Transport Phenomena of Non-Newtonian Fluids

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Lecture - 36
Quasi-Steady Analysis of Simultaneous HT and MT

Welcome to the MOOC's course Transport Phenomenon of Non-Newtonian Fluids. The title of today's lecture is Quasi Steady Analysis of Simultaneous Heat and Mass Transfer. In the previous lecture, we have taken a case where simultaneous heat transfer, mass transfer and reaction are taking place.

So, under such conditions how to apply the quasi analysis and get the time dependent properties as well those things we have seen. So, in this lecture we will be taking a couple of cases where only heat and mass transfer occurring and then we apply the quasi steady analysis approach to get the required information, ok.

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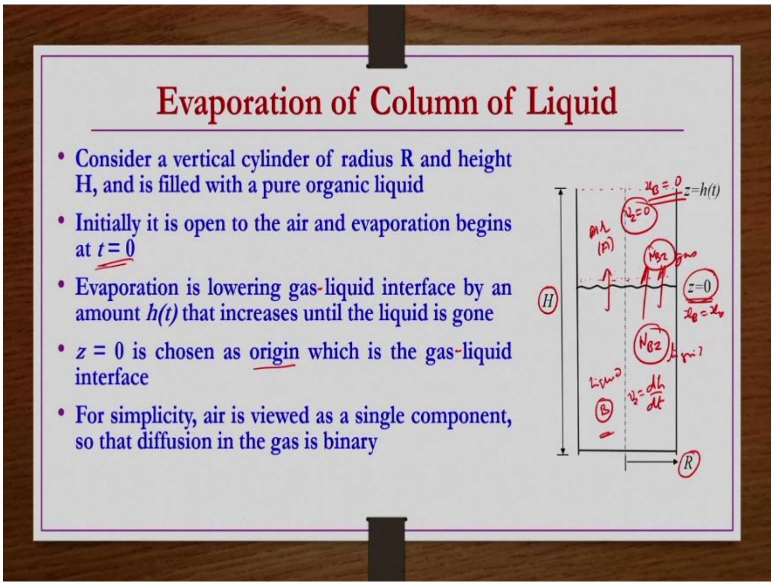

Evaporation of Column of Liquid

- Consider a vertical cylinder of radius R and height H, and is filled with a pure organic liquid
- Initially it is open to the air and evaporation begins at $t = 0$
- Evaporation is lowering gas-liquid interface by an amount $h(t)$ that increases until the liquid is gone
- $z = 0$ is chosen as origin which is the gas-liquid interface
- For simplicity, air is viewed as a single component, so that diffusion in the gas is binary

So, first problem we will be discussing evaporation of column of liquid right. So, here we have a cylindrical column whose radius is R and then height is H right. So, in this one initially a liquid which is organic in nature so, that is filled in so, that we are designating with B and then outside there is air.

So, that we are designating as A right liquid is B and then air is A and then we are assuming the air is single component material we are not we are not taking into account the composition of the air. So, that to avoid the diffusion between multi component. So, now, we have pure liquid and then air. So, that only we can have a binary diffusion case, right.

So, initial interface is taken at $z = 0$ as shown here right. So, now, initially it is opened to the air and the evaporation begins at $t = 0$ right. Evaporation is lowering the gas liquid interface by an amount h (t) that increases until the liquid is completely gone.

So, initially it was here. So, then gradually the interface is falling down. So, let us say at certain time t the height is $z = h(t)$ and then completely when complete evaporation takes place the height would be H ok, interface height should be said there should not be any interface after it is completely evaporated; however, till the last layer we can call it as H, ok.

So, then $z = 0$ is chosen as a origin which is the gas liquid interface ok. So, this is the origin we are taking for our coordinate system. For simplicity air is viewed as a single component so, that diffusion in the gas is binary only. So, we do not need to worry about the diffusion of multi component systems because if you take oxygen, nitrogen separately and then if any other gas is present in the air then the system will not be binary anyway ok. So, that is the reason for simplicity we are taking air is as a single component.

So, now initially the concentration now the actually you know evaporation of liquid B is taking place right evaporation of liquid B is taking place ok so; that means, we need to find out N_{Bz}; z direction you know it is taking place right, we need to have the boundary conditions as well. So, the evaporation whatever is taking place. So, then that is diffusing out. So, that near the interface the organic whatever is there that is very less amount only the organic vapours mostly it is occupied by the air right.

So, at $z = 0$ whatever the concentration of x_B is there. So, that we call x_0. And then $z = h(t)$ whatever the concentration of B should be there that should be 0 because we are saying that you know most of the organic whatever is there that is only small amount of organic is evaporating and then that is at the interface only right.

So, rest most of the remaining space is occupied with the air so; obviously, at the top which is you know $z = h(t)$ location as per this schematic your x_B would be 0. So, we need two boundary condition need to be understand right. And then this N_{Bz} is there in the liquid

N_{Bz} towards liquid side whatever is there is very much higher compared to the N_{Bz} towards the gas side.

Because liquid B is evaporating so; obviously, towards the liquid side whatever is there. So, the flux would be higher right and then we are going to do the interfacial balance as we have done for the previous problem where we have taken the quasi steady analysis. So, then this understanding is required in simplifying the interfacial flux balance equation right.

And then also here v_z the liquid whatever is there that height is decreasing so; that means, there is a motion. So, that $\frac{dh}{dt}$ would be there towards the liquid or the v_z in the other side would be 0 because that gas is not moving ok. So, these understanding is required to simplify the interfacial balance equations, right.

