Computer Aided Applied Single Objective Optimization
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Lecture – 32
Branch and Bound for MILP

Welcome in the last session we had seen the simplex method for solving linear programming problem. In this session we will be looking into the Branch and Bound method for solving integer linear programming problem. The branch and bound method solves a series of linear programming problems to solve the underlying integer linear programming. So, we will be discussing only about integer linear programming right, but the concepts what we discussed for integer linear programming can be extended directly for mixed integer linear programming problems.

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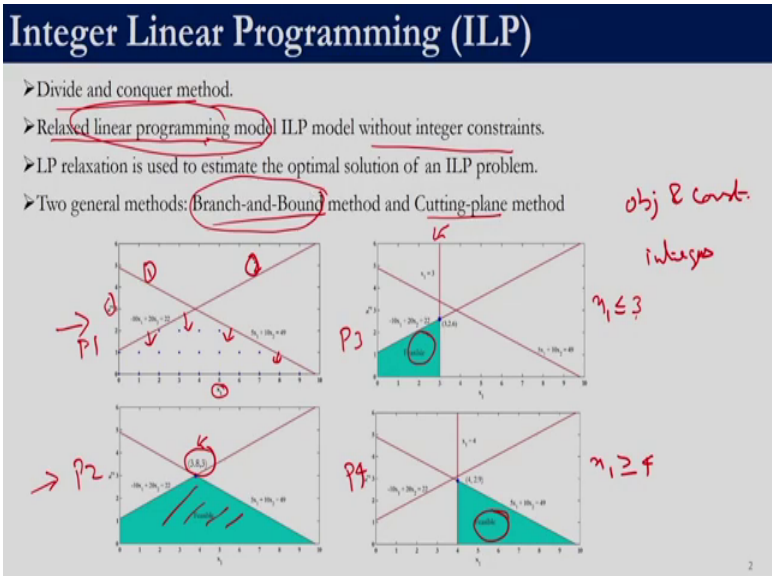

So, for an integer linear programming the objective as well as the constraints need to be linear right and the variables can take only integer values. So, there is no non-linear constraint. In branch and bound integer linear programming problems are solved by a divide and conquer method. So, for example, consider this problem. So, these two are our constraints, let us say this is constraint 1 and this is constraint 2 and this region is the feasible region of constraint 1 and 2.

So, our solution basically lies over here right. Only thing is that the decision variable x 1 and x 2 are integers. So, it can take only the values denoted by the dots right. So, it cannot take any other value because the variables need to be integer. So, to solve this problem what we will do is we will first relax the integer requirement right. So, that is called as relaxed linear programming model right. So, wherein we relax the integer requirement so, the problem actually has the constraint that the variables can take only integer, but we will relax that requirement. So, that is why we have a relaxed linear programming model right.

So, when we do that through this problem P 1 becomes P 2 right. So, now, it becomes a linear programming problem. And we can use simplex method to find the solution of P 2. Now, when we find a solution of P 2, if the solution of P 2 happens to be an integer solution right. So, in this case it is not x 1 is equal to 3.8 and x 2 is equal to 3, but if it happens to be an integer we can stop there because this solution which we have got is already an integer right, but if the solution is not integer right.

So, for example, over here 3.8 and 3.3 then what we will do is we will divide this region. So, we will divide this shaded region into two parts right. So, we will get problem P 3 and problem P 4. So, problem P 3 has this additional constraint that x 1 should be less than equal to 3. So, this is the line x 1 equal to 3, but the constraint is x 1 should be less than equal to 3 whereas, in problem 4 we have another constraint x 1 is greater than or equal to 4 right.

So, this shaded region what we see over here has been divided into 2. One region is x 1 less than equal to 3 and the other region is x 1 greater than or equal to 4. Now, we will search separately in this region and also we will search in this region right. So, that is why the term divide and conquer is used. So, there are various methods to solve integer linear programming the commonly used methods are a branch and bound and cutting plane method. So, we will be discussing only the branch and bound method. So, even here we will only be discussing the preliminaries of branch and bound method in view of the nature of this course.

