0 h square by D L y star del C star del X star minus J h over D del C star del y star is equal to del square C star del y star square.

Now I we replays h by equivalent diameter D by 4, that we have already seen earlier what is the definition of equivalent diameter, and how it is connected to channel half height. So, if you do that these becomes 3 by 16 u 0 D square by D L y star del C star del X star minus 1 by 4 J de by D del C star del y star is equal to del square C star del y star square. Now if you can look into this equation the right hand side is completely non dimensional, that means all the quantities on the left hand side become non dimensional. Now I thing we have always talked or discussed earlier these issue that u 0 de square D L is nothing but Reynolds times smite time de by L. You can open up the Reynolds number you can open up smite number you can find out that will be getting u 0 D square by D L and we define the non dimensional flux as J D by D so these becomes non dimensional flux this is becomes Reynolds times smite times de by L and all of them are non dimensional so you get a set of non dimensional quantities.

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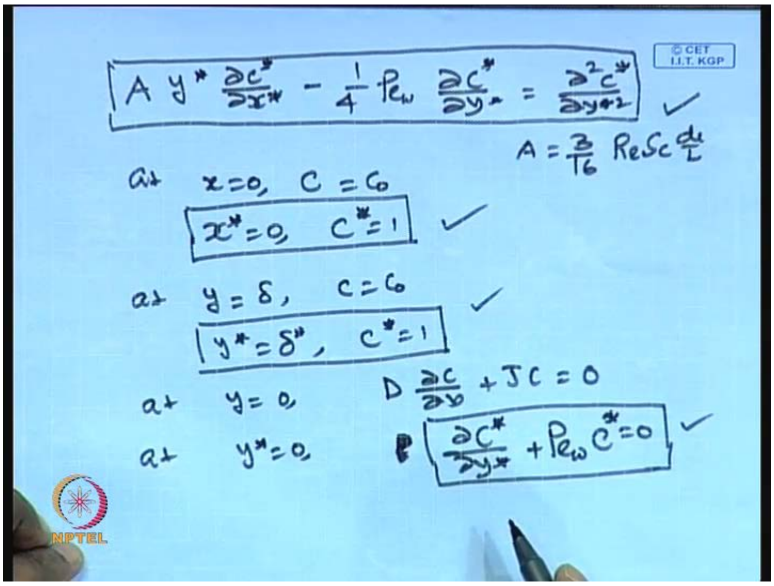

$$\boxed{A\, y^* \frac{\partial c^*}{\partial x^*} - \frac{1}{4} Pe_w \frac{\partial c^*}{\partial y^*} = \frac{\partial^2 c^*}{\partial y^{*2}}} \checkmark$$

$$A = \frac{3}{16} Re\, Sc \frac{d_e}{L}$$

at $x = 0, \quad C = C_0$

$$\boxed{x^* = 0, \quad c^* = 1} \checkmark$$

at $y = \delta, \quad c = C_0$ ✓

$$\boxed{y^* = \delta^*, \quad c^* = 1}$$

at $y = 0 \qquad D \frac{\partial c}{\partial y} + J c = 0$

at $y^* = 0 \qquad \boxed{\frac{\partial c^*}{\partial y^*} + Pe_w c^* = 0} \checkmark$

So we write this expression as a y star del C star del X star X star minus 1 by 4 p w del C star del y star is equal to del square C star del y star square, were a is equal to nothing but 3 by 16 Reynolds smite D by L so this is the non dimensional governing equation as earlier. I do not thing there is any change of these in a non dimensional version with the earlier version that we have talked about. So, we require to have boundary conditions at X is equal to 0 C was equal to 0 that means X star is equal to 0 C star is equal to 1 so that is 1 boundary initial condition and the other boundary conditions are at y is equal to delta C is equal to C naught. Now we can put since we are doing a doing integral analysis, now will be doing a integral analysis, now integral in an integral analysis we can really put the value of know the boundary condition at y equal to delta.

So, at y is equal to delta that is the exact boundary condition this is more accurate compare to y is equal to infinity C is equal to C naught right, that y is equal to delta C is equal to C naught. Therefore, at y star is equal to delta star divided by h C star becomes 1. This is the second boundary condition and the third boundary condition will be at y is equal to 0 you have the mixed boundary condition D del C del y plus JC will be equal to 0. So, this is the convective this is the mixed boundary condition that is coming, so to the convective flux towards the membrane plus diffusive flux away from the membrane will be equal to 0 at the steady state, and these was if you remember this was the governing equation of the one dimensional problem.

Make it non dimensional y star is equal to 0 this becomes this becomes del C star del y star plus pw C star will be equal to 0. So, this is the governing equation 1 2 3 these are the 3 boundary condition and with these well defined problem and we can solve this equation now whole analysis will be very simplified these case because C M the membrane surface concentration is now constant that is C g C M is nothing but C g which is constant so because of these simplicity we can we can go for a for a integral method solution that means we can assume the some concentration profile some kind of concentration profile.

