Transport Phenomena of Non-Newtonian Fluids
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Lecture - 37
Quasi-Steady Analysis of Simultaneous HT and MT - II

Welcome to the MOOCs course Transport Phenomena of non-Newtonian Fluids. The title of today's lecture is Quasi-Steady Analysis of Simultaneous Heat and Mass Transfer, part II. In this lecture also we will be taking two problems which we will be studying or you know analysing using pseudo steady state analysis or quasi steady analysis. The first one is freezing of salt water.

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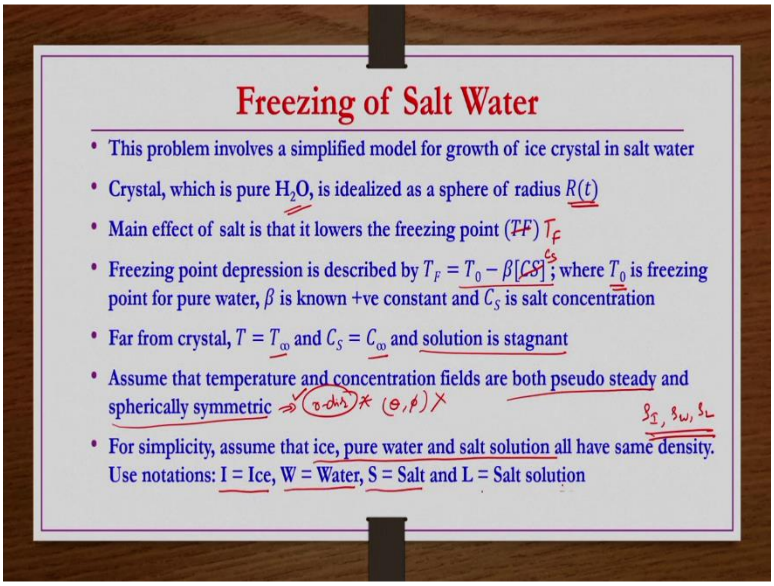

Freezing of Salt Water

- This problem involves a simplified model for growth of ice crystal in salt water
- Crystal, which is pure H_2O, is idealized as a sphere of radius $R(t)$
- Main effect of salt is that it lowers the freezing point (~~TF~~) T_F
- Freezing point depression is described by $T_F = T_0 - \beta[\cancel{CS}]$ C_S; where T_0 is freezing point for pure water, β is known +ve constant and C_S is salt concentration
- Far from crystal, $T = T_\infty$ and $C_S = C_\infty$ and solution is stagnant
- Assume that temperature and concentration fields are both pseudo steady and spherically symmetric $\Rightarrow$ ✓ r-dir ✱ (θ, ϕ) ✗
- For simplicity, assume that ice, pure water and salt solution all have same density. ρ_I, ρ_W, ρ_L Use notations: I = Ice, W = Water, S = Salt and L = Salt solution

So, here this problem involves a simplified model for growth of ice crystal in salt water. We have an ice crystal of certain radius R and then it is in a salt water. So, the purpose is that the growth of this ice crystal how is it going to take place in salt water that is what we are going to have the problem. So, in that case, so, if we in order to know the change in radius with respect to time etcetera how to do the analysis.

If you wanted to know the temperature profile etcetera in the liquid phase then how to do all those analysis with that is what we are going to do here. Crystal which is pure H_2O is idealized as a sphere of radius R function of time t; that means, it is increasing as growth of crystal is taking place.

Main effect of salt is that it lowers the freezing point. Whatever freezing point T_F is there. Now, that would be lowered if you add or if you do this crystallization salt water rather than in pure water right. Freezing point depression is described by $T_F = T_0 - \beta\, C_S$, where T_0 is freezing point for pure water, beta is non positive constant and C_S is salt concentration ok.

So, this T_0 is freezing point for pure water right, but this crystallisation or growth of ice crystal whatever you wanted to do little rapidly faster. So, then rather crystallizing in pure water, what you do? You do in salt water. So, that by using that salt water this freezing point will decrease and then the depression in freezing point is given by this expression.

Far from crystal the temperature T_∞ and then concentration C_∞ are known and then solution is stagnant, that is the other condition. So, these things are known to us. C_∞ and T_∞ are known for which are far away from the crystals.

Assume that temperature and concentration fields are both pseudo steady and spherically symmetric. When we say spherically symmetric then what we can say? Whatever the changes in Species concentration which has a species conservation and then change in temperature etcetera because of solving the energy equation that we get the all those things would be waiting in only r direction.

So, we do not need to worry about θ and then ϕ direction variations. We need to worry variations in r direction only as long as the sphere is symmetrically spherical. For simplicity, assume that ice, pure water and salt solution all have the same density. So, ρ_I, ρ_W, ρ_L all these are equal to each other that is what it mean by. Notations I we are using for ice phase, W for water, S for salt alone, L for salt solution ok. So, they are different fine.

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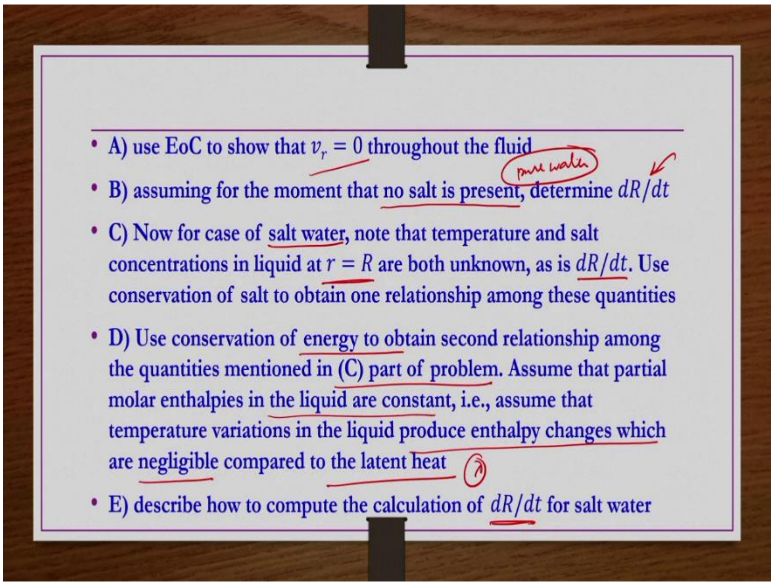

- A) use EoC to show that $v_r = 0$ throughout the fluid
- B) assuming for the moment that no salt is present, determine dR/dt pure water
- C) Now for case of salt water, note that temperature and salt concentrations in liquid at $r = R$ are both unknown, as is dR/dt. Use conservation of salt to obtain one relationship among these quantities
- D) Use conservation of energy to obtain second relationship among the quantities mentioned in (C) part of problem. Assume that partial molar enthalpies in the liquid are constant, i.e., assume that temperature variations in the liquid produce enthalpy changes which are negligible compared to the latent heat λ
- E) describe how to compute the calculation of dR/dt for salt water

Now, the first part of the question is we have to prove that $v_r = 0$ throughout the fluid by simplifying the equation of continuity. Then second part assuming for the moment that no salt is present that is you take you are taking pure water. Then what is the expression for $\frac{dR}{dt}$? How this R is increasing with respect to time as the crystallization process continuous?

Now, for case of salt water note that temperature and salt concentration in liquid at r = R are both known, right. So, temperature and then salt concentration at the surface are known and as is $\frac{dR}{dt}$. Use the conservation of salt to obtain one relationship among these quantities that is among $dR\ dt$ T_0 and then C_S at the surface etcetera for these relating these quantities we have to develop one relation.

Then use conservation of energy to obtain second relationship among these quantities mentioned in part C of the problem. Assume that partial molar enthalpy is in the liquid are constant. Assume the temperature variations in the liquid produce enthalpy changes which are negligible compared to the latent heat λ. Describe how to compute the calculation of $\frac{dR}{dt}$ for salt water that is the last part of the question.

So, we take one by one as we have done in the previous two lectures and then try to obtain these things. Problem looks lengthier, but the sequence when we go step by step it looks very simple and straight forward because we have already done couple of problems like this.

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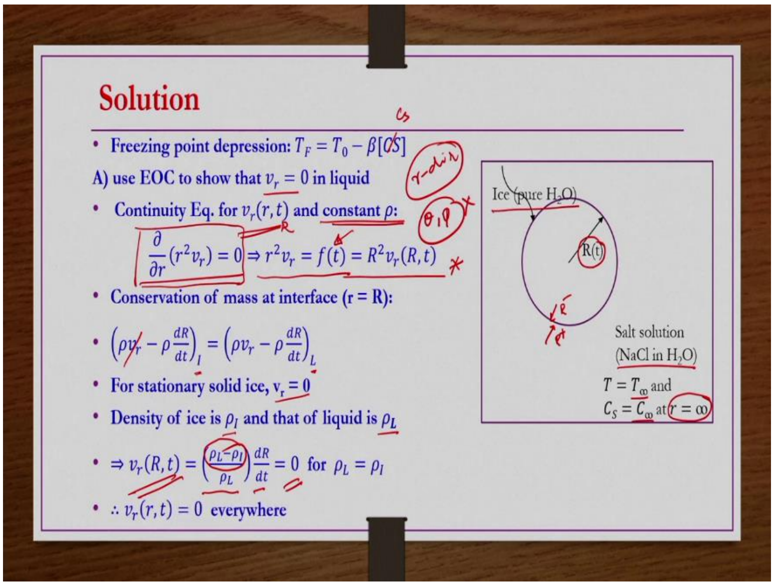

Solution

- Freezing point depression: $T_F = T_0 - \beta[\overset{C_S}{\cancel{C}S}]$

A) use EOC to show that $v_r = 0$ in liquid

r-dir

- Continuity Eq. for $v_r(r,t)$ and constant ρ: θ, φ ✗

$$\boxed{\frac{\partial}{\partial r}(r^2 v_r) = 0} \Rightarrow r^2 v_r = f(t) = R^2 v_r(R,t) \quad *$$

- Conservation of mass at interface (r = R):
- $\left(\rho \cancel{v_r} - \rho \frac{dR}{dt}\right)_I = \left(\rho v_r - \rho \frac{dR}{dt}\right)_L$
- For stationary solid ice, $v_r = 0$
- Density of ice is ρ_I and that of liquid is ρ_L
- $\Rightarrow v_r(R,t) = \left(\frac{\rho_L - \rho_I}{\rho_L}\right)\frac{dR}{dt} = 0$ for $\rho_L = \rho_I$
- $\therefore v_r(r,t) = 0$ everywhere

$T = T_\infty$ and

$C_S = C_\infty$ at $r = \infty$

So, schematically if you take the system, we have a spherical crystal whose radius is R initially and then it is function of time it gradually increases. This crystal is of pure water ok and it is in a salt solution. Far away from the crystal that is at r = ∞, T_∞ and C_∞ that is temperature and then salt concentration are known at far away distance from the crystal right.

Now, freezing point depression this is what is given. This is C_S. $T_F = T_0 - \beta\, C_S$. C_S is salt concentration. Now, we use equation of continuity to show that $v_r = 0$ in the liquid. So, continuity equation v_r now we understand only r coordinate variations are only there because of the spherically symmetric condition that is given.

