**Transport Phenomena of Non-Newtonian Fluids**
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**Lecture - 41**
**Thermal and Concentration Boundary Layer Thickness of Non-Newtonian Fluids**

Welcome to the MOOCs Course Transport Phenomena of Non-Newtonian Fluids. The title of today's lecture is Thermal and Concentration Boundary Layer Thickness of Non-Newtonian Fluids. Before going into the details of today's lecture, what we will be doing? We will be having a kind of recapitulation of what we have seen in last two classes.

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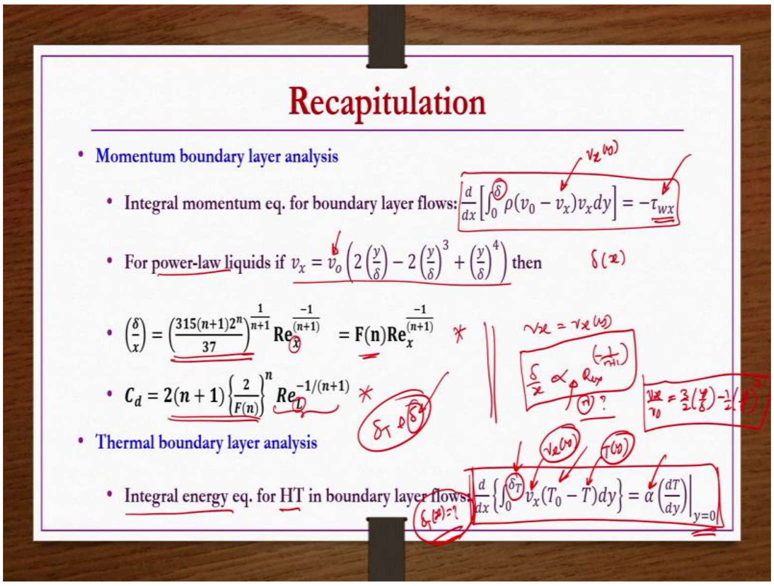


So, in last two classes we have seen basics of a momentum boundary layer and then their analysis. And then, for the momentum boundary layer or the flow is taking place boundary layer flows are taking place and then the conditions are isothermal conditions, that is only momentum transfer is occurring. So, for that boundary layer flows what is the integral momentum equation, how to derive that one those details we have seen.

And then finally, we got this expression integral momentum equation for boundary layer flows which is valid for both Newtonian and non-Newtonian fluids right, because the nature of the fluid rheology will come into the picture through this τ information right ok. Then, after having this information subsequently what we have seen? We have developed different expressions or we have taken different expressions for the velocity profile as function of y and then we try to obtain what is the momentum boundary layer thickness δ, that is what we have seen right.

So, we have taken two different types of velocity profiles and then for Newtonian, power law fluids, and then Bingham plastic fluids we developed the momentum boundary layer thickness δ. Let us say, fluid is power law fluid and then $v_x$ is given by this expression $v_0\left(2\left(\frac{y}{\delta}\right)-2\left(\frac{y}{\delta}\right)^3+\left(\frac{y}{\delta}\right)^4\right)$; $v_0$ is free stream velocity, δ is momentum boundary layer thickness, y is the normal distance or distance normal to the surface right.

From the surface how in the vertical direction y value increasing that is what it right. And, then we understand in all these analysis δ is function of x. So, what is the δ for this velocity profile? If the fluid is power law liquid is nothing but this one. This is what we got. Then subsequently, we also got the drag coefficient for the same case right.

Then we have seen that this, if you change the v expression. Let us say if you take some other expression for v; $v_x$ then what you do? You get mostly $\frac{\delta}{x}$ you know as you know proportional to the $Re_x^{\frac{1}{n+1}}$, if it is a power law fluid. And then only this constant and then functions of n, whatever are there they are only changing otherwise mostly they are remaining same.

Because, we have taken other velocity profile also $\frac{v_x}{v_0}$ is equals to you know $\frac{3}{2}\frac{y}{\delta}-\frac{1}{2}\left(\frac{y}{\delta}\right)^3$ this is what we have seen right. So now, this expression when you have used we got the similar expression only this constant has been changed. And then this F (n) function is that is changing that is a different that is what we have seen ok.

But, $Re_x^{\frac{1}{n+1}}$ is remaining same and then $C_d$ also is proportional to the $Re_L^{\frac{1}{n+1}}$ that also remaining same only this constants are changing. Or, the functions of n whatever are there they are changing they are constant if it is a Newtonian fluid they are function of n if it is a power law fluid ok.

$Re_x$ is the local Reynolds number and then $Re_L$ is the overall Reynolds number based on the length of the plate that we have taken. So, these analysis we have taken a flow parallel to flat plate, horizontal flat plate ok.

Then we have also seen few basics of a thermal boundary layer, and then we have seen the analysis, and then we have seen how to develop the integral energy equation for the boundary layer flows if there is a heat transfer. That is if the surface of the solid plate and then fluid entering or the fluid which is flowing over this plate or at different temperatures then heat transfer it also take place between the boundary layer right.

So, then thermal boundary layer would be developing both thermal and then momentum boundary layer would be developing simultaneously right. So, but we have taken the case where they are developing independently or developing independent of each other. Because we assume the physical properties are independent of a temperature gradient.

Than under such conditions the integral energy equation for heat transfer in boundary layer flows we got this expression right. Here, the $v_x$ is nothing but the velocity distribution, $v_x$ changing in y direction, how it is changing? T is nothing but the temperature distribution T function of y.

This $T_0$ is nothing but the free stream temperature, $\delta_T$ is nothing but thermal boundary layer thickness, $\delta$ is the momentum boundary layer thickness, $\delta_T$ is thermal boundary layer thickness, this $\alpha$ is thermal diffusivity $\frac{k}{\rho c_p}$ right. And then this $\frac{dT}{dy}$ is evaluated at the surface of the plate that is y = 0 location.

So, this is what we have seen in last two classes. Now in this lecture, what we are going to see? We are going to use this equation integral energy equation for heat transfer in boundary layer flows and then try to obtain what is this $\delta_T$ as function of x. That we can do, when we know what is this $v_x$ and as function of y and then what is this T as function of y. This $v_x$ as function of y we have already seen in the previous lecture how to get and all that.

Temperature profile also we try to obtain in a similar way by assuming a third degree polynomial for this temperature distribution, then find out those constants and all that we are going to do. Then substituting them here and then finally simplifying to get this $\delta_T$ as function of x.

But this is going to be much more complicated than what we have done for $\delta$ that is momentum boundary layer thickness in the previous lecture. The derivation is very lengthy as well as the complicated one, so we have to be very careful.

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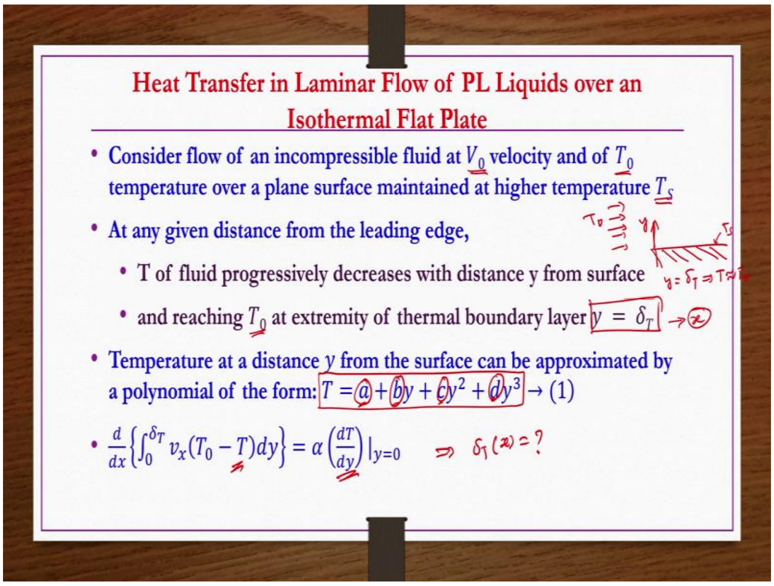


So, heat transfer in laminar flow of power law liquids over an isothermal flat plate. Flat plate; plate is isothermal condition, but the overall system is having non isothermality, because the fluid temperature is $T_0$ and then plate temperature is $T_s$. There is a temperature difference and obviously, heat transfer is there and then that heat transfer occurring in laminar boundary layer.

So, then that in within that laminar boundary layer constraint if both momentum and heat transfer are taking place. So then how this momentum boundary layer thickness is changing in the flow direction that is as function of x that is what we are going to derive here. Consider flow of an incompressible fluid at free stream velocity $V_0$ which is at temperature $T_0$ over a plane surface maintained at higher temperature $T_s$ right. Surface temperature and then free stream fluid temperature are provided.

Now, at any given distance from the leading edge, temperature of fluid progressively decreases with distance y from the surface because we have taken $T_s$ is higher than the $T_0$, so right so then this is the plate, so this is at $T_s$ at higher temperature and this is y direction. So, as we move away in the y direction right this $T_0$, the fluid is at that lower temperature $T_0$, so then temperature gradually decreased from $T_s$ to $T_0$ at infinite distance in the y direction ok.

That infinite distance in reality we cannot have, so then we are taking that location $\delta_T$ which is function of x by which, when y become; when y = $\delta_T$ then the temperature becomes approximately equals to $T_0$ ok. And, reaching $T_0$ at extremity of thermal boundary layer that is at y = $\delta_T$. Actually, theoretically it should occur at infinite distance, but that is not possible in reality.

So, that is the reason we are having this boundary layer concept that is defining a region of fluid or enclosure of a fluid in which the gradients are important or beyond that one the gradients are negligible. So, that boundary whatever is there in which the concentration gradients or temperature gradients are existing, that we call as a boundary layer flow.

