Transport Phenomena of Non-Newtonian Fluids
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Lecture - 09
Rotational Viscometers - II

Welcome to the MOOCs course Transport Phenomena of Non-Newtonian Fluids, the title of this lecture is Rotational Viscometers part II. In the previous lecture we started discussing about the working principles for rotational viscometers. In the rotational viscometers category, we started with concentric cylinder rheometers right. For that case we have developed equation for the shear stress calculations and then we have also developed equation for a shear rate calculation as well.

So, we will be having a recapitulation of whatever the things that we have discussed in the previous lecture.

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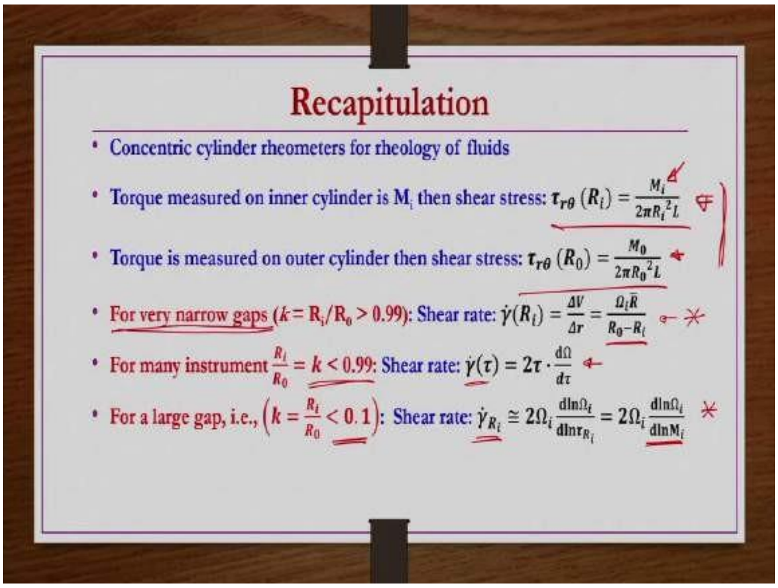

When you use the concentric cylinder rheometers for measuring the rheology of a fluid then the shear stress that you can measure by using the torque. If you know the torque then you can measure the shear stress easily by $\frac{M_i}{2\pi R_i^2 L}$; where M_i is nothing but torque and then R_i is the radius of the inner cylinder. This equation you can use when you are measuring; when the torque is measured on inner cylinder ok. This L is nothing but the height of the cylinder ok.

So, if the torque is measured on outer cylinder then shear stress you can calculate by using this equation $\frac{M_0}{2\pi R_0{}^2 L}$; M_0 is nothing but the torque measured on the outer cylinder. And then R_0 is nothing but the radius of the outer cylinder, L is nothing but the height of the cylinder, right.

So, this concentric cylinder the annular space whatever is there that is very narrow. Then what is the shear rate expression? If that gap is very large then what is the expression for the shear rate those things we have seen.

For very narrow gaps that is when $\frac{R_i}{R_0}$ is greater than 0.99. That means, almost they are touching to each other then what we can say? We can say that the curvature effects would be negligible. And then avoiding the curvature effects we got that shear rate is nothing but $\frac{\Omega_i \bar{R}}{R_0 - R_i}$. $\bar{R}$ is nothing but the midpoint distance between R_0 and R_i that is $\frac{R_0 + R_i}{2}$.

But however, it is always not possible to have such narrow gaps, but majority of the concentric cylinder rheometers commercially are available. They are having k value less than 0.99 right. So, it is not true for all the cases, but majority of them will have k less than 0.99. So, then you cannot say that the curvature effect is negligible and then you start using this equation.

This equation you can use only when you say the curvature effect is negligible, that you can see when the gap between outer cylinder and inner cylinder is very very small negligible gap is there right. When the gap is sufficiently large enough, then shear rate expression we obtain as $\dot{\gamma}\ (\tau) = 2\tau . \frac{d\Omega}{d\tau}$.

τ we already obtained either of these expressions. So, τ is known and then Ω rotational velocity; at what rotational velocity the cylinder is rotating that anyway we know experimentally. So, you know all the quantities in the right-hand side. So, then left-hand side shear rate you can find out right.

So, it is a generalized one where the k is less than 0.99 ok. But for a very large gap case when k is smaller than 0.1 then we obtained this $\dot{\gamma}\ (R_i)$ that that is nothing but $\dot{\gamma}$ at inner cylinder of radius R_i. So, that is $\dot{\gamma}_{R_i} \cong 2\Omega_i . \frac{d \ln \Omega_i}{d ln \tau_{R_i}}$. Since τ is directly proportional to torque.

This equation can also be written as $2\Omega_i.\frac{d\ln\Omega_i}{d\ln M_i}$. This is what we can have ok.

So, now shear stress equation you are having only say one equation irrespective of the gap ok. So, but anyway you do not go for a very large gap in general, but moderately moderate gaps in general we use right. However, we have developed the equation. So, shear stress is anyway irrespective of the gap we can use any of this equation. But when you wanted to calculate the corresponding shear rates then you first calculate what the value of k is and then accordingly you have to choose an equation.

So, if gap is very narrow so, then you can use this equation for shear rate, if gap is very less then you can use the last equation for the shear rate; but sometimes we may have intermediate gap also. So, then what expression for shear rate should we use; that is what we are going to see now in this lecture.

For fairly narrow gap that is when 0.5 less than k less than 1. So, that is k between 0.5 and 1 then what should we do?

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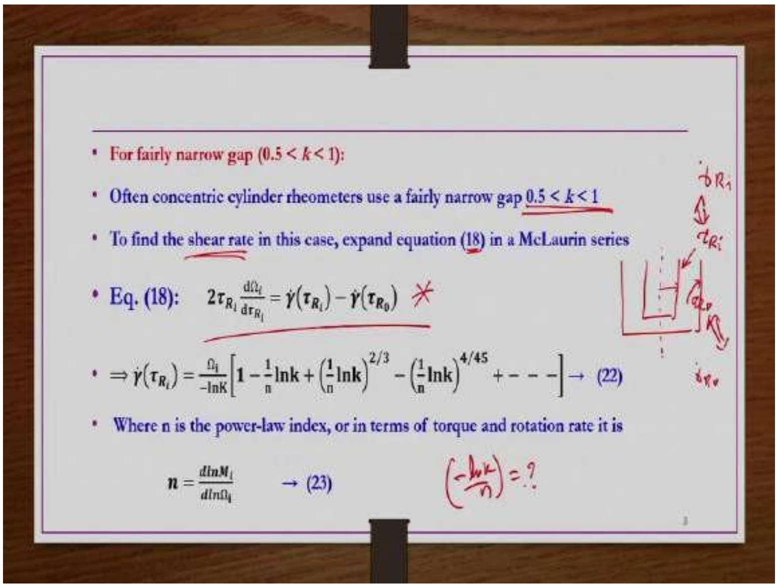

So, often concentric cylinder rheometers are using such fairly narrow gap; you cannot say very narrow, you cannot say very large, but in between these regions. So, then what we have to do? In order to find the shear rate in this case where the gap is fairly narrow then whatever the equation 18 in the previous lecture that we have derived we have to apply

McLaurin series for that equation. What is the equation number 18? Is nothing but this equation that is $2\tau_{R_i}.\frac{d\Omega_i}{d\tau_{R_i}} = \dot{\gamma}\left(\tau_{R_i}\right) - \dot{\gamma}\left(\tau_{R_0}\right)$.

Recapitulating again, τ_{R_i} suffix R_i is nothing but the shear stress measured on the inner cylinder surface, τ_{R_0} is nothing but shear stress measured on the surface of the outer cylinder ok. So, that is the difference right geometries that we have like you know this concentric cylindrical geometry.

So, this is R_i so, if you are measuring shear stress on the surface then we call it τ_{R_i}, if you are measuring shear stress on the surface here then we call it τ_{R_0} ok. Corresponding gamma dots are nothing but $\dot{\gamma}_{R_i}$ and $\dot{\gamma}_{R_0}$ that is what we have seen these are the notations that we are using last class and then this class as well.

So, now, this equation when you apply the McLaurin series this equation number 22 you get for $\dot{\gamma}\left(\tau_{R_i}\right)$ or $\dot{\gamma}_{R_i} = \frac{\Omega_i}{-lnk}\left[1 - \frac{1}{n}lnk + \left(\frac{1}{n}lnk\right)^{2/3} - \left(\frac{1}{n}lnk\right)^{4/45}\right]$. This is what we have.

Now in this equation Ω_i is known already to us right. Ω_i is already known through the experimental you know experimentation at what rotational velocity are you rotating the inner cylinder right; k you know ratio between radius of inner cylinder and radius of outer cylinder that is k that also we know right. What is n? n is nothing but $\frac{dln\, M_i}{d\ln\Omega_i}$ ok. It is nothing but power law index, but in terms of torque and rotational rate we can write $n = \frac{dln\, M_i}{d\ln\Omega_i}$.

Now this equation you know the series goes on. So, how many terms should we include? After how many terms we can truncate the series? That is another important question right. That depends on the value of $-\frac{1}{n}lnk$; how much value is it having? So, accordingly we have to take number of terms in the series ok.