So, θ and then ϕ we will not be taking for anything whether it is conservation of mass, conservation of momentum, conservation of energy or conservation of species transfer whatever is there. So, we take only r coordinate variations because it has been mentioned in the problem that symmetrically spherical crystal is there.

And then under such conditions this v_r or temperature or concentration variations whatever are they will be function of r only with respect to the space, but in addition to the space it is also dependent on the time. So, v_r as function of r and t we have to find out and then when the density is constant. Why constant? Because it has mentioned that densities of all three are equal.

So, continuity equation if you write you know in spherical coordinate $\frac{\partial}{\partial r}(r^2 v_r) = 0 + \frac{1}{r sin\theta}\frac{\partial}{\partial\theta} v_\theta \sin\theta$ would also be there $+ \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta} v_\phi \sin\theta$ these terms should also be there, but you know we are not taking these terms right, v_ϕ terms or θ terms we are not taking. So, that is a reason we are writing only this part ok.

This we understand by this now because we have already solved so many problems. So, in the continuity equation I am writing only the term which is having you know r coordinates are the function of r that term only we are writing v_θ $v_\phi$terms I am not writing because it is clearly mentioned and then we have already solved so many problems, we know how to do that one.

So, we get $\frac{\partial}{\partial r}(r^2 v_r) = 0$; that means, $(r^2 v_r)$ is equal to constant, but that constant is function of time at constant is function of time, but it is independent of r ok, independent of space, but dependent on the time ok. So, if you apply boundary condition at r = R then again we get the same thing $R^2 v_r$ (R, t) is again f (t) ok.

So, but now this from here we cannot prove that v_r is 0 everywhere. So, we have to leave it as it is now, but we write conservation of mass at the interface that at r = R right. So, then towards the crystal side whatever is there that we are writing R^- towards the liquid side whatever is there and that we are writing R^+ both of them are at the surface only.

Towards the crystal side we are writing indicating R^- quantities or towards the liquid side we are writing quantities at R^+ or R^+ quantities R_I, R_L whatever we are taking. So, $\left(\rho v_r - \rho\frac{dR}{dt}\right)_I = \left(\rho v_r - \rho\frac{dR}{dt}\right)_L$ right. In the ice there is no velocity. So, $v_r$is 0. Now, here ρ_I, this is ρ_I, this is ρ_L, this is also ρ_L. So, then what we get?

By applying the stationary solid ice v_r is 0 then remaining part ice density you write ρ_I and then that of liquid you write ρ_L then simplify then you get $v_r = \left(\frac{\rho_L - \rho_I}{\rho_L}\right)\frac{dR}{dt}$. And then that should be 0 because $\rho_L - \rho_I = 0$ because in the problem it has been mentioned that you know densities of all three are equal to each other.

So that means, v_r is 0 in the entire liquid ok. So, that is the first part is done. First part is done that we have to show $v_r = 0$ in the liquid.

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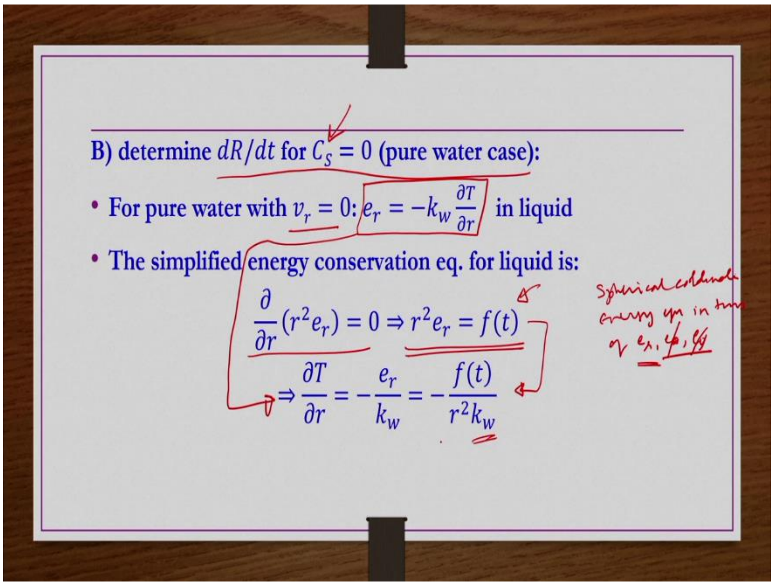

B) determine dR/dt for $C_S = 0$ (pure water case):

- For pure water with $v_r = 0$: $\boxed{e_r = -k_w \frac{\partial T}{\partial r}}$ in liquid
- The simplified energy conservation eq. for liquid is:

$$\frac{\partial}{\partial r}(r^2 e_r) = 0 \Rightarrow r^2 e_r = f(t)$$

$$\Rightarrow \frac{\partial T}{\partial r} = -\frac{e_r}{k_w} = -\frac{f(t)}{r^2 k_w}$$

Spherical coordinate
energy eqn in terms
of e_r, e_ϕ, e_θ

Now, we take the second part of the problem. Determine $\frac{dR}{dt}$ for pure water case when there is no salt. When there is no salt added to the liquid then it is a pure water. So, then C_S is 0, because C_S indicates the salt concentration, right. For pure water with $v_r = 0$ then energy equation e if you write you get only this $-k_w \frac{\partial T}{\partial r}$ or e_r would be having the only conduction terms other terms will not be there right because the v_r is 0. So, the convection terms are anyway not there ok.

And then system is so small, there is no velocity etcetera. So, the viscous dissipation, etcetera all those terms would also be not there or work done by pressure etcetera those terms are not there right. The simplified energy conserve equation for liquid if you write that is $\frac{\partial}{\partial r}(r^2 e_r) = 0$ would be there right. In spherical coordinates you write energy equation in terms of e quantities that is e_r e_θ e_ϕ etcetera.

So, then e_θ if are any way not there and then e_r terms are also only variation with respect to you know r direction are there that also only conduction terms are there. Now, convection side terms are not there because v_r is 0 right. So, then we get $\frac{\partial}{\partial r}(r^2 e_r) = 0$ right.

Now, again here again if you integrate $r^2 e_r$is equals to you will get a constant, but that constant is function of time. It is a constant with respect to r or with respect to space it is a constant, but with respect to time it is function of time some unknown function f (t) let us not worry about what it is.

So, now from this equation what we can write? We can write $\frac{\partial T}{\partial r} = -\frac{e_r}{k_w}$ right and then from here we can write e_r is nothing but $\frac{f(t)}{r^2}$ and in minus and divide by k_w as it is right. So, this if you solve, you get the temperature profile ok.

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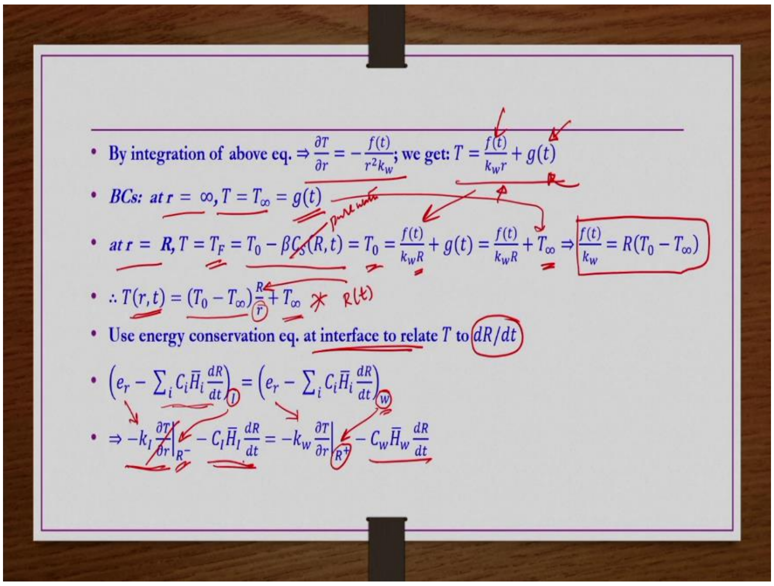

- By integration of above eq. $\Rightarrow \frac{\partial T}{\partial r} = -\frac{f(t)}{r^2 k_w}$; we get: $T = \frac{f(t)}{k_w r} + g(t)$ R
- *BCs:* at $r = \infty, T = T_\infty = g(t)$ pure water
- at $r = R, T = T_F = T_0 - \beta \cancel{C_S}(R,t) = T_0 = \frac{f(t)}{k_w R} + g(t) = \frac{f(t)}{k_w R} + T_\infty \Rightarrow \boxed{\frac{f(t)}{k_w} = R(T_0 - T_\infty)}$
- $\therefore T(r,t) = (T_0 - T_\infty)\frac{R}{r} + T_\infty$ * R(t)
- Use energy conservation eq. at interface to relate T to $\boxed{dR/dt}$
- $\left(e_r - \sum_i C_i \bar{H}_i \frac{dR}{dt}\right)_{Ⓘ} = \left(e_r - \sum_i C_i \bar{H}_i \frac{dR}{dt}\right)_{Ⓦ}$
- $\Rightarrow -k_I \frac{\partial T}{\partial r}\Big|_{R^-} - C_I \bar{H}_I \frac{dR}{dt} = -k_w \frac{\partial T}{\partial r}\Big|_{R^+} - C_w \bar{H}_w \frac{dR}{dt}$

So, now this equation we integrate first ok. Then we get T = $\frac{f(t)}{k_w r} + another\ constant\ g(t)$. So, this is also constant with respect to r, but it is function of some function of time. What it is we do not know right. Now, we have two boundary condition.

It has been mentioned that you know at the surface the concentrations are known, far away concentration or anyway known that both of them at the surface temperature is known and then far away from the crystal surface also at r = r_∞ also the temperature is known that is given.

So, at r = r_∞, T = T_∞ is given. If r = ∞ that if you substitute in this equation. So, 1/∞ is 0. So, then first term is gone. So, T_∞ = g (t) right. Other boundary condition at r = R, T = T_F freezing point which is given as $T_0 - \beta\ C_S(R,t)$, but it is a pure water. Pure water case we are doing problem B.

Problem C is for the salt water case. So, for problem B pure water, so, salt concentration is 0. So, T_F is nothing but T_0 and then this you substitute here. So, you get $\frac{f(t)}{k_w R} + g(t)$. So, that is $\frac{f(t)}{k_w r} + g(t)$ is nothing but T_∞ from the first boundary condition. So, $\frac{f(t)}{k_w}$ we can write $R(T_0 - T_\infty)$. So that means, this $\frac{f(t)}{k_w}$ is now known and then g (t) is also known to us.

So, $\frac{f(t)}{k_w}$ is nothing but $\frac{R(T_0 - T_\infty)}{r}$ is there and + g (t) is nothing but T_∞. So, this is the relation for the temperature as function of time. You may be thinking that right side everything is function of r, there is no time function. So, that is not true because this R whatever is there that R is function of time. So, that is the reason here also we have written T (r, t) ok.

Now, use energy conservation equation at interface to relate T to $\frac{dR}{dt}$. Actually, our purpose of question B is not to get temperature profile, but we have to get relation for $\frac{dR}{dt}$ and then that relation that should be related to T. What we have done now? T we have related to R, but we have to relate T to $\frac{dR}{dt}$. So, that is the reason we are writing energy conservation equation at the interface, at the interface.