And then that is also changing in the x direction function of x that is flow direction, that is at the leading edge boundary layer thickness is 0 and then as x increases boundary layer thickness increases whether it is momentum boundary layer or thermal boundary layer, that is what we have seen. So now what we assume? Temperature profile is having third degree polynomial of this form $T = a + by + cy^2 + dy^3$.

Now, if this a b c d you can find out then, obviously you can you know use the final temperature profile in this momentum in this integral energy equation here. And then get the $\frac{dT}{dy}$ at y = 0 and then substitute here and then finally solve this equation $\delta_T$ has function of x right. So, first what we have to do? We have to find out this a, b, c, d constants. So obviously, we need to have boundary conditions.

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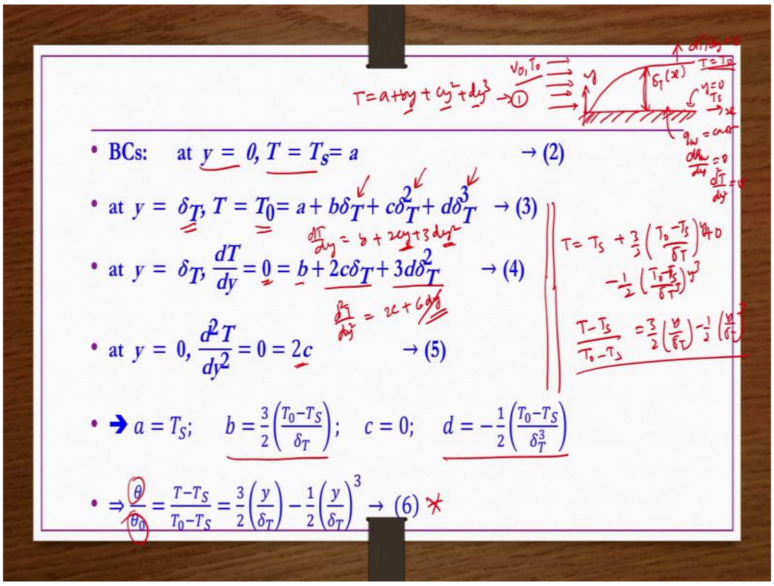

So, boundary conditions what we are having? We are actually, if you recollect we have a you know solid plane like this; flat plate the fluid is coming and then approaching the plate and then flowing like this. So, this is vertical direction is y direction, horizontal direction is x direction. So, this is at y = 0 location, temperature $T = T_s$ that is given, and there is a thermal boundary layer within the laminar flow region, forget about the momentum boundary layer as if now that is not required right.

So, this thermal boundary layer thickness changes with x that is increases with x in the flow direction, because as x direction increases the fluid element experiences more retardation and all that.

That we have already seen. So, at this $y = \delta_T$ onwards $T = T_0$ and then $\frac{dT}{dy} = 0$. Temperature becomes equal to the free stream temperature, free stream temperature is $T_0$, free stream velocity is $V_0$ right. And then, from this point onwards at $y = \delta_T$ onwards, $\frac{dT}{dy} = 0$; that is gradient becomes 0.

And then at the solid surface here, what we have? We have constant flux dq is constant this is what we know. So, $\frac{dq_w}{dy}$ should be 0, and then $q_w$ is nothing but $-k\frac{dT}{dy}$ so; that means, $\frac{d^2T}{dy^2} = 0$ at y = 0. So, at y = 0 we have two boundary condition, at $y = \delta_T$ we have two boundary conditions for the temperature.

So, we use those four boundary conditions to get this four these four constants from this equation number 1. So, at y = 0, $T = T_s$, if you substitute, T what we have taken $T = a + by + cy^2 + dy^3$. which is nothing but our equation number 1. So, now here if you substitute y = 0, so you get right hand side only a and then left answer $T = T_s$ then at $y = \delta_T$, that is at the thermal boundary layer location $T = T_0$. So, here in place of y right side we are substituting $\delta_T$. So, then we have this equation.

Then, at $y = \delta_T$ $\frac{dT}{dy}$ is 0. That is the other boundary condition at $y = \delta_T$. So, $\frac{dT}{dy}$ is what? $\frac{dT}{dy}$ is nothing but $b + 2cy + 3dy^2$. So now, here in place of c you substitute $\delta_T$ here. So then you get $b + 2c\delta_T + 3d\delta_T^2 = 0$, because $\frac{dT}{dy} = 0$ at $y = \delta_T$. And now, at y = 0 $\frac{d^2T}{dy^2} = 0$.

We are having $\frac{d^2T}{dy^2}$ is what it is nothing, but $2c + 6dy$. So now, here if you substitute y = 0, so then this is gone, that is 2 c = 0. That means, c = 0 and then a = $T_s$ and then remaining this equations 3 and 4 if you solve for b and d constants, b you will get $\frac{3}{2}\left(\frac{T_0 - T_S}{\delta_T}\right)$ and d you get $-\frac{1}{2}\left(\frac{T_0 - T_S}{{\delta_T}^3}\right)$ right.

So, when you substitute these equations in this equation number 1, what you get? T = a is nothing but $T_S$ and then b is nothing but $\frac{3}{2}\left(\frac{T_0 - T_S}{\delta_T}\right)$, c is nothing but 0. And then d is nothing but $-\frac{1}{2}\left(\frac{T_0 - T_S}{{\delta_T}^3}\right)$. So that means, $\frac{T - T_S}{T_0 - T_S}$ is nothing but $\frac{3}{2}$. This y is already here and then $y^3$ is here.

So, $\frac{3}{2}\frac{y}{\delta_T}, -\frac{1}{2}\left(\frac{y}{\delta_T}\right)^3$ you are getting. So, that is you get here this one. So, $T - T_S$ we are designating by θ, $T_0 - T_S$ we are designating by $\theta_0$ right. So, this is temperature profile you got already, now you need to get $\frac{dT}{dy}$ at y = 0.

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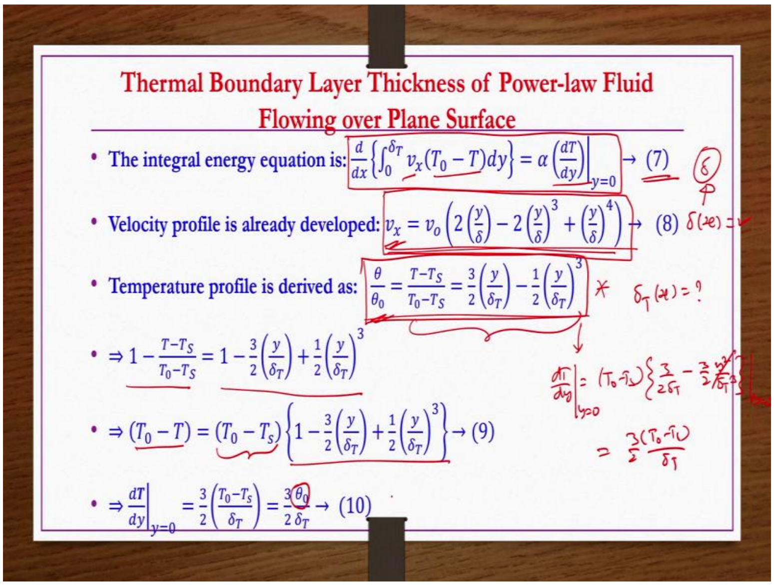


So, thermal boundary layer thickness of power law fluid flowing over a plain surface we take, here integral energy equation is this one, fine. Velocity profile already we have taken this one in the previous class because, we have to we cannot do the momentum boundary layer derivation again for a different velocity profile. Because this equation when you substitute $T_0 - T$ and $\frac{dT}{dy}$ and $v_x$ in this equation number 7, you also find that δ terms are there.

So, that that δ derivation we should have already done. So, for this velocity profile this δ; that is momentum boundary layer thickness as function of x, this is already we have done in previous class, so that we can adopt as it is. So, temperature profile just now we have taken this one. So remember, whatever the derivation that we are going to do in order to get this δ as function of x, now that is valid for these two velocity profiles and then temperature profiles given by equation numbers, this equation and this equation.

If you change any of these two equations or both the equations, the derivation is going to be different and then final results are going to be different. Especially those function of n is going to be different. May be Reynolds power one-third and Prandtl power 1 by 2 etcetera, those terms should be there, they will be as it is, they may not be changing.

So, so you have to be very careful you know which velocity profile you are doing because we are doing it for two different velocity profiles and then two different temperature profiles. Because in the at the end of this class we are going to take a different velocity profile and then see the answers; final answers of the $\delta_T$ ok anyway. So now, if you take this velocity profile and then this temperature profile what is $\delta_T$ as function of x that we will see first.

So, we need to have $T_0 - T$, so what we do? This equation both sides we multiply by minus 1 and then add plus 1 either side, so $1 - \frac{T-T_s}{T_0-T_s} = 1 -$ this one. So, that is when you do the LCM you get $T_0 - T_s - T + T_s$. So that is, $T_0 - T$ you get $\frac{T_0-T}{T_0-T_s}$.

So, that divided by $T_0 - T_s$ that I have taken to the right hand side. The terms of right hand side are as it is right, then we also need $\frac{dT}{dy}$. So, this expression, if you do the $\frac{dT}{dy}$ this expression what you get? $\frac{dT}{dy} = (T_0 - T_s)\left\{\frac{3}{2\delta_T} - \frac{3}{2}\frac{y^2}{{\delta_T}^3}\right\}$. This is what you get and then if you do $\delta_T$ at y = 0 then what will happen this y terms are there, so second term is gone only thing that you get $\frac{3}{2}\frac{T_0-T_s}{\delta_T}$.

So, $T_0 - T_s$ we are writing as $\theta_0$. So now, in order to substitute or simplify this equation number 7, we need $v_x$ we need $T_0 - T$ and then we need $\frac{dT}{dy}$ at y = 0 all of them we got it now right. By equation 8, 9 and 10, this equation 8, 9, 10 we are going to substitute in equation number 7 now, right.