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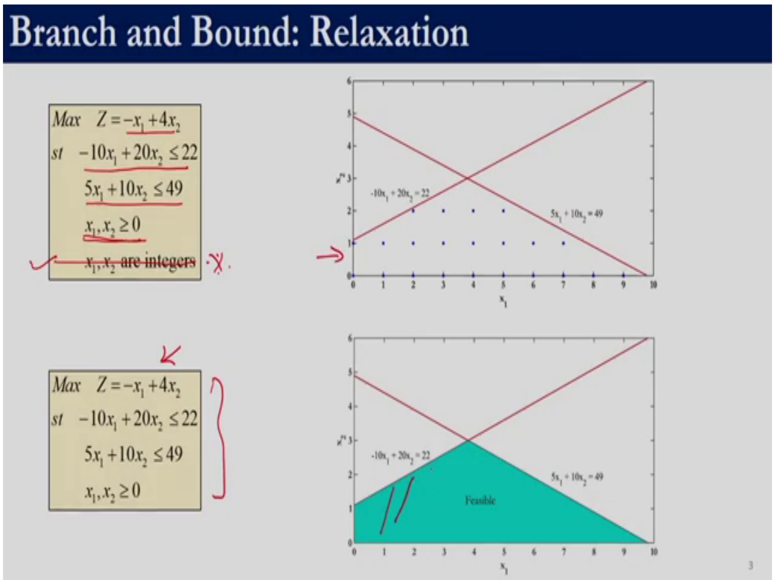

Now, let us take an example right. So, our problem is to maximize minus x 1 plus 4 x 2 right. And subject to the constraint that minus 10 x 1 plus 20 x 2 is less than equal to 22, 5 x 1 plus 10 x 2 is less than equal to 49 and x 1 and x 2 need to be greater than equal to 0. In addition we also have this requirement that x 1 and x 2 are integers, if this requirement was not there then it would be straightforward to solve this problem using the simplex method which we discussed in the previous session right.

So, but since we have this requirement what we will first do is we will ignore this requirement right and we will solve this particular problem. So, x 1 and x 2 are integers. So, that is ignored. So, we will solve this problems. So, this problem if we see it is a relaxed problem because we have relaxed the constraint that x 1 and x 2 are integers. As discussed a little while back instead of searching in this discrete search space what we will be doing is? We will be searching in this feasible region right.

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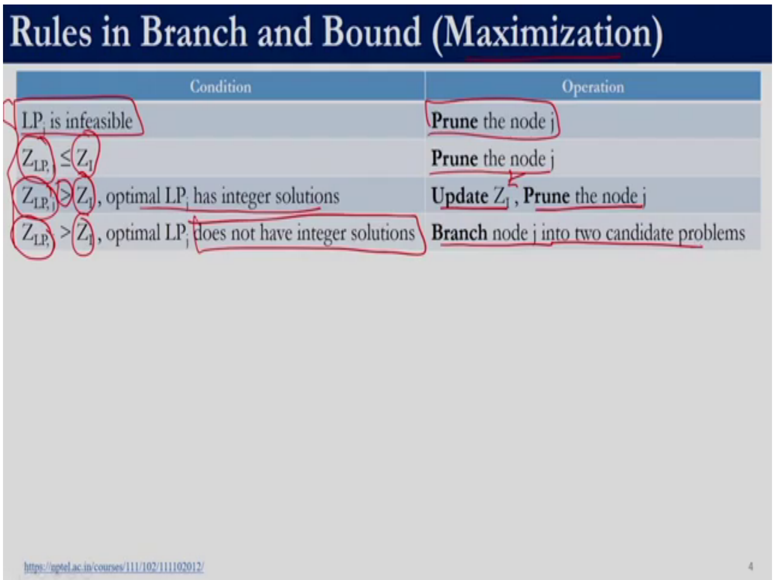

Before solving that example, these are the rules that we need to follow in branch and bound right. Remember these rules are written for a maximization problem these rules state that, if a particular problem is infeasible right. If a particular node is infeasible then we will no longer explore that node if at any node it happens that the value of the relaxed problem is less than the best integer that has been discovered. So, far then we will prune the node, but if it happens that the LP at any particular node gives a better objective function value.

Since its a maximization problem we have this greater than symbol. And we are saying that Z LP has a better objective function value. And if the solution if the decision variables are integer then we need to update this value of Z I and prune that particular node; however, if the objective function value of a linear programming problem at any particular node is better than Z I, but the decision variables are not integer right. then we do not need to update the value of Z I, but we need to branch on that particular node to create two additional problems right.

So, now we will see this with an example, with an example these rules will become a lot more clearer.