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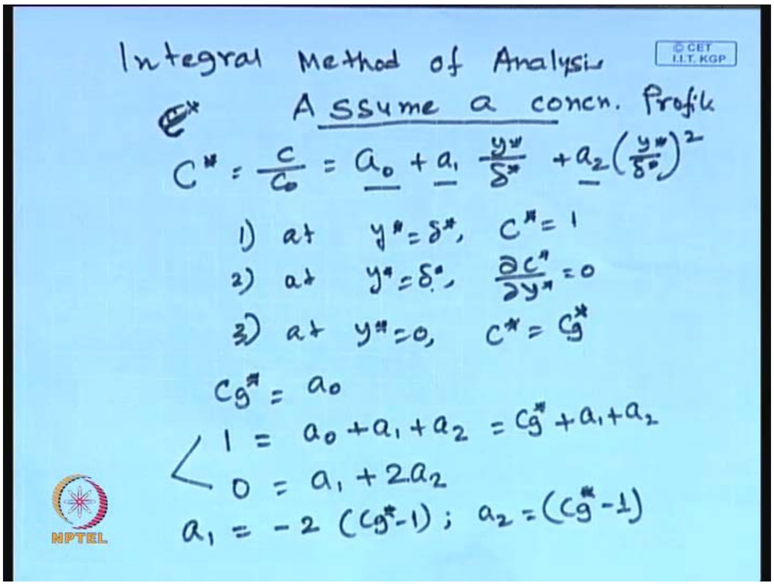

Integral Method of Analysis

Assume a concn. Profile

$C^* = \frac{C}{C_0} = \underline{a_0} + \underline{a_1} \frac{y^*}{\delta^*} + \underline{a_2} \left(\frac{y^*}{\delta^*}\right)^2$

1) at $y^* = \delta^*$, $C^* = 1$

2) at $y^* = \delta^*$, $\frac{\partial C^*}{\partial y^*} = 0$

3) at $y^* = 0$, $C^* = C_g^*$

$C_g^* = a_0$

$1 = a_0 + a_1 + a_2 = C_g^* + a_1 + a_2$

$0 = a_1 + 2a_2$

$a_1 = -2\,(C_g^* - 1)\,; \quad a_2 = (C_g^* - 1)$

So we take request to an integral method of analysis, we assume a concentration profile C star is equal to C by C naught a 0 plus a 1 y star by del star plus a 2 y star by del star square. Now you can look into and any text book and see that they said this is an approximate method, integral method is always an approximate method. Why it is approximate? It is approximate, because you are assuming a concentration profile we do not know whether this concentration profile really exists in your system physically or not, you are assuming something. Therefore, it is always an approximate method but we have we can see that later on that these really similar is a problem very nicely, and we assume a third order polynomial.

Why not second order polynomial? Why not fourth order polynomial? Simply because and these constant coefficients are unknown coefficients, we are going to find out these coefficients a 0 a 1 a 2. Why third order polynomial? Because there are 3 conditions one can have that these as concentration profile we satisfy what are the 3 conditions. That means the condition number 1, at the edge of boundary layer C is always equal to C naught right, that means y star is equal delta star C star is equal to 1. The second boundary condition at y star is equal to delta star del C star del y star becomes 0 .These 2 are called the basic boundary conditions of any boundary layer.

Even if it is a case of heat transfer then at y star is equal to delta T star T star would have been equal 2 1 and del T star del y star would have been vanish their these 2 have to be satisfied. Number 3 in this particular problem in this particular case, at y star is equal to 0 my C star is always equal to C g star. Because you are talking about gel layer controlling filtration the concentration of the membrane surface will be always constant and that constant is C g star so that is known to us. Therefore, since you will be having 3 conditions 3 boundary conditions, you can have 3 undetermined coefficients in your profile.

That means you can go up to a second order polynomial. You can go up to a third order polynomial, fourth order polynomial or higher order polynomials but all of them give you spurious solution. So, ultimately it will be reverted to third order second order polynomial with 3 coefficients. Now what are what is the next task is how to evaluate? Now have to evaluate a 0 a 1 and a 2 appropriately for this particular system. So, let us first find out a 0 will be utilizing the condition number 3 that means y star is equal to 0 C star is equal to equal to C g star.

That means C g star is equal to a 0 y equal to 0 means this will be half, this will be half C g star becomes y a 0. If you use the second boundary the first boundary condition y star is equal to delta star C star equal to 1 will be having 1 is equal to a 0 plus a 1 plus a 2 right if you use the other 1 boundary condition that y star equal to delta star del C star del y star will be equal to 0 will be getting 0 is equal to a 1 plus 2 a 2 a 0 is known. So, these becomes C g star plus a 1 plus a 2. Now you have 2 equations and 2 unknown a 1 and a 2 and if you really find it out a 1 trans out to be minus 2 C g star minus 1 a 2 trans out to be C g star minus 1, now probably you are in a position to write down the concentration profile to its full extent.