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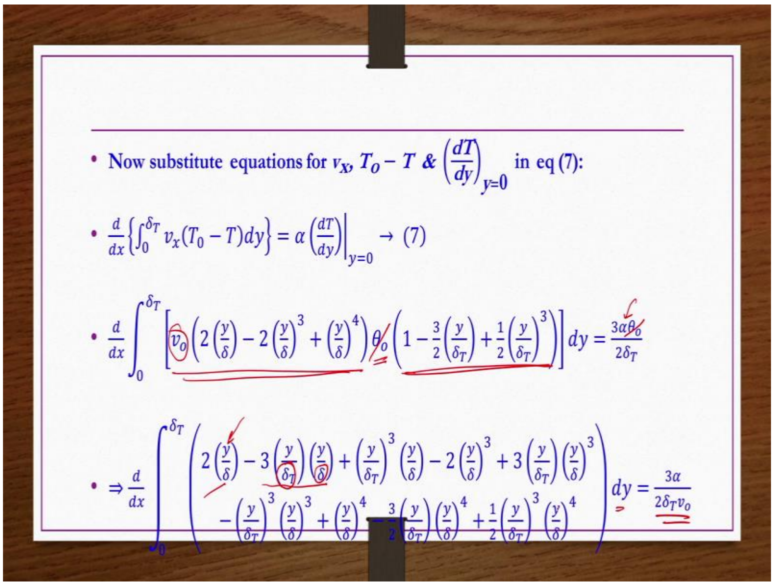


Equation number 7 is rewritten here. So, $\frac{d}{dx}\left\{\int_0^{\delta_T} v_x(T_0 - T)dy\right\}$ is that is $\theta_0$ multiplied by this one, dy = $\frac{dT}{dy}$ at y = 0 is nothing but 3 $(T_0 - T_s)$, that is $\frac{\theta_0}{2\,\delta_T}$ and then multiplied by α.

So, what we can do now? This $\theta_0$, this $\theta_0$ we can cancel out. This $v_0$ is a constant free stream velocity constant velocity that we can take to the right hand side. So and then remaining terms here whatever are there; these two terms we are multiplying and then we are writing like you know expanding these terms like; $2\frac{y}{\delta} - \frac{3}{2}\frac{y}{\delta}\left(2\frac{y}{\delta}\right)$, so you get $3\frac{y}{\delta_T}\ \frac{y}{\delta}$.

So, you have to be very careful about $\delta_T$ and δ right. So, both the terms are there here right. Next step what we do? Next step we integrate. When we integrate with respect to y, so in place of y here you get y square by 2, here you have y square so $\frac{y^3}{3}$ you get here $y^3$. So, $y^4$ you get like that all the terms you get right.

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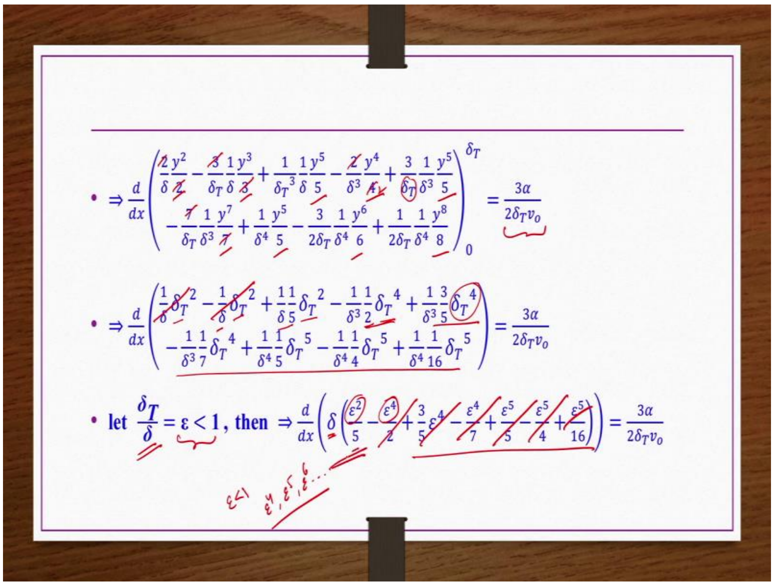


So, that when you do $\frac{y^2}{2}, \frac{y^3}{3}, \frac{y^5}{5}, \frac{y^4}{4}, \frac{y^5}{5}, \frac{y^7}{7}, \frac{y^5}{5}, \frac{y^6}{6}$ and then $\frac{y^8}{8}$ are there, that is what we get after integration remaining terms are constant we are keeping as it is. Limits 0 to $\delta_T$ and then right hand side term as it is, we are not doing anything right.

Next term what we are doing? We are substituting the limits right. So, here this 2 and this 2 is cancelled out. So, $\frac{1}{\delta}y^2$, now ${\delta_T}^2$ next term is this 3 this 3 cancelled out. So, $\frac{1}{\delta}$ as is as it is and then in $y^3$ so $\frac{{\delta_T}^3}{{\delta_T}^2}$. Next term, $\frac{1}{\delta}\frac{1}{5}$ as it is then $\frac{{\delta_T}^5}{{\delta_T}^3}$, so ${\delta_T}^2$ you get, so next step here 2 1's are 2 2's are.

So, $\frac{1}{\delta^3}\frac{1}{2}$ as it is and then $y^4$, so after substituting the limits ${\delta_T}^4$ then here $\frac{1}{\delta^3}, \frac{3}{5}$ as it is. And this $y^5$ is now $\frac{{\delta_T}^5}{\delta_T}$, so then you get ${\delta_T}^4$. So, like that remaining terms also done here like this and then we take $\frac{\delta_T}{\delta} = \varepsilon$ which is less than 1, that is for the Prandtl numbers are very large or the higher Reynolds number, high Prandtl number flows, where the boundary layer thickness are thinner.

And then especially thermal boundary layer thickness is smaller compared to the momentum boundary layer thickness, that case we are taking. So, then here $\frac{d}{dx}$ and then from all the terms, what we are taking you know $\delta$ we are taking common. So, so this term and then this term are same with plus and minus sign, so then that is 0. So, remaining terms now here δ if you take common, you know then what we get already $\frac{1}{\delta}$ is there. So, $\frac{1}{\delta}$ another $\frac{1}{\delta}$ will be there. So, $\delta^2$ would be there.

So, $\frac{{\delta_T}^2}{\delta^2}$ I am writing $\frac{\varepsilon^2}{5}$. Then here this term, $\frac{1}{2}\frac{{\delta_T}^4}{\delta^4}$ that is $\varepsilon^4$ I can write. Like that remaining terms also I can write like this here right. Next step what I do? Since, ε is less than 1, $\varepsilon^4$; $\varepsilon^5$, $\varepsilon^6$ and so on. So higher power terms would be negligible. Would be having small contribution compared to the compare to the ε or $\varepsilon^2$ term. So, what we can do we can strike of these terms and then take only $\varepsilon^2$ terms, then what we have?

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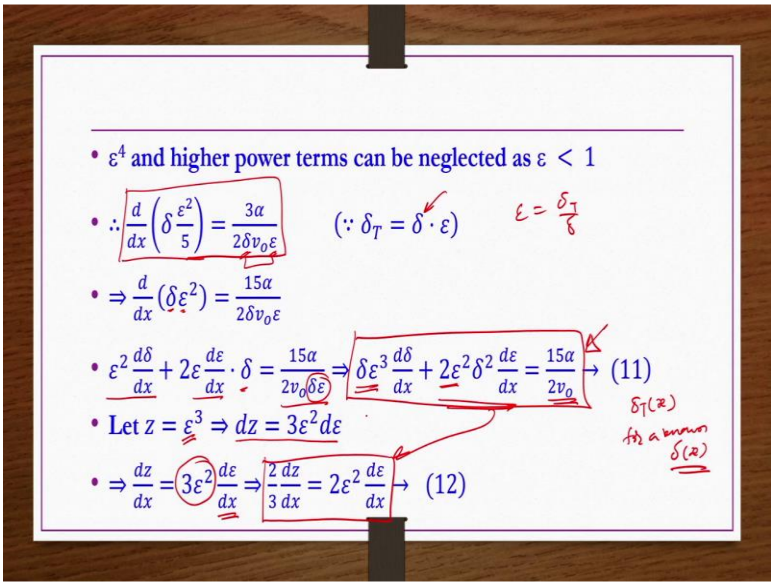


$\frac{d}{dx}\left(\delta\frac{\varepsilon^2}{5}\right) = \frac{3\alpha}{2\delta v_0 \varepsilon}$; in place of $\delta_T$ I am writing δ multiplied by ε, because ε we have defined as $\frac{\delta_T}{\delta}$. So, now this is the simplified equation. So, this equation you can simplify to get $\delta_T$ as function of x, because for the velocity profile whichever we have selected what is δ is also known.

We have done derivation in previous class that we can adopt here right. So, before that what we do? We further simplify this equation. So, this 5 I have taken to the right hand side. So, then I have $\frac{15\alpha}{2\delta v_0 \varepsilon}$ now this δ and then ε both of them are function of x.

So, when you differentiate this one, you get $\varepsilon^2 \frac{d\delta}{dx} + 2\varepsilon \frac{d\varepsilon}{dx}$ multiplied by δ as it is equals to right hand side term as it is. Next what I am doing? This whatever you know δ ε in the right hand side is there, so that I am taking to the left hand side.

So, that I have $\delta\ \varepsilon^3 \frac{d\delta}{dx} + 2\varepsilon^2\delta^2 \frac{d\varepsilon}{dx} = \frac{15\alpha}{2v_0}$. So, right hand side now completely a constant ok. Now, let z = $\varepsilon^3$ now here this is all a mathematical simplification in order to solve this equation. This equation we have to solve to get $\delta_T$ as function of x, for a known δ as function of x, δ is known as of now. We have already done the derivation, so we adopt it as it is.