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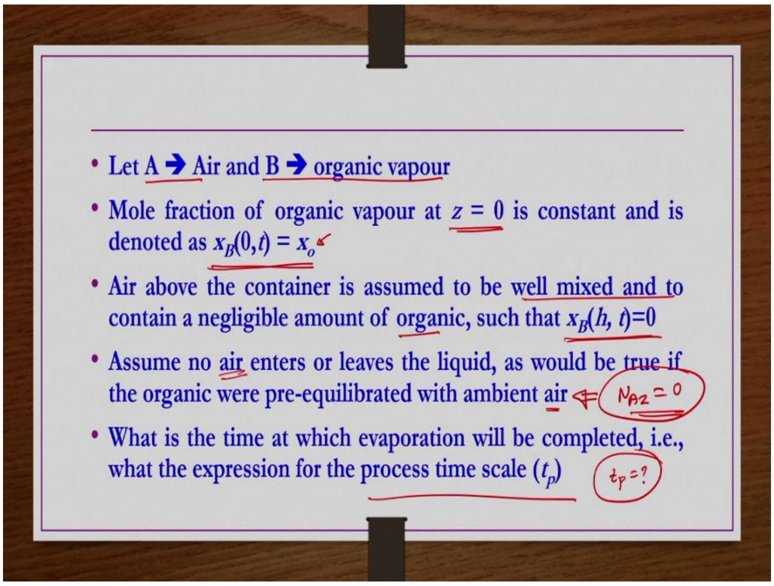

- Let A → Air and B → organic vapour
- Mole fraction of organic vapour at $z = 0$ is constant and is denoted as $x_B(0,t) = x_o$
- Air above the container is assumed to be well mixed and to contain a negligible amount of organic, such that $x_B(h, t)=0$
- Assume no air enters or leaves the liquid, as would be true if the organic were pre-equilibrated with ambient air $\Leftarrow$ $\boxed{N_{Az} = 0}$
- What is the time at which evaporation will be completed, i.e., what the expression for the process time scale (t_p) $\boxed{t_p = ?}$

Additional details if you see we are taking air as A and then organic vapour as B. Mole fraction of organic vapour at z = 0 is constant and it is x_0 actually it should be time of actually it should be function of time as well not only position. So, x_B (0, t) is x_0 ok.

Air above the container is assumed to be well mixed and contain a negligible amount of organic such that x_B (h, t) = 0 that is at the top layer at which there is no organic or very little organic is there mostly it is air is there, ok. Assume no air enters or leaves the liquid, as would be true if the organic were pre-equilibrated with the ambient air.

So, this is the additional conditions. So, what does it mean by? If no air is entering or leaving; that means, flux of this component is 0 this component is designated as A and then it is we are flux we are taking in the z direction. So $N_{Az} = 0$ that is for the air N_{Az} has to be 0. So, in the combined momentum flux equation when you write so, there you can substitute this $N_{Az} = 0$ as per the statement.

And then what is the time at which evaporation will be completed that is the expression for the process time scale t_p we have to find out ok. So, since the time you are finding out. So, you have to have go for you must go for you know so called quasi steady analysis or completely time dependent approach completely time dependent approach is not possible analytically doing. So, then that is the reason we are going for quasi steady analysis.

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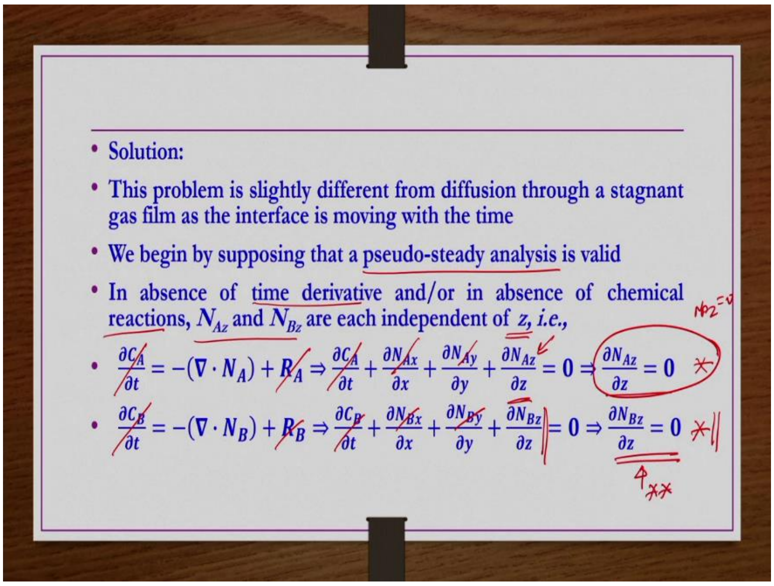

- Solution:
- This problem is slightly different from diffusion through a stagnant gas film as the interface is moving with the time
- We begin by supposing that a pseudo-steady analysis is valid
- In absence of time derivative and/or in absence of chemical reactions, N_{A_z} and N_{B_z} are each independent of z, i.e.,

$N_{Bz} = 0$

- $\frac{\partial \cancel{C_A}}{\cancel{\partial t}} = -(\nabla \cdot N_A) + \cancel{R_A} \Rightarrow \frac{\partial \cancel{C_A}}{\cancel{\partial t}} + \frac{\partial \cancel{N_{Ax}}}{\partial x} + \frac{\partial \cancel{N_{Ay}}}{\partial y} + \frac{\partial N_{Az}}{\partial z} = 0 \Rightarrow \frac{\partial N_{Az}}{\partial z} = 0 \quad *$
- $\frac{\partial \cancel{C_B}}{\cancel{\partial t}} = -(\nabla \cdot N_B) + \cancel{R_B} \Rightarrow \frac{\partial \cancel{C_B}}{\cancel{\partial t}} + \frac{\partial \cancel{N_{Bx}}}{\partial x} + \frac{\partial \cancel{N_{By}}}{\partial y} + \frac{\partial N_{Bz}}{\partial z} = 0 \Rightarrow \frac{\partial N_{Bz}}{\partial z} = 0 \quad *\|$

$**$

So, what we do if you recollect we have done a similar problem diffusion through stagnant gas film where a liquid A is evaporating and then mixing with gas B and then diffusing. So, what is the you know expression for the mole fraction $\frac{x_A - x_{A\delta}}{x_{A0} - x_{A\delta}} =$ that expression we have derived right.