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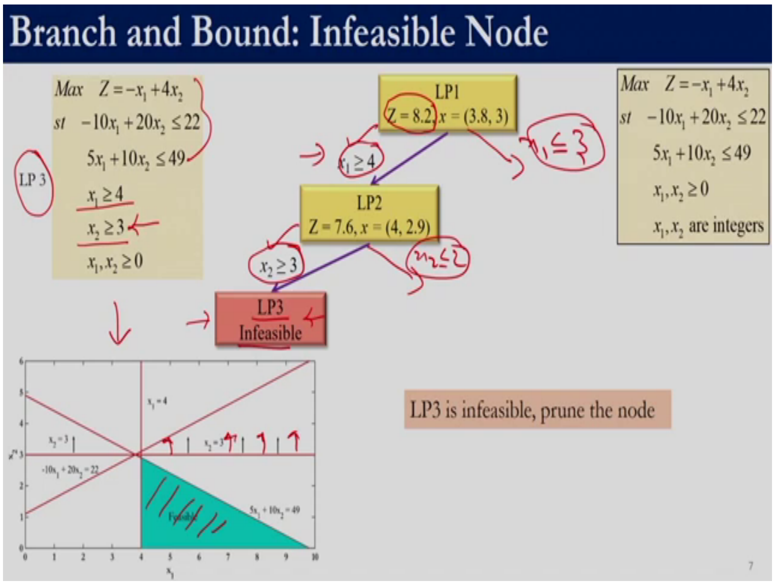

Now, let us label this problem as LP 1 right. So, it is the same problem which was given to us except for the fact that we have removed the requirement that both the variables x 1 and x 2 need to be integers right. So, now, it is a linear programming problem right. If we solve this linear programming problem we would get a solution 3.8 comma 3.

So, x 1 is equal to 3.8 and x 2 is equal to 3 with an objective function value of 8.2. So, this was our first linear programming problem which we have solved. Since we are going to solve multiple linear programming problems we are just labeling it right. So, now, this solution is not an integer solution right. So, this solution is not acceptable to us had it been integer we would right away stop because whatever solution we got actually satisfies our problem that x 1 and x 2 are integers right.

Since this is not integer what we will have to do this we will have to divide the region right. So, what we are doing is since x 1 is 3.8 we are going to divide this region into 2 right. So, we will divide this region into this one and this one right. So, one is this thing where x 1 greater than or equal to 4 and the other region is x 1 is less than equal to 3 right and then we will separately search in both of this region.

So, let us first take this constraint right. So, since this is 3.8. Let us say we add this particular constraint x 1 greater than or equal to 4 right. So, this is our original problem over here all of it is included over here right, except for this x 1 comma x 2 are integers in addition to this constraint we also add this constraint x 1 is greater than or equal to 4 right. So, if we add this constraint to our problem right. So, the solution of LP 2 can never be better than the solution of LP 1.

So, 8.2 gives us a bound right so, we will not be able to definitely better this 8.2 because now, we are solving a problem which is more constrained than LP 1. So, now, if we solve this particular problem LP 2 now, it is again a linear programming problem when we solve this linear programming problem using simplex method which we have discussed. The optimal solution that we would get is x 1 is equal to 4, x 2 is equal 2.9 with an objective function value of 7.6.

So, that basically is over here. So, this is our solution for LP 2. So, had it been an integer value we would have got a solution which is integer right, but we would still not stop because remember there is another region which we have not explored. So, the other region is x 1 is less than or equal to 3 right, but if we had got an integer over here we would no longer branch this one right, but since it is not an integer what we will have to do is again we will have to branch right.

So, this time we will branch on the variable x 2 right. So, we have 2.9 one region is x 2 greater than or equal to 3. The other region is x 2 is less than or equal to 2. So, right now we are considering this region x 2 greater than or equal to 3, over here also we have this pending wherein x 1 is less than or equal to 3. So, when we add this constraint right to the original problem along with this constraint this particular node is coming through this path right.

So, this constraint has to be added this constraint has to be added. And so we have this from the problem since we had previously added this constraint x 1 greater than equal to 4. Now, we add this constraint x 2 greater than or equal to 3 and since it is a new linear programming problem let us label it as LP 3. So, this linear programming problem is graphically depicted over here right. So, here if we see this is our feasible region where as this particular constraint x 2 greater than or equal to 3 indicates that the feasible region is this.

So, now, we have constraints which are conflicting right. So, we do not have a feasible space at all when we solve this problem we would get a infeasibility right. So, LP 3 is infeasible. So, once we know that this is infeasible we just prune this node right. So, no branching is to be done below this right. So, far what we know is there cannot be a solution which is better than 8.2 right and we do not have a single solution which is integer, but we still have some search space for example, this search space we have not looked into and similarly this branch we have not looked into right.