So, if you take z = $\varepsilon^3$, dz = 3 $\varepsilon^2 d\varepsilon$, but here we need 2 $\varepsilon^2 \frac{d\varepsilon}{dx}$, so from here dz = 3 $\varepsilon^2$ dε if I do divided by dx, I get $\frac{dz}{dx}$ in the left hand side, right hand side $\frac{d\varepsilon}{dx}$ I will be having and then 3 $\varepsilon^2$ as it is. But I need 2 $\varepsilon^2$ $\frac{d\varepsilon}{dx}$ for this term to substitute here right.

So that, what I am doing, now both sides I am multiplying by 2/3, so that $\frac{2}{3}\frac{dz}{dx}$ would be nothing but 2 $\varepsilon^2 \frac{d\varepsilon}{dx}$. $\varepsilon^3$ is nothing but z. So, those things now we substitute here.

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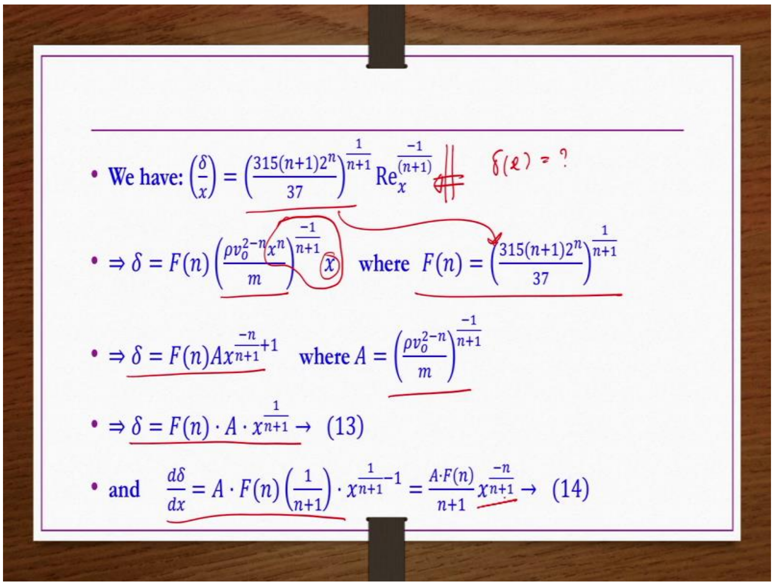


Before substituting we also enlist what is δ, because we need to substitute δ. Also δ as function of x this we have derived in the previous lecture. So, we are taking directly, so δ from this one is nothing, but this is function n we are calling. So that function n multiplied by $Re_x$ for the power law fluid is nothing, but $\left(\frac{\rho v_0^{2-n} x^n}{m}\right)^{\frac{-1}{n+1}}$ and then left hand side x whatever is there that we have taken right hand side.

F (n) is nothing but this one, this quantity we are calling F (n) right. So, we also need $\frac{\mathrm{d}\delta}{dx}$ also to substitute in the previous equation. So now, δ simply in a simple way I am writing A. F (n) whatever the other than x power these terms are there.

So, other than these terms I am writing that as A. F (n), A is nothing but $\left(\frac{\rho v_0^{2-n}}{m}\right)^{\frac{-1}{n+1}}$. So, the remaining term $x^{\frac{1-n}{n+1}}$, that is what I am having, so that is δ = F (n) A $x^{\frac{1}{n+1}}$. Now $\frac{\mathrm{d}\delta}{dx}$ from here you get A F (n) $\frac{1}{n+1}$ $x^{\frac{1}{n+1}-1}$, that is $x^{\frac{-n}{n+1}}$.

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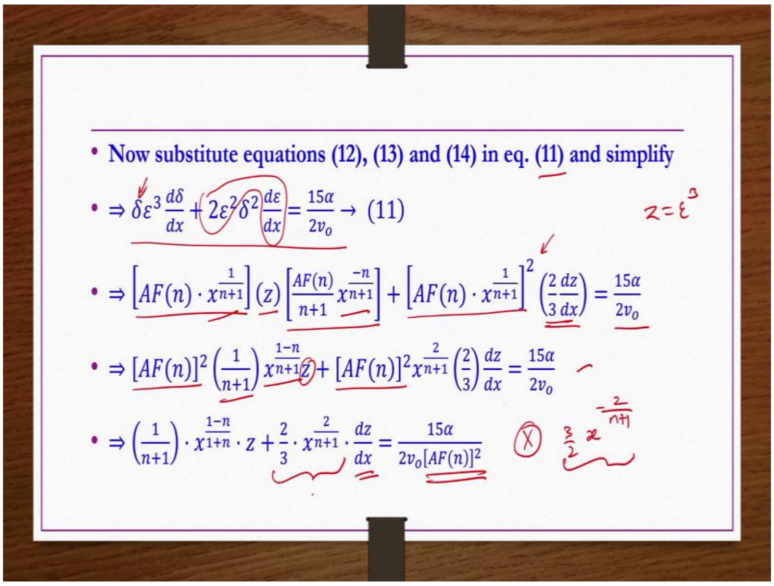


So, now we have all the quantities to substitute in the equation number 11, so now, those equation numbers 12, 13, 14 we are going to substitute in equation number 11. This is equation number 11. δ is A F (n) whatever x power is there, $\frac{1}{n+1}$ we are writing as it is, $\varepsilon^3$ is nothing but z, $\frac{\mathrm{d}\delta}{dx}$ we got $\frac{AF(n)}{n+1} x^{\frac{-n}{n+1}}$ + $\delta^2$ is nothing, but $AF(n)x^{\frac{1}{n+1}}$. So, square is there then remaining terms are you know 2 $\varepsilon^2 \frac{\mathrm{d}\varepsilon}{dx}$ is nothing but $\frac{2}{3}\frac{\mathrm{d}z}{dx}$ right.

Right hand side as it is, z is nothing but $\varepsilon^3$ that is what we are taking. So, this equation what we are trying to do? We are now here the combining the x terms. So, $x^{\frac{1}{n+1}}$ you will be having this A F (n) we are having 2 times. So, $[A\,F(n)]^2$ and then $\frac{1}{n+1}$ z is as it is $x^{\frac{1-n}{n+1}}$ we are having. Here, $[A\,F(n)]^2$ we are writing and then $x^{\frac{2}{n+1}} + \frac{2}{3}\frac{dz}{dx}$ right hand side as it is.

So, next step this $[A\,F(n)]^2$ that we have taken to the right hand side remaining terms are as it is. Now, I want this one this equation in the next step I want only $\frac{dz}{dx}$. I do not want any multiplication of this thing for $\frac{dz}{dx}$ term. So, for that what I am doing? I am multiplying this equation both sides by $\frac{3}{2}x^{\frac{-2}{n+1}}$. This is what I am doing, so that this will become on multiplying this and this will become 1, I have $\frac{dz}{dx}$ and then remaining terms we are writing like this.

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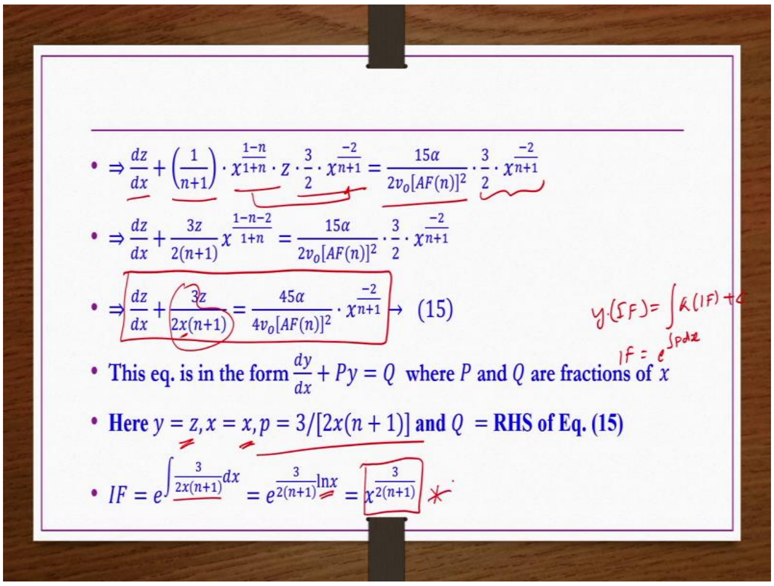


So, $\frac{dz}{dx} + \frac{1}{n+1}x^{\frac{1-n}{1+n}}$ and then z were as it is this $\frac{3}{2}x^{\frac{-2}{n+1}}$ we are multiplying. Right hand side this term is as it is and then this is the new term that we are multiplying alright. So, next step what we do? We further simplify this equation by combining these two x terms.

So, $x^{\frac{1-n-2}{1+n}}$ that is, what you get? $-n-1$ that is you get $x^{-1}$ so that is I am writing in the denominator as x here. Remaining all other terms are as it is. So, now, this is differential equation which is having a standard form $\frac{dy}{dx} + Py = Q$ where P and Q are functions of x.

So, that the solution of this equation will become y multiplied by integration factor is equals to integral of Q multiplied by integration factor plus integration constant. So, what is the integration factor? I F is nothing, but $e^{\int Pdx}$. So now here, that we do compare, so y = z here x is nothing, but x and then P is nothing but whatever this function other than z whatever this is there. So, that is P and then right hand side term is entirely is Q.

So, here integration factor $e^{\int Pdx}$ P is this one. So, when you do the integration of this one $\int 3/2(n+1)$ is as it is, integral of $\frac{dx}{x}$ is nothing but ln x right. So, that is I can write $e^{\frac{3}{2(n+1)}lnx}$. So, e to the power of ln that will be cancel out only whatever $x^{\frac{3}{2(n+1)}}$ is there that would be there as integration factor. So, the solution would become here.