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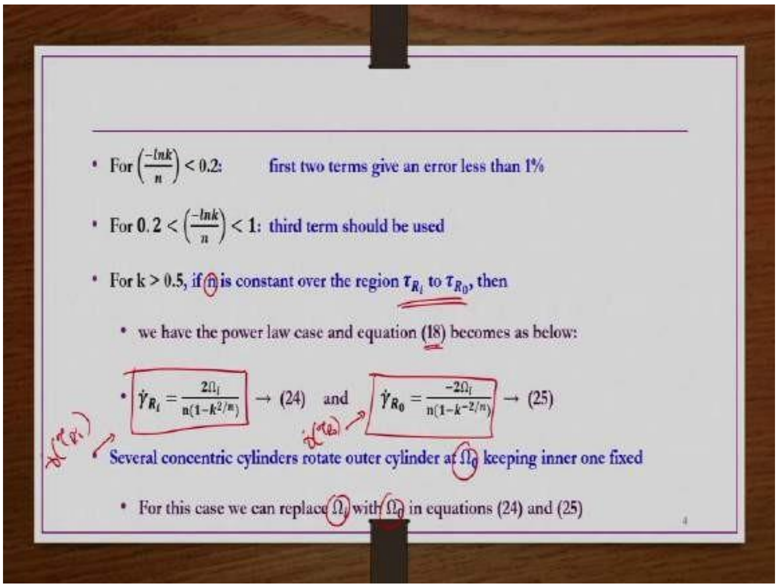

If $-\frac{1}{n} lnk$ is less than 0.2 then first two terms give an error of less than 1 percent, so you can include only first two terms that is sufficient. But, $-\frac{1}{n} lnk$ is between 0.2 and 1 then third term should also be included in the series previous equation number 22.

And then for k greater than 0.5 if n is constant; if n is constant over the entire region of τ_{R_i} to τ_{R_0}, then we have the power law case and the equation 18 becomes as below. We are not going to simplify this equation, but anyway you can do it in a straightforward.

So, $\dot{\gamma}_{R_i}$is nothing but $\dot{\gamma}\left(\tau_{R_i}\right)$ $\dot{\gamma}\left(\tau_{R_0}\right)$is nothing but $\dot{\gamma}\left(\tau_{R_0}\right)$ ok. That is what the notation that we are having ok. So, $\dot{\gamma}_{R_i}$ is nothing but $\frac{2\Omega_i}{n(1-k^{2/n})}$ and then $\dot{\gamma}_0$ is nothing but $\frac{-2\Omega_i}{n(1-k^{-2/n})}$.

So, these are the equations for the shear rate depending on the gap which equation should be used that now we should be very careful. Because now anyway we have the knowledge whether it is narrow gap, whether it is large gap, whether it is fairly narrow gap; so, then we have the equations for the shear rate. Shear stress equation we have already obtained irrespective of the gap ok.

So, this is what about the concentric cylinder rheometers if you are using to measure the shear stress and shear rate ok. Now, we will be discussing about how to measure the normal stress using the concentric cylinders right.

However, before going into those topics what do you how to observe? It is not necessary that always the inner cylinder is rotating, many a time's inner cylinder is fixed and then outer cylinder is rotating. Sometimes it is also possible that both inner and then outer cylinders are rotating at different speeds.

Let us say if inner cylinder is fixed and then outer cylinder is rotating, then simply what you have to do? You have to you know replace Ω_i with Ω_0. Ω_0 is nothing but the velocity at which outer cylinder is rotating ok. Then you can use these two equations for shear rate, right.

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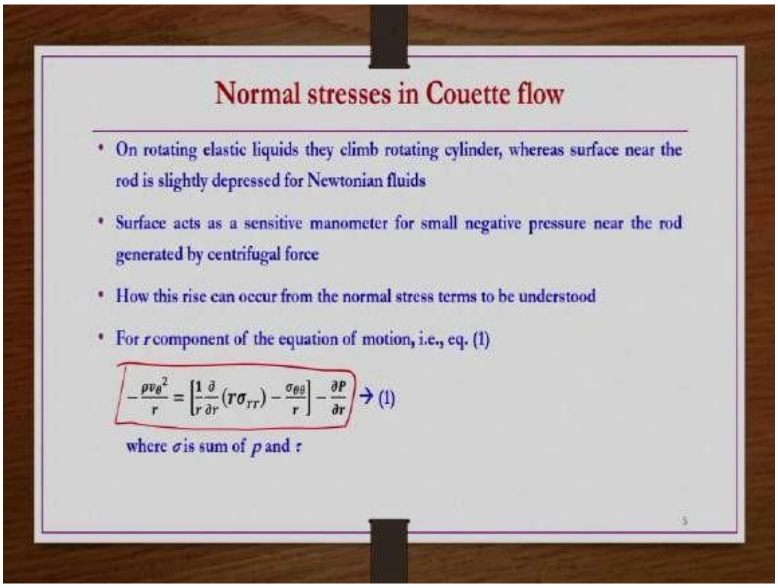

Now, we will be discussing how to measure normal stresses in Couette flow or concentric cylinder geometry whatever is there; concentric cylindrical geometry when the narrow gap is there. So, then that geometry in that geometry when there is a flow then that flow is often called as a Couette flow.

So, normal stresses in Couette flow that is normal stresses and concentric cylinder geometry that we are going to discuss now. What we have seen that on rotating elastic liquids or viscoelastic liquids they climb the rotating cylinder; that we already seen when we were discussing about the classification of non-Newtonian fluids.

However if the fluid is Newtonian then we can see slight depression at the surface; near the surface that you know the liquid Newtonian fluid surface becoming slightly depressed. Whereas the viscoelastic surface is climbing the rod on rotation. So, that we know, but why this climbing of rod takes place? That also we have discussed because of the normal stresses. So, can we prove that mathematically now? That is what we are going to see now.

Surface this, whatever the liquid surface climbing the rod that acts as a sensitive manometer for small negative pressure near the rod generated by the centrifugal force on rotating the cylinder. How this rise can occur from the normal stress terms to be understood now.

So, now in the previous lecture what we have done? We have done simplification of r, theta and z components of equations of motion in cylindrical coordinates. And then each equation has given some kind of information; one was giving information about the normal stress or you know providing a simplified equation to get the normal stresses. One equation providing the simplified equation to get the information about the shear stress. Another equation was relating the hydrostatic pressure and then gravitational force and all those kind of things.

So, r component of equation of motion yesterday we simplified we got a simplified expression which provide information about the normal stresses. So, that equation I have rewritten here again. In the previous lecture we have simplified r component of a momentum equation for the case of concentric cylinder. So, this is what we got; this equation we got ok.

So, now this equation we make use in order to get the information about the normal stresses. So, here sigma is nothing but the sum of P and τ that now we substitute here.

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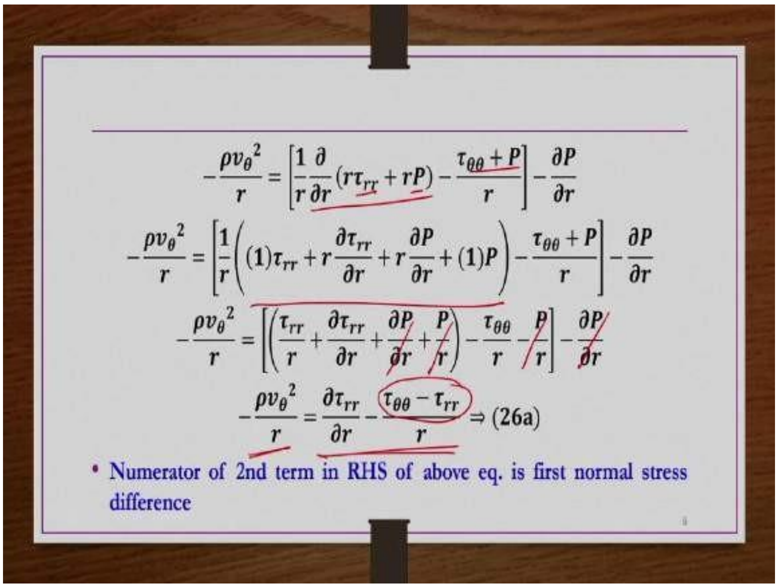

In place of sigma we are substituting τ + P. So, then we have σ_{rr} is nothing but τ_{rr} + P right and then $\sigma_{\theta\theta}$ is nothing but $\tau_{\theta\theta}$ + P. So, this one is there right; $-\frac{\partial P}{\partial r}$ is anyway is there. So, when you differentiate this particular term then you have τ_{rr} + r $\frac{\partial \tau_{rr}}{\partial r} + r\frac{\partial P}{\partial r} + P$ and this term as it is last term is also as it is.

Further, if you bring this r inside of the parentheses then you get $\frac{\tau_{rr}}{r} + \frac{\partial \tau_{rr}}{\partial r} + \frac{\partial P}{\partial r} + \frac{P}{r} - \frac{\tau_{\theta\theta}}{r} - \frac{P}{r}$. These two terms are coming from here. So, then what we have? This $\frac{\partial P}{\partial r}$, this $-\frac{\partial P}{\partial r}$, this $+\frac{P}{r}$, this $-\frac{P}{r}$ can be cancelled out.

So, then finally, we have $-\frac{\rho v_\theta^2}{r}$ in the left-hand side remaining as it is that should be $= \frac{\partial \tau_{rr}}{\partial r} - \frac{\tau_{\theta\theta}-\tau_{rr}}{r}$. This is what we have. And then this information $\tau_{\theta\theta} - \tau_{rr}$ is nothing but the first normal stress difference ok.