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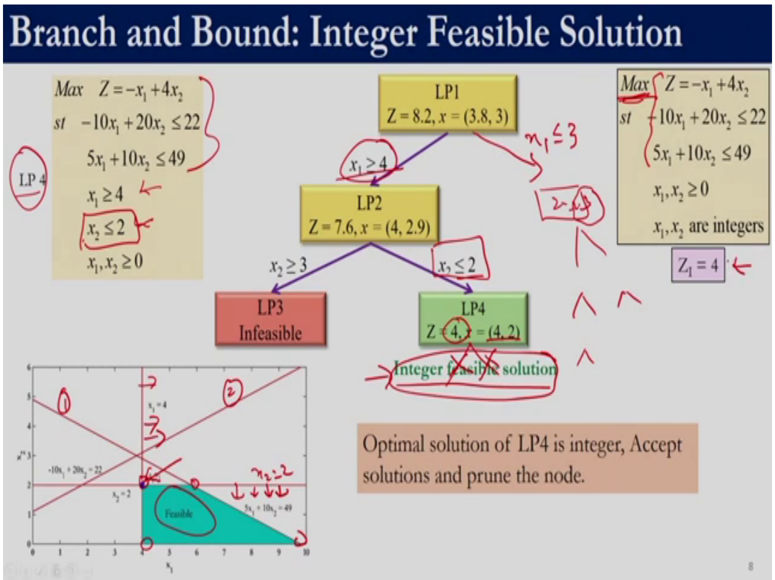

So, now we look into what happens if x 2 is less than equal to 2, so now when we have this constraint right. So, again the new linear programming problem that we have let us term it as LP 4 we need to have all these 3 equations as such right. And then this x 2 is less than equal to 2 is coming after this adding this constraint right. So, we need to still add that constraint x 1 is greater than or equal to 4. And x 2 is less than or equal to 2 right.

So, now this is a linear programming problem we again convert it into standard form and then when we solve it we get a solution 4 comma 2, x 1 is equal to 4 x 2 is equal to 2 with an objective function value of 4 right. So, graphically if we see over here these two constraints right. This constraint 1 and this constraint 2 come from the problem definition itself. This x 1 greater than 4 this region is coming because we are branching for x 1 greater than equal to 4 right?.

And then we have this constraint new constraint which we have added right x 2 less than equal to 2. So; that means, this particular line x 2 equal to 2 right. So, this is the feasible region for x

2 is equal to 2. And considering the other constraints the feasible region is over here right. So, the optimal solution lies at one of those four vertices over here. So, the optimal solution will either lie here or here.

So, in this case on solution you would see that the optimal solution lies at 4 comma 2 right, which is this particular point. And the objective function value at that particular point is 4. So, now, we have an integer feasible solution. So, now, we have a solution which satisfies all these requirements right, but we are not supposed to stop over here because there is another branch which we have not explored which is x 1 is less than equal to 3 right.

So, right now what we can say is to the right of this x 1 greater than equal to 4, the best solution that is possible is 4. So, again now there is no need to branch on this LP 4 right because we have already obtained the integer feasible solution. So, now, we have this information that Z I is equal to 4 right. So, if we get any node. So, for example, if when we are branching this right. So, let us say if we get something over here which says Z is equal to 2.3 right.

Then anything below that no matter how deep we need to go to get an integer solution it will be not better than 2.3. And here we are solving a maximization problem right? So, we do not need to explore it right. So, this bound Z I is equal to four can help us to prune nodes also.

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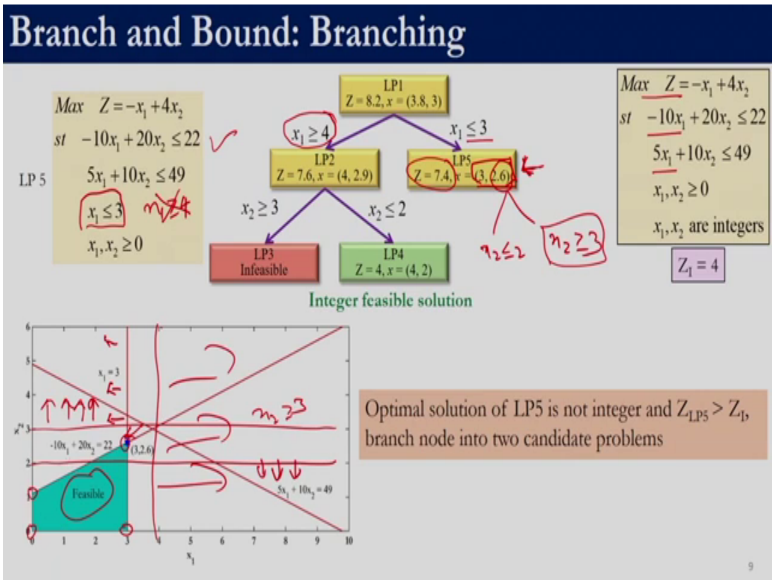

Let us see what happens if we branch on this that x 1 is less than equal to 3. So, over here when we have x 1 less than equal to 3 right. So, the new problem is the objective function over here these two constraint and x 1 less than equal to 3. Remember we are not supposed to add this constraint x 1 greater than or equal to 4 right so, that is only for this particular branching. So, previously we had searched this region right. So, the search in this region is over now we are only searching to the left of x 1 equal to 3 or x 1 less than equal to 3. So, we should not add that constraint x 1 greater than or equal to 4 is not to be added.