So, that expression if you remember this problem is quite similar to that one, but while solving that problem that was a first problem we discussed in our mass transfer part of the course we assume that you know interface is constant, there also I clearly specified that we are going to take a similar problem where interface is not fixed as a constant it is varying. So, now, here that same problem similar if not the same we cannot say the same problem similar problem we are taking by considering the moving interface as well ok.

Now, we assume this pseudo-steady analysis is valid for this problem again. Then in the absence of time derivative and or in the absence of chemical reactions both N_{Az} and N_{Bz} are independent of z how it is, that you can realise when you simplify the species conservation equation. So, for species C this is the equation. So, now, in the case of there is no time derivative and then in the absence of reaction we have this equation right.

So, here time derivative we are not taking right and then N_{Az} is there only in the z direction only N_{Az} is there N_{Ax} N_{Ay} are not there and then N_{Az} is varying only in z direction. So, $\frac{\partial N_A}{\partial N_{Az}} = 0$; that means, N_{Az} is independent of z direction is not it? So, this is going to be very useful in further simplifying the equations similarly for the B component you do no time derivative no reaction.

So, B flux also it is there only in the z direction right. So, $\frac{\partial N_{Bz}}{\partial z} = 0$. So, this is in fact, very much essential compared to this one because this one we anyway know that N_{Az} is 0 from the problem statement. So, this one is going to be very much useful in solving you know subsequent balance equations.

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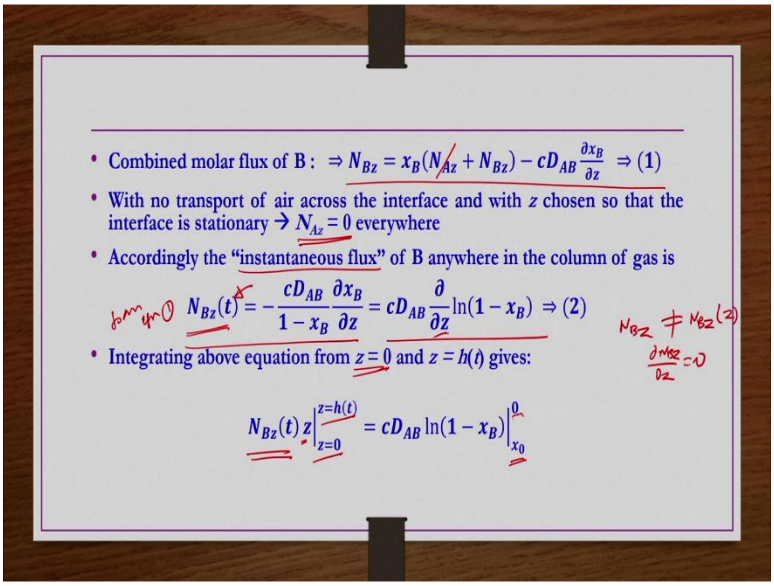

- Combined molar flux of B : $\Rightarrow N_{Bz} = x_B(\cancel{N_{Az}} + N_{Bz}) - cD_{AB}\frac{\partial x_B}{\partial z} \Rightarrow (1)$
- With no transport of air across the interface and with z chosen so that the interface is stationary → $N_{Az} = 0$ everywhere
- Accordingly the "instantaneous flux" of B anywhere in the column of gas is

from eqn (1) $N_{Bz}(t) = -\frac{cD_{AB}}{1-x_B}\frac{\partial x_B}{\partial z} = cD_{AB}\frac{\partial}{\partial z}\ln(1-x_B) \Rightarrow (2)$

$N_{Bz} \neq N_{Bz}(z)$

$\frac{\partial N_{Bz}}{\partial z} = 0$

- Integrating above equation from $z = 0$ and $z = h(t)$ gives:

$$N_{Bz}(t)\, z\Big|_{z=0}^{z=h(t)} = cD_{AB}\ln(1-x_B)\Big|_{x_0}^{0}$$

So, now combined molar flux for B; we are writing. Why we are writing for B why not A? Because now here B is the organic liquid which is evaporating so, for that only we are writing this combined flux equation. So, $N_{Bz} = x_B(N_{Az} + N_{Bz}) - cD_{AB}\frac{\partial x_B}{\partial z}$ N_{Az} is 0 in this case as per the problem statement. Because no air enters or leaves the interface so; obviously, N_{Az} is 0. So, with no transport of air across the interface and with z chosen so, that the interface is stationary N_{Az} is 0 everywhere.

So, now accordingly the instantaneous flux of B anywhere in the column of gas we can get by simplifying this equation number 1 that is nothing but N_{Bz} at t instantaneous because N_{Bz} is varying with time also because concentration is varying with time interface is varying with time. So, then; obviously, the flux of B component which is there that will also be varying with the time.

So, now, this N_{Bz}we cannot write just N_{Bz} we have to write N_{Bz} function of time should be $= \frac{-cD_{AB}}{1-x_B}\frac{\partial x_B}{\partial z}$ which is coming from equation 1 only from equation 1 only, but we are specifically writing N_{Bz} as function of time because now we realise that the interface is moving so; obviously, N_{Bz} should be changing with the time. So, the same thing we can write this equation as $cD_{AB}\frac{\partial}{\partial z}\ln(1 - x_B)$, right.

Now, this equation if you integrate because what we understand? Already we understand that N_{Bz} is not function of z. By simplifying the species conservation equation in the previous slide where we get $\frac{\partial N_{Bz}}{\partial z} = 0$. So, when you integrate this one we are going to integrate with respect to z. So, N_{Bz}as function of time can be taken as or can be treated as constant because we are integrating with respect to z. And then at z = 0 what is x_B it is x_0 and z = h (t) what is x_B it is 0 that also we have seen.

So, $N_{Bz}\,(t)z|_{z=0}^{z=h(t)} = cD_{AB}\ln(1 - x_B)$ would be there. So, after integration; integration of z is z, integration of dz is z and then z = 0 to z = h (t) and then integration of $d\ln(1 - x_B)$ would be $\ln(1 - x_B)$ limits when x = 0, x_B is equals to a constant which is say x_0 and then z = h (t) then $x_B = 0$ that also we have seen, right.