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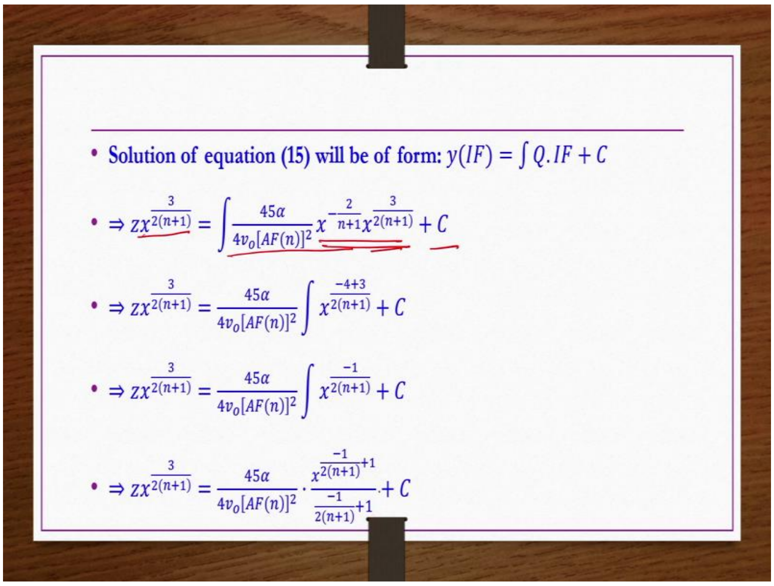


z multiplied by this I F is equals to integral of this is Q multiplied by I F + C right. So, now, this here these two terms we are combining x terms we are combining. So that we can do the integration remaining terms we are keeping as it is. So, here we get $x^{\frac{-1}{2(n+1)}}$; and then when you do the integration of this one $\frac{x^{\frac{-1}{2(n+1)}}}{\frac{-1}{2(n+1)}}$ right.

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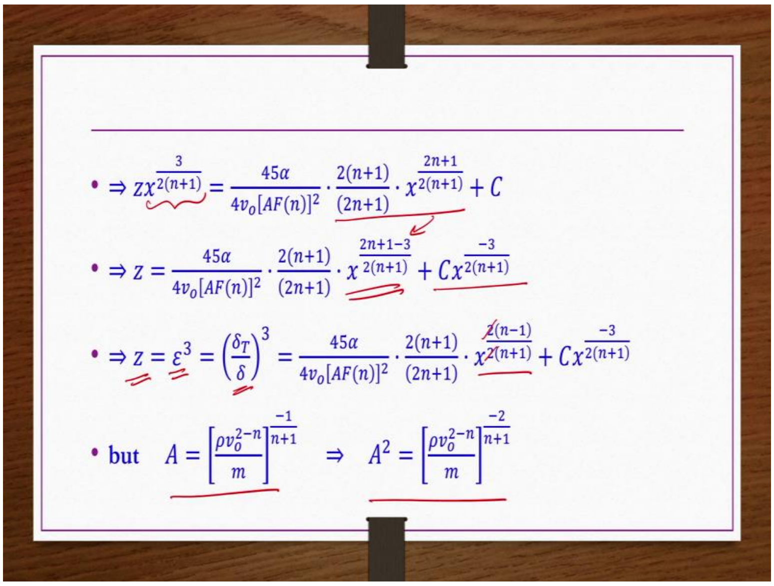


So, next step, what we have, this is what we have. Constant C is as it is. So, next step what I am doing is, whatever x power this is there that also I am taking to the right hand side. So, $x^{\frac{-3}{2(n+1)}}$ is there. So, that and then these term we are combining together. So, that $x^{\frac{2n+1-3}{2(n+1)}}$, I can write and then C is being multiplied by $x^{\frac{-3}{2(n+1)}}$ right.

So, now here this is nothing, but when you do the simplification you get 2 n − 2, that is 2 multiplied by $(n-1)$ 2 2 you can cancel out so that you have this term right. Remaining terms are as it is. We are not doing anything with the remaining terms. Now, this z we got which is nothing, but $\varepsilon^3$ and then $\varepsilon^3$ is nothing but $\left(\frac{\delta_T}{\delta}\right)^3$, because ε is nothing, but $\frac{\delta_T}{\delta}$.

So, still the solution is not complete we are only in the halfway because this $\delta^3$ also we have to take to the other side and then do the lot of simplification; though it looks like almost we are at the end of the derivation, but still half the way we have to go. So, in this equation just recapitulating this A is nothing but this one.

So, $A^2$ would be this one. Why? Because, we will be substituting all these things, and then writing terms again now in terms of Reynolds number or Prandtl numbers or both for that reason now we are going to substitute this A here.

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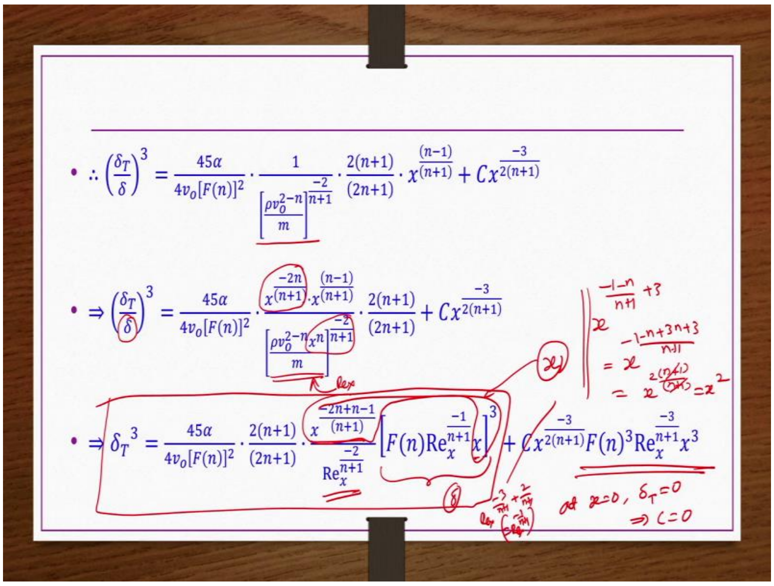


So, this A square I have written $\left(\frac{\rho v_0^{\ 2-n}}{m}\right)^{\frac{-2}{n+1}}$. So, power $\frac{-2}{n+1}$ is there, all other terms are as it is. Then, this term what I am doing? I am multiplying $x^{\frac{-2}{n+1}}$ and then dividing by $x^{\frac{-2n}{n+1}}$. So, that in the denominator $x^{\frac{-2}{n+1}}$, I can write and then that $x^n$if it is combined with $\frac{\rho v_0^{\ 2-n}}{m}$.

So, that is $\frac{\rho v_0^{\ 2-n} x^n}{m}$ you get and then whole power $\frac{-2n}{n+1}$ you get, and this is nothing but your Re x for power law liquids local Reynolds number ok. Rest all other terms are as it is. So, that when you write; so $Re_x^{\frac{-2}{n+1}}$ we are having and then whatever this $\delta^3$ is there that also I have taken to the right hand side.

So, δ is nothing but F (n) $Re_x^{\frac{-1}{n+1}} x$. This is nothing but δ, so $\delta^3$ is there. So, the same thing is coming here also to the constant. So, but this we do not need to worry because at x = 0, $\delta_T$ = 0. So, from applying this leading edge boundary condition you get C = 0. So, all this term altogether the second term in the RHS would be 0. So, we can take only this term and then further do the simplification.

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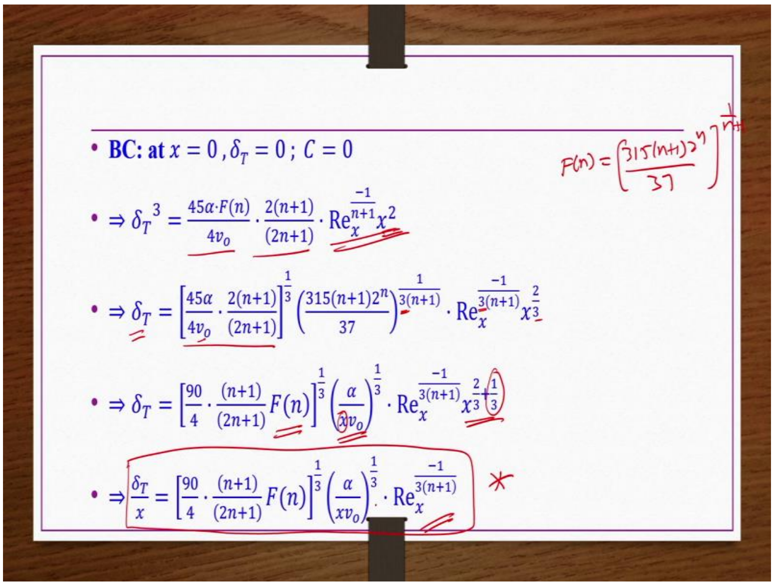


So, when you do that one, you have this one this term. Because here, this when you combining these x terms $x^3$ and then this $x^{\frac{-1-n}{n+1}}$ and then this plus 3 when you combine it you get $x^{\frac{-1-n+3n+3}{n+1}}$. So, that is $x^{2n+2}$; that is 2 multiplied by $\frac{n+1}{n+1}$. So, this $n+1,\ n+1$ cancelled out, so you get $x^2$.

So, these two terms; when you combine this term and then this $x^3$ term you get $x^2$ from this simplification. So, rest all other terms are as it is. So, that $x^2$ is coming here. Similarly, this $Re_x^{\frac{-3}{n+1}}$ and then multiplied by; $Re_x$ terms also if you simplify $Re_x^{\frac{-3}{n+1}}$ and then $\frac{2}{n+1}$ you get $Re_x^{\frac{-1}{n+1}}$.