So, further we can simplify this equation.

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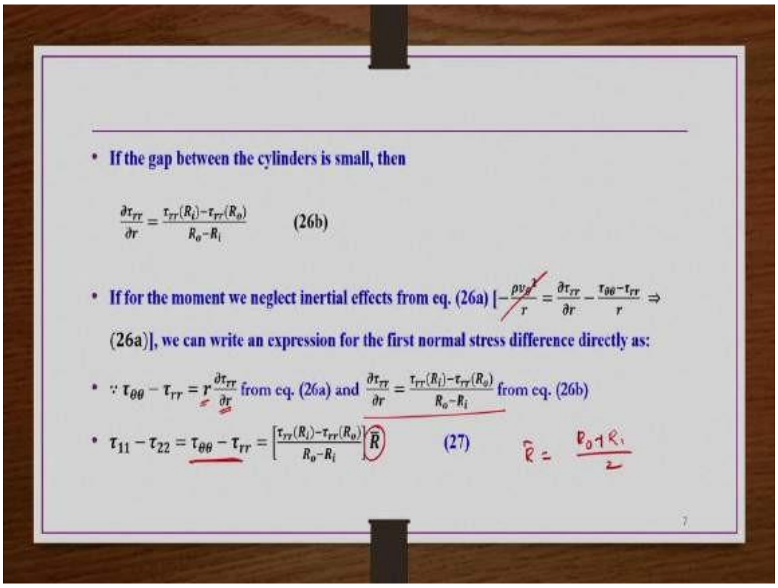

If the gap between the cylinder is small then what can we see? Whatever this $\frac{\partial \tau_{rr}}{\partial r}$ is there we can apply Taylor expansion and then we can write this equation as $\frac{\tau_{rr}(R_i)-\tau_{rr}(R_0)}{R_0-R_i}$. That is what we can simply write right without any difficulty.

So, from this equation 26 after striking off the inertial term then we can have $\tau_{\theta\theta} - \tau_{rr} = r\frac{\partial \tau_{rr}}{\partial r}$. And then from equation 26 b we have $\frac{\partial \tau_{rr}}{\partial r} = \frac{\tau_{rr}(R_i)-\tau_{rr}(R_0)}{R_0-R_i}$. This is what we have.

So, now in place of $\frac{\partial \tau_{rr}}{\partial r}$ we will be writing this expression and then in place of r we have to select the position at which we are measuring the normal stress. So, then these R usually we take $\bar{R}$ that is midpoint between R_i and then R_0. So, first normal stress difference $\tau_{\theta\theta} - \tau_{rr}$ you can find out from this expression easily without any difficult, ok.

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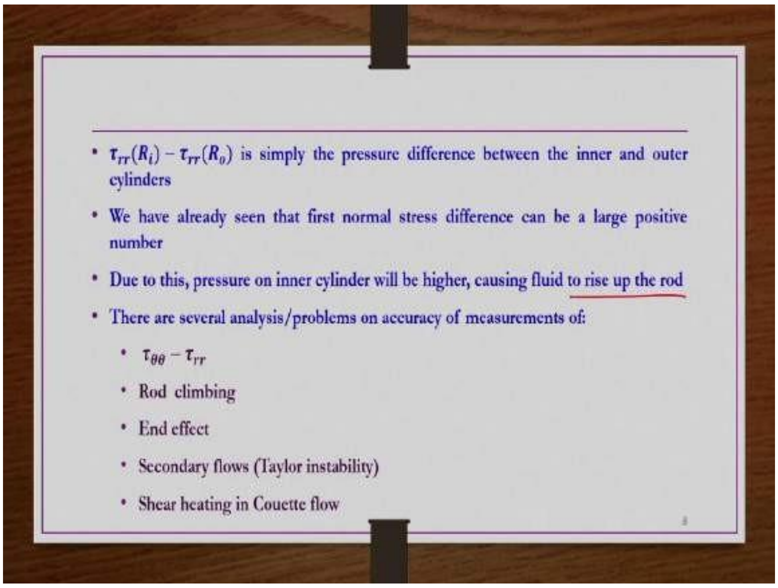

$\tau_{rr}(R_i) - \tau_{rr}(R_0)$ is simply the pressure difference between the inner and outer cylinders. And then we have already seen that first normal stress difference can be largely positive number when we have taken example of viscoelastic fluids.

So, then due to this pressure on inner cylinder will be higher causing the fluid to rise up the rod. That is the region you know the rising of the viscoelastic fluid, along the rotating rod or rotating cylinder in this case is taking place because of this normal stress; normal stress differences ok.

So, this is all about how to measure the normal stress; normal stress differences, shear stress, shear rate etcetera using the concentric cylinder rheometers. Now there are some issues in a with respect to the accuracy of these equations, because of the several other issues like you know end effects, slip effects, etcetera as we have seen in the capillary viscometer also.

Some of them that one should be worried in the case of concentric cylinders are you know what is this first normal stress difference how much it is then rod climbing effect, then end effect, then secondary flows because of Taylor instabilities, and then shear heating in Couette flow etcetera these are the some issues one can think of.

But however, we are not going into the details of all of these topics because we are concentrating we supposed to concentrate on the applied rheology. So, since we are talking about applied rheology some basics are required. So, then that is what we are discussing. We do not need to go into the details of all these things anyway.

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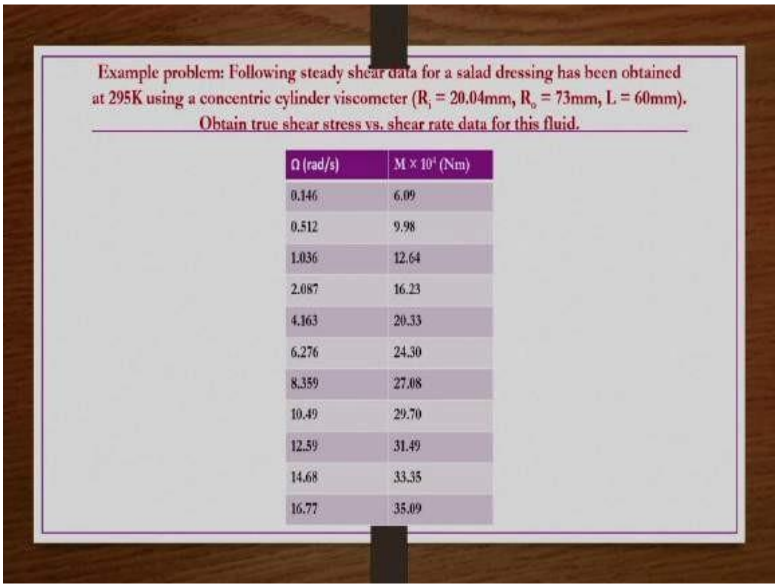

Example problem: Following steady shear data for a salad dressing has been obtained at 295K using a concentric cylinder viscometer (R_i = 20.04mm, R_o = 73mm, L = 60mm). Obtain true shear stress vs. shear rate data for this fluid.

Ω (rad/s)	$M \times 10^4$ (Nm)
0.146	6.09
0.512	9.98
1.036	12.64
2.087	16.23
4.163	20.33
6.276	24.30
8.359	27.08
10.49	29.70
12.59	31.49
14.68	33.35
16.77	35.09

So, now we take an example problem. Following steady shear data for a salad dressing has been obtained at 295 Kelvin using a concentric cylinder viscometer, R_i 20.04 mm, R_0 73 mm, L 60 mm; obtain the true shear stress versus shear rate data for this fluid. What is the data is given? Data is given this *Ω* in radians per second is given and then torque in Newton meters it is given, ok.

So, if omega is given torque is given so, and then you can find out the shear rate, shear stress respectively. So, torque you can use to get the shear stress expression, *Ω* you can use to get the shear rate expression, but in order to get the shear rate expression we have to be specifically clear about what is the value of k, then only we have we can select the corresponding equation for shear rate.

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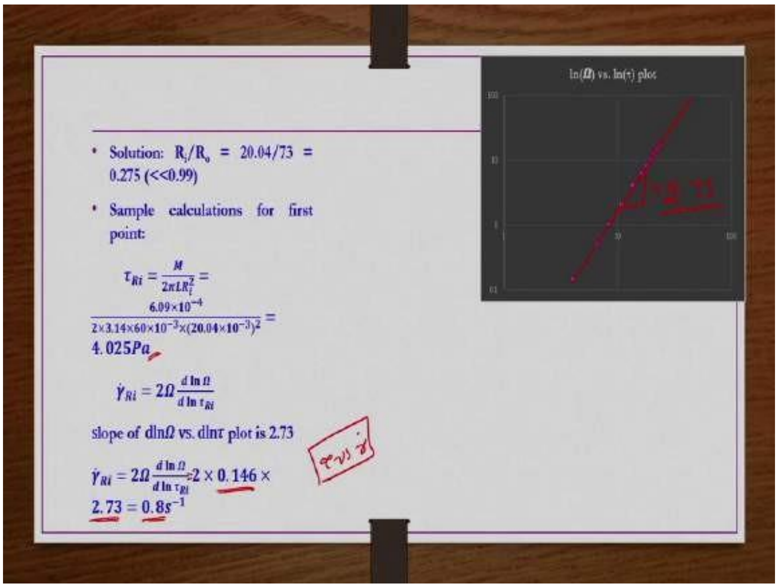

So, what we see? $\frac{R_i}{R_0}$ in the case in this case point $\frac{20.04}{73}$ that is 0.275 that is very much smaller than 0.99. So, then we cannot say that the curvature effect is negligible, but it is more than the 0.1. So, then we cannot say that the gap is very large. So, then what we have to do? We have to take a case where fairly narrow gap equations are provided right.