So, when we have this linear programming problem right we transform it into the conventional form add slack variables and if we solve the optima would lie at any of this four vertices. So, in this case the optima lies at this particular point 3 comma 2.6 and the objective function value over there is 7.4 right. So, previously we discussed that had it been 4 or less than 4 we do not need to explore it right, but here we have 7.4. So, we cannot prune this node as of now. So, since this is not an integer solution the next step is to again branch on this variable x 2 right.

So, x 2 less than equal to 2 and x 2 greater than or equal to 3. So, x 2 greater than or equal to 3 if you see we will have this line. And this region is what would be feasible for x 2 greater than or equal to 3 right whereas, the other constraints point to this particular region. And hence this is going to be infeasible node whereas, x 2 less than equal to 2 might give us an integer feasible solution because still there is something to be searched over here right?.

So, now let us branch on to x 2 greater than or equal to 3 right. So, x 2 greater than or equal to 3.

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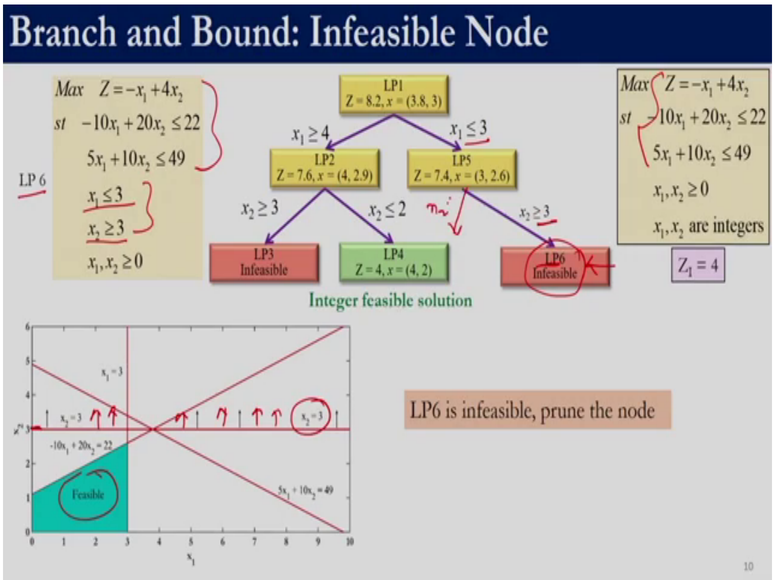

So, this is the line x 2 is equal to 3. So, this region indicated by the arrows is what is x 2 greater than or equal to 3 right. So, as we can see there is a conflict that this constraint points to this region above this x 2 equal to 3 line whereas, the other constraints point to this region as feasible region right.

So, when we solve this problem we will get infeasible right. So, here we write LP 6 as infeasible still we have not completed our search because remember we have not explored this one x 2 being less than equal to 2.

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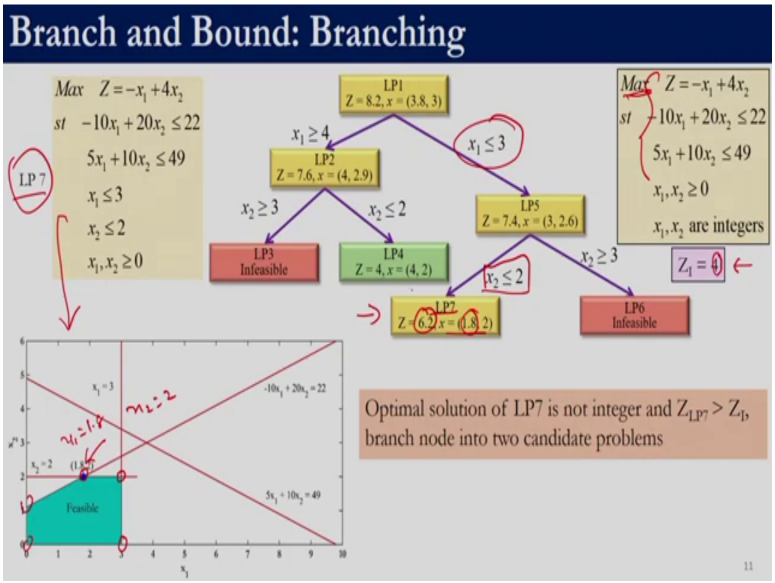

Now, let us look at that particular branch. So, when we have this additional constraint x 2 less than equal to 2. Right now to our original problem we need to add this constraint x 1 less than equal to 3, and this constraint x 2 less than equal to 2.