So, now crystal side ice we are writing W and then water side liquid side now it is pure water. So, then W we are writing. So, $e_r - \sum C_i \overline{H}_i \frac{dR}{dt}$ to towards the ice side should be balanced by $e_r - \sum C_i \overline{H}_i \frac{dR}{dt}$ towards the water side should be balanced at interface right.

So, now this e_r is nothing but $-k\frac{\partial T}{\partial r}$. Now, again we are replacing this I by R^- just to indicate that it is towards the ice side. And then this W we are replacing by R^+ indicating towards the liquid side or water side now because it is a pure water. So, this we can write e_r is nothing but only conduction term is there here no, other contribution is there.

Convection term and then viscous dissipation contribution etcetera are not there. Whatever because of the concentration or enthalpy changes etcetera are there they are we have already written them here. So, e_r is having only conduction contribution that is $-k_I \frac{\partial T}{\partial r}|_{R^-} - C_i \overline{H}_i \frac{dR}{dt}$ as it is. You may be thinking that you know what about that N_{ir} or N_{ix} terms N_i in this case N_{ir} terms or N_{Sr} terms etcetera why they are not coming.

They will be coming if we have you know dilute system or more than one component is there. Now, its two phases, but only one component H_2O, pure water and then water crystal made up pure water. So, other those other terms are also not coming into e_r term. They will be coming in the C problem where we are doing the same this B problem for the case of salt water or salt solution right. So, this is what we have.

Now, towards this right side also what we have? e in place of e_r we have only $-k_w \frac{\partial T}{\partial r}$ and this is $C_w \overline{H}_w \frac{dR}{dt}$. There is no summation because only one component is there ok. So, now, towards the ice side temperature is constant from this centre to the interface. Temperature is constant in the crystal; obviously, it has to be. So, that is the reason $\frac{dT}{dR}$ is 0.

So, now what we can write? From here $-k_w \frac{\partial T}{\partial r} = C_w \overline{H}_w \frac{dR}{dt} - C_I \overline{H}_I \frac{dR}{dt}$ we can write. So, that we have written here right.

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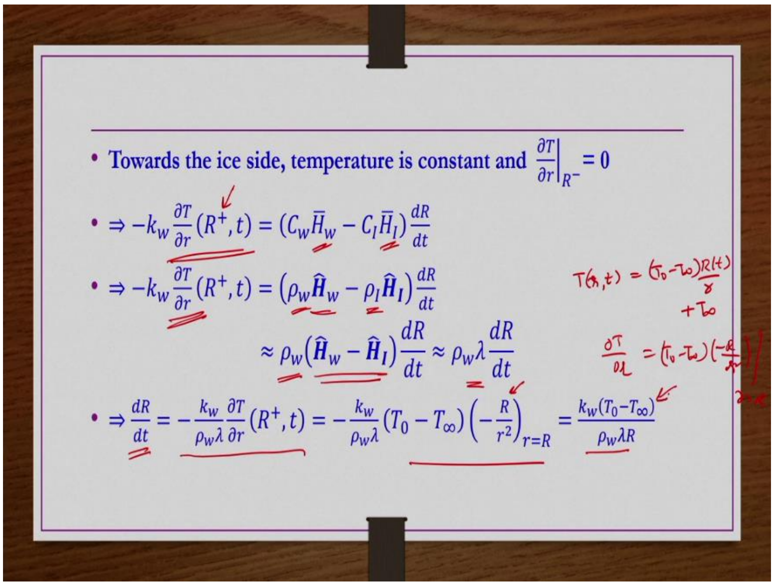

- Towards the ice side, temperature is constant and $\left.\frac{\partial T}{\partial r}\right|_{R^-} = 0$
- $\Rightarrow -k_w \frac{\partial T}{\partial r}(R^+, t) = (C_w \overline{H}_w - C_I \overline{H}_I) \frac{dR}{dt}$
- $\Rightarrow -k_w \frac{\partial T}{\partial r}(R^+, t) = (\rho_w \widehat{\boldsymbol{H}}_w - \rho_I \widehat{\boldsymbol{H}}_I) \frac{dR}{dt}$

$$\approx \rho_w (\widehat{\boldsymbol{H}}_w - \widehat{\boldsymbol{H}}_I) \frac{dR}{dt} \approx \rho_w \lambda \frac{dR}{dt}$$

- $\Rightarrow \frac{dR}{dt} = -\frac{k_w}{\rho_w \lambda} \frac{\partial T}{\partial r}(R^+, t) = -\frac{k_w}{\rho_w \lambda}(T_0 - T_\infty)\left(-\frac{R}{r^2}\right)_{r=R} = \frac{k_w (T_0 - T_\infty)}{\rho_w \lambda R}$

$T(r,t) = (T_0 - T_\infty)\frac{R(t)}{r} + T_\infty$

$\frac{\partial T}{\partial r} = (T_0 - T_\infty)\left(-\frac{R}{r^2}\right)\Big|_{r=R}$

So, but this $\frac{dT}{dR}$ at R^+ ok. Now, this molar enthalpies we have written. The concentrations are not known. C_w, C_I etcetera are not given, but ρ is given. So, that is the reason this molar enthalpies, partial molar enthalpies we are writing you know per unit mass enthalpies specific enthalpy.

So, $\rho_w \widehat{H}_w - \rho_I \widehat{H}_I$ for the partial molar and then tildes per specific per unit mass ok and then this the $\frac{dR}{dt}$ fine. So, now, densities are also constant it has been mentioned. So, that we can take up ρ_w and then multiplied by this one. So, now, the change in enthalpies whatever are there because of the temperature variations, temperature variations are shown in the left hand side.

So, change in enthalpies are very small that is I had mentioned. Compared to the latent heat this change in enthalpy is very small. So, then this we can replace by λ here. So, we get $-k_w \frac{\partial T}{\partial r} = \rho_w \lambda \frac{dR}{dt}$. So, $\frac{dR}{dt}$ relation we want. So, that $\frac{dR}{dt}$ we are keeping one side, rest everything we are writing other side.

In the previous slide we got T = or T (r, t), we got it as $(T_0 - T_\infty)\frac{R\,(t)}{r} + T_\infty$ we had. So, if you do $\frac{\partial T}{\partial r}$ here you get $(T_0 - T_\infty)\left(-\frac{R}{r^2}\right)|_{r=R}$. So, that we have written here right, R that is function of time. So, as it is we are writing here.

So, this equation we can write, you know when you substitute r = R, you will get $\frac{1}{R}$. So, that is $\frac{k_w(T_0 - T_\infty)}{\rho_w \lambda R}$. This is the relation for $\frac{dR}{dt}$ ok.

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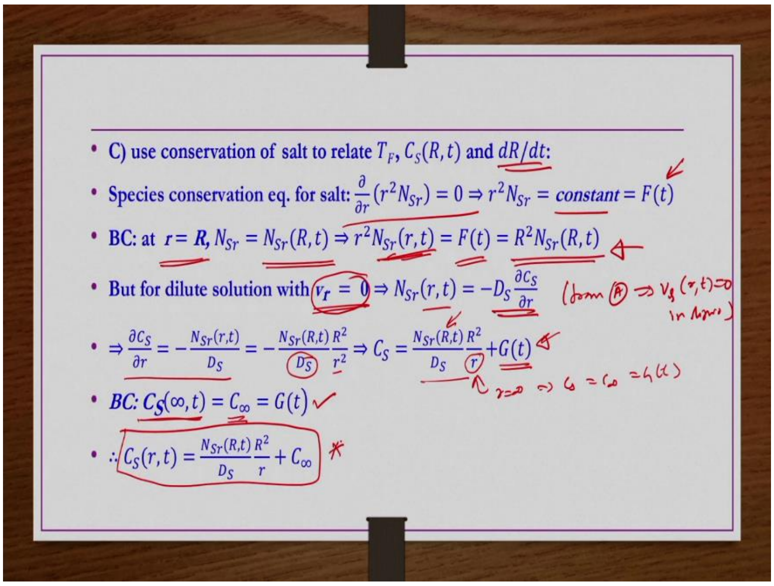

- C) use conservation of salt to relate T_F, $C_S(R,t)$ and $\underline{dR/dt}$:
- Species conservation eq. for salt: $\frac{\partial}{\partial r}(r^2 N_{Sr}) = 0 \Rightarrow r^2 N_{Sr} = \underline{constant} = F(t)$
- BC: at $\underline{r = R}$, $\underline{N_{Sr} = N_{Sr}(R,t)} \Rightarrow r^2 \underline{N_{Sr}(r,t)} = \underline{F(t)} = \underline{R^2 N_{Sr}(R,t)}$
- But for dilute solution with $\boxed{v_r = 0} \Rightarrow N_{Sr}(r,t) = -D_S \frac{\partial C_S}{\partial r}$ (from Ⓐ ⇒ $v_s(r,t) = 0$ in liquid)
- $\Rightarrow \frac{\partial C_S}{\partial r} = -\frac{N_{Sr}(r,t)}{D_S} = -\frac{N_{Sr}(R,t)}{D_S}\frac{R^2}{r^2} \Rightarrow C_S = \frac{N_{Sr}(R,t)}{D_S}\frac{R^2}{r} + G(t)$
- $r = \infty \Rightarrow C_S = C_\infty = G(t)$
- BC: $C_S(\infty, t) = C_\infty = G(t)$ ✓
- $\therefore \boxed{C_S(r,t) = \frac{N_{Sr}(R,t)}{D_S}\frac{R^2}{r} + C_\infty}$ *

Now, the same thing we do by taking a salt solution ok. Till now we are taking ice crystal in pure water for the problem B. Now, for problem C that same pure ice water crystal whatever is there that we are taking in salt solution because by adding some salt depression of freezing point is taking place. So, crystal growth can be little faster ok.

So, for that case what is $\frac{dR}{dt}$ that we have to do. Species conservation equation for salt solution we write, you get $\frac{\partial}{\partial r}(r^2 N_{Sr}) = 0$; that means, $r^2 N_{Sr}$ is equals to constant which is function of time, but independent of r with respect to r it is constant, but it is function of time ok. So, this we are getting. Here also $N_{S\theta}$ and then $N_{S\phi}$ etcetera those terms would also be there, but we are not writing them.

Why we are not writing? Because we already know that spherically symmetric system is there. So, variations are there only in r direction, θ and ϕ direction those variations are not there. Boundary condition at r = R, it has been mentioned that the flux is known. Let us say that flux is N_{Sr} (R, t) right.

So that means, r^2 and N_r (r, t) = F (t) that should be equal to $R^2 N_{Sr}(R, t)$. Now, this equation is also of no use until and unless what is this relation that we find out, what is N_{Sr} (r, t) that is what we have to find out then only we can use this equation.

So, for but dilute solution with $v_r = 0$, if v_r is 0 that means, there is no bulk motion. If there is no bulk motion in the combined flux you will be having only diffusive flux terms only be there right. So, due to the bulk motion whatever the flux contribution is there that is not there. So, that is N_{Sr} at $N_{Sr}(r, t)$ is nothing but $-D_S \frac{\partial C_S}{\partial r}$ because $v_r = 0$ in the liquid that we have seen problem A.