So, that is sample calculations shear stress $\tau_{R_i} = \frac{M}{2\pi {R_i}^2 L}$ this we know first data point M is given as 6.09 into 10^{-4} Newton meters $2\pi L$ is 60 mm R_i is 20.04 mm. So, when you substitute these numbers and do the simplification you get τ_{R_i} 4.025 Pascal.

So, for this k value of 0.275 appropriate equation of shear rate is $2\Omega \frac{\mathrm{d}\ln\Omega}{\mathrm{d}\ln\tau_{R_i}}$. That means, for each M_i value what is the corresponding τ_{R_i}value we have to calculate as we have calculated for the first point; Ω corresponding to that one what is the Ω value is already given.

So, then Ω versus τ_{R_i}data whatever is there that you have to plot on a log-log scale and then get the slope of that curve. So, that comes out to be 2.73. When you plot ln Ω versus ln τ you get a straight line like this and then when you get the slope of this one you will get it as 2.73.

So, first point $\dot{\gamma}_{R_i}$ is nothing but $2\Omega\frac{\mathrm{dln}\Omega}{\mathrm{dln}\tau_{R_i}}$ is equals to 2 multiplied by Ω first data point omega is 0.146 and then $\frac{\mathrm{dln}\Omega}{\mathrm{dln}\tau_{R_i}}$ is nothing but 2.73. So, $\dot{\gamma}_{R_i}$ 0.8 second inverse for the first data point. Likewise we have to do for all the data points then you have τ versus $\dot{\gamma}$ information. That information is provided here in the tabular format ok.

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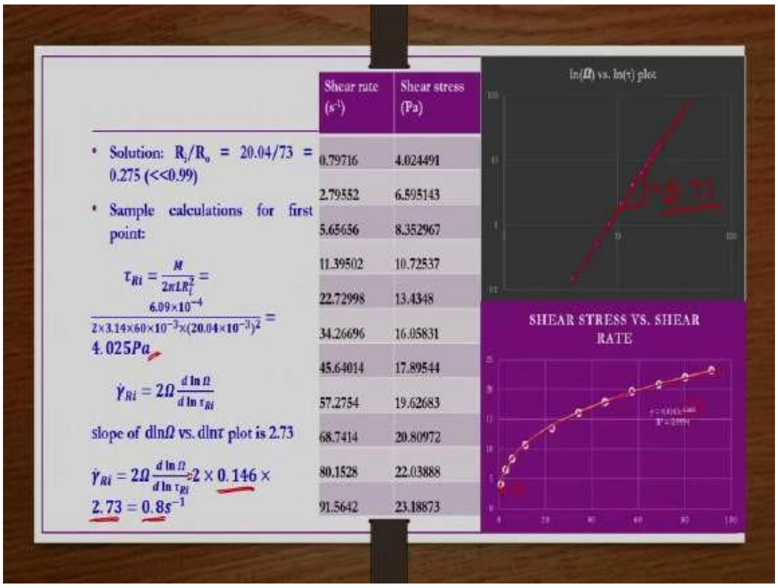

So, now when you plot it what you can realize that. This material is slightly viscoplastic because the tau naught is there, but it is very small value approximately 3 Pascal or something like that ok; 4 Pascal or something like that. And then this n value is coming out to be 0.3661. So, it is a Herschel-Bulkley fluid with mild yield stress value ok.

So, now we have done example problem also how to use the equations of a concentric cylinder rheometer to obtain the rheological behaviour of any given fluid, if you know the omega versus the rotational velocity versus torque information from the experimental data ok.

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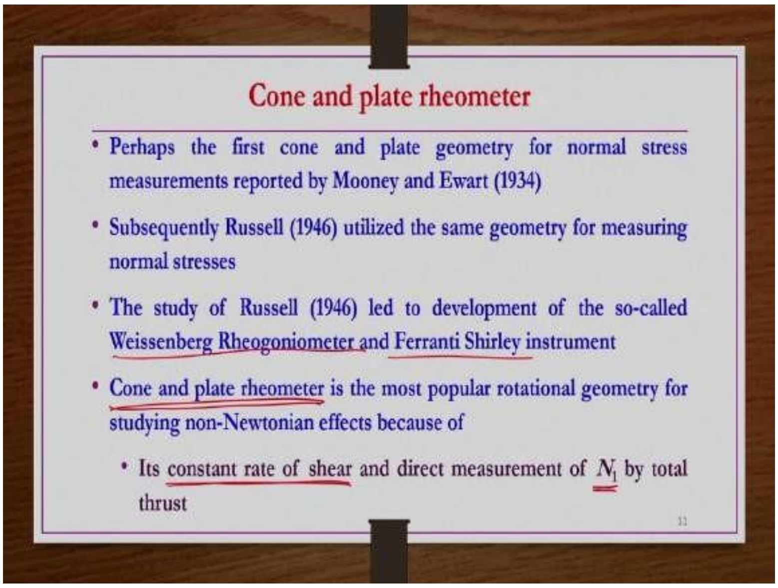

Now, we take another geometry cone and plate rheometer ok. So, perhaps this is the first geometry used to obtain the normal stress differences for viscoelastic fluids and then that has been reported by Mooney and then Ewart. Mooney and Ewart in 1934, but however, this work extensively carried forward by Russell to measure the normal stresses. And then because of work by Russell the present Weissenberg Rheogoniometer, Ferranti Shirley instrument etcetera have been developed. That is purely because of the Russell's work ok.

So, in the current scenario cone and plate rheometer is the best option if you wanted to know the rheological behaviour of non-Newtonian fluids. Why? Because one reason is the constant shear rate and the geometry you can maintain the shear rate constant and then direct measurement of first normal stress differences; first normal stress difference by a total thrust. So, because of these two reasons so this is the best rheometer for studying the rheology of non-Newtonian fluids.

So, now, that we are going to see how to develop the equations. So, like you know in the concentric cylinder case, we develop the equation for shear stress as function of torque and then shear rate as function of rotational velocity normal stresses as function of the pressure differences etcetera all those things we have seen.

How we got that information? We got that information by simplifying the momentum equations. So, same thing we here also we are going to do. We are going to simplify the equations of motion and then try to get the simplified equations for a cone and plate geometry of a certain constraints. So, what are those constraints, what are the, how the geometry looks like that is what we are going to see now.

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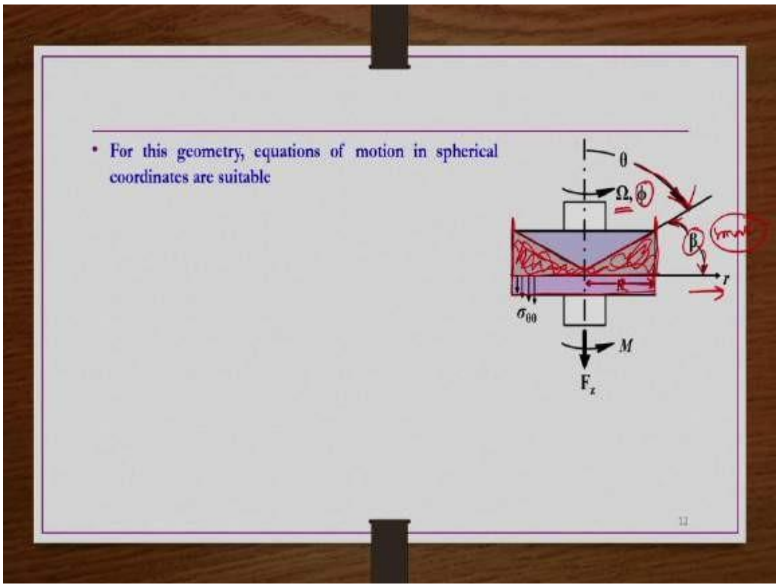

Since it is a cone and plate geometry. So, there is a plate and then there is a cone kind of arrangement is there something like that ok. So now, here the plate is having this kind this plate is here right the radius of this plate is R. Now a cone is placed on this geometry something like this ok. So, this is your cone right the cone angle is β ok.

Now, what we have to realize? So, let us say let us say this is the cone you are putting like this upside down like this right. So now, I am just putting up like this to have a kind of clarity. So now, this direction it is rotating that direction of rotation whatever is there that is ϕ direction right and then θ direction is in this direction ok. So, this position and this position is the radial direction ok.

So, now the rotation is in the ϕ direction rotation rotating; it the cone is rotating in the ϕ direction at a velocity Ω right. θ is measured from the top, so we can see the arrow here. So, this arrow indicates that θ is measure is measured from this top like this. Cone angle is only β small very small cone angle we in general use right. So, now, this direction is here r direction, ok.

So, once it is clear. So, then whatever the fluid is there that is being confined in between this cone and plate right when the cone is rotating deformation in the fluid will take place because of the rotation of the cone ok. So now, once you once there is a deformation, so, the corresponding shear rate corresponding shear stress we have to measure ok.

Since here the cone angle, whatever this cone angle is there that is a very very small in general the cone angle is taken very small. So, that you know whatever the shear rate etcetera is there. So, that can be maintained constant ok.