So, the problem which will get we term it as LP 7. Again that would be a linear programming problem. Graphical representation of this problem is given over here. So, here we can see that we have 1, 2, 3, 4, 5, 5 vertices. So, the optimal solution is going to lie at one of this 5 vertices. So, when we solve it by simplex method we would observe that this is the point x 1 is equal to 1.8. And x 2 is equal to 2 is the optimal solution for this linear programming problem LP 7 with an objective of 6.2.

So, right now what we know is best integer value which we have is 4 right. Over here what we have is objective function is 6.2 and we are solving a maximization problem right. So, there is no guarantee that below this we will not get a better solution right. So, still we need to branch because here this variable is not an integer variable, had we got an integer value over here we could have compared that value with Z I and whichever is better we could have taken that as the final solution, but since this is not an integer solution we need to now branch on x 1.

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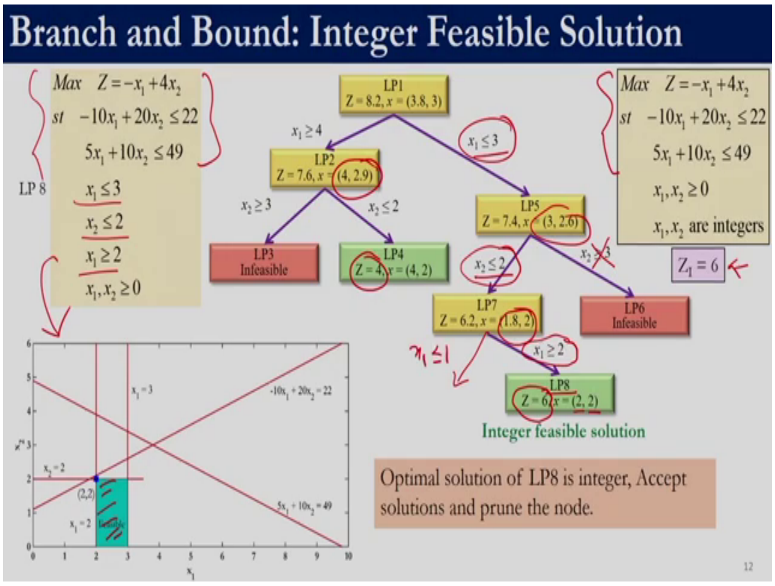

So, let us look at x 1 greater than or equal to 2. So, now, let us term our new problem to be LP 8 right. So, LP 8 will contain these 3 equations right. In addition to that it will have this constraint x 1 less than equal to 3, x 2 less than equal to 2 and x 1 greater than or equal to 2 right. It will not have this constraint because we are not exploring that branch we are exploring this this branch x 1 less than equal to 3, x 2 less than equal to 2 and x 1 greater than equal to 2.

So, we have those 3 additional constraint in addition to those 3 equations of the problem right. So, this again shows the graphical representation of this particular problem. So, now, this is our feasible region as indicated over here when we solve this particular problem using simplex method we get a solution x 1 is equal to 2, x 2 is equal to 2 with a objective function value of Z is equal to 6 right.

So, now the best obtained integer solution is Z I is equal to 6. Previously it was equal to 4 right, because that was the best integer solution nowhere else did we get an integer solution right. So, as in when we get an integer solution we compare it with previously obtained integer solution and update this variable. So, over here if you see we have Z I is equal to 6 and we are solving a maximization problem right. So, since this is 1.8 one region was x 1 greater than or equal to 2, the other region is 1 is less than or equal to 1.

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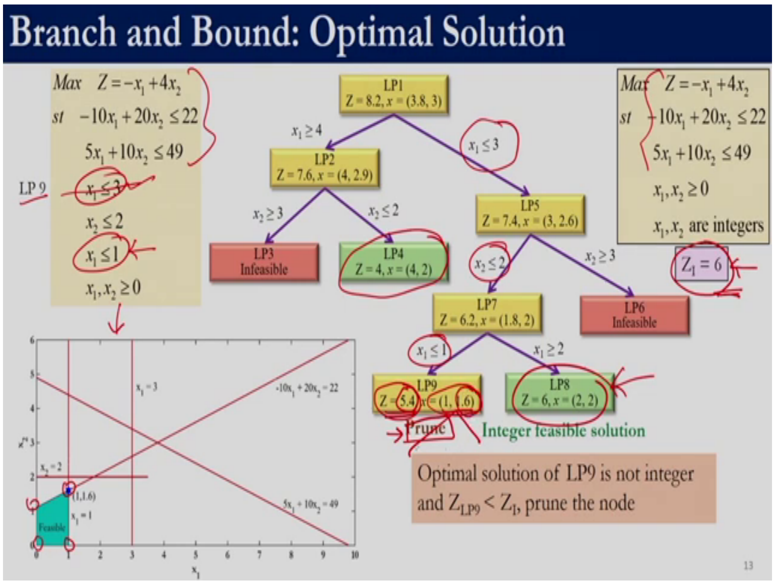