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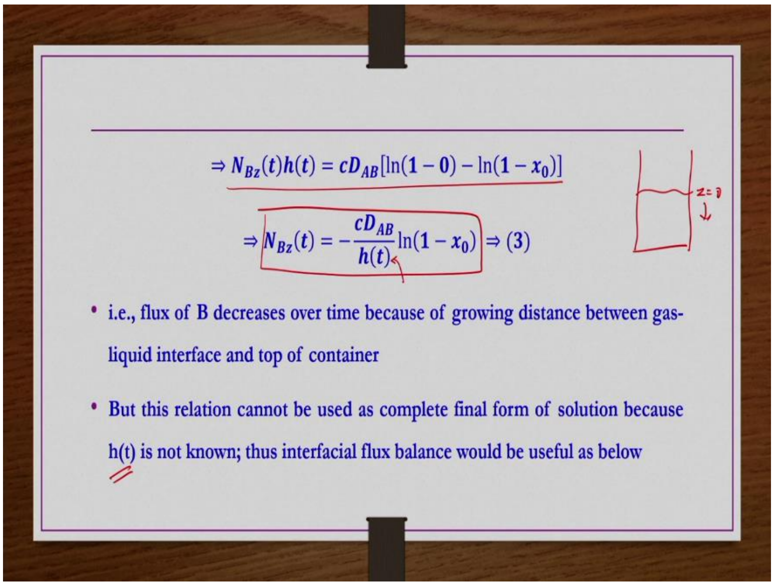

$$\Rightarrow \underline{N_{Bz}(t)h(t) = cD_{AB}[\ln(1-0) - \ln(1-x_0)]}$$

$$\Rightarrow \boxed{N_{Bz}(t) = -\frac{cD_{AB}}{h(t)}\ln(1-x_0)} \Rightarrow (3)$$

- i.e., flux of B decreases over time because of growing distance between gas-liquid interface and top of container
- But this relation cannot be used as complete final form of solution because h(t) is not known; thus interfacial flux balance would be useful as below

When you substitute the limits and then simplify you get N_{Bz} is equals to this and then finally, N_{Bz} at some given t time t is $-\frac{cD_{AB}}{h(t)}\ln(1-x_0)$ right. So, now this is the final flux equation for component B if you simplify this equation you can get the probably required time to get the complete evaporation done. So, that we can do only when this $h(t)$ is known right so, otherwise we cannot use this equation that we are going to do anyway.

So, now what this equation indicates the flux of B decreases over time because of growing distance between gas and liquid interface and top of the container ok. So, initially you know this is the origin that we have taken. So, this z = 0 gradually decreasing with increasing time. So, that is what it means by so; obviously, the flux of B decreasing over time. But, this relation cannot be used as complete final form of solution because $h(t)$ is not known thus interfacial flux balance would be useful as below that we are going to do now.

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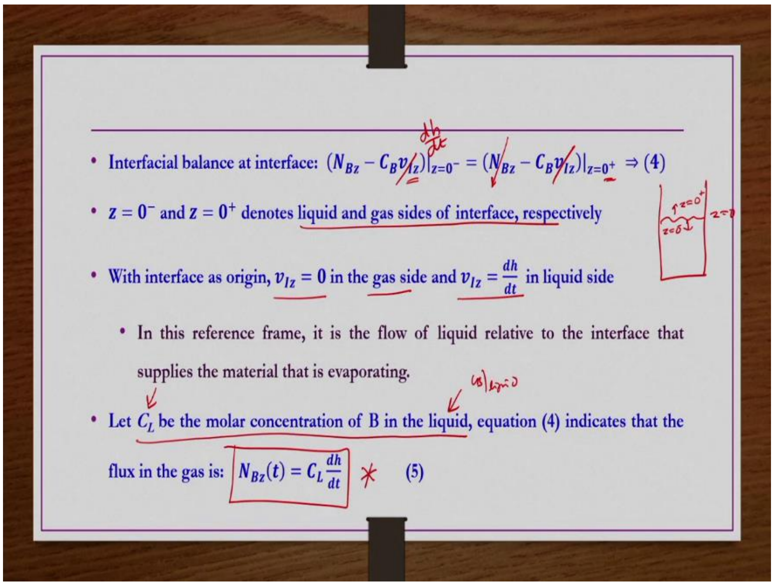

- Interfacial balance at interface: $(N_{Bz} - C_B \overset{\frac{dh}{dt}}{\cancel{v_{Iz}}})|_{z=0^-} = (N_{Bz} - C_B \cancel{v_{Iz}})|_{z=0^+} \Rightarrow (4)$
- $z = 0^-$ and $z = 0^+$ denotes liquid and gas sides of interface, respectively
- With interface as origin, $v_{Iz} = 0$ in the gas side and $v_{Iz} = \frac{dh}{dt}$ in liquid side
  - In this reference frame, it is the flow of liquid relative to the interface that supplies the material that is evaporating.
- Let C_L be the molar concentration of B in the liquid, equation (4) indicates that the flux in the gas is: $\boxed{N_{Bz}(t) = C_L \frac{dh}{dt}}$ * (5)

$C_B|_{liquid}$

So, interfacial balance that we do for the flux of B then $(N_{Bz} - C_B v_{Iz})$ we are writing just indicating interface at z = 0^- and that should be balanced by $(N_{Bz} - C_B v_{Iz})$ at z = 0^+ location. So, this container this is what we are having initially. So, this is z = 0 towards the liquid side whatever layer is there that we are calling z = 0^-, towards the vapour side whatever is there that we are calling z = 0^+ the same location, but which side are we taking, right

Now, towards the vapour side there is no velocity. So, this term is 0 because you know its air is stationary right and then the vapour B is also very small quantity in the gas that is what it has been mentioned. So, compared to the flux in the liquid side this flux should be very small.