So, this constant multiplied by $Re_x^{\frac{-1}{n+1}}x^2$ you will be getting. So, that is what right. So now, we take cube root either side so that we can have δ. So, this all these constants whatever are there. So and then functions of n we are writing together and then taking cube root; that is power 1, 3.

We are taking one-third. F (n) is there, whatever F (n) is nothing but this 315, F (n) is nothing but $\left(\frac{315(n+1)2^n}{37}\right)^{\frac{1}{n+1}}$. So, cube root of this one, so this 3 is there, and then $Re_x$ also this 1/3 is there and then x also 1/3 is there right.

Now, this 45 multiplied by 2 $\frac{90}{4}.\frac{(n+1)}{(2n+1)}$and then this F (n) also I am writing here again once again, no problem whole power 1/3. And then this $\left(\frac{\alpha}{v_0}\right)^{1/3}$ as it is. Now, I am multiplying by $(x)^{1/3}$ and then dividing by $(x)^{1/3}$, so that here I can have $\left(\frac{\alpha}{xv_0}\right)^{1/3}$. And then whatever $(x)^{2/3}$ and then $(x)^{1/3}$ is there that I can write.

$(x)^{\frac{2}{3}+\frac{1}{3}}$ that is x. So, that this x I can take to the left hand side, so that I can write $\frac{\delta_T}{x}$ is equals to this form right. So, this more or less you can take it as a final solution, but still we have to do some more simplification, because we know for a Newtonian fluid the you know thermal boundary layer thickness are subsequent Nusselt numbers that we you have.

So, they are having $Re_x^{\frac{-1}{3}}$ and then Prandtl number $\frac{-1}{2}$ etcetera those terms are there. So, $Re_x$ terms are there, Prandtl number terms are not here. So, that that modification we are going to do now.

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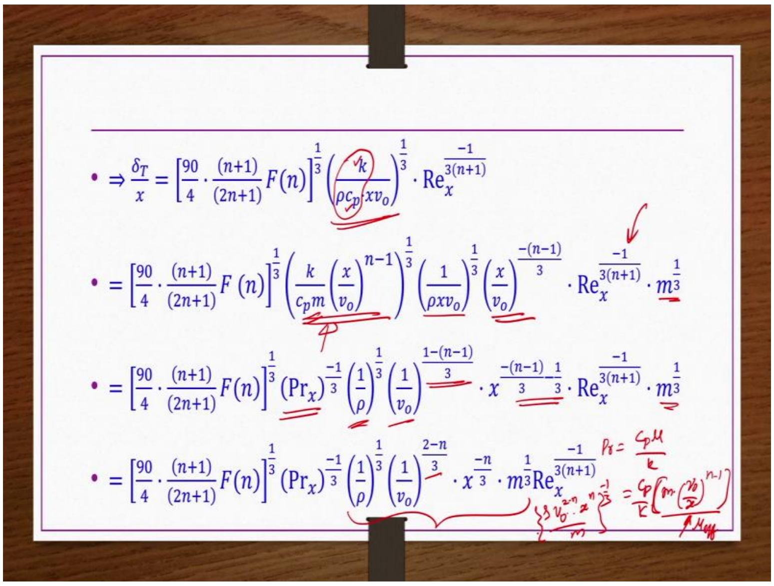


So, that equation previous equation what I have done? A in place of $\frac{\alpha}{xv_0}$ we were having, so in place of α I am writing $\frac{k}{\rho c_p}$ remaining all terms are as it is right. Next step what I am doing? I am writing $\frac{k}{c_p m}$ here and then $\left(\frac{x}{v_0}\right)^{n-1}$ and then the power 1/3 is there. So, the remaining what k I have taken $c_p$ I have taken from this parenthesis terms. So, the remaining terms are $\frac{1}{\rho x v_0}$, so $\left(\frac{1}{\rho x v_0}\right)^{1/3}$.

So here, but additionally, what I am doing? I am taking $\left(\frac{x}{v_0}\right)^{n-1}$ and then 1/m. So, the inverse of those terms I have to multiplied. So, $\left(\frac{x}{v_0}\right)^{-\frac{n-1}{3}}$ and then $m^{1/3}$, I am multiplying ok. Whereas, $Re_x$ term is as it is.

Why am I writing this one? Because, this is nothing but Prandtl number inverse for a power law fluid right. So, here this in place of $(Pr_x)^{-1}$ we can write, so that is $(Pr_x)^{-1/3}$. This is also local Prandtl number because it is defined as x right. Prandtl number for Newtonian fluid what you have $\frac{c_p\mu}{k}$.

So, in place of $\frac{c_p}{k}$as it is in place of μ what you can write m then $v_0$ by distance $(x)^{n-1}$, this is you can write it as effective viscosity. This is nothing but mu effective for a power law fluids. So, that is whatever $\frac{c_p m}{k}$ and then $\left(\frac{v_0}{x}\right)^{n-1}$is there that is nothing but Prandtl number.

So, we are here we are having inverse of Prandtl number and whole power 1 by 3, so $(Pr_x)^{-1/3}$. And then remaining terms I am just expanding and then combining any similar terms are there. So, $\left(\frac{1}{\rho}\right)^{1/3}$, this $\left(\frac{1}{v_0}\right)^{1/3}$ here I am having and then here $\left(\frac{1}{v_0}\right)^{-1-(n-1)/3}$ are there.

So, then power (1 − n – 1)/3 I am here writing and then here, $(x)^{-(n-1)/3}$ is there and divided by $(x)^{1/3}$, is there, so then power $\frac{-1}{3}$ I am writing here; m power as it is I am writing, $(m)^{1/3}$ as it is I am keeping why because. So, when I further do this is nothing but I get 2 $\left(\frac{1}{v_0}\right)^{2-n/3}$and then this $(x)^{-n/3}$. And then this $(m)^{1/3}$.

So, then here if 1 by 3 whole power 1 by 3 is there, so what I can do? From here I am having $\left(\frac{\rho v_0^{2-n} x^n}{m}\right)^{\frac{-1}{3}}$ I can write, is not it right. So, when I am writing these combine these terms combining joining together like this; $\left(\frac{\rho v_0^{2-n}x^n}{m}\right)^{\frac{-1}{3}}$. So, now, again this is nothing but $Re_x$. So, $Re_x^{\frac{-1}{3}}$ you are having right, so that we are substituting here.

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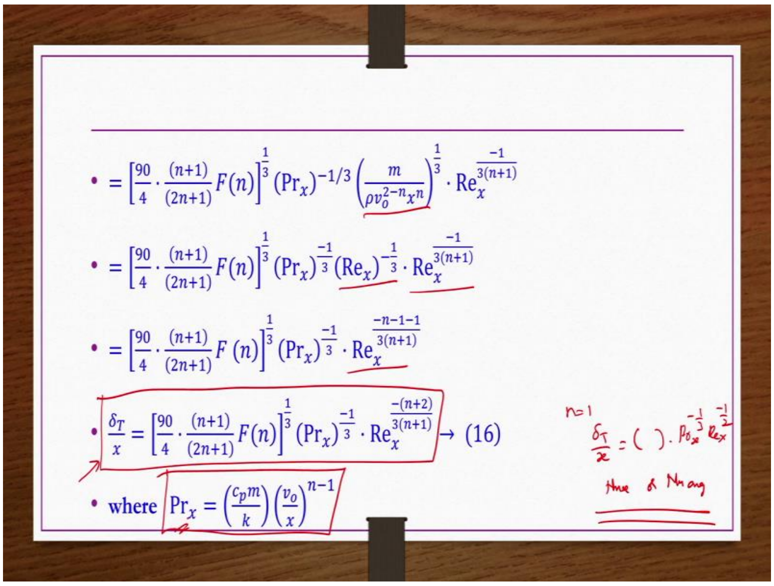


So, this term I am writing $Re_x^{\frac{-1}{3}}$, and then this $Re_x^{\frac{-1}{3(n+1)}}$. So, now, when you combined them you get $Re_x^{\frac{-(n+2)}{3(n+1)}}$ right. So, this is the final expression that we have right. So now, here if n is equals to, if n = 1 then what is this $\frac{\delta_T}{x}$ is some constant Prandtl power $(Pr_x)^{-1/3}$; and then $Re_x^{\frac{-1}{2}}$ you get.

Because, whatever a $Nu_x$ or a $Nu_{avg}$ etcetera are that your expressions that are there they are obtained from after using this expression in the those definition right. That is we are going to do anyway now. So here, as I already mentioned, $\frac{c_p m}{k}\left(\frac{v_0}{L}\right)^{n-1}$ is nothing, but local Prandtl number fine.

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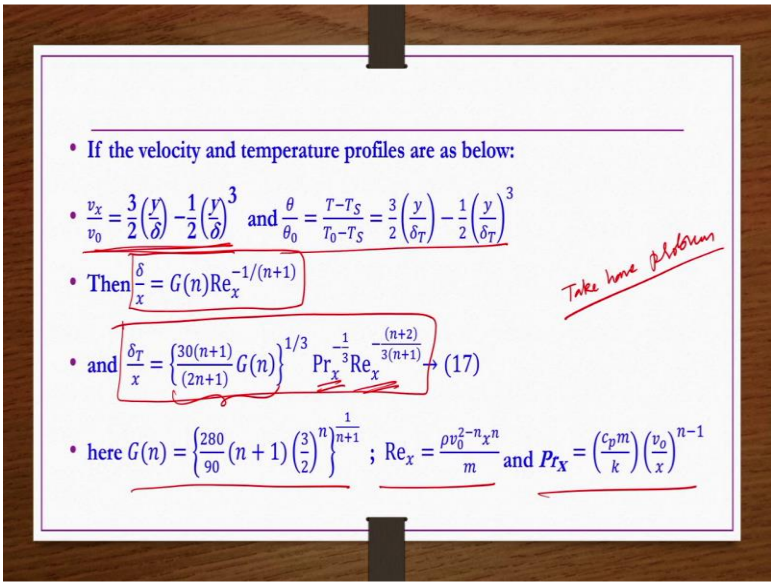


So, if the velocity and temperature profile are taken as differently. So, the temperature profile we are keeping same as whatever we have derived today's lecture, but the velocity profile we are taking this one different one. In the derivation until now whatever we have done the $\frac{v_x}{v_0}$ is a different expression. So now, I am changing only the velocity profile, but temperature profile I am keeping same. Then, do the same analysis $\frac{\delta}{x}$ you get this one which we have already derived in yesterdays lecture at least.