So, now we have added this constraint right. So, to our original 3 equations over here we need to write we need to add x 1 is less than equal to 1, x 2 is less than equal to 2 and x 1 is less than equal to 3. Now, if we look into these two constraint we can say that this constraint is a redundant constraint right.

Because here with this constraint we are limiting the value the maximum value of x 1 to be 1. So, there is no point of saying x 1 is less than or equal to 3 right. So, in this case we can remove this equation and solve using simplex method. So, for this LP 9 this is the graphical representation. Now, these 4 are the vertices right and the optimal solution lies at 1 comma 1.6 and it has an objective function value of 5.4.

Now, we already have an integer solution over here right which has an objective function value of 6 right. So, there is no point of branching this further right, because even if we branch even if we were to get an integer solution it will not be better than this 5.4. Remember we are solving a maximization problem. So, there is no need to branch on this variable because we use this information that the best integer solution which we have is 6 and anything below this would not be better than 5.4 right.

So, since we already have 6 and anything below this would not be better than 5.4 enhance better than 6 there is no need to explore further we can prune that node. And now, if see we have explored all the branches and the best integer solution which we currently have is Z I is equal to 6 which is corresponding to LP 8. Remember if we had been satisfied with this LP 4 that we have got an integer solution we would have missed out on this solution Z is equal to 6.

So, therefore, it is necessary that we should not stop as soon as we get an integer solution they still need to explore the rest of the branches right, when we say we need to explore the rest of the branches we need to make use of this information. So, right now we may say that we can branch on x 2 right that is true we can branch on x 2, but we are not going to get a better solution than 6. This is how we use branch and bound method to solve an integer linear programming problem.

If we have a mixed integer linear programming problem we would still employ the same thing, but we would not be branching on those variables which are continuous we would be only branching on those variables which are integers.

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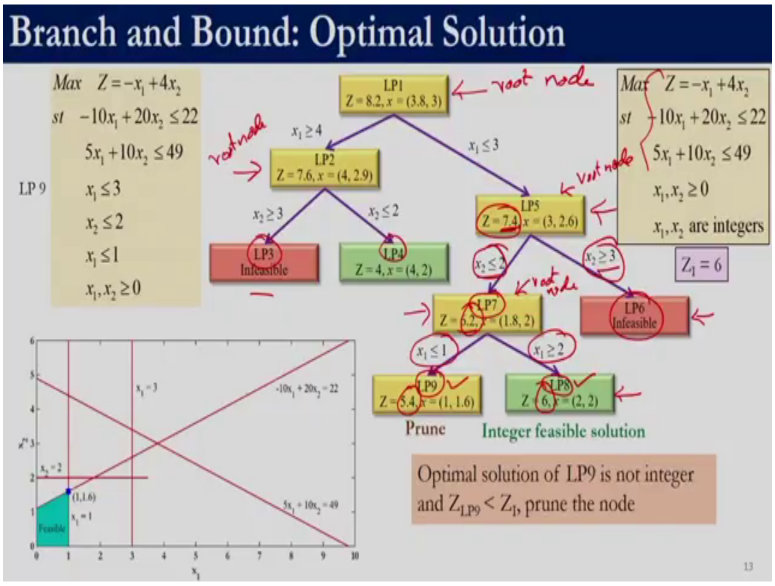

So, here if we see that all the branches below a particular node would have only solutions which are inferior to the root node. So, here this is the root node right. So, any solution below this right will not have objective function better than 8.2. So, in this case since it is a maximization problem it will not be greater than 8.2 right. So, this is the root node for these two problems for LP 3 and LP 4. So, LP 3 and LP 4 even without solving we can say that their objective function value will not be greater than 7.6 right.

So, for these two nodes this is the root node. Similarly over here for LP 5 if we consider LP 5 as the root node, anything below this root node will not have a objective function value greater than 7.4 right. So, here if we say it is 6.2 this is an infeasible problem here it is 6 and here it is 5.4 right because LP 5 is a relaxed problem when compared to LP and LP 6 right, because in LP 6 and LP 7 we have added these two constraints.