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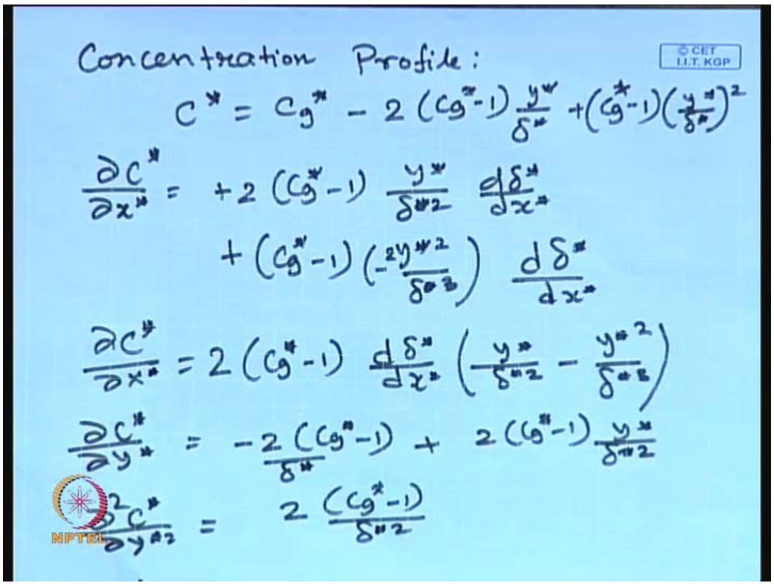

Concentration Profile:

$$C^* = C_g^* - 2(C_g^*-1)\frac{y^*}{\delta^*} + (C_g^*-1)\left(\frac{y^*}{\delta^*}\right)^2$$

$$\frac{\partial C^*}{\partial x^*} = +2(C_g^*-1)\frac{y^*}{\delta^{*2}}\frac{d\delta^*}{dx^*} + (C_g^*-1)\left(-\frac{2y^{*2}}{\delta^{*3}}\right)\frac{d\delta^*}{dx^*}$$

$$\frac{\partial C^*}{\partial x^*} = 2(C_g^*-1)\frac{d\delta^*}{dx^*}\left(\frac{y^*}{\delta^{*2}} - \frac{y^{*2}}{\delta^{*3}}\right)$$

$$\frac{\partial C^*}{\partial y^*} = -\frac{2(C_g^*-1)}{\delta^*} + 2(C_g^*-1)\frac{y^*}{\delta^{*2}}$$

$$\frac{\partial^2 C^*}{\partial y^{*2}} = 2\frac{(C_g^*-1)}{\delta^{*2}}$$

Let us write down the concentration profile within the boundary layer. The concentration profile becomes C star is equal to C g star minus 2 C g star minus 1 y star by del star plus C g star minus 1 y star by del star square. So, these becomes the concentration profile where C g star is no longer a variable it is a constant value. Now what we are going to do? Like the similarity solution we take the derivative of del C star del X star, we take the derivative of del C star del y star, we take the derivative of del square C star del y star square and substitute them in the governing equation.

Let us, take derivative del C star del X star now in these what is independent quantity that is delta star delta star is only the independent quantity. So, C g star derivative of the since it is the constant it will be there will be nothing minus 2 C g star minus 1 y star delta star becomes delta star square minus there will be plus sign here it becomes D delta star d x star plus C g star minus 1 y star square and delta star to the minus 2 minus sign here so it there will be minus 2 will be there and it will be delta star cube and will be having D delta star d x star right. So, in a in a neat form you can write down as to C g star minus 1 that will be common D delta star d x star that will be also common. In the bracket will be having y star by delta star square minus y star square divided by delta star that is the expression of del C star del X star.

Similarly, we can get the del C star del y star what is del C star del y star more term coming from the first term will be having minus 2 C g star minus 1 divided by delta star. That is from the first term and the second term will be plus 2 C g star minus 1 y star by del star square what is del square C star del y star square delta is a function of X only. So, there is no contribution from the first term there will be contribution from the second term only and will be getting 2 C g star minus 1 divided by delta star square. So, basically these are the 3 derivatives that you will be getting out of it and now what I am going to do and I am going to substitute these 3 expression del C star del X star del C star del y star del square C star del y star square into the governing equation.