From a problem A, we have already proved that v_r (r, t) = 0 in liquid. So, in the N_{Sr}only diffusive flux contribution would be there, convective flux contribution would not be there. So, $N_{Sr}(r, t)$ is $-D_S \frac{\partial C_S}{\partial r}$. This is again still not complete because what is the C_S function of r that also we did not know.

So, for that reason we have to go for interfacial balance equations anyway that we will do. So, from this equation $\frac{\partial C}{\partial r}$ we can write $-\frac{N_{Sr}(r,t)}{D_S}$. $N_{Sr}(r, t)$ from this equation what we can write? $N_{Sr}(R, t)\frac{R^2}{r^2}$ we can write and divided by D_S is as it is.

We get $C_S = N_{Sr}$ this is at $N_{Sr}(R, t)$. So, it is a constant. It is changing in radial direction, but at the interface it is constant. So, we have to treat it as a constant divided by D_S and then R^2 and then integration of $\int -\frac{1}{r^2}$ is $+\frac{1}{r}+$ constant integration constant. That is constant with respect to r, but it is constant that constant whatever is there that is function of time. So, G (t) here right. So, this we have to find out.

We know at far away distance from the crystal that is at r = ∞ concentration of salt is known that is C_∞. So, here if you substitute r = ∞ then C_S would be C_∞, so that means, G (t) = C_∞ because $\frac{1}{\infty}$ is 0 here. So, G (t) = C_∞ that we know. So, what we get from here? C_S as function of r, t is nothing but given by this expression right.

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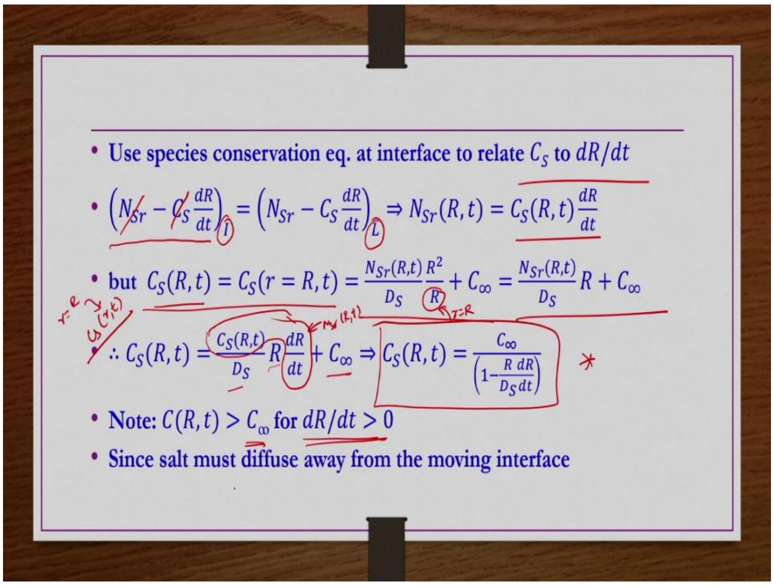

- Use species conservation eq. at interface to relate C_S to dR/dt
- $\left(\cancel{N_{Sr}} - \cancel{C_S}\frac{dR}{dt}\right)_{I} = \left(N_{Sr} - C_S\frac{dR}{dt}\right)_{L} \Rightarrow N_{Sr}(R,t) = C_S(R,t)\frac{dR}{dt}$
- but $C_S(R,t) = C_S(r = R,t) = \frac{N_{Sr}(R,t)}{D_S}\frac{R^2}{R} + C_\infty = \frac{N_{Sr}(R,t)}{D_S}R + C_\infty$
- $\therefore C_S(R,t) = \frac{C_S(R,t)}{D_S}R\frac{dR}{dt} + C_\infty \Rightarrow \boxed{C_S(R,t) = \frac{C_\infty}{\left(1-\frac{R}{D_S}\frac{dR}{dt}\right)}}$ *
- Note: $C(R,t) > C_\infty$ for $dR/dt > 0$
- Since salt must diffuse away from the moving interface

Now, use species conservation equation at interface to relate $C_S \frac{dR}{dt}$ because the previous equation C_S as function of r, t that we got and then that is in terms of r as function of time is there right in this slide here. So, now, here C_S is related to R right. This R is related to time, but we want this C_S related to $\frac{dR}{dt}$. So, that we wanted to do. So, that we can get here by using the species conservation equation at the interface.

So, now here you can see $N_{Sr} - C_S\frac{dR}{dt}$ towards the ice side should be balanced by $N_{Sr} - C_S\frac{dR}{dt}$ should be balanced by liquid side right. So, in the ice side or the crystal whatever is there ice crystal is there of it is pure water. So, C_S is 0. So, if there is no salt in the crystal phase, so, then obviously, its flux would also be 0 in the crystal side. So, what we have here? $N_{Sr} = C_S \frac{dR}{dt}$.

So, now this C_S (r, t) we know, but now here it is as function of R, t that is what we have. So, the previous slide what we got? C_S (r, t) we got it as $\frac{N_{Sr}(R,t)}{D_S}$ and $\frac{R^2}{r^2}$ was there. So, in place of this r you are writing R because you want $C_S(R,t)$. So, in the previous expression previous slide whatever $C_S(r,t)$ was there. So, this r you replace by R.

So, get $C_S(R,t)$. So, then we have this one. So, that is this one right. So, what we have here? $C_S(R,t)$. = $\frac{N_{Sr}(R,t)}{D_S R}$ + C∞ was there, but $N_{Sr}(R,t)$we do not know right. So, what we know? It is nothing but $C_S(R,t) \frac{dR}{dt}$.

So, $C_S(R,t) \frac{dR}{dt}$in place of, in place of $N_{Sr}(R,t)$ that we have written. And then remaining this $\frac{R}{D_S}$ + C∞ terms are as it is right. Then what you can write? $C_S(R,t)$ terms one side remaining terms other side if you write you get the relation between C_S and $\frac{dR}{dt}$ by given way this one, fine.

So, now here C at R, t would be very much greater than C∞ for $\frac{dR}{dt}$ if it is large greater than 0 according to this equation that is what we can understand. Species salt must diffuse away from the moving interface that is the reason that has to be $\frac{dR}{dt}$ has to be positive.

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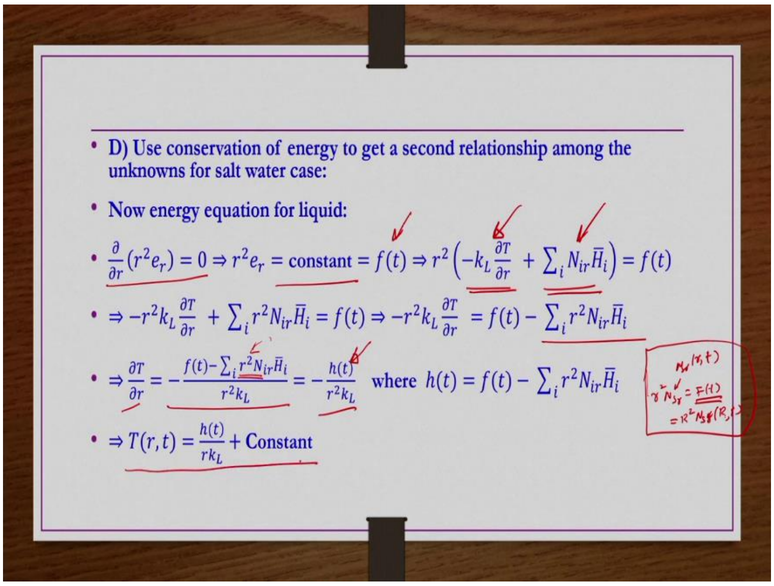

- D) Use conservation of energy to get a second relationship among the unknowns for salt water case:
- Now energy equation for liquid:
- $\frac{\partial}{\partial r}(r^2 e_r) = 0 \Rightarrow r^2 e_r = \text{constant} = f(t) \Rightarrow r^2\left(-k_L \frac{\partial T}{\partial r} + \sum_i N_{ir}\bar{H}_i\right) = f(t)$
- $\Rightarrow -r^2 k_L \frac{\partial T}{\partial r} + \sum_i r^2 N_{ir}\bar{H}_i = f(t) \Rightarrow -r^2 k_L \frac{\partial T}{\partial r} = f(t) - \sum_i r^2 N_{ir}\bar{H}_i$
- $\Rightarrow \frac{\partial T}{\partial r} = -\frac{f(t) - \sum_i r^2 N_{ir}\bar{H}_i}{r^2 k_L} = -\frac{h(t)}{r^2 k_L}$ where $h(t) = f(t) - \sum_i r^2 N_{ir}\bar{H}_i$
- $\Rightarrow T(r,t) = \frac{h(t)}{r k_L} + \text{Constant}$

$N_{sr}(r,t)$

$r^2 N_{sr} = F(t)$

$= R^2 N_{sr}(R,t)$

Now, the fourth question or fourth part of the problem use conservation of energy to get a second relationship among the unknowns for salt water case. So, energy equation for liquid if you write you get though by $\frac{\partial}{\partial r}(r^2 e_r) = 0$ because $e_\theta e_\phi$ terms we are not taking. And then $r^2 e_r$ should be constant upon integration that constant is constant with respect to r, but it is function of time. So, f (t) we are writing.

So, but now e_r here in the previous case when pure water was there only conduction part was there right. Flux because of the mass transfer etcetera those terms were not there. So, now, those terms are being added here because this case we are doing for the salt solution case. Problem C and D are for the salt water case salt solution case. So, this term is being added that is $\sum N_{ir}\bar{H}_i = f(t)$.

So, $r^2 k_L \frac{\partial T}{\partial r}$ these term $\sum r^2 N_{ir}\bar{H}_i = f(t)$. What we are writing? This $\sum$ term is also we are taking to the right-hand side right then $\frac{\partial T}{\partial r}$ we are writing one side and rest all other term we are taking other side right.

So, this $f(t) - \sum_i r^2 N_{ir}\bar{H}_i$ we are writing h (t). This $r^2 N_{ir}$ is constant right. From the species conservation equation simplification previous slide that we have what we got? $r^2 N_{Sr} = F(t) = R^2 N_{Sr}$, this is at R, t, this is at N_{Sr} (r, t).

So, this is what get. So, $r^2 N_{Sr}$ only for salt part we have written there, but that is true for all the cases. We get the same thing. So, that is a function of time only. So, combining these terms and then writing as one simple function of time is not going to affect the further integration that we are going to do because this entire term though here by appearance it is looking r^2 term is also there.

So, how can we take as a constant because we have to integration, we have to do with integration next step that you may be thinking, but that $r^2 N_{Sr}$ or $r^2 N_{ir}$ is constant and that is function of time only as per the previous slide we have done the species conservation equations simplification right. So, this is what we have. Now, you integrate this one $\mathrm{T}\ (\mathrm{r}, \mathrm{t}) = \frac{h(t)}{r k_L} +$ constant. So, we have to find out two constants h (t) and then that constant.