So now, here the cone angle that is taken very small in general for most of the cone and plate geometry, because the purpose is that when the β cone angle is very small. So, for one thing is that you can avoid the curvature of it and another thing that you know the shear rate can be maintained constant ok. So, the gap is narrow, so then we can say that shear rate almost remains constant. So, that is the advantage.

So, assumptions here flow is rotating in free direction.

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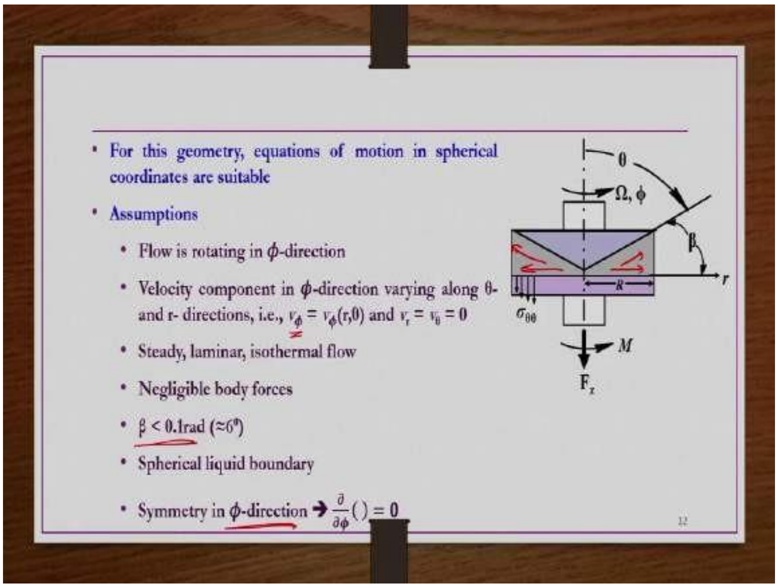

So, the only velocity component existing in ϕ direction so, that is v_ϕ is there v_ϕ is there. So, this ϕ now function of r and θ because this fluid is whatever you say that is because of the rotations is moving towards the r direction as well as you know it is moving towards the theta direction also because of the cone shape ok. So, we cannot say that v_ϕ is function of r only it is also function of θ ok.

So, and then, but however, compared to v_ϕ v_r v_θ 0. Then flow is steady laminar and isothermal, negligible body forces and then β cone angle is very small 0.1 radiance that is approximately 6 degrees or less. So, such small cone angles in generally used in order to maintain the constant shear rate.

Then spherical liquid boundary is there because the cone is can be best represented by the spherical coordinates. And then symmetry in ϕ direction, so $\frac{\partial}{\partial\varphi}$ if anything is 0 ok. So, these are the constraints.

So now, using these constraints we are going to simplify r, θ, ϕ components of equations of motion so that we get some simplified equation. So, that we get some simplified equations which we can use to get the information about shear stress, shear rate and then normal stress differences etcetera.

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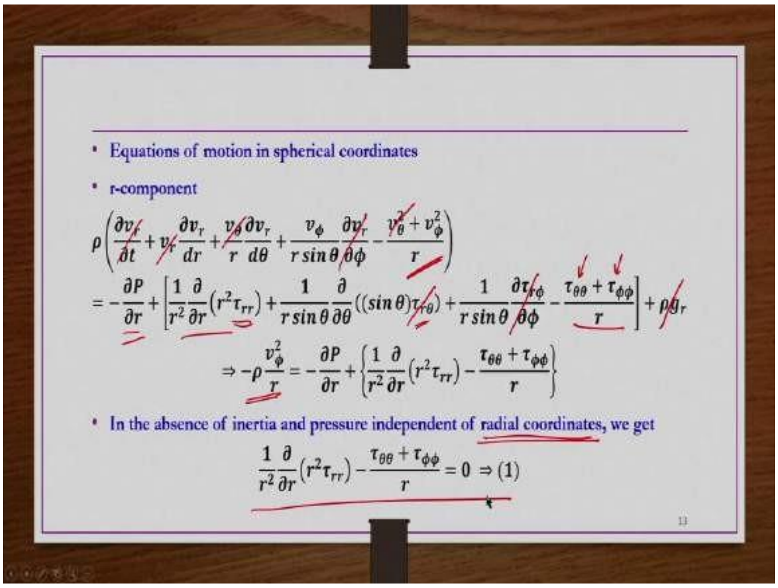

So, let us start with r component. So, r component of equation of motion in spherical coordinates this is what we are having. So now, this equation we are going to apply the constraints that we have a seen in the previous slide.

So, it is a steady flow. So, first term is 0 v_r is 0 v_θ is 0 v_ϕ is not 0, but $\frac{\partial}{\partial\varphi}$ of anything is 0; because of this symmetry, v_θ is 0 and it is remaining here. Pressure we do not know as of now; normal stress components τ_{rr} we are not separating we are not removing because we are trying to develop equations which can also be used for the viscoelastic fluids, ok.

So, shear stress which component is existing? In this slide so, previous slide the rotation and ϕ direction is there. So, then only $\tau_{\theta\phi}$ component of shear stress is only existing other component of shear stress are not existing ok. So, that is the constraint. So, then because of that one this is anyway 0.

So, because of symmetry this term is 0 ok. So, this we cannot cancel out this we cannot cancel out and then we are not taking any gravity forces. So, then this equation we are having simplified equation ok. So, this term, this term and this term in the right-hand side only $-\rho \frac{v_\phi^2}{r}$ on the left-hand side these terms are remaining.

So, now in the case of inertia in such kind of cone and plate geometries usually rotational speed are not very high that you need to convert; that you need to worry about the inertial forces etcetera. But however, if there is higher speed so, then these are terms should be included.

So, let us say if you have the case, where the inertial forces are; inertial terms and then pressure terms are independent of radial coordinates then we can have this final equation ok. So, this is the equation we can use in order to get information about the normal stress differences.

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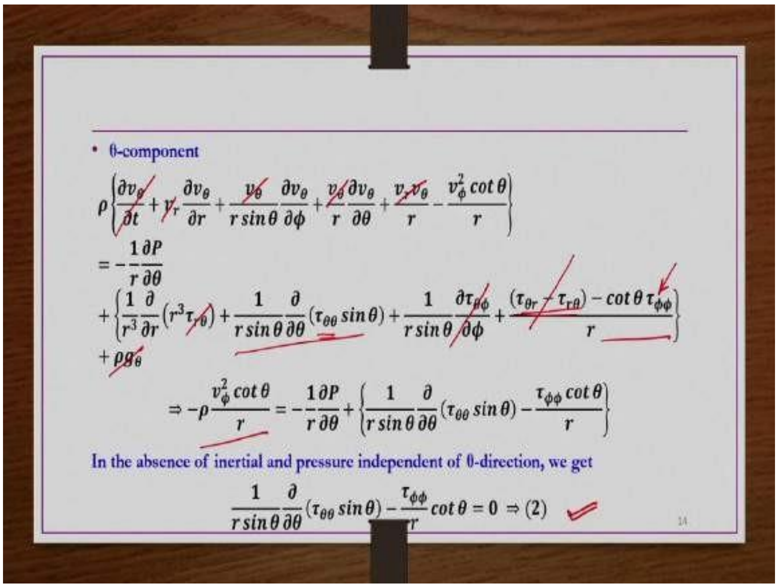

Then θ component of equation of motion is this one in spherical coordinates. So, now, we apply the constraints. So, first term is 0 because steady state v_r is 0 v_θ is 0 v_θ

is 0; v_r v_θ is both are 0 and this is not 0. So, this term would be there. Pressure we do not know anything, this shear stress term is not there. So, extra stress components we cannot cancel out because of symmetry this term is 0.

And then these two terms are identically equal to each other as at least for a laminar flow conditions. So, we cannot, we can strike out that term because the difference is 0, but this term we cannot strike up because of this one we have to worry about extra stress component. And then body forces we are not taking so, this is 0. So, then what we have? This equation we are having a right-hand side three terms left-hand side one term it is there ok.

And the absence of inertia in theta direction and then pressure independent of θ direction so, then this equation we can write like this. So, from here also we get some information about the normal stresses ok. That is the reason cone and plate geometry is the best option if you wanted to find out normal stress differences for a viscoelastic fluid; for viscoelastic fluid.

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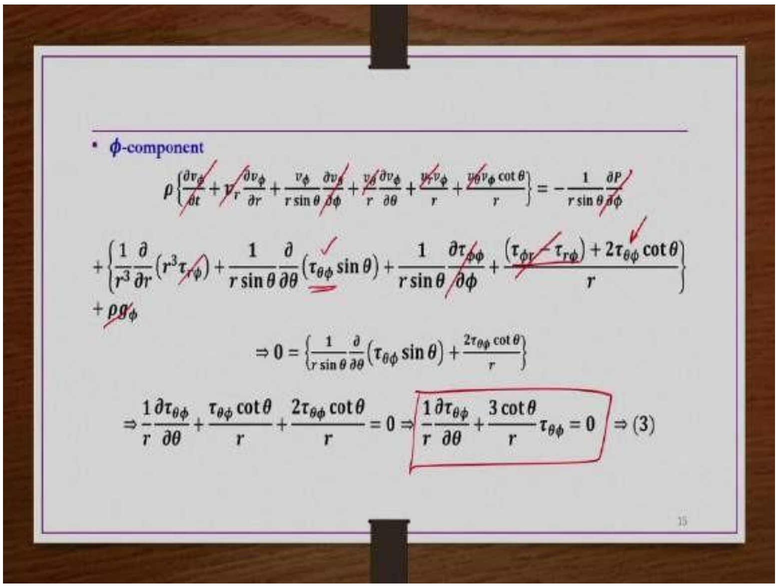

So, ϕ component; so, this is the equation now applying the constraints of the problem. So, this is steady state so 0, so v_r is 0, because of symmetry this term is 0, v_θ is 0, v_r is 0 v_θ is 0. So, left-hand side we do not have any terms and then because of symmetry this term is 0.