So, now, if we compare LP 7, LP 8 and LP 9 this is the root node right. So, this has an objective function value of 6.2. So, neither LP 8 nor LP 9 would have a value better than 6.2, because LP 7 is a relaxed problem when compared to LP 9 and LP 8 right. So, LP 8 and LP 9 have an additional constraint here it is x 1 greater than or equal to 2 and here it is x 1 less than or equal to 1. So, we can use this information to prune some of the nodes.

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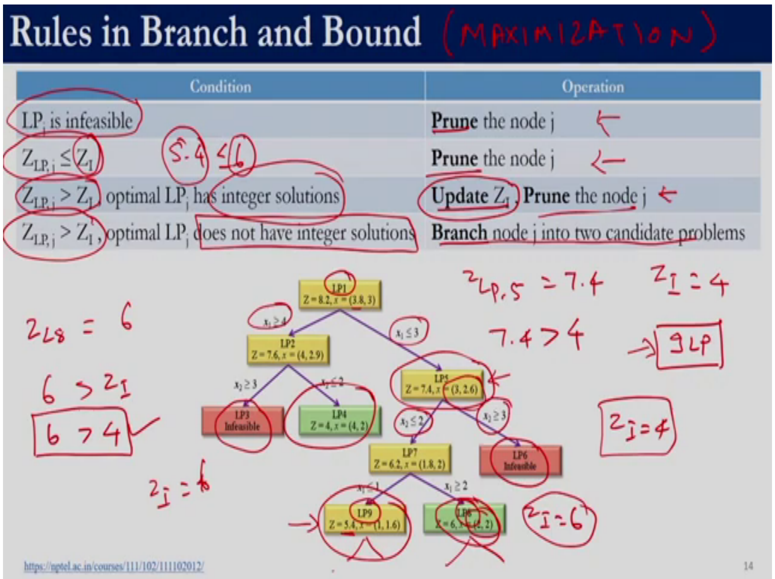

So, this gives a complete picture of what we solved right. So, we had one integer programming problem involving two variables, but we ended up solving 9 linear programming problems. This can help you understand that though the number of variables and constraints may be few in an integer linear programming problem it can become computationally very intensive because we solve a large number of linear programming problems.

So, if we consolidate the operations which we have done. So, we were either pruning a particular node right, when we say pruning a particular node we do not explore further into it or we were branching on a particular node right or we were pruning or we were updating the best integer solution which we have found.

So, these are the 3 operations which were broadly implementing. A node can be prune under 3 condition right. If the linear programming itself is infeasible then we can prune the node. So, for example, here LP 3 became infeasible. So, we did not branch on this right similarly here LP 6 become infeasible. So, we did not branch on this right. So, in that case we prune the node.

We also prune a particular node if its objective function value is lower. Than this rules which we have over here is for a maximization problem if we get a solution for a linear programming problem which has an objective function value less than or equal to the best integer which we have so far. So, for example, when we were solving LP 9 if you remember we already had the information that Z I is equal to 6 right. Since ZI is equal to 6 and this linear programming problem itself is 5.4 right.

So, over here this 5.4 is less than or equal to si6x right. We have an integer solution which has an objective of 6 and we have a node whose solution is 5.4 right. So, even if we branch on that particular node we are not going to get a better solution. So, we prune that node in that case right. So, here also we did not explore it is not like this was infeasible this was feasible, but we did not branch on this we just pruned this particular branch because we have a better solution right.

So, that is the second case right. If you have a linear programming problem whose objective function value is greater than the best that we have right. So, when we were solving LP 8 Z LP 8 is 6 right. So, the 6 is greater than Z I right, Z I is 4 when we were solving this node. So, since this condition satisfied what we did is? We updated the ZI right. So, Z I if you remember we updated it from 4 to 6 and pruned that node.

So, we do not need to explore this node further because we already have integer solutions right, 2 comma 2 is an integer solution right. So, that is our third condition. So, the fourth condition is what we commonly encountered. So, for example, when we solved LP 1 we did not get an integer solution. So, then what we do is branch node j into 2 candidate problems. So, x 1 greater than or equal to 4 and x 1 less than or equal to 3.

The same thing happened over here Z LP 5 is 7.4 right and Z I which we had is when we were solving LP 5 right. So, Z I that time what we had is 4 right. So, since this 7.4 is greater than 4. It means that if we further explore this LP 5 we might get a better solution, but since these are not integer’s right. What we will have to do? Branch it into 2.

So, over here we branched it into x 2 less than equal to 2 and x 2 greater than or equal to 3. These are the basic rules of branch and bound, with these rules we are in a position to solve integer linear programming problems or mixed integer linear programming problems. So, with that we will conclude the session on integer linear programming.

Thank you.