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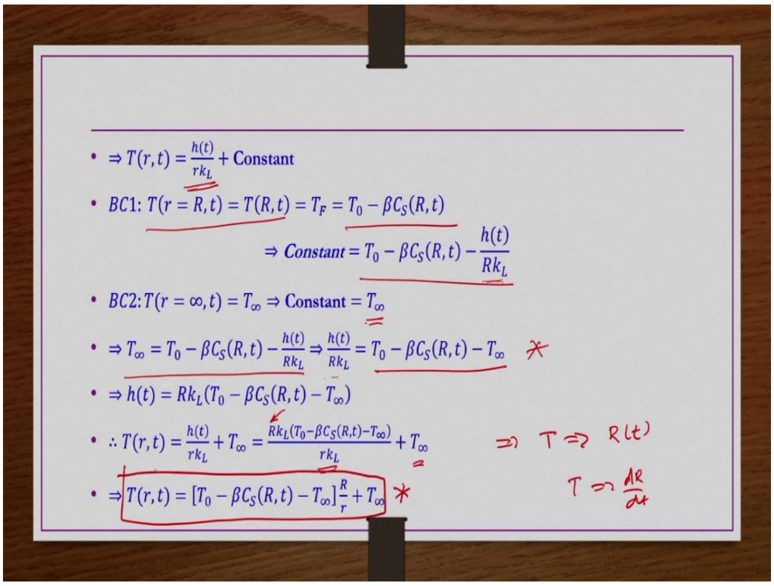

- $\Rightarrow T(r,t) = \frac{h(t)}{r k_L} + \text{Constant}$
- $BC1{:}\ T(r=R,t) = T(R,t) = T_F = T_0 - \beta C_S(R,t)$

  $\Rightarrow Constant = T_0 - \beta C_S(R,t) - \frac{h(t)}{R k_L}$
- $BC2{:} T(r=\infty,t) = T_\infty \Rightarrow \text{Constant} = T_\infty$
- $\Rightarrow T_\infty = T_0 - \beta C_S(R,t) - \frac{h(t)}{R k_L} \Rightarrow \frac{h(t)}{R k_L} = T_0 - \beta C_S(R,t) - T_\infty$ *
- $\Rightarrow h(t) = R k_L (T_0 - \beta C_S(R,t) - T_\infty)$
- $\therefore T(r,t) = \frac{h(t)}{r k_L} + T_\infty = \frac{R k_L (T_0 - \beta C_S(R,t) - T_\infty)}{r k_L} + T_\infty$
- $\Rightarrow \boxed{T(r,t) = [T_0 - \beta C_S(R,t) - T_\infty]\frac{R}{r} + T_\infty}$ *

$\Rightarrow T \Rightarrow R(t)$

$T \Rightarrow \frac{dR}{dt}$

So, one boundary condition in the fourth problem it has been mentioned that the at surface the temperature is known that is at r = R temperature is known that is mentioned in the problem, D part of the problem clearly. So, T (R, t) is known. So, that is $\mathrm{T_F}$ and then that is given in terms of $T_0 - \beta\ C_S(R, t)$ ok. So, now, here when you apply this boundary condition here at constant should be $T_0 - \beta\ C_S(R, t) - \frac{h(t)}{R k_L}$.

Other boundary condition at r = ∞, T = $\mathrm{T_\infty}$. So, in this equation if you substitute r = ∞ you get constant = $\mathrm{T_\infty}$. So, for constant we have two relations, this relation and this relation.

When you equate them together you have this relation. So, from here $\frac{h(t)}{Rk_L}$ you get this one. So, we have constant as a T∞ in $\frac{h(t)}{Rk_L}$ also we got it by this relation.

So, h t is also known and then that constant we have written as a constant only that is also known. So, that we substitute here T (r, t). h (t) is nothing but $\frac{Rk_L(T_0-\beta\ C_S(R,t)-T_\infty)}{rk_L}$ is as it is $+\ T_\infty$ fine. So, now, this is T related to R function of time right. Whatever R's are there they are all function of time, but we wanted to relate this T to $\frac{dR}{dt}$.

So, for that what we do? We will be doing in the next slide you know by writing the balance at the interface ok. So, temperature distribution as function of space and time is given by this expression finally.

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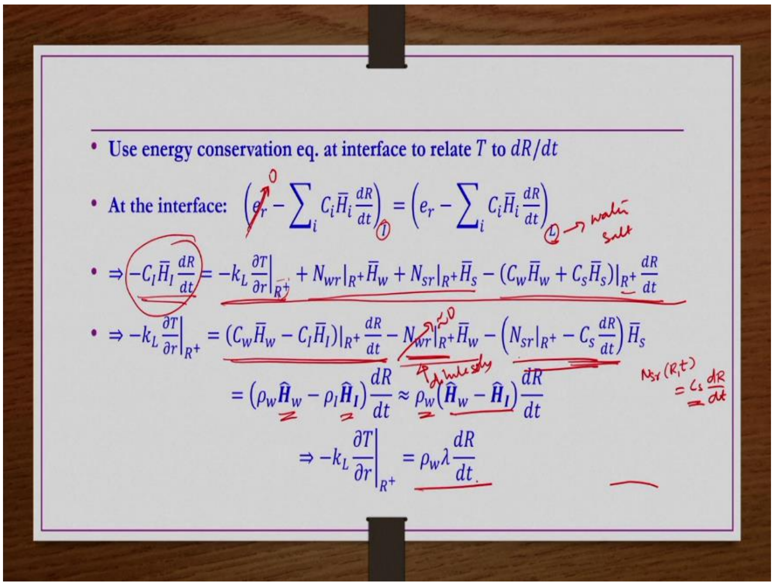

- Use energy conservation eq. at interface to relate T to dR/dt
- At the interface: $\left(\cancel{e_r}^{\,0}-\sum_i C_i\bar{H}_i\frac{dR}{dt}\right)_{Ⓘ}=\left(e_r-\sum_i C_i\bar{H}_i\frac{dR}{dt}\right)_{Ⓛ} \to$ water salt
- $\Rightarrow -C_I\bar{H}_I\frac{dR}{dt}=-k_L\frac{\partial T}{\partial r}\Big|_{R^+}+N_{wr}|_{R^+}\bar{H}_w+N_{sr}|_{R^+}\bar{H}_s-(C_w\bar{H}_w+C_s\bar{H}_s)|_{R^+}\frac{dR}{dt}$
- $\Rightarrow -k_L\frac{\partial T}{\partial r}\Big|_{R^+}=(C_w\bar{H}_w-C_I\bar{H}_I)|_{R^+}\frac{dR}{dt}-\overset{\approx 0}{\cancel{N_{wr}}}|_{R^+}\bar{H}_w-\left(N_{sr}|_{R^+}-C_s\frac{dR}{dt}\right)\bar{H}_s$

dilute soln

$N_{sr}(R,t)=C_s\frac{dR}{dt}$

$$=\left(\rho_w\hat{\boldsymbol{H}}_w-\rho_I\hat{\boldsymbol{H}}_I\right)\frac{dR}{dt}\approx\rho_w\left(\hat{\boldsymbol{H}}_w-\hat{\boldsymbol{H}}_I\right)\frac{dR}{dt}$$

$$\Rightarrow -k_L\frac{\partial T}{\partial r}\Big|_{R^+}=\rho_w\lambda\frac{dR}{dt}.$$

So, use conservation equation at interface to relate temperature to $\frac{dR}{dt}$. When we do we have $\left(e_r-\sum_i C_i\bar{H}_i\frac{dR}{dt}\right)_I=\left(e_r-\sum_i C_i\bar{H}_i\frac{dR}{dt}\right)_L$, liquid side and then ice side right. So, here ice side this e_r/e_r whatever is there $\frac{\partial T}{\partial r}$ is 0 because whether it is conduction or anything whatever no changes are taking place in the ice crystal side because the temperature is constant in the entire crystal from the center to this interface ok.

So, remaining one is that in the ice side we have only ice, no salt nothing is there ice crystal is there. So, $-\sum_i C_i \overline{H}_i \frac{dR}{dt}$ only we are writing without any summation because in the crystal side we have only, we have only ice crystal no other component is there. In the liquid side e_r is having you know conduction term then whatever the energy changes because of the species flux terms and then because of the enthalpy variations in terms.

And then in the liquid side we have liquid is having both water and then salt. So, w, S quantities are there. $N_{wr}|_{R^+}\overline{H}_w + N_{Sr}|_{R^+}\overline{H}_S - (C_w\overline{H}_w + C_S\overline{H}_S)|_{R^+}\frac{dR}{dt}$ right and then conduction to $-k_L \frac{\partial T}{\partial r}$.

So, liquid side whatever are there that those quantities we are indicating with R^+. So, at liquid side all these quantities there when you expand this equation you get, but ice side we have only this one. All other terms would be 0 because ice is pure water crystal ok and then temperature variations are not there. So, conduction term should be 0, flux term should be 0 right and then pure component is there. So, that is what we have only this component of ice, only pure ice is there. So, only this part is there right.

So, now next step what we do? We write this $-k_L \frac{\partial T}{\partial r}$ one side and then this $C_w\overline{H}_w \frac{dR}{dt}$ and then the $C_i\overline{H}_i \frac{dR}{dt}$ that we are combining together right and then multiplied by $\frac{dR}{dt}$. Remaining $-N_{wr}|_{R^+}\overline{H}_w$ and then $-N_{Sr}|_{R^+}\overline{H}_S$ and then this minus of minus $+C_S \frac{dR}{dt}$ term as it is here fine.

So, we have found just now in the one of the previous slide N_{Sr} (R, t) is nothing but $C_S \frac{dR}{dt}$ at R, t. So, these two terms together 0 because this is equal to this one and then one is having plus another one is minus. So, this is 0 together right and then dilute solution, it is a dilute solution. So, flux of water would be negligible. Flux of salt would be higher, but flux of water would be very negligible because it is a dilute solution right.

So, this again concentration to C_w C_I are not known, but ρ_w ρ_I are non are known. So, then we are writing this enthalpy also per unit mass units. So, these remaining two terms are anyway 0. So, densities are also equal to each other. So, then ρ_w we are taking common. And then problem statement it has been given that the change in enthalpies because of the temperature variations are there very negligible compared to the latent heat. So, then this we can write ρw λ $\frac{dR}{dt}$ fine.