So, this component of shear stress is not existing, this component of shear stress is existing because of symmetry this term is 0, these two are equal to each other. So, then the difference is 0 and then this component of shear stress is existing. So, we cannot cancel out, body forces we are not taking into the account.

So, then we have this equation simplified equation, further you do the differentiation and simplify. So, then you get this equation $\frac{1}{r}\frac{\partial \tau_{\theta\varphi}}{\partial \theta} + \frac{3cot\theta}{r}\tau_{\theta\varphi} = 0$. So now, this equation will give us some information about you know shear stress ok.

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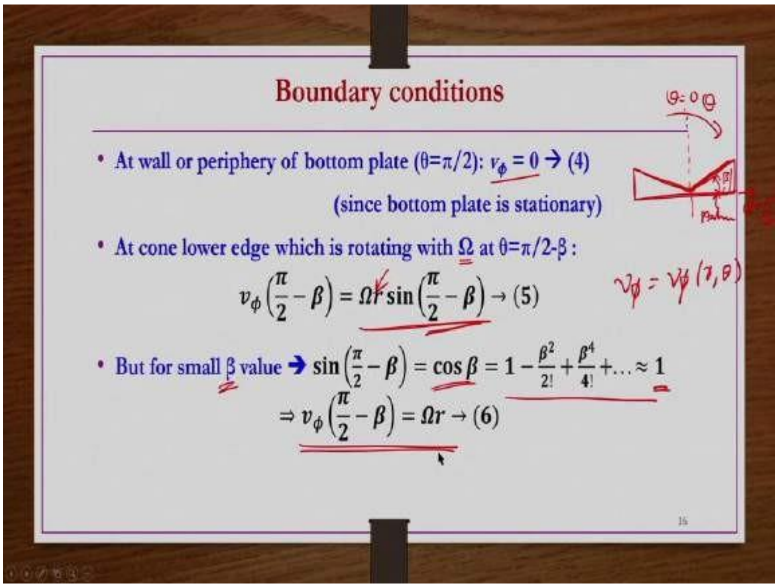

So, now boundary conditions: At the wall is periphery of bottom plate. So, what we have? We have a bottom plate like this within this one we have a cone like this, arranged like this right something like this. So now, in this case what we have taken? We have taken the bottom plate whatever this is there that is stationary, right.

So obviously, v_ϕ should be 0 and then angle you know how are we measuring θ? So, this is the theta direction, θ we are measuring from top. So, θ is equals to 0 is this one and then this line is nothing but θ = $\frac{\pi}{2}$. So, at θ = $\frac{\pi}{2}$ v_ϕ = 0 because bottom plate is stationary.

But at lower edge of the cone at this cone the velocity is not 0 because the cone is rotating with certain velocity right. So, that is at θ = $\frac{\pi}{2} - \beta$ because this cone angle is β, this entire is $\frac{\pi}{2}$. So, $\frac{\pi}{2} - \beta$ is the location nothing but the lower edge of the cone.

So, cone lower edge is rotating with Ω. So, at θ = $\frac{\pi}{2} - \beta$ we have Ω r $\sin\left(\frac{\pi}{2} - \beta\right)$ as a kind of velocity component, because the cone angle β is very small. So, then you can get this value anyway. So, v_ϕ at θ = $\frac{\pi}{2} - \beta$ it is not multiplied, but it is a θ = $\frac{\pi}{2} - \beta$. So, that you get this information.

We have started with like v_ϕ is function of r and θ right. So, now, you can see r component is there and then this is $\frac{\pi}{2} - \beta$ is nothing but theta. So, it is function of both r and θ ok. So however, when β is very small $\sin\left(\frac{\pi}{2} - \beta\right)$you can write it as cos β, when you expand it you will get $1 - \frac{\beta^2}{2!} + \frac{\beta^4}{4!} + \cdots$and so on so, that you can write is approximately equals to 1 if β is very small.

So that means, $\sin\left(\frac{\pi}{2} - \beta\right) \cong 1$ if β is small. So, then we can write v_ϕ at θ = $\frac{\pi}{2} - \beta$ is nothing but Ω r.

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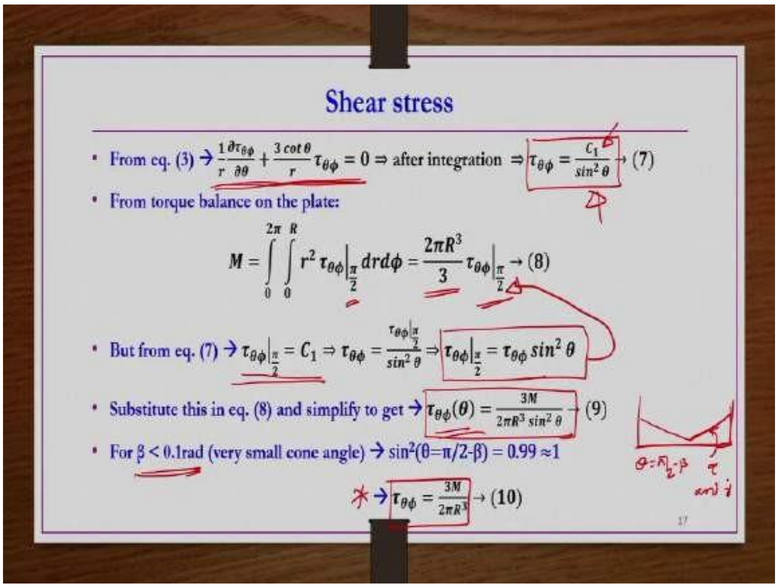

So, the velocity boundary conditions so for the velocity we got now we see shear stress information how to get. So, equation number 3; this is what the equation number 3 that is when we simplified the phi component of equation of motion in spherical coordinates, then we got this equation just a couple of slides before that is equation number 3. So, now, this equation if you do the integration you will get $\frac{C_1}{sin^2\theta}$ right.

So, now this equation can be useful only when you get this C_1 information that we can get from the experiments if you know the torque. If you know the torque, from the torque balance torque $M = \int_0^{2\pi}\int_0^R r^2\tau_{\theta\varphi}|_{\frac{\pi}{2}}$ that is at what location are you measuring dr dϕ $= 2\pi R^3$ now integration after you do $\frac{2\pi R^3}{3}$.

So, 0 to R limits if you substitute you know you will be having 0 to R $\frac{R^3}{3} - \frac{0^3}{3}$. So, that is $\frac{2\pi R^3}{3}\tau_{\theta\varphi}|_{\frac{\pi}{2}}$ this we do not know; still we do not know. But we can make use of this equation to get that expression. So, from equation 7 $\tau_{\theta\varphi}|_{\frac{\pi}{2}}$ is nothing but C_1 because sin $\frac{\pi}{2}$ is nothing but it is 1.

So, $sin^2\frac{\pi}{2}$ would be 1. So, $\tau_{\theta\varphi}|_{\theta} = \frac{\pi}{2}$ is nothing but C_1. So that means, $\tau_{\theta\varphi} = \frac{\tau_{\theta\varphi}|_{\frac{\pi}{2}}}{sin^2\theta}$ or $\tau_{\theta\varphi}|_{\theta} = \frac{\pi}{2}$ is nothing but $\tau_{\theta\varphi}.sin^2\theta$.

So, this you can, using this equation number 8 so that you get $\tau_{\theta\varphi}(\theta) = \frac{3M}{2\pi R^3 sin^2\theta}$ right. After substituting this one in equation number 8 we have written only $\tau_{\theta\varphi}(\theta)$ one side and rest all other terms you have taken to the other side. So, then we get $\frac{3M}{2\pi R^3 sin^2\theta}$.

Now, again β is very small in general very small cone angle. So, $sin^2\theta|_{\frac{\pi}{2}-\beta}$; now θ is nothing but at what location are you measuring the shear stress? That is the reason that is important. So, you are measuring at this location at the lower edge of the cone at the lower edge of the cone, at this location you are measuring τ and $\dot{\gamma}$ also of course that we are going to do. So, this location is nothing but $\theta = \frac{\pi}{2} - \beta$.

So in this equation if you substitute $\theta = \frac{\pi}{2} - \beta$. So, then whatever the equation that you will get that is nothing but $\frac{3M}{2\pi R^3}$ is the expression for shear stress. So, shear stress expression we already got for the case of you know cone and plate geometry ok.