Because yesterday for this profile as well as the other profile we have developed the momentum boundary layer thickness, so this we know. So, $\frac{\delta_T}{x}$ when you do, you get this expression. You can see Prandtl power and then Reynolds number powers they are not changing; only this constants or function of n they are only changing right.

So, where here G (n) is this one, and then $Re_x$ is this one, Prandtl x is this one, same thing. So, this you take it as take home problem, because again if we do the entire thing it will take one class again. So, this at least similarly the way that whatever derivation that we have done today similarly you can do and then you get this one right.

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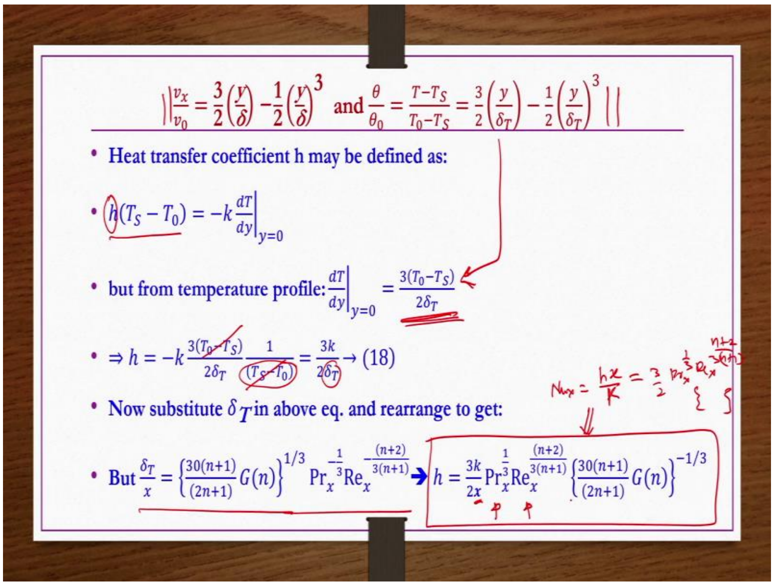


So now, for this velocity profile and then temperature profile heat transfer coefficient h if you wanted to know. So, h $\Delta T = -k\frac{dT}{dy}|_{y=0}$ that balance if you do the heat transfer by the convection and then heat transfer by the conduction at the interface, then you can find out what is this h, $\frac{dT}{dy}$already you got it as $\frac{3(T_0-T_s)}{2\delta_T}$ from this expression right.

This is what you get by substituting $\frac{dT}{dy}$ and then substituting y = 0 this is what you got. You get we already done also a few slides before. So, then h you will be getting $-k\frac{3(T_0-T_s)}{2\delta_T}$; whatever the left hand side $T_s - T_0$ was there that we brought to the right hand side. So, this is cancel; $\frac{3k}{2\delta_T}$ you are having.

So, $\delta_T$ for this combination of velocity profile and temperature profile is nothing, but this one. So, h you will be getting this one. After substituting $\delta_T$ here then rewriting, so you will get this one. So now, here from this equation what you do? h, x and k of right hand side terms you take to the left hand side. $\frac{hx}{k}$ if you do then you get $\frac{3}{2}(Pr_x)^{1/3}Re^{\frac{n+2}{3(n+1)}}$ and then this whatever the constant as it is. So, this $\frac{hx}{k}$is nothing but $Nu_x$ local Nusselt number.

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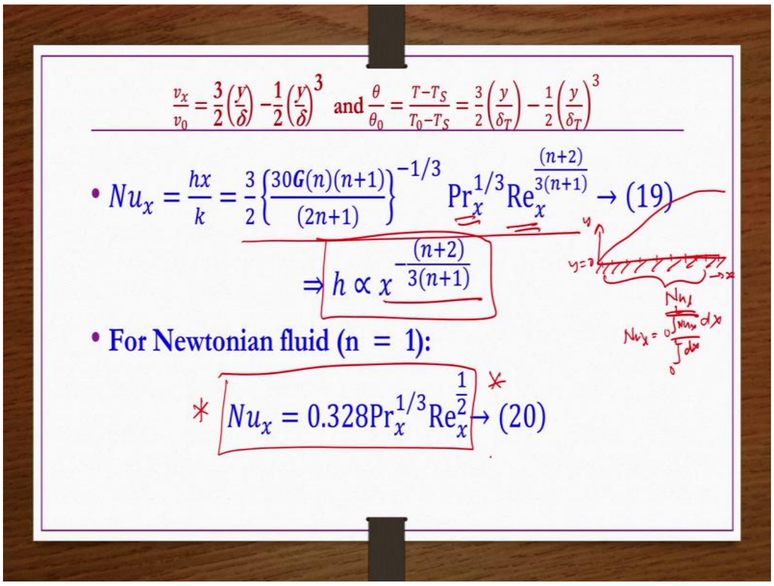


So, that is what we have written here. So, that is the local Nusselt number is this one $\frac{hx}{k}$, rearranging the previous equation that is it. So, this is the local Nusselt number that is local Nusselt number in the sense, this is the plate we are having this is y = 0 and then y is equals to you know gradually increasing this is the boundary layer is having.

So, at different x values x direction what is the Nusselt number here along the plate surface, that you wanted to know then you have to use this equation ok. Then, you have to use this equation along the plate length. As the x value increasing how the Nusselt number is increasing that you can get from this number.

But, if you wanted to know the overall Nusselt number average for the over overall entire length of the plate then you have to do the integration of this one. And then you have to do the integration of this one and then divide by integration of the distance dx and then integration 0 to l.

This is what we have to do, this we are doing in the next slide here anyway. So, from here this equation 19 what we are finding h is proportional to the $x^{\frac{n+2}{3(n+1)}}$. How do you get? So, now, here all these terms whatever this h is equals to this expression is there. Now Prandtl also Prandtl number also you expand $Re_x$ also you expand, so here whatever this x powers are there that you get as this one.

So, h is proportional to the $x^{-\frac{n+2}{3(n+1)}}$ that is what you and that is what you will get you can do the simplification. So, if you substitute n = 1, in this equation 19 you get Nusselt number is equals to this one. You might remember this correlation for Nusselt number for a flat plate that you might have by hearted or remembering in the fluid mechanics or heat transfer course. So, the derivation is coming like this.

We have we have done for the power law case as a generalized 1, so that n = 1 I can substitute I can get the Newtonian solution as well. So, that by 1 derivation I can get the two solutions. So, in this equation 19 if you substitute n = 1 this is what you get.

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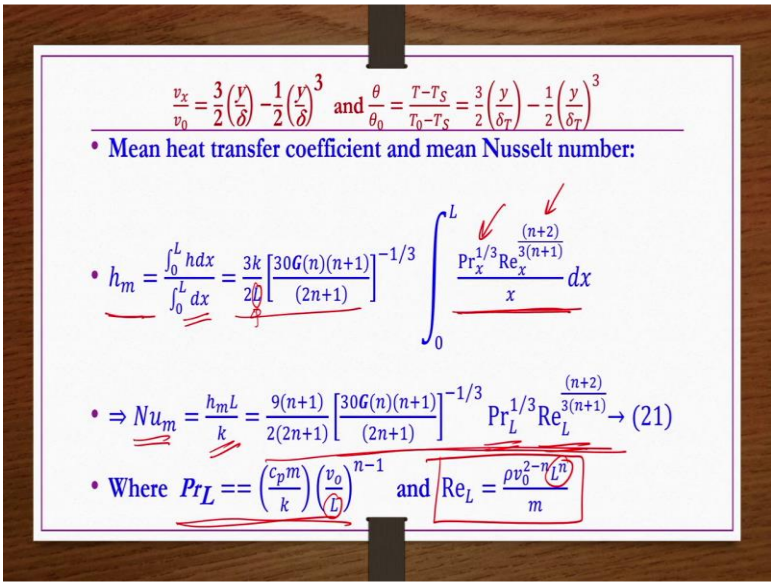


So, if you wanted to know the mean heat transfer coefficient and mean Nusselt number, $h_m$ that is mean heat transfer coefficient is $\frac{\int_0^L hdx}{\int_0^L dx}$ you have to do. h already you got this expression right ok, except this L that is the x. So now, x terms are there only in this ${Pr_L}^{1/3}$ and then $Re_x^{\frac{n+2}{3(n+1)}}$ and divided by x right.

Whole divided by $\int_0^L dx$ that is coming out L that we have written like this. So now here again, you expand this expressions for a Prandtl's Prandtl x and then $Re_x$ and then you write x terms, remaining terms you keep constant and then do the integration, substitute the limits 0 to L. Then finally, you get this expression whatever the $\frac{9k(n+1)}{2n+1}$ and then all this one you get.

So, that equation you rearrange $h_m$, that k/L you take to the left hand side. So, $\frac{h_m L}{k}$ if you write you get $Nu_m$, so that is nothing but this one. Now here, after substituting the limits what you get? x place in place of x you will be getting L. So, then this Prandtl numbers and then Reynolds number would be based on the entire length of the plate ok. So, that is $Re_L = \frac{\rho v_0^{2-n} L^n}{m}$, not $x^n$, $L^n$ and then $Pr_L$ you will get $\frac{c_p m}{k}\left(\frac{v_0}{L}\right)^{n-1}$ not $x^{n-1}$.