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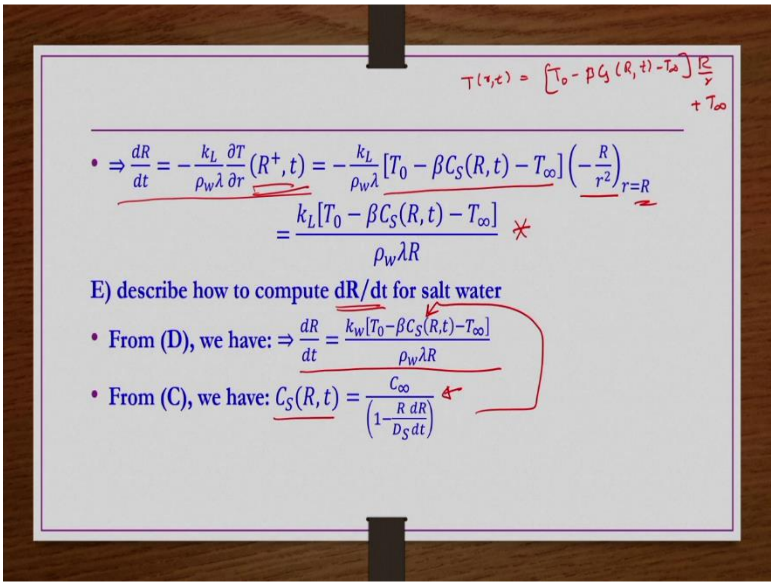

$T(r,t) = [T_0 - \beta C_S(R,t) - T_\infty]\frac{R}{r} + T_\infty$

- $\Rightarrow \frac{dR}{dt} = -\frac{k_L}{\rho_w \lambda}\frac{\partial T}{\partial r}(R^+,t) = -\frac{k_L}{\rho_w \lambda}[T_0 - \beta C_S(R,t) - T_\infty]\left(-\frac{R}{r^2}\right)_{r=R}$

$$= \frac{k_L[T_0 - \beta C_S(R,t) - T_\infty]}{\rho_w \lambda R} \quad *$$

E) describe how to compute dR/dt for salt water

- From (D), we have: $\Rightarrow \frac{dR}{dt} = \frac{k_w[T_0 - \beta C_S(R,t) - T_\infty]}{\rho_w \lambda R}$
- From (C), we have: $C_S(R,t) = \frac{C_\infty}{\left(1 - \frac{R}{D_S}\frac{dR}{dt}\right)}$

So, that equation $\frac{dR}{dt}$ you right one side remaining term that is $-\frac{k_L}{\rho_w \lambda}\frac{\partial T}{\partial r}(R^+,t)$ ok. This is what we have. Now, the previous slide temperature distribution what we got? We got T (r, t) is nothing but $[T_0 - \beta\, C_S(R,t) - T_\infty]\left(\frac{R}{r}\right) + T_\infty$ (Refer Time: 40:58), alright.

So, from here $\frac{\partial T}{\partial r}$ if you do, you get that whatever this one multiplied by $\left(-\frac{R}{r^2}\right)$ you will get and then you have to substitute r = R because this is this $\frac{\partial T}{\partial r}$ is it R^+ ok. So, now, when you do this one, so, what you have? You have $\frac{k_L}{\rho_w \lambda R}[T_0 - \beta\, C_S(R,t) - T_\infty]$ as an expression for $\frac{dR}{dt}$ ok.

So, now next part last part of the question is that come how to compute $\frac{dR}{dt}$ for salt water. So, that is $\frac{dR}{dt}$ these expression we got for $\frac{dR}{dt}$ for salt water case ok, but how to evaluate this one? So, for that from D part, we have already $\frac{dR}{dt}$ is this relation right. Here in terms of temperature we are getting you know I mean like in $\frac{dR}{dt}$ related to temperature and then R as function of time. So, it is here already we have and then from C part $C_S(R,t)$ we have already this one.

So, this C_S you substitute here right and then take all $\frac{dR}{dt}$ terms one side and then solve this equation to get the final $\frac{dR}{dt}$ in terms of known quantities you will get.

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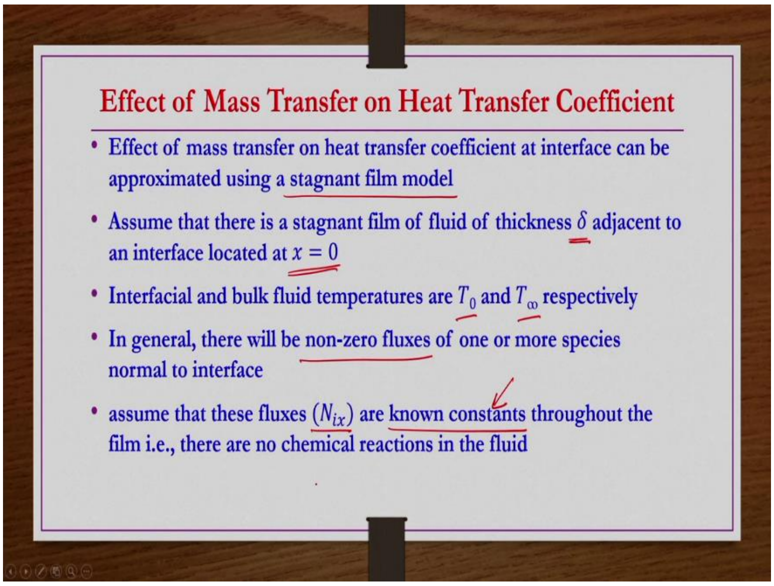

Effect of Mass Transfer on Heat Transfer Coefficient

- Effect of mass transfer on heat transfer coefficient at interface can be approximated using a stagnant film model
- Assume that there is a stagnant film of fluid of thickness δ adjacent to an interface located at $x = 0$
- Interfacial and bulk fluid temperatures are T_0 and T_∞ respectively
- In general, there will be non-zero fluxes of one or more species normal to interface
- assume that these fluxes (N_{ix}) are known constants throughout the film i.e., there are no chemical reactions in the fluid

So, now we take another problem, effect of mass transfer on heat transfer coefficient. Here effect of mass transfer on heat transfer coefficient at interface can be approximated using a stagnant film model. Assume that there is a stagnant film of fluid of thickness δ adjacent to an interface located at x = 0, fine.

Interfacial and bulk fluid temperatures T_0 and T_∞ respectively are known they are given that is at x = 0, T_0 and x = δ, T_∞. In general, there will be non 0 fluxes of one or more species normal to interface. Assume that these fluxes N_{ix} are known constants they are known they are constants, and they are known ok throughout the film there are no chemical reactions in the fluid.

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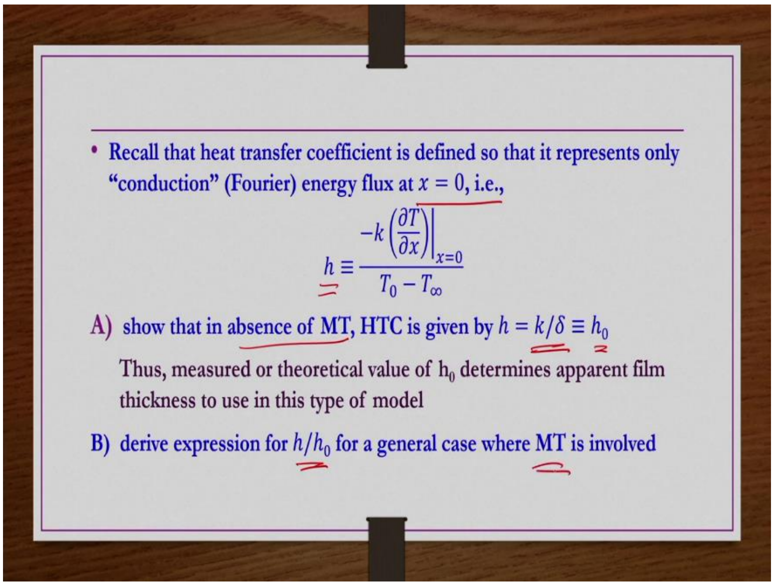

- Recall that heat transfer coefficient is defined so that it represents only "conduction" (Fourier) energy flux at $x = 0$, i.e.,

$$h \equiv \frac{-k\left(\frac{\partial T}{\partial x}\right)\Big|_{x=0}}{T_0 - T_\infty}$$

A) show that in absence of MT, HTC is given by $h = k/\delta \equiv h_0$

Thus, measured or theoretical value of h_0 determines apparent film thickness to use in this type of model

B) derive expression for h/h_0 for a general case where MT is involved

So, recall that heat transfer coefficient is defined so that it represents only conduction Fourier energy flux at x = 0 is $\frac{-k\left(\frac{\partial T}{\partial x}\right)|_{x=0}}{T_0 - T_\infty}$ that is nothing but heat transfer coefficient. Now, what you have to find out or prove? In the absence of mass transfer HTC is given by k/δ which is a constant because film thickness is constant for given system k is constant.

So, that constant let us say h_0 that is the first part of this problem. The second part of the problem derive an expression for h/h_0 for a general case when there is a mass transfer as well right. So, these two problems we have to take.

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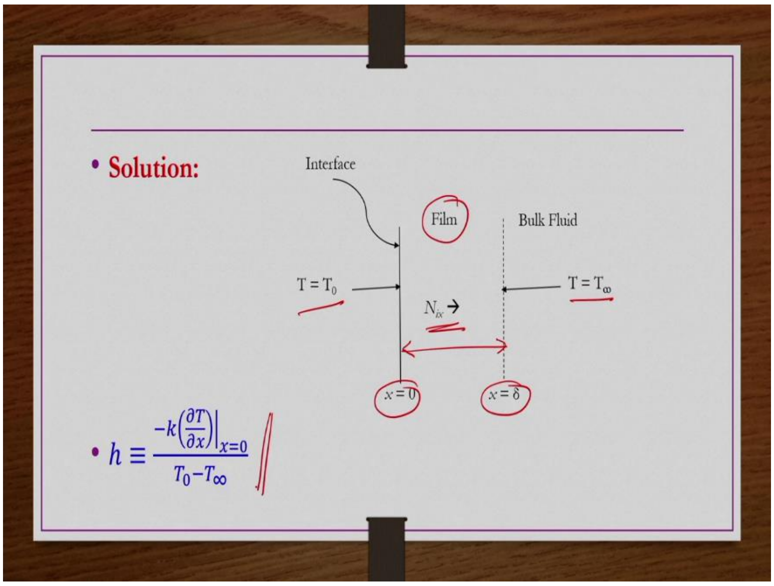

- **Solution:**
- $h \equiv \frac{-k\left(\frac{\partial T}{\partial x}\right)\Big|_{x=0}}{T_0 - T_\infty}$

Pictorially now, we have an interface which is located as x = 0 at which temperature is equal to T_0 right. Next to this interface there is a film. Film thickness is δ and then at the boundary between film thickness and bulk fluid that is at x = δ temperature is T_∞.

So, between these two limits that is x = 0 to x = δ that is within the film this flux are constant and they are known ok. They are moving from x = 0 to x = δ side right. So, here the h is defined like this.