So, this can be cancelled out compared to the flux towards the liquid side. So, that is what and then liquid side the velocity is nothing but $\frac{dh}{dt}$ that we know because interface is moving down with respect to time, right. So, then z = 0^- and z = 0^+ denotes liquid and gas sides of interface respectively with the interface as the origin $v_{Iz} = 0$ in the gas side and $v_{Iz} = \frac{dh}{dt}$ in the liquid side.

In this reference frame it is the flow of liquid relative to the interface that supplies the material that is evaporating. So, now if you take C_L be the molar concentration of B in the liquid that is C_B in liquid side whatever is there that we are calling now C_L, right. Then this equation 4 you will be getting $C_{Bz} = C_L \frac{dh}{dt}$ remember we are writing 0^+ 0^- or 0^+ locations like that in the previous problem also only for doing the balance required balance to in order to write the required balance.

Once the required balance is balanced equation has been written and then simplified then we remove all those things because it is the same location at the interface only thing that here in this case $z = 0^-$ indicates towards the liquid and $z = 0^+$ indicates towards the gas side ok, but it is at the same z = 0.

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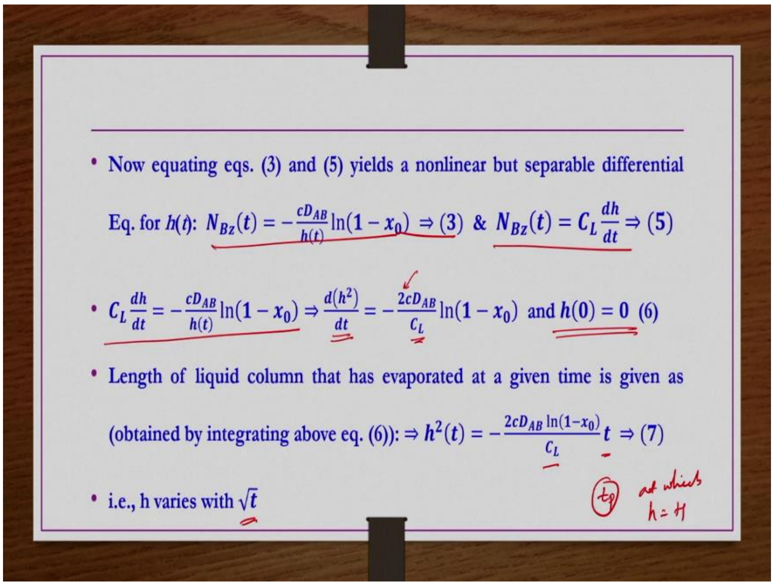

- Now equating eqs. (3) and (5) yields a nonlinear but separable differential Eq. for $h(t)$: $\underline{N_{Bz}(t) = -\frac{cD_{AB}}{h(t)}\ln(1-x_0) \Rightarrow (3)}$ & $\underline{N_{Bz}(t) = C_L\frac{dh}{dt}} \Rightarrow (5)$
- $\underline{C_L\frac{dh}{dt} = -\frac{cD_{AB}}{h(t)}\ln(1-x_0)} \Rightarrow \frac{d(h^2)}{dt} = -\frac{2cD_{AB}}{C_L}\ln(1-x_0)$ and $\underline{h(0) = 0}$ (6)
- Length of liquid column that has evaporated at a given time is given as (obtained by integrating above eq. (6)): $\Rightarrow h^2(t) = -\frac{2cD_{AB}\ln(1-x_0)}{C_L}t \Rightarrow (7)$
- i.e., h varies with $\sqrt{t}$

$\textcircled{t_p}$ at which $h = H$

So, now this is the flux equation. Now, this equation number 3 that previously we solved and then this equation number 5 just we got because of the interfacial balance equation. Then if you integrate both of them what you get $C_L \frac{dh}{dt} = -\frac{cD_{AB}}{h(t)}\ln(1-x_0)$ and then initial condition what is that? At t = 0 h = 0 ok.

So, now, this equation if you integrate $h^2(t) = -\frac{2cD_{AB}\ln(1-x_0)}{C_L}t$ that is what you get right; that means, h varies with $\sqrt{t}$, right. But, now we need to find out t_p; t_p is the time at which this h becomes H, h as function of time whatever you say that becomes H that is complete liquid has evaporated. So, if the complete liquid has evaporated; that means, height is the height of the column that is H.

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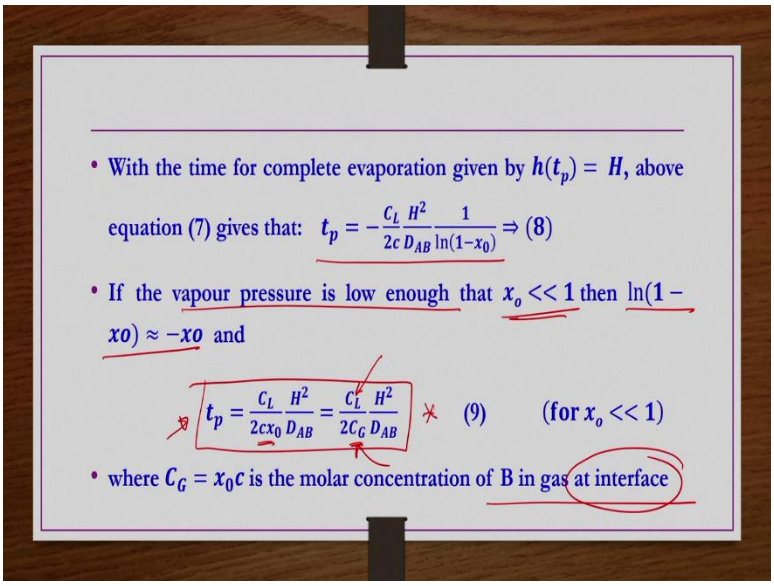

- With the time for complete evaporation given by $h(t_p) = H$, above equation (7) gives that: $t_p = -\frac{C_L}{2c}\frac{H^2}{D_{AB}}\frac{1}{\ln(1-x_0)} \Rightarrow (8)$
- If the vapour pressure is low enough that $x_o << 1$ then $\ln(1-xo) \approx -xo$ and

$$\boxed{t_p = \frac{C_L}{2cx_0}\frac{H^2}{D_{AB}} = \frac{C_L}{2C_G}\frac{H^2}{D_{AB}}} \quad * \quad (9) \qquad (\text{for } x_o << 1)$$

- where $C_G = x_0 c$ is the molar concentration of B in gas at interface

So, that if you substitute here you get $t_p = -\frac{C_L}{2c}\frac{H^2}{D_{AB}}\frac{1}{\ln(1-x_0)}$ ok. Now, if the vapour pressure is low enough then x_0 would be usually very very smaller than 1 under such conditions we can write $\ln(1 - x_0) \approx -x_0$.