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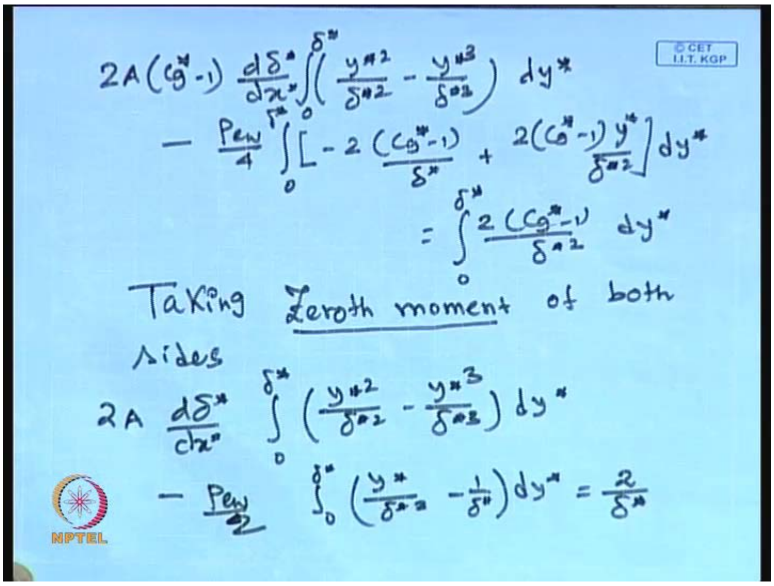

$$2A(C_g^*-1)\frac{d\delta^*}{dx^*}\int_0^{\delta^*}\left(\frac{y^{*2}}{\delta^{*2}}-\frac{y^{*3}}{\delta^{*3}}\right)dy^* - \frac{Pe_w}{4}\int_0^{\delta^*}\left[-2\frac{(C_g^*-1)}{\delta^*}+\frac{2(C_g^*-1)y^*}{\delta^{*2}}\right]dy^* = \int_0^{\delta^*}\frac{2(C_g^*-1)}{\delta^{*2}}dy^*$$

Taking Zeroth moment of both sides

$$2A\frac{d\delta^*}{dx^*}\int_0^{\delta^*}\left(\frac{y^{*2}}{\delta^{*2}}-\frac{y^{*3}}{\delta^{*3}}\right)dy^* - \frac{Pe_w}{2}\int_0^{\delta^*}\left(\frac{y^*}{\delta^{*2}}-\frac{1}{\delta^*}\right)dy^* = \frac{2}{\delta^*}$$

And see what you get? If you do that you will be getting 2 a C g star minus 1 D delta star d x star y star square divided by delta star square becomes square because there was y star previously. So, these becomes y star cube divided by delta star cube minus P e w by 4 the whole thing in the brackets. So, minus 2 C g star minus 1 divided by delta star plus 2 C g star minus 1 y star by del star square bracket end is equal to 2 C g star plus 1 delta star square. So, you will be getting these equation with 3 terms. Now these equation can be you can you can (( )) or eliminate the y star dimension by multiplying both side by D y star and integrating across the boundary layer thickness 0 to delta star.

If you do that you can average out or you can (( )) eliminate the variation y star and ultimately what you will get will be getting governing equation of delta star only that is a standard form of integral analysis in any boundary layer probably you must (( )) derivation for the in case of hydro dynamic boundary layer other course is second year or third year. So, what I will do? I will multiply both side by D y star so multiply both sides by D y star and integrate across the boundary layer thickness.

Now a is constants C g star is constant delta star is a sole function of X so detailed as the (( )) that function of X only so it has nothing to do with the y star so I integrate between 0 to delta star there will be constant again p w is a function of X only so 0 to delta star and from 0 to delta star so I integrate across the boundary layer thickness and do while (( )) the y dimension so I will be getting governing equation in X now multiplying both side by d y star means you are multiplying both side y 2 the power 0 y star 2 the power 0 into d y star this is known this is also known as taking the first moment taking the 0th moment.

This is also equivalent to the (()) except size that means multiply both side and integrate across the boundary layer thickness is equivalent to these statement that taking 0th moment of both sides is another statement, or you can take the first you can always do that you can take the first moment of both the side that means, you can multiply both side by y 2 the power 1 y star 2 the power 1 d y star and integrate across. But if you take a first moment on of the will be landing up we can identity, because you will be basically exacting a decrease of freedom. Now if you there are you know certain conditions of the boundary layer analysis where there is need to take 0th order moment as well as the first order moment but not in these case so if you anyone is interested we can discussed it later on when to take first order moment or something like that or higher order moments.

But for this case typically the 0th order moment is good enough so if you really do that and in fact a 2 times C g star minus 1 will be cancel down both the sides and the whole thing will be becomes very simplified. So, these becomes 2 a D delta star d x star 0 to delta star will be getting y star square divided by delta star square minus y star cube divided by delta star cube d y star minus p w by p w by 2 p w by 4 and it should multiplied by 2 therefore, you will be getting 0 to delta star y star by del star square minus 1 by delta star d y star and here all these things will be coming out and it will be basically delta star is the function of X so it will be integral of d y star will be delta star so there will be 1 delta star will be cancelled out and will be getting 2 by delta star only.

I have eliminated C star minus 1 from all the sides now there rest of the X y is very simple the integration of this will be y star cube by 3 so it will be del delta star cube by 3 so it will be delta by 3 minus delta by 4 so it becomes delta by 12 and similarly, you will be getting minus half here minus 1 by 2 here, so I am just omitting ask only 1 single step and writing the final expression.