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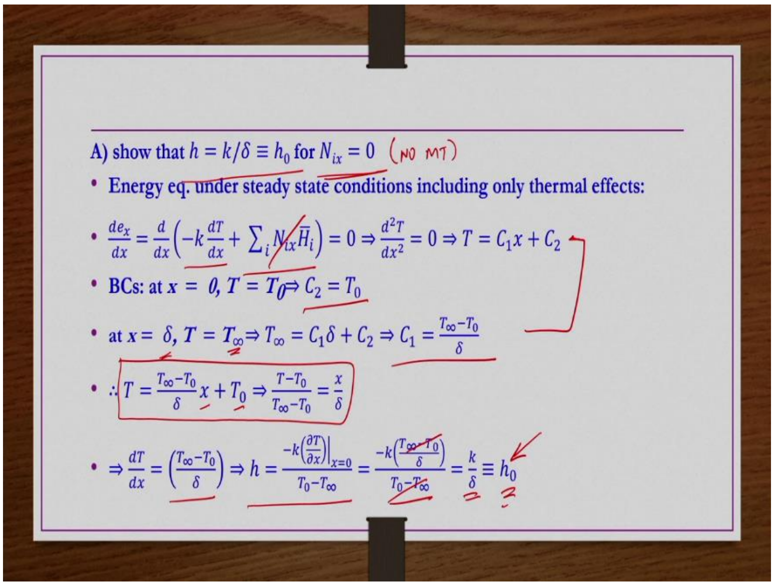

A) show that $h = k/\delta \equiv h_0$ for $N_{ix} = 0$ (NO MT)

- Energy eq. under steady state conditions including only thermal effects:
- $\frac{de_x}{dx} = \frac{d}{dx}\left(-k\frac{dT}{dx} + \sum_i \cancel{N_{ix}}\bar{H}_i\right) = 0 \Rightarrow \frac{d^2T}{dx^2} = 0 \Rightarrow T = C_1x + C_2$
- BCs: at $x = 0,\ T = T_0 \Rightarrow C_2 = T_0$
- at $x = \delta,\ T = T_\infty \Rightarrow T_\infty = C_1\delta + C_2 \Rightarrow C_1 = \frac{T_\infty - T_0}{\delta}$
- $\therefore \boxed{T = \frac{T_\infty - T_0}{\delta}x + T_0 \Rightarrow \frac{T - T_0}{T_\infty - T_0} = \frac{x}{\delta}}$
- $\Rightarrow \frac{dT}{dx} = \left(\frac{T_\infty - T_0}{\delta}\right) \Rightarrow h = \frac{-k\left(\frac{\partial T}{\partial x}\right)\Big|_{x=0}}{T_0 - T_\infty} = \frac{-k\left(\frac{\cancel{T_\infty - T_0}}{\delta}\right)}{\cancel{T_0 - T_\infty}} = \frac{k}{\delta} \equiv h_0$

For first case when there is no mass transfer we have to find out this h is a constant. Show that h = k/δ = h_0 for N_{ix} = 0. N_{ix} is 0 that means, no mass transfer case the problem 1, right. So, energy equation under steady state conditions including only thermal effects if you write $\frac{\partial e_x}{\partial x} = 0$ you will be getting.

This is a Cartesian coordinate system now we are having and then e_x is having only conduction and then mass flux terms are there or energy associated because of the mass flux whatever is there. So, $\frac{\partial}{\partial x}\left(-k\frac{\partial T}{\partial x} + \sum_i N_{ix}\overline{H}_i\right)$ and then this variations in temperature and flux whatever are there they are also only x direction we are taking as per the problem confinement ok.

So, if it is 0 then from here this actually this N_{ix} is anyway 0 for the case of problem A when there is no mass transfer. So, we get $\frac{\partial^2 T}{\partial x^2} = 0$ then T = C_1 x + C_2 at T = 0, T = T_0. So, C_2 = T_0 at T is at x = δ T = T_∞. So, T_∞ = C_1 δ + C_2 C_1 is $\frac{T_\infty - T_0}{\delta}$.

So, now this C_2 and then C_1 if you substitute here then you get T = C_1. C_1 is $\frac{T_\infty - T_0}{\delta}$ (x + C_2). C_2 is T_0 that is $\frac{T - T_0}{T_\infty - T_0} = \frac{x}{\delta}$. Now, we need $\frac{\partial T}{\partial x}$ right. So, this equation $\frac{\partial T}{\partial x}$ you do. You get $\frac{T_\infty - T_0}{\delta}$ multiplied by 1.

And then h definition is given by this $\frac{-k\frac{\partial T}{\partial x}|_{x=0}}{T_0 - T_\infty}$. $-k\frac{\partial T}{\partial x}$ is nothing but $\frac{T_\infty - T_0}{\delta}$ and then whole divided by $T_0 - T_\infty$ is as it is. So, this $T_\infty - T_0$ this $T_0 - T_\infty$ cancelled out with $-$. So, $-$ of $- + \frac{k}{\delta}$.

So, which is a constant, that constant we are calling h_0 as per the problem statement. So, this is the first part, exactly similar way we have to do the second part, but there we cannot cancel out this N_{ix} because in the second case we are taking mass transfer also included in the system.

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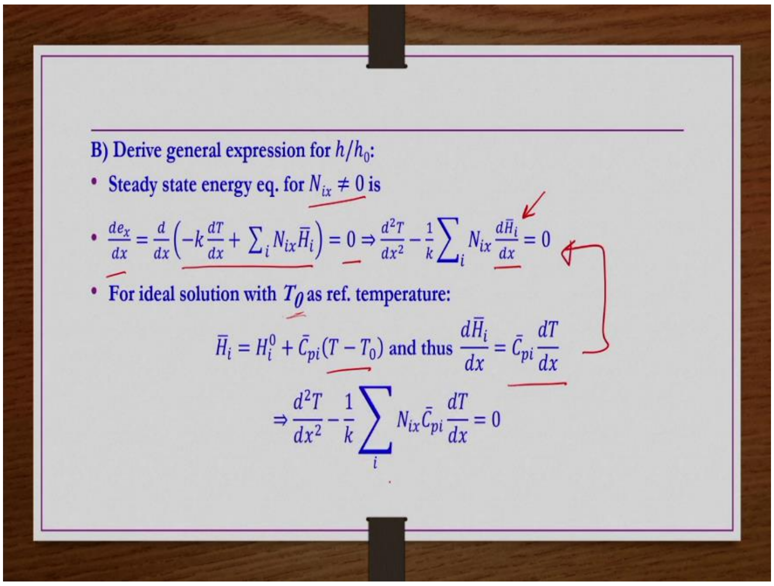

B) Derive general expression for h/h_0:

- Steady state energy eq. for $N_{ix} \neq 0$ is
- $\frac{de_x}{dx} = \frac{d}{dx}\left(-k\frac{dT}{dx} + \sum_i N_{ix}\bar{H}_i\right) = 0 \Rightarrow \frac{d^2T}{dx^2} - \frac{1}{k}\sum_i N_{ix}\frac{d\bar{H}_i}{dx} = 0$
- For ideal solution with T_0 as ref. temperature:

$$\bar{H}_i = H_i^0 + \bar{C}_{pi}(T - T_0) \text{ and thus } \frac{d\bar{H}_i}{dx} = \bar{C}_{pi}\frac{dT}{dx}$$

$$\Rightarrow \frac{d^2T}{dx^2} - \frac{1}{k}\sum_i N_{ix}\bar{C}_{pi}\frac{dT}{dx} = 0$$

So, $N_{ix} \neq 0$, but it is constant and then constant is known that is what it is given in the problem. So, energy equation here the $\frac{\partial e_x}{\partial x} = 0$ and then e_x is nothing but conduction term and then energy associated because of the mass transfer. So, that is $\sum_i N_{ix}\bar{H}_i$ that is also known right.

So, this equation we can what we can do? We can divide by minus k both sides and then differentiate with respect to $\frac{d}{dx}$. Then you get $\frac{d^2T}{dx^2} - \frac{1}{k}\sum_i N_{ix}\frac{d\bar{H}_i}{dx} = 0$ right. So, for ideal solution with T_0 as a reference temperature you know $\bar{H}_i$ we have to define because in general this $\bar{H}_i$ are not known right.

But these $\bar{H}_i$ are related to the $\bar{C}_{pi}$ etcetera, so, which are known for a given system from the standard text books. So, that is the reason we write $\bar{H}_i = \bar{H}_{i0} + \bar{C}_{pi}\Delta T$. So, from here $\frac{d\bar{H}_i}{dx}$ is nothing but $\bar{C}_{pi}\frac{dT}{dx}$, so, that you can substitute here. Then you get $\frac{d^2T}{dx^2} - \frac{1}{k}\sum_i N_{ix}\bar{C}_{pi}\frac{dT}{dx} = 0$. This equation we have to solve right to get the temperature profile then only we can get the h definition.

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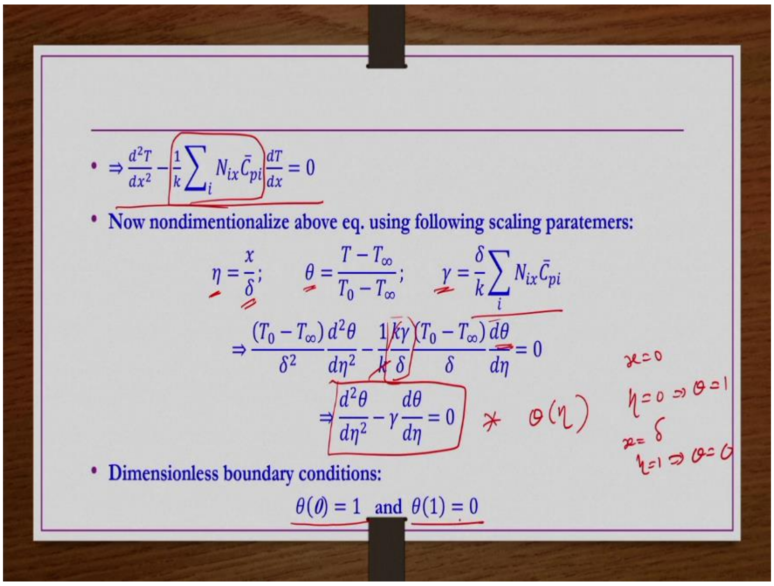

- $\Rightarrow \frac{d^2T}{dx^2} - \boxed{\frac{1}{k}\sum_i N_{ix}\bar{C}_{pi}}\frac{dT}{dx} = 0$
- Now nondimentionalize above eq. using following scaling paratemers:

$$\eta = \frac{x}{\delta}; \qquad \theta = \frac{T - T_\infty}{T_0 - T_\infty}; \qquad \gamma = \frac{\delta}{k}\sum_i N_{ix}\bar{C}_{pi}$$

$$\Rightarrow \frac{(T_0 - T_\infty)}{\delta^2}\frac{d^2\theta}{d\eta^2} - \frac{1}{\cancel{k}}\frac{\cancel{k}\gamma}{\delta}\frac{(T_0 - T_\infty)}{\delta}\frac{d\theta}{d\eta} = 0$$

$$\boxed{\Rightarrow \frac{d^2\theta}{d\eta^2} - \gamma\frac{d\theta}{d\eta} = 0} \quad * \quad \theta(\eta)$$

$x = 0$

$\eta = 0 \Rightarrow \theta = 1$

$x = \delta$

$\eta = 1 \Rightarrow \theta = 0$

- Dimensionless boundary conditions:

$$\theta(0) = 1 \text{ and } \theta(1) = 0$$

So, now non-dimensionalizing this above equation using following parameters we are doing for simplicity of integration only. So, this x/δ we are writing as η and $\frac{T-T_\infty}{T_0-T_\infty}$ we are writing as θ and whatever $\frac{\delta}{k}\sum_i N_{ix}\bar{C}_{pi}$ which is a constant that we I am writing as a γ. This is all a constant because N_{ix} is constant throughout the film that is mentioned and this constants are known that is also mentioned.