So, this equation we can write like this $t_p = \frac{C_L}{2cx_0}\frac{H^2}{D_{AB}}$ right because now $\ln(1 - x_0)$ is nothing but $-x_0$ and then this cx_0 we can write C_G which is nothing but molar concentration of B in the gas, but at the interface.

So; that means, if you know the molar concentration of component B in the liquid side and then if you know the molar concentration of the same component B in the gas side then you can find out you know total evaporation time using this equation. Because in general D_{AB} for such for majority of the systems are known or available or we can measure, ok.

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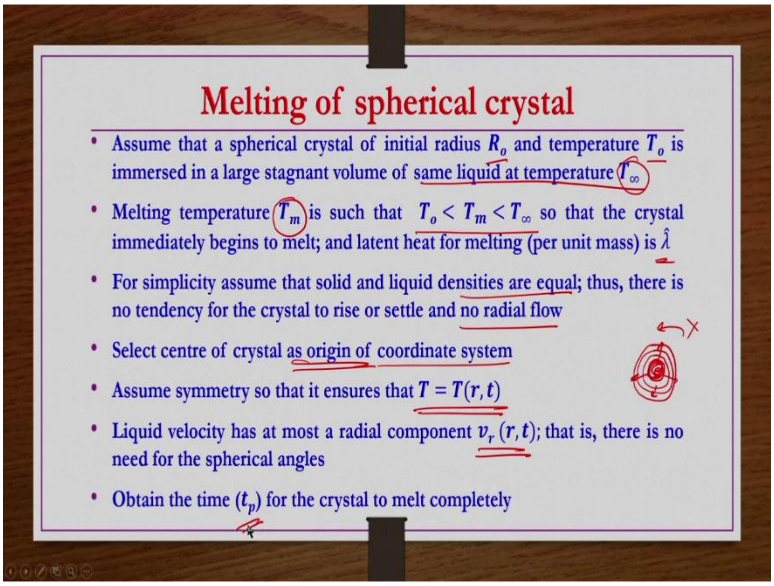

Melting of spherical crystal

- Assume that a spherical crystal of initial radius R_o and temperature T_o is immersed in a large stagnant volume of same liquid at temperature T_∞
- Melting temperature T_m is such that $T_o < T_m < T_\infty$ so that the crystal immediately begins to melt; and latent heat for melting (per unit mass) is $\tilde{\lambda}$
- For simplicity assume that solid and liquid densities are equal; thus, there is no tendency for the crystal to rise or settle and no radial flow
- Select centre of crystal as origin of coordinate system
- Assume symmetry so that it ensures that $T = T(r,t)$
- Liquid velocity has at most a radial component $v_r(r,t)$; that is, there is no need for the spherical angles
- Obtain the time (t_p) for the crystal to melt completely

Now, we take another problem melting of spherical crystal ok we have a spherical crystal it is having the initial radius R_0 it is gradually you know melting. So, R_0 whatever initial radius is there that gradually decreases. So, R is function of time ok. So, how much time it required for the complete melting of the spherical crystal that is what we are going to see in this problem ok.

Assume that a spherical crystal of initial radius R_0 and then temperature T_0 is immersed in a large stagnant volume of same liquid at temperature $T\ \infty$; that means, the crystal is let us say if you have taken the you know water as a system. So, in the water you have the ice spherical crystal of radius R_0 ok.

So, that is the material is of same whether it is you know whichever material you are taking water or some organic liquid the crystal and then liquid phase both are of them of both of them are of same material only they are only that they are in different phases right.

And then liquid at temperature T_∞ the volume of liquid is very large and it is also stagnant. Melting temperature T_m of that crystal is such a way that T_m is in between T_0 and T_∞ right. So, that the crystal immediately begins to melt and latent heat for melting per unit mass is lambda tilde.

For simplicity assume that the solid and liquid densities are equal. So, that you know there is no question of rising or settling of this crystal and there is no radial flow also that is also given. So, select centre of crystal as the origin of coordinate system. So, we are taking origin as the centre of the spherical crystal whatever has been provided assume symmetry so, that temperature is function of r direction and then time only because it is a quasi steady problem, ok.

Liquid velocity has at most a radial component that is v_r and function of r and t that is there is no need for this spherical angles; that means, you know whatever the let us say this is the crystal is there. So, the liquid you know vibrations like you know moving the velocities is like that or contours should also be like that spherically moving in the radial direction, right. As we move in the radial direction. So, there will be no effect of θ or ϕ directions etcetera. Obtain the time for the crystal to melt completely that is the question.