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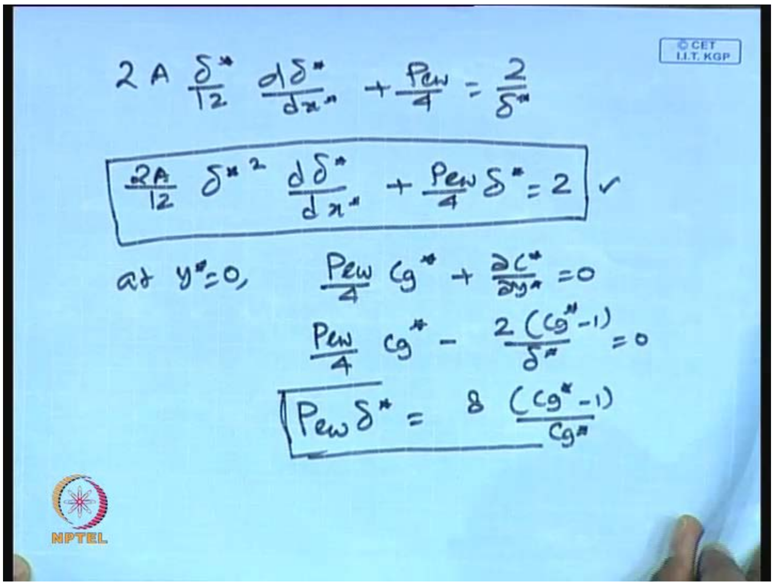

$$2A\frac{\delta^*}{12}\frac{d\delta^*}{dx^*}+\frac{Pe_w}{4}=\frac{2}{\delta^*}$$

$$\boxed{\frac{2A}{12}\delta^{*2}\frac{d\delta^*}{dx^*}+\frac{Pe_w}{4}\delta^*=2}\ \checkmark$$

at $y^*=0$,

$$\frac{Pe_w}{4}C_g^*+\frac{\partial C^*}{\partial y^*}=0$$

$$\frac{Pe_w}{4}C_g^*-\frac{2(C_g^*-1)}{\delta^*}=0$$

$$\boxed{Pe_w\delta^*=8\frac{(C_g^*-1)}{C_g^*}}$$

You can always do the final expression becomes 2 A delta star by 12 D delta star d x star plus p w by 4 is equal to 2 by delta star so you can multiply both side by delta star so you will be getting 2 a by 12 delta star square D delta star d x star plus p w by 4 delta star is equal to 2. So, governing equation now results into this expression now there is a trick here, the trick is that since p w is the function of X you can estimate p w in terms of delta star from a separate set of form a separate equation. What is that equation? The boundary condition at y is equal to 0 if you look into the boundary condition at y is equal to 0 that means at y star is equal to 0 the boundary condition was p w by 4 C g star plus del C star del y star is equal to 0.

Now what we can do? We can substitute the derivative del C star del y star in terms of delta and evaluate it at y star is equal to 0 the what you will get is that you just look into this derivative we have already done evaluate the derivative at y star equal to 0 will be getting p w by 4 C g star minus 2 C g star minus 1 divided by delta star is equal to 0. So, on simplification you will be getting p w by delta star is nothing but 8 times C g star minus 1 divided by C g star. This is very important relationship these says that permeate flux multiplied by the thickness of concentration boundary layer is a constant. What is that constant? 8 times C g star minus 1 divided by C g star in most of the cases C g star is much higher than 1 therefore, this will this will be almost these whole right hand side we can approximated by 8 only. So, what we can do next? Next will just substitute these p w these expression into the governing equation. So, basically what I am going to do here from these boundary condition and evaluating a value of p w and substring that here the governing equation.

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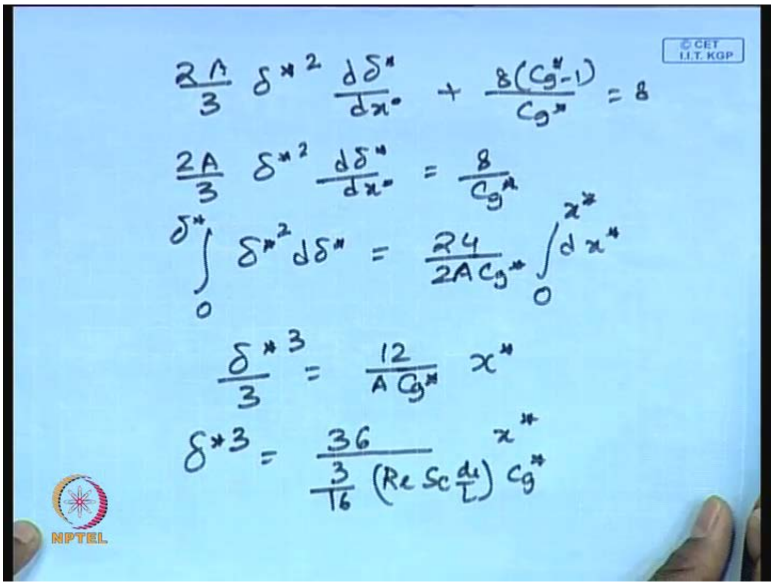