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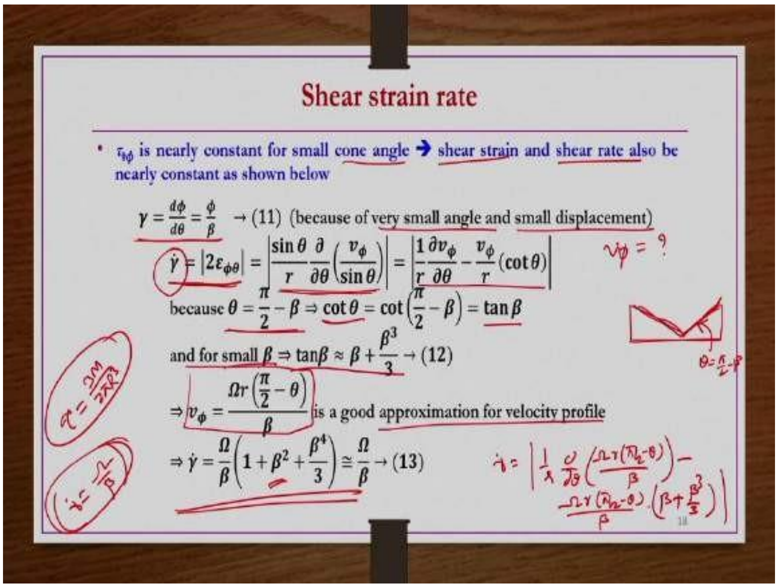

Now, we try to get expression for shear strain rate. So, $\tau_{\theta\varphi}$ is nearly constant for small cone angle that we have seen. So, then obviously, shear strain and shear rate will also be constant nearly, because that β is very small less than or equals to 6 degrees in general, ok. And then that is how that only that much are even smaller beta angle cone and plate rheometers are you know manufactured in general, ok.

So, what we have? Shear strain gamma that rotation in the ϕ direction and then variation is in the θ direction. So, then what we have; $\frac{d\varphi}{d\theta} = \frac{\varphi}{\beta}$ because of very small angle and then small displacement as well. So, from the components of rate of deformation we know that $\dot{\gamma}$ is nothing but $|2\epsilon_{\varphi\theta}|$.

So, this from the transport phenomenon book this quantity modulus quantity that we know that it is $\left|\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\varphi}{sin\theta}\right)\right|$. Any transport phenomenon book you can see the extra stress component for non-Newtonian fluid, so then you can get this component. When you expand this one, so then you will get this equation.

So, now what you have to have in order to get the shear rate expression? You need to know v_φ then only you can get it. So, that we have already seen, but once again we see. So now, $\theta = \frac{\pi}{2} - \beta$ only we wanted to know the rheology of the material because we wanted to know the shear rate and then shear stress at the lower edge of cone which is nothing but $\theta = \frac{\pi}{2} - \beta$ location at this lower edge of the cone right.

So, at $\theta = \frac{\pi}{2} - \beta$ $cot\theta$ is nothing but tan β and then for small β tan β is nothing but $\beta + \frac{\beta^3}{3}$ and so on so, that series goes on. So, then we need only for small β. So, then we can just write it as it is best approximation for velocity profile should be this one that is $v_\varphi = \frac{\Omega r\left(\frac{\pi}{2}-\theta\right)}{\beta}$ which are same.

So, then approximately it gives a linear profile with respect to r almost linear profile with respect to r and then θ components have been also been brought into the picture. So, now, this expression you write here in the in place of cot θ you write tan β here and then in place of tan β you write whatever this expression.

So, let us do the simplification $\dot{\gamma}$ would be $\left|\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\Omega r\left(\frac{\pi}{2}-\theta\right)}{\beta}\right) - \frac{\Omega r\left(\frac{\pi}{2}-\theta\right)}{\beta}\left(\beta + \frac{\beta^3}{3}\right)\right|$.

So, this you further do the simplification and then you get this expression $\frac{\Omega}{\beta}\left(1 + \beta^2 + \frac{\beta^4}{3}\right)$. So, which approximately you can write β by which approximately you can write it as $\frac{\Omega}{\beta}$ because β is very small so, β^2, β^4terms would be negligible.

So, now we got shear rate expression also $\dot{\gamma} = \frac{\Omega}{\beta}$ whereas, shear stress expression we got it as a $\frac{3M}{2\pi R^3}$. So, shear stress and shear rate expressions we already got by using cone and plate rheometers. By using cone and plate geometry you know shear stress and shear rate expressions are these things which we can use in order to get the rheology of the material. So, now we discuss something about normal stresses how to measure them.

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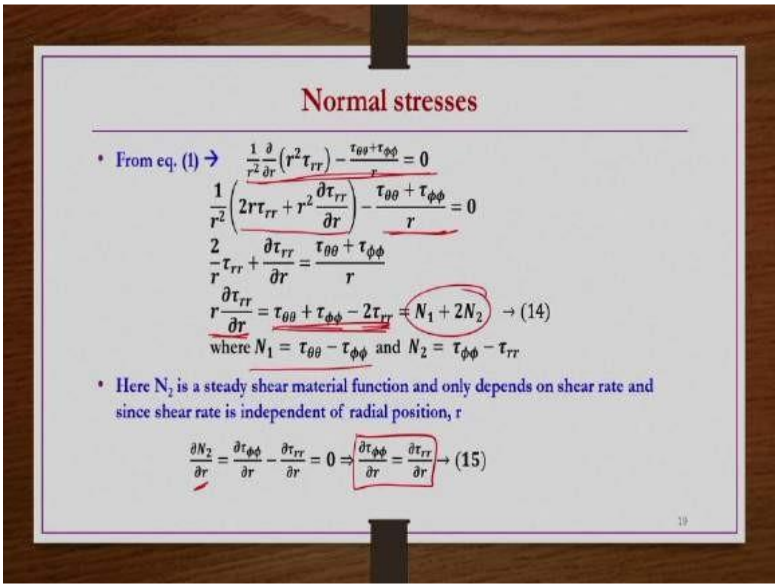

From equation number 1 that is r component of equation of motion after simplifying this is what we got this right. $\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr}) - \frac{\tau_{\theta\theta}+\tau_{\varphi\varphi}}{r} = 0$. So, now, when you do the differentiation of this term then you get $\left(2r\tau_{rr} + r^2\frac{\partial \tau_{rr}}{\partial r}\right)$ and then this term is as it is.

So, now when you bring $\frac{1}{r^2}$ within inside the parenthesis then you get $\frac{2}{r}\tau_{rr} + \frac{\partial \tau_{rr}}{\partial r}$. And then whatever this quantity that we are taking to the right-hand side so, then$\frac{\tau_{\theta\theta}+\tau_{\varphi\varphi}}{r}$. So, this equation what how we write?

We write only $r\frac{\partial \tau_{rr}}{\partial r}$ one side by multiplying this equation both sides by r then what we have right-hand side? $\tau_{\theta\theta} + \tau_{\varphi\varphi}$you will be having. Left-hand side you will be having 2 τ_{rr} so that will bring it to the right-hand side. So, then $-2\tau_{rr}$.

So, this expression this is nothing but $N_1 + 2N_2$; where $N_1 = \tau_{\theta\theta} - \tau_{\varphi\varphi}$ and then $N_2 = \tau_{\theta\theta} - \tau_{rr}$ ok. Here N_2 is steady shear material functions and only depends on shear rate and since shear rate is independent of radial position.

So, this will also be independent of radial position. N_2 will also be independent of radial position. That means, $\frac{\partial N_2}{\partial r}$ should be 0. So, $\frac{\partial \tau_{\varphi\varphi}}{\partial r} - \frac{\partial \tau_{rr}}{\partial r} = 0$. That means, we have $\frac{\partial \tau_{\varphi\varphi}}{\partial r} = \frac{\partial \tau_{rr}}{\partial r}$.

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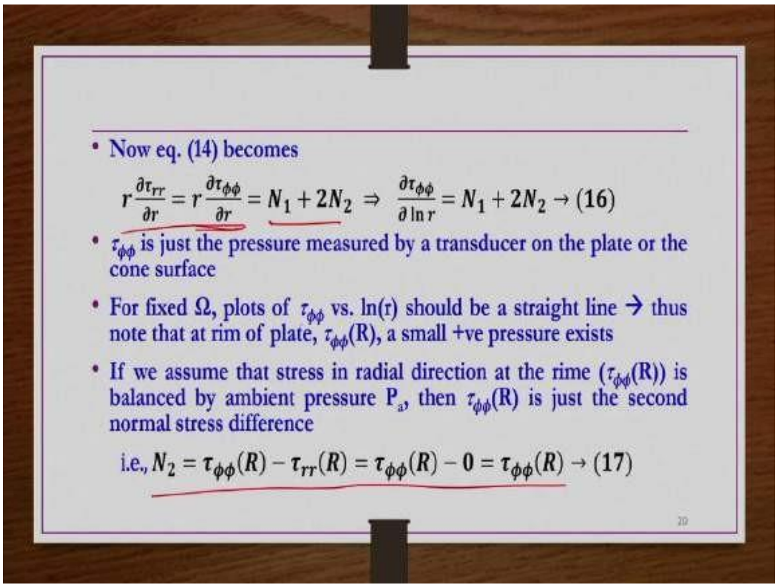

So, now equation 14 this is what we have ok; equation 14 is $r\frac{\partial \tau_{rr}}{\partial r}$.. So, in place of $r\frac{\partial \tau_{rr}}{\partial r}$ we can write $r\frac{\partial \tau_{\varphi\varphi}}{\partial r}$ we can write. So, when we write that one we get equation 14 like this right-hand side $N_1 + 2N_2$ is as it is. So, that you know here now we can write here $\frac{\partial \tau_{\varphi\varphi}}{\partial lnr} = N_1 + 2N_2$ we can write.