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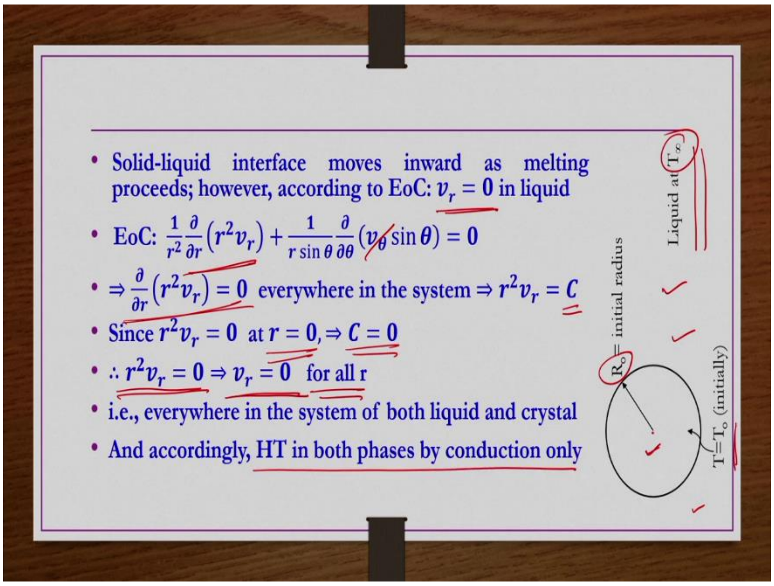

- Solid-liquid interface moves inward as melting proceeds; however, according to EoC: $v_r = 0$ in liquid
- EoC: $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\cancel{v_\theta}\sin\theta\right) = 0$
- $\Rightarrow \frac{\partial}{\partial r}\left(r^2 v_r\right) = 0$ everywhere in the system $\Rightarrow r^2 v_r = C$
- Since $r^2 v_r = 0$ at $r = 0, \Rightarrow C = 0$
- $\therefore r^2 v_r = 0 \Rightarrow v_r = 0$ for all r
- i.e., everywhere in the system of both liquid and crystal
- And accordingly, HT in both phases by conduction only

So, now pictorially we have a spherical crystal of radius R_0 initial radius R_0 right it is initially at temperature $T = T_0$ it is suspended in a liquid of the same material and then the liquid is at some other temperature T_∞ ok. Solid-liquid interface moves inward as melting proceeds; however, according to equation of continuity $v_r = 0$ in the liquid that we can prove how to prove. So, let us say equation of continuity in spherical coordinates we write. So, I have written only the r and θ components phi component also we can similarly write, ok.

So, but now v_r is function of r ok, but v_θ is not there so, then what we understand from here? $\frac{\partial}{\partial r}(r^2 v_r) = 0$ everywhere in the system, right. So, this should be valid everywhere in the system that is inside here and then surrounding liquid also because it is a continuity equation; that means, $r^2 v_r = C$ right.

Now, if you substitute r = 0 that is at the centre $r^2 v_r$ should be 0, right so; that means, obviously, C should be 0 because of the C = 0 $r^2 v_r = 0$. So, that $v_r = 0$ for all values of r ok it is very important otherwise we cannot able to solve the problem without this information that is everywhere in the system of both liquid and crystal v_r is 0 and then; obviously, heat transfer in both phases would be by conduction only.

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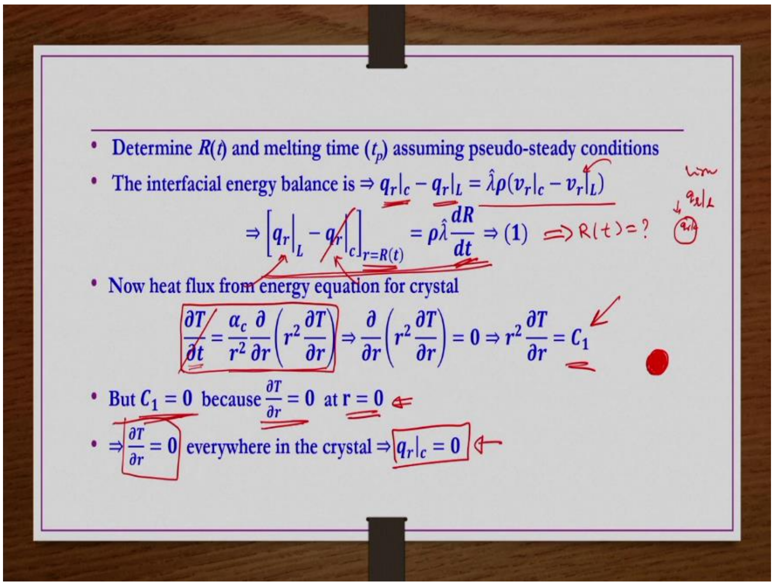

- Determine $R(t)$ and melting time (t_p) assuming pseudo-steady conditions
- The interfacial energy balance is $\Rightarrow q_r|_c - q_r|_L = \hat{\lambda}\rho(v_r|_c - v_r|_L)$

$$\Rightarrow \left[q_r\Big|_L - \cancel{q_r}\Big|_c\right]_{r=R(t)} = \rho\hat{\lambda}\frac{dR}{dt} \Rightarrow (1) \Rightarrow R(t) = ?$$

- Now heat flux from energy equation for crystal

$$\boxed{\cancel{\frac{\partial T}{\partial t}} = \frac{\alpha_c}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)} \Rightarrow \frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = 0 \Rightarrow r^2\frac{\partial T}{\partial r} = C_1$$

- But $C_1 = 0$ because $\frac{\partial T}{\partial r} = 0$ at $r = 0$
- $\Rightarrow \boxed{\frac{\partial T}{\partial r} = 0}$ everywhere in the crystal $\Rightarrow \boxed{q_r|_c = 0}$

So, now the question is that determine R as function of time and then melting time t_p assuming pseudo steady conditions right. So, the interfacial balance if you do. The q_r in the crystal side. So, that is what we are having here also we have a crystal and then there is a surrounding liquid.

So, in the crystal side what is the flux $q_r|_c$ and then what is the flux at liquid side that is $q_r|_L$ that should be balanced by $\hat{\lambda}\rho(v_r|_c - v_r|_L)$ right this we should have the understanding and then these two are right at the interface right, not somewhere far away in the liquid or not somewhere towards the centre of the crystal right on the crystal surface, but one quantity is towards the crystal another quantity is towards the liquid side fine.

Now, this equation here the liquid side whatever is there liquid side because the crystal is gradually melting. So, because of that one there is a motion or you know disturbance in the liquid side. So; obviously, in the liquid side this velocity whatever is there that is nothing but $\frac{dR}{dt}$.

So, now, $q_r|_L - q_r|_c$ is related to $\frac{dR}{dt}$ by this balance equation so; that means, $q_r|_L$ and then $q_r|_c$ that is the heat flux towards the liquid and then towards the crystal side, but at the interface at r = R (t) if you find; you can find out what is this R function of time once it is there. So, then you can find out the t_p also.