$$\frac{2A}{3}\delta^{*2}\frac{d\delta^*}{dx^*} + \frac{8(C_g^*-1)}{C_g^*} = 8$$

$$\frac{2A}{3}\delta^{*2}\frac{d\delta^*}{dx^*} = \frac{8}{C_g^*}$$

$$\int_0^{\delta^*}\delta^{*2}\,d\delta^* = \frac{24}{2AC_g^*}\int_0^{x^*}dx^*$$

$$\frac{\delta^{*3}}{3} = \frac{12}{AC_g^*}x^*$$

$$\delta^{*3} = \frac{36}{\frac{3}{16}\left(Re\,Sc\,\frac{d_i}{L}\right)C_g^*}x^*$$

That is always we can do so if you do that what you will be getting is that will be getting 2 A by 3 delta star square D delta star by d x star plus 8 times C g star minus 1 divided by C g star is equal to 8. Now take the these term on the right hand side and C what you get 2 A by 3 delta star square D delta star square d x star and if you take this on the right hand side and do a simplification will be getting 8 by C g star. Now what now everything is you can get on integration of delta star over X and get an get an expression. So, what you can do delta star square d delta star is nothing but 24 by 2 a C g star d x star a if you remember it is 3 by 16 Reynolds smite D by L 24 by 2 is nothing but 12 and C g star is constant. So, you can integrate out over the over the length 0 to X star and X is equal to 0 there is no existence of boundary layer so it is 0 to delta star so delta star by cube is nothing but 12 by A C g star X star and these becomes delta star cube is equal to 36 by A now you can write the expression of A 3 by 16 Reynolds smite d e by L rest to the power Reynolds smite D by L times C g star X star so you can get an expression of delta star out of these so if you if you just simplify these expression and 1 can get the expression of delta star as a function of X star.

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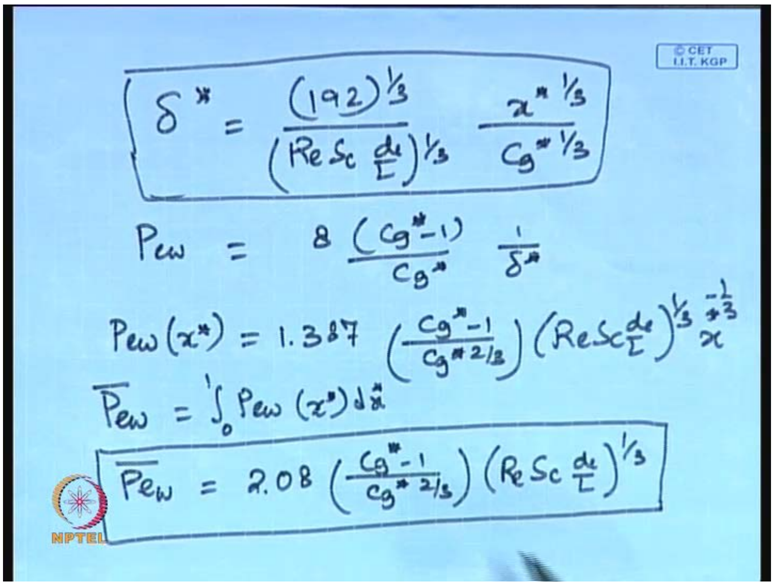

$$\boxed{\delta^* = \frac{(192)^{1/3}}{\left(Re\,Sc\,\frac{d_e}{L}\right)^{1/3}}\,\frac{x^{*\,1/3}}{C_g^{*\,1/3}}}$$

$$P_{ew} = 8\,\frac{(C_g^*-1)}{C_g^*}\,\frac{1}{\delta^*}$$

$$P_{ew}(x^*) = 1.387\left(\frac{C_g^*-1}{C_g^{*\,2/3}}\right)\left(Re\,Sc\,\frac{d_e}{L}\right)^{1/3} x^{*\,-1/3}$$

$$\overline{P_{ew}} = \int_0^1 P_{ew}(x^*)\,dx^*$$

$$\boxed{\overline{P_{ew}} = 2.08\left(\frac{C_g^*-1}{C_g^{*\,2/3}}\right)\left(Re\,Sc\,\frac{d_e}{L}\right)^{1/3}}$$

So delta star becomes now the expression of delta star 192 rest to the power 1 upon 3 divided by Reynolds smite d e by L rest to the power 1 upon 3 X star to the power 1 upon 3 C g star rest to the power 1 upon 3 this is the expression of delta star. Now there is a similarity to these analysis to the osmotic pressure analysis that we have done by using similarity solution if you remember there also the delta star was varying as a function of X star to the power 1 upon 3 here also it is varying as a as a function of X to the power 1 upon 3, Although the coefficients are different but here also delta star is inversely proportional to Reynolds smite d e by L that simply means, if you increase the Reynolds number the thickness of boundary layer will be small that means if you increase the turbulence in your system the growth of the boundary layer will be arrested.