So, this is just a pressure measured by a transducer on the plate or on the cone surface. So, for fixed Ω plots of $\tau_{\varphi\varphi}$versus ln r should be straight line. So obviously that means at rim of plate that is at r = R $\tau_{\varphi\varphi}$ at r = R should have a positive pressure; small positive pressure should be there ok.

If we assume that stress in radial direction at the rim is balanced by ambient pressure P_a then $\tau_{\varphi\varphi}(R)$ is the second normal stress difference, and then that we can get using this equation.

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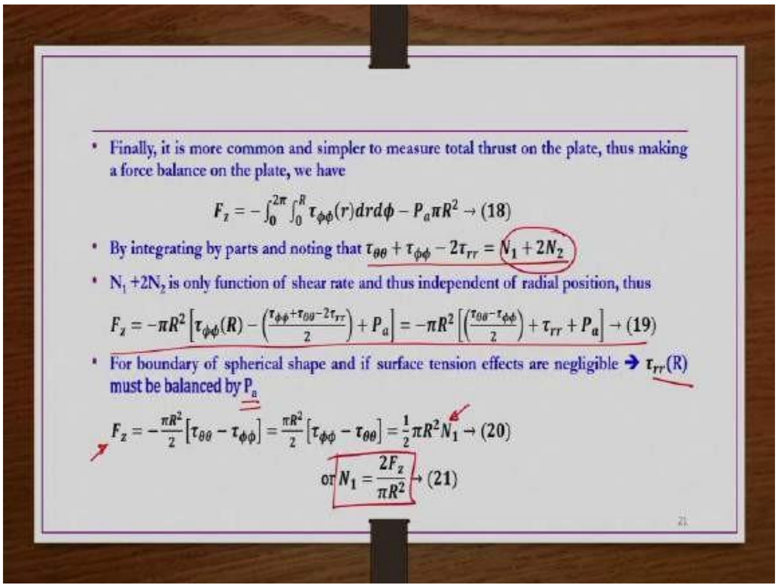

You know it is simpler to measure total thrust and plate. Thus, by making a force balance on the plate we can have force due to the thrust F_z is equals to this one, P_a is the ambient pressure. So now, by integrating this equation above equation by parts and then applying these results whatever that we got in our previous slide.

And then obviously, we know that $N_1 + 2N_2$ is function of shear rate and then it is independent of radial position because shear rate is almost constant that is what we have seen. So, then this quantity $N_1 + 2N_2$ should also be constant as at least irrespective of the radial position. So, then when you apply that one and then do the simplification this is what you get the expression for the thrust, ok.

So, if boundary of spherical shape and if surface tension effects are negligible so, then this quantity must be replaced by P_a. So, then we have $F_z = \frac{\pi R^2 N_1}{2}$. So, that directly first normal stress difference can be obtained by the thrust which you can measure by experimentally without any difficulty ok; or $N_1 = \frac{2F_z}{\pi R^2}$.

So, that is about how to measure the normal stress differences, shear rate and then shear stress using cone and plate geometry ok. Now, we take an example problem before winding up this lecture.

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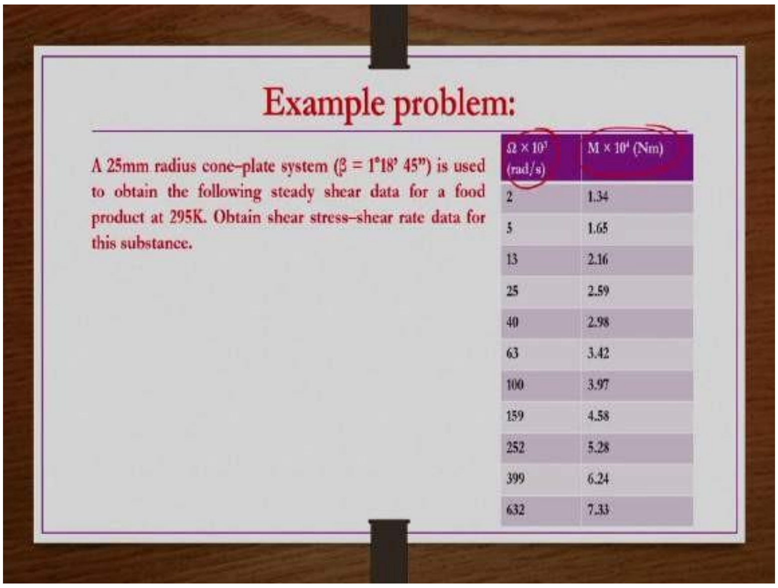

So, a 25 mm radius cone plate β 1° 18' 45" is used to obtain the following steady shear data for a food product at 295 Kelvin. Obtain shear stress versus shear rate data for this substance. And then the given information is that rotational velocity and then torque is given, ok.

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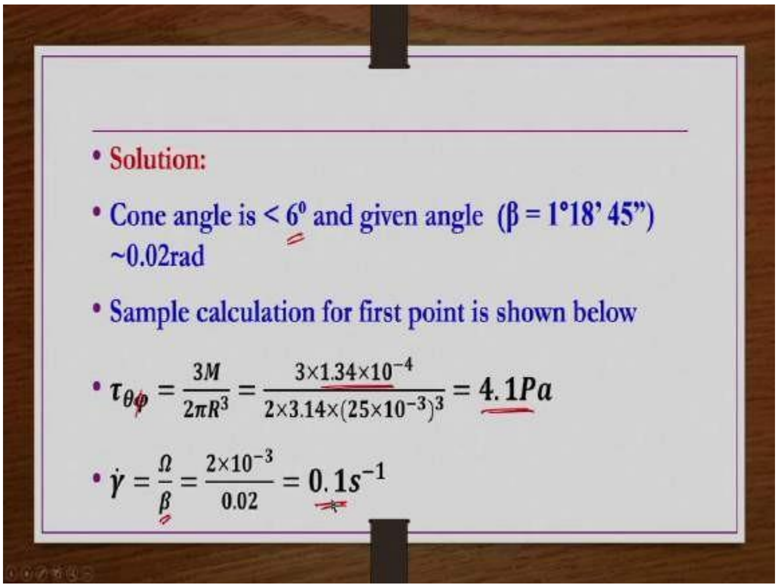

So, solution cone angle is less than 6 degrees ok. So, which is approximately 0.02 radians whatever it is given? So, then what we can do? We can for the first data point $\tau_{\theta\varphi}$

is nothing but $\frac{3M}{2\pi R^3}$ M for the first data point is 1.34 into 10^{-4} Newton meters and then R is nothing but 25 mm. So, then we get 4.1 Pascals.

Similarly, shear rate omega by beta because beta is very small. So, first data points Ω is nothing but 2 x 10^{-3} and then β is nothing but 0.02 radians. So, then $\dot{\gamma}$ you get 0.1 s^{-1}.

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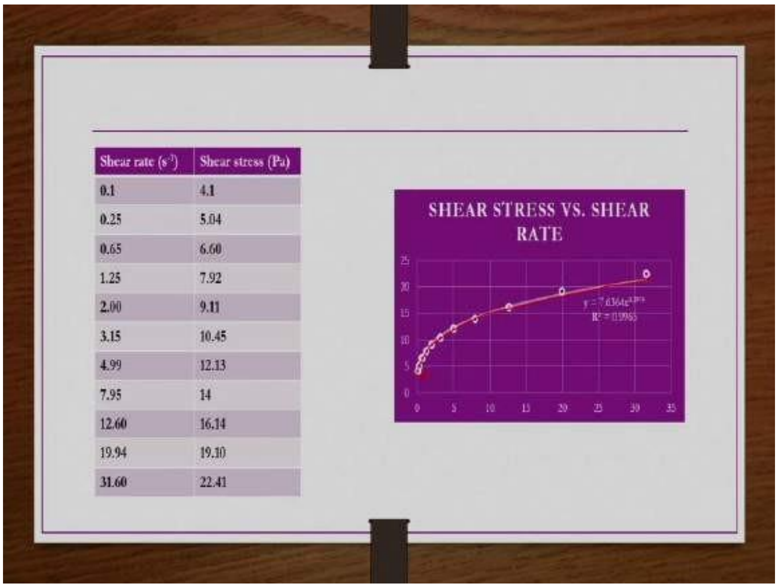

So, likewise for all the data points you obtain shear rate and then shear stress then you tabulate them like this here. So, then you get shear stress versus shear rate curve like this. Herschel-Bulkley fluid nature with a small yield stress that is what you can get.

So, that is what how to get rheological behavior of unknown fluid by using cone and plate geometry right. We have seen the equations for shear stress, shear rate and then normal stress as well. But we have done this when β cone angle is very small, then you can maintain the constant shear rate that is the advantage of this geometry.

Another advantage of this geometry is that you know you can directly measure the first normal stress difference by thrust. Thrust you can directly get from the experimental calculation. So, then $N_1 = \frac{2F_z}{\pi R^3}$ that you can use to get the first normal stress difference.

So, that is the reason the cone and plate geometries are found to be very much useful to study the non-Newtonian behavior to study the rheological behavior of unknown fluids which are expected to have non-Newtonian behavior ok.

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So, the references for this lecture are given here. The most of the slides were prepared from this book Rheology Principles Measurements and Application by Macosko. Some example problems have been taken from this reference book by Chhabra and Richardson. Other reference books may also provide useful insights about these topics.

Thank you.