So, that in this relation except $\frac{\partial T}{\partial x}$ whatever you say that entire thing is a constant right. So, now, this when you use this expression to convert this equation in terms of non dimensionalized variables then we get first equation $(T_0 - T_\infty)\partial^2\theta$ divided by, ∂^2T you will get $(T_0 - T_\infty)\partial^2\theta$ and then dx^2 you get $\delta^2 d\eta^2$ and then $-\frac{1}{k}$, whatever $\sum_i N_{ix}\bar{C}_{pi}$ is nothing but $\gamma\frac{k}{\delta}$ right.

And then dt from here dt is nothing but $(T_0 - T_\infty)$ d θ and then dx is nothing but δ d η = 0. So, here this k this k is cancelled out anyway. So, this equation both sides you multiply by $\frac{\delta^2}{T_0-T_\infty}$. Then you have this equation $\frac{d^2\theta}{d\eta^2} - \gamma\frac{d\theta}{d\eta} = 0$ right.

So, this equation you have to solve to get θ as function of η right. So, you need two boundary conditions for θ for different η values right. η = 0, x = 0; that means, $\eta = 0$ and then x = 0 T = T_0. So, θ should be 1 and then x = δ $\eta = 1$. So, θ should be and then x = δ θ is equal to nothing but T is nothing but T_∞. So, $T_\infty - T_\infty$ is 0 by $T_0 - T_\infty$ is 0.

So, boundary conditions θ at η = 0 is 1 and θ at η = 1 is 0 right. So, this equation when you integrate first you get $\frac{d\theta}{d\eta} = C_1 e^{\gamma\eta}$

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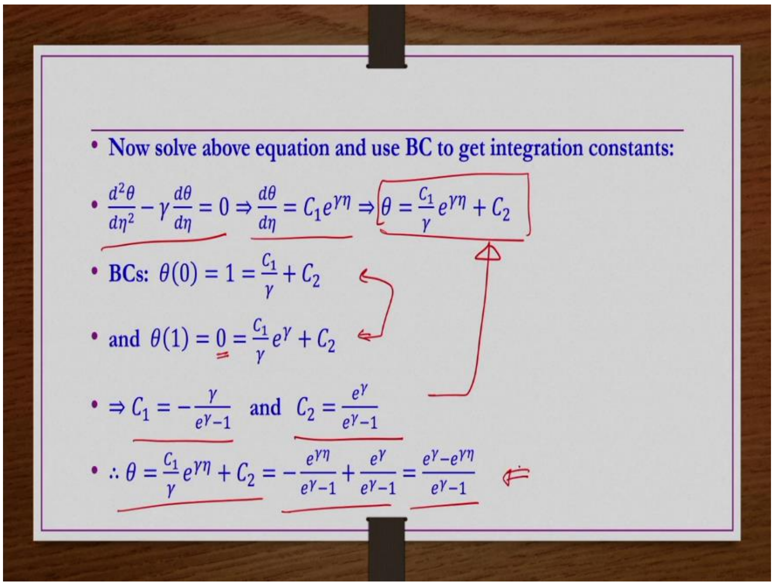

- Now solve above equation and use BC to get integration constants:
- $\frac{d^2\theta}{d\eta^2} - \gamma\frac{d\theta}{d\eta} = 0 \Rightarrow \frac{d\theta}{d\eta} = C_1 e^{\gamma\eta} \Rightarrow \boxed{\theta = \frac{C_1}{\gamma}e^{\gamma\eta} + C_2}$
- BCs: $\theta(0) = 1 = \frac{C_1}{\gamma} + C_2$
- and $\theta(1) = 0 = \frac{C_1}{\gamma}e^{\gamma} + C_2$
- $\Rightarrow C_1 = -\frac{\gamma}{e^{\gamma}-1}$ and $C_2 = \frac{e^{\gamma}}{e^{\gamma}-1}$
- $\therefore \theta = \frac{C_1}{\gamma}e^{\gamma\eta} + C_2 = -\frac{e^{\gamma\eta}}{e^{\gamma}-1} + \frac{e^{\gamma}}{e^{\gamma}-1} = \frac{e^{\gamma}-e^{\gamma\eta}}{e^{\gamma}-1}$

Then second time when integrate it you get $\theta = \frac{C_1}{\gamma}e^{\gamma\eta} + C_2$. This C_1 C_2 are integration constants. I have written directly you can do by steps ok. So, first boundary condition when you substitute that is when η = 0 θ = 1. So, 1 = $\frac{C_1}{\gamma}e^0$ is 1 + C_2.

And then when η = 1 θ = 0. So, 0 = $\frac{C_1}{\gamma}e^{\gamma} + C_2$. You have two equations two constants. When you solve you get C_1 is equal to this one, C_2 is equal to this one. I have written directly, you can do them, very simple, straight forward.

So, this C_1 C_2 you can substitute here in this equation to get θ that is this one. $\theta = \frac{e^{\gamma}-e^{\gamma\eta}}{e^{\gamma}-1}$, right. So, this is nothing but temperature distribution in non-dimensional form right.

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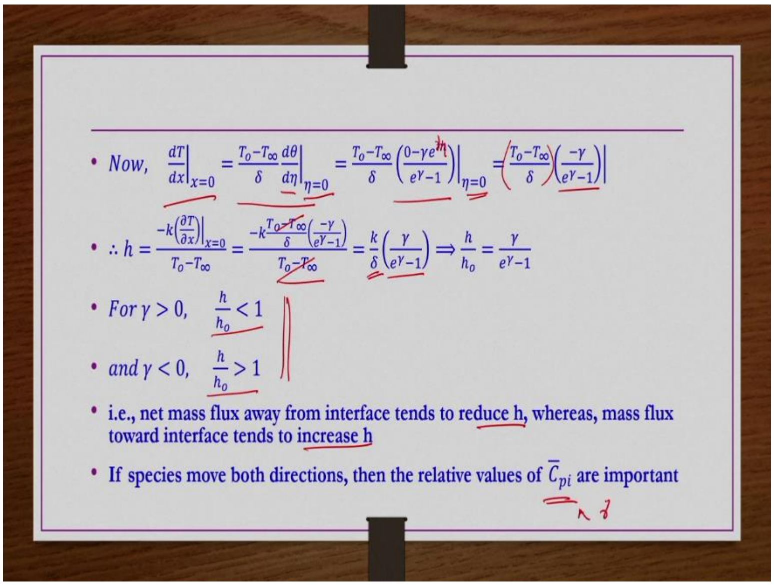

- Now, $\left.\frac{dT}{dx}\right|_{x=0} = \frac{T_0-T_\infty}{\delta}\left.\frac{d\theta}{d\eta}\right|_{\eta=0} = \frac{T_0-T_\infty}{\delta}\left.\left(\frac{0-\gamma e^{\gamma\eta}}{e^\gamma-1}\right)\right|_{\eta=0} = \left(\frac{T_0-T_\infty}{\delta}\right)\left(\frac{-\gamma}{e^\gamma-1}\right)\Big|$
- $\therefore h = \frac{-k\left(\frac{\partial T}{\partial x}\right)\big|_{x=0}}{T_0-T_\infty} = \frac{-k\frac{\cancel{T_0-T_\infty}}{\delta}\left(\frac{-\gamma}{e^\gamma-1}\right)}{\cancel{T_0-T_\infty}} = \frac{k}{\delta}\left(\frac{\gamma}{e^\gamma-1}\right) \Rightarrow \frac{h}{h_0} = \frac{\gamma}{e^\gamma-1}$
- For $\gamma > 0$, $\frac{h}{h_0} < 1$
- and $\gamma < 0$, $\frac{h}{h_0} > 1$
- i.e., net mass flux away from interface tends to reduce h, whereas, mass flux toward interface tends to increase h
- If species move both directions, then the relative values of $\bar{C}_{pi}$ are important

So, but we want $\frac{dT}{dx}$ because in the relation that $h = -\frac{-k\left(\frac{\partial T}{\partial x}\right)|_{x=0}}{T_0-T_\infty}$ then that we have to do. So, $\frac{dT}{dx}$ is 0; that means, in terms of θ it is nothing but $\frac{T_0-T_\infty}{\delta}\frac{d\theta}{d\eta}$.

In place of dT you can write $T_0 - T_\infty$ d θ. In place of dx you can write δ d η and then if x = 0 that means, η = 0. So, previous slide we just got a θ as function of η that you differentiate with respect to η. So, then you will get this expression right, $\frac{0-\gamma e^{\gamma\eta}}{e^\gamma-1}$.

Now, here η = 0 if you substitute here e^0. So, that is 1. So, $\frac{-\gamma}{e^\gamma-1}$ and then remaining constant is as it is ok. So, $h = \frac{-k\left(\frac{\partial T}{\partial x}\right)|_{x=0}}{T_0-T_\infty}$ and this − k as it is. $\left(\frac{\partial T}{\partial x}\right)|_{x=0}$ is nothing but $\frac{\frac{T_0-T_\infty}{\delta}\frac{\gamma}{e^\gamma-1}}{T_0-T_\infty}$.

This T_0/T_∞ this T_0/T_∞, we can cancelled out and then minus and minus multiplied. So, then, + $\frac{k}{\lambda}$ multiplied by $\frac{\gamma}{e^\gamma-1}$. So, $\frac{k}{\delta}$ is nothing but h₀ that we know. So, $\frac{h}{h_0} = \frac{\gamma}{e^\gamma-1}$. And then this γ we already know it is a constant.

If it is a positive constant from this relation $\frac{h}{h_0} < 1$. If it is a negative constant then $\frac{h}{h_0} > 1$, right. So, that is net mass flux away from interface tends to reduce h whereas, the mass flux towards the interface tends to increase the heat transfer coefficient h according to these two conditions. If species move both direction, then relative values of $\bar{C}_{pi}$ are important because the $\bar{C}_{pi}$ terms are coming into the picture in γ definition ok.

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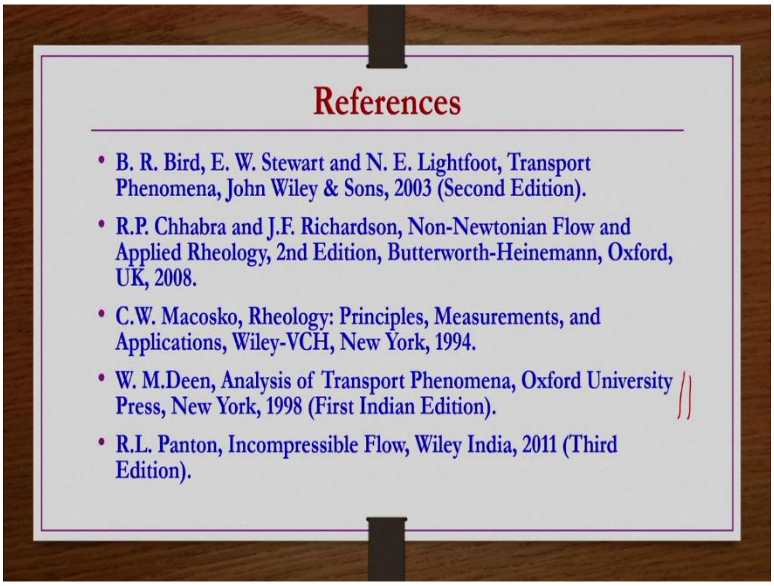

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The references for this lecture are provided here. So, you can find these problems in this reference book Analysis of Transport Phenomena, by Deen.

Thank you.