Now let us look into the expression of p w the permeate flux right p w will had p w is nothing but 8 C g star minus 1 divided by C g star times 1 by delta star that has been expression of permeate flux. Now we put the because you have now we got already the explicit expression of delta star as a function of X star if you substitute it here will be getting how my permeate flux is varying as a function of X star. If you do that will be getting 1 point I am just writing the final expression 1.387 C g star minus 1 divided by C g star rest to the power 2 by 3 bracket Reynolds smite d e by L rest to the power 1 upon 3 X star to the power minus 1 upon 3 .

So, these expression give you how my permeate flux is varying as a function of X. Now of course, the most important thing you are you are concern is what is the length average permeate flux. So, I can do a length averaging over this and can get the length average permeate flux that means p w bar is nothing but 0 to 1 integral p w X star d x star that means I can carry out the integral and find out what is the length averaged permeate flux. Now if you do that you will be getting 2.08 C g star minus 1 divided by C g star rest to the power 2 by 3 Reynolds smite d e by L rest to the power 1 upon 3.

That is the expression of length averaged permeate flux and under the gel layer control filtration. Now if you remember that the, if you talked about the permeate flux that is arising out of film theory, it was J is equal to K L n C g by C naught. Now under certain circumstances under certain assumptions these equation boils down to the to the almost to the form of the film theory equation, but we take different coefficients so I am going to do a mathematical trick on this expression.

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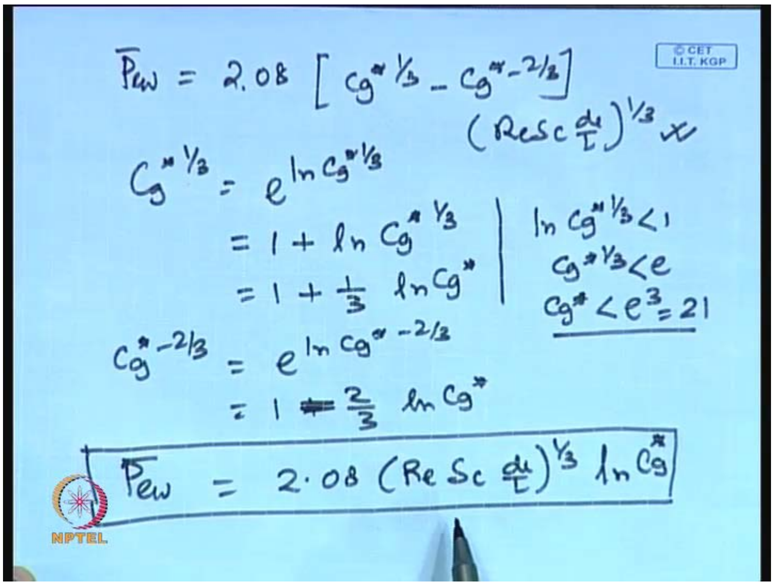

$$\bar{P}_{ew} = 2.08\left[C_g^{*\,1/3} - C_g^{*\,-2/3}\right]\left(Re\,Sc\,\frac{d_e}{L}\right)^{1/3} x$$

$$C_g^{*\,1/3} = e^{\ln C_g^{*\,1/3}} = 1 + \ln C_g^{*\,1/3} = 1 + \frac{1}{3}\ln C_g^{*}$$

$$\ln C_g^{*\,1/3} < 1$$

$$C_g^{*\,1/3} < e$$

$$\underline{C_g^{*} < e^{3} = 21}$$

$$C_g^{*\,-2/3} = e^{\ln C_g^{*\,-2/3}} = 1 - \frac{2}{3}\ln C_g^{*}$$

$$\boxed{\bar{P}_{ew} = 2.08\left(Re\,Sc\,\frac{d_e}{L}\right)^{1/3}\ln C_g^{*}}$$

And see what you get? So, let be let be rewrite these equation the length average permeate flux as 2.08 C g star 1 upon 3 minus C g star minus 2 upon 3 times Reynolds smite d e by L rest to the power 1 upon 3 so I write down C g star to the power 1 upon 3 as exponential L n C g star to the power 1 upon 3. We always as possible now what are you do it is in the form of e to the power X, so I will just open up this exponential series keeping the first order term assuming the higher order tern will be much less than equal to less than 1. So, I writing write these as 1 plus L n C g star to the power 1 upon 3 neglecting the higher order terms that means that means L n C g star to the power 1 upon 3 is much less than 1 so these can be written as 1 plus 1 upon 3 L n C g star.