So, they are the Reynolds number and then Prandtl number for the entire geometry, not the local one fine. So, that is all about the thermal boundary layer thickness and then corresponding heat transfer coefficients. And then Nusselt numbers both average heat transfer coefficient and then mean heat transfer coefficients and then average Nusselt numbers, local Nusselt numbers etcetera those derivations we have done.

With that we complete the thermal boundary layer thickness of non-Newtonian fluids as well right. Now, what we do? We see concentration boundary layer thickness of non-Newtonian fluids. So, but we are not going to the do the all the derivation as we have done for the momentum and thermal bounty layer cases. We take a thermal boundary layer case and then analogously we write a concentration boundary layer case right. So, mass transfer in laminar boundary layer flow of power law liquids.

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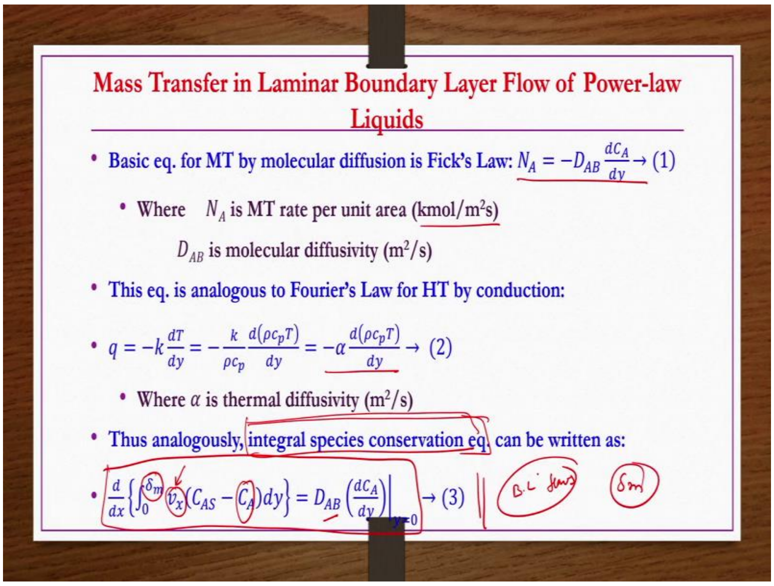

Basic equation for molecular diffusion by Fick's law is nothing, but minus $D_{AB}\,\frac{dC_A}{dy}$. I am writing $N_A$ some books write you know $G_a$ and all that do not worry thats a notation. So, $N_A$ is mass transfer rate per unit area that is mass flux if you are writing in moles then it is molar flux kilo mole per meter square second. $D_{AB}$ is molecular diffusivity which is in meter square per second.

And then, analogous this is analogous 2 Fourier's Law of Heat conduction. So, that we already know that it as $q = -\alpha \frac{d(\rho c_p T)}{dy}$ right. So now, when you compare equation number 1 and 2, they are similar to each other they are similar to each other. So, only thing that in place of $\alpha$ you can write $D_{AB}$ in place of $\rho$ $c_p$ T you can write $C_A$ ok. Then integral equation in a similar way we can write like this, integral species conservation equation.

So, what we have done? In place of $v_x$ it is as it is. In place of $T_s - T$ what we are writing? We are writing $(C_{AS} - C_A)dy$ ok. And then in place of $\delta_T$ we are writing $\delta_m$, that is concentration boundary layer thickness $\delta_m$. And then in right side in place of a $\alpha$ we are writing $D_{AB}$ in place of $\frac{dT}{dy}$ we are writing $\frac{dC_A}{dy}$. Because, here $\rho$ $c_p$ would also be there, but the that $\rho$ $c_p$ is there in the left hand side also, so that has been cancelled out that is it ok.

So then, analogously species conservation equation integral species conservation equation for boundary layer flows. This is only for the boundary layer flows, we can write like this. So, now similarly, as we did for the thermal boundary layer case if you know $v_x$, if you know $C_A$ then you can find out the concentration boundary layer thickness $\delta_m$ as well ok. So, but we are not going to do that one because the equations are similar, so their solutions would also be similar.

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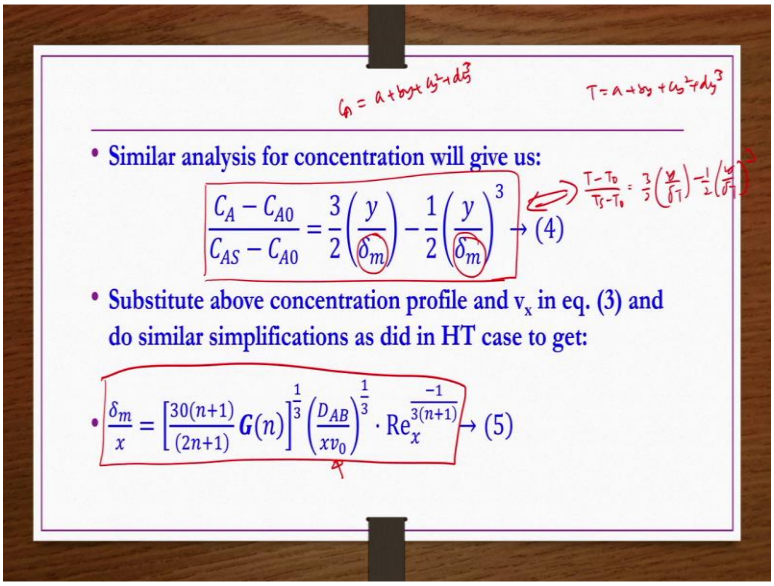


So, temperature difference what we had? We had taken $\frac{T-T_0}{T_S-T_0} = \frac{3}{2}\frac{y}{\delta_T} - \frac{1}{2}\left(\frac{y}{\delta_T}\right)^3$, we got by taking $T = a + by + cy^2 + dy^3$ expression or third degree polynomial expression for the temperature. Similarly, for the concentration also if you do, if you take third degree polynomial expression; that is $C_A = a + by + cy^2 + dy^3$.

And then apply the 4 boundary conditions to get y = 0 and to get y = $\delta_m$ and then simplify, so you get exactly the similar expression like a temperature profile, only thing that in place of $\delta_T$ you have $\delta_m$ and in place of T you have c ok. So obviously, the results are going to be similar. So, $\frac{\delta_m}{x}$ you are going to have this expression, only thing that in place of Prandtl number you get now Schmidt number right.

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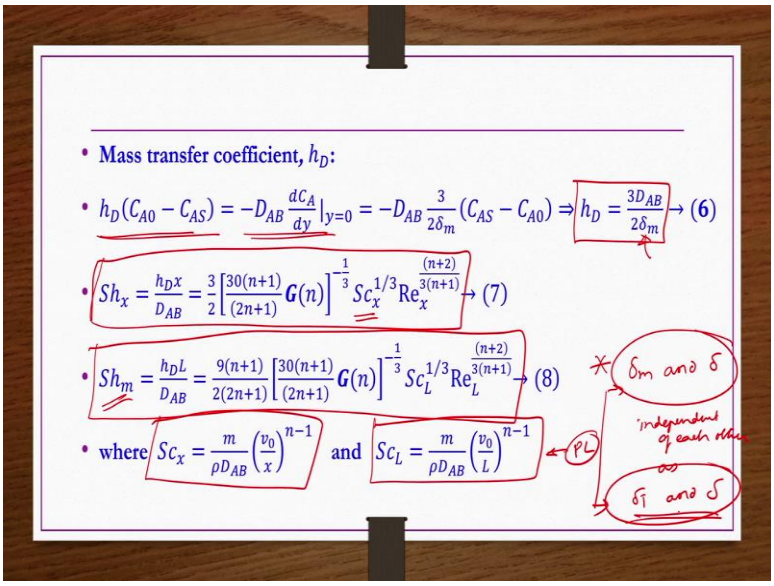


So, mass transfer coefficient also $h_D$, in a similar way if you do balance, that is mass transfer by convection is balanced by the mass transfer by diffusion, then you have an expression for $h_D$ that is given by this one and δ m just we have written equation number 5 previous slide that you substitute here. So, you get that one. And then you do all the simplification rearrangement so then you get the local Sherwood number, which is similar to local Nusselt number in the case of thermal boundary layer flows.

So, that is what you get here, but you have to careful that here now you get this Schmidt number rather than the Prandtl number, because mass transfer boundary layer that we are considering. So, average Sherwood number along the length of the plate along the entire length of the plate from x = 0 to x = l the average Sherwood number would be this one right. So, where here $Sc_x$ is the local Schmidt number given by $\frac{m}{\rho D_{AB}}\left(\frac{v_0}{x}\right)^{n-1}$, this is again for power law liquid ok.

And then $Sc_L$ is nothing by the overall Schmidt number for the entire geometry based on the length of the plate and then this is also for the power law liquids right. Remember, here also one more important thing that $\delta_m$ and δ are independent of each other. As we have considered in the case of $\delta_T$ and δ ok, then only these results are valid. Because we are writing analogously for this case with the comparison with the $\delta_T$ case ok.

So, $\delta_T$ case that we have derived assuming that δ that is assuming that the momentum and thermal boundary layers are independent of each other. Similar way, in the case of a concentration boundary layers as well the concentration boundary layer and then momentum boundary layer are developing independent of each other or they are not interfering each other.

The concentration gradients are maintained such a way right. Like you know, in the case of thermal boundary layer case the temperature difference or temperature gradients are maintained such a way that they are not disturbing the or they are not changing the physical property significantly right. So, this is the last lecture of the course.

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The references are provided here, for this course. But, the most of the details I got from these reference books, but any way derivations are not available any of the books. We have to do carefully right.

Thank you so much for your kind cooperation throughout the course.