So, what is the limit? L n the limit is these L n C g star to the power 1 upon 3 is less than 1.What is that? That means C g star to the power 1 upon 3 is less than e that means C g star is less than e cube and if you remember the value of e is around 2.7 cube of that will be around 3 around ninety twenty right it will be around twenty. So, in fact the its value exact value will be is equal to twenty 120.78 something like that, so it will be roughly equal to 21 so if you C g star is less than 21 then whatever I am doing that is valid in most of the cases C g star is less than 21 and whatever we are doing that will be valid if your C g star trans out to be around 5 hundred or fifty then whatever I am doing is not valid that in that case this will be the expression of length average permeate flux.

So, I can write down the C g star to the power minus 2 by 3 as e to the power L n C g star to the power minus 2 by 3 again doing same thing you can write it as 1 minus 2 by 3 L n C g star once we do that we substitute the governing equation and you will see C g star to the power 1 upon 3 minus C g star to the minus 2 by 3 1 and 1 will be getting cancelled out and 1 by 3 L n C g star plus 2 by 3 once L n C g star becomes only L n C g star so these becomes length average permeate flux p w 2.08 Reynolds smite D e by L rest to the power 1 upon 3 L n C g star means C g by C naught right L n C g star now these expression as the grasping similarity with the film theory equation if you remember now if I write down these 2 equation.

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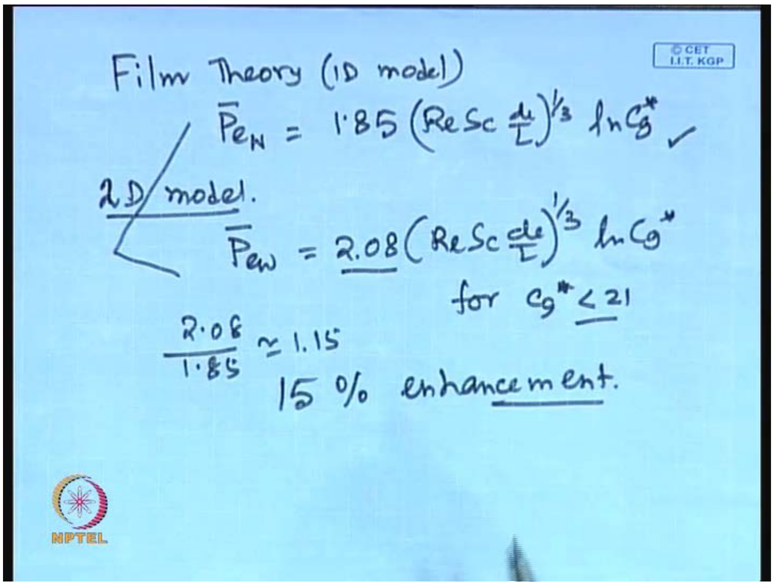

Film Theory (1D model)

$$\overline{Pe}_N = 1.85 \left(Re\, Sc\, \frac{d}{L}\right)^{1/3} \ln C_g^* \quad ✓$$

2D model.

$$\overline{Pe}_w = \underline{2.08} \left(Re\, Sc\, \frac{d_e}{L}\right)^{1/3} \ln C_g^*$$

for $C_g^* \underline{<21}$

$$\frac{2.08}{1.85} \simeq 1.15$$

15 % enhancement.

One after another you can understand the one-dimensional film theory model that gives p w is equal to K L n C g star by C naught, that means length average permeate flux was should have been 1 point Sherwood number times L n C g star Sherwood number was 1.85 Reynolds smite d e by L rest to the power 1 upon 3 L n C g star C g by C naught. So, the non dimensional form these becomes the mass transfer coefficient becomes Sherwood number, if you remember the film theory equation it was p w is equal to K L n C g star and if you make J 0 the permeate flux J as non dimensional K becomes Sherwood number and Sherwood number under laminar flow condition becomes 1.85 Reynolds smite d e L rest to the power 1 upon 3 for the channel flow.

And the 2 dimensional model becomes p w bar is equal to 2.08 Reynolds smite d e L rest to the power 1 upon 3 L n C g star for of course, this is there is the validity of the equation what is validity C g star is less than 21 .Now you can see the difference of 2 the under this condition let say these 2 expression are exactly identical. Now only there is difference in the coefficient, and this coefficient is higher and how much is higher if you if you if you write 2.08 if you evaluate 2.08 divided by 1.85 these become around 1.15. That means 15 percent enhancement. What does that mean? This is the same thing I am trying to (()) on I am trying to (()) every time from the very beginning when you are talking about the film theory.

But assume we are assuming a constant thickness of concentration boundary layer which is not true. Actually the thickness is growing initially the thickness is less than it is later on when developing after long distance it will be fully developed where film theory equation is valid. That means initially you are neglecting the permeate flux so if you use the film theory equation you will be definitely under predicting permeate flux and how was the how you quantify the under prediction the under prediction will be in the tune of 15 percent. So, if you use the two dimensional model developing mass transfer boundary layer or concentration boundary layer the there will be a definite integral mass transfer coefficient and that will be in the order of 15 percent more than the 1 dimensional film theory equation. So, I will stop here and then take up this issue in the next class. Thank You.