So, this equation we can use only when we know what is $q_r|_L$ $q_r|_c$. So, for that reason what we do we will be writing a balance equation or energy balance equation whatever is there or the energy equation that we have derived previously that we simplify. So, we have to write that for the liquid as well as the crystal separately. So, now, heat flux from energy equation for crystal we can find out.

So, what we have already realised that here you know only conduction is mode of heat transfer previous slide that we have seen because v_r is 0, v_θ, v_ϕ are anyway 0 that statement given they are not under concentration no spherical angles need to be considered. So, v_r is function of r and then that v_r also we found it as 0, if v_r is 0 there would not be any convention there would only be conduction.

So, in the energy equation temporal term and then conduction terms have been included. Conduction terms also only in the radial direction we have included because heat conduction is taking place only in a radial direction. So, remaining terms we have not written directly because now we have done several problems almost we are end of the course. So, then you can understand this way right.

So, that is the reason directly I have written right. So, now, temperature as function of time we are not considering because the time factor we have considered in the balance equation interfacial balance equation right. Because it is not completely time dependent solution it is a quasi steady solution ok.

Then here whatever α_c we do not need to worry. So, then what we get $\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) = 0$ $\left(r^2 \frac{\partial T}{\partial r}\right) = C_1$; that means, C_1 should be 0 because $\frac{\partial T}{\partial r} = 0$ at r = 0 right. So, this is the crystal that we have we are doing for the crystal only at the centre what is the temperature right.

So, that we do not know actually, but then what we can say? We can say $\frac{\partial T}{\partial r}$ at the $\frac{\partial T}{\partial r} = 0$ at the centre. So, that centre is nothing but at r = 0 location. So, at r = 0 $\frac{\partial T}{\partial r}$ is 0. So, then; obviously, C_1 should be 0 in this equation.

If C_1 is 0 then what we get $\frac{\partial T}{\partial r}$ would be 0 because C_1 is 0 because C_1 is 0. So, $r^2 \frac{\partial T}{\partial r} = 0$; that means, $\frac{\partial T}{\partial r} = 0$; that means, everywhere in the crystal $q_r|_c$ is 0 right. So, that is what we got. So, $q_r|_c$is 0. So, now, you take off this $q_r|_c$ from this equation number 1 because now you realise that $q_r|_c = 0$, then what you have? $q_r|_L = \rho \, \hat{\lambda} \frac{dR}{dt}$ right.

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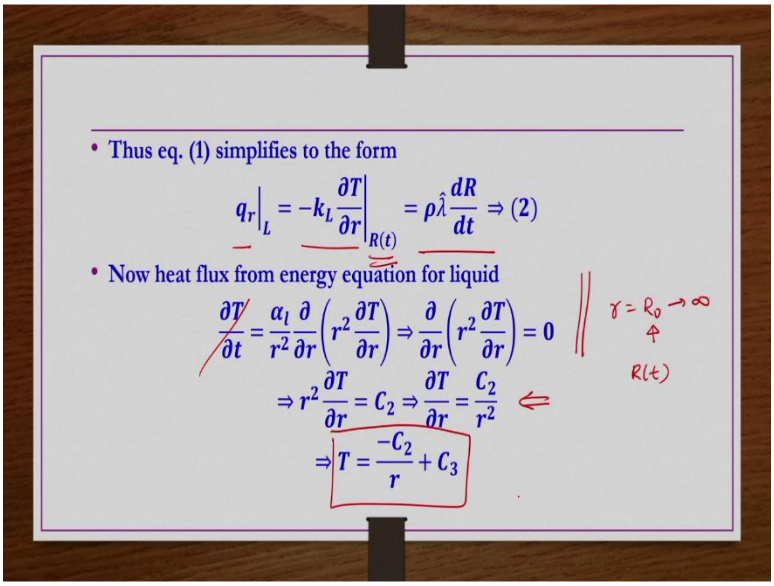

- Thus eq. (1) simplifies to the form

$$\underline{q_r}\Big|_L = \underline{-k_L \frac{\partial T}{\partial r}}\Big|_{\underline{\underline{R(t)}}} = \underline{\rho \hat{\lambda} \frac{dR}{dt}} \Rightarrow (2)$$

- Now heat flux from energy equation for liquid

$$\cancel{\frac{\partial T}{\partial t}} = \frac{\alpha_l}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) \Rightarrow \frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) = 0 \quad \Big\| \quad r = R_0 \to \infty$$

$$\Rightarrow r^2 \frac{\partial T}{\partial r} = C_2 \Rightarrow \frac{\partial T}{\partial r} = \frac{C_2}{r^2} \Leftarrow$$

$$\Rightarrow \boxed{T = \frac{-C_2}{r} + C_3}$$

↑

R(t)

And then this $q_r|_L$ is nothing but $-k_L \frac{\partial T}{\partial r}$ at r = R (t). So, now, we have to find out what is $\frac{\partial T}{\partial r}$ for the liquid side. So, similar energy equation we have to write for the liquid side. So, that heat flux from energy equation for liquid for the liquid side also the energy equation is nothing but having only conduction terms.

Because convection terms are not there v_r is 0 and then conduction also and then conduction also it is there only in the r direction θ and ϕ direction we do not need to consider that is given in the problem. So, simplified energy equation would be this one and then temperature variation with respect to time we are not taking because it is a quasi steady, right.

So, from here $\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = 0$; that means, $\left(r^2\frac{\partial T}{\partial r}\right) = C_2$ or $\frac{\partial T}{\partial r} = \frac{C_2}{r^2}$. Now, you cannot say you know you apply that r is equals to you know zero boundary condition and say here also the constant is this C_2 and then all that.

Because, this equation now is valid for $r = R_0$ to ∞ far away from the crystal then, but this R_0 is changing we cannot say that at R_0 $T = T_0$ and all that; that boundary condition we cannot take because this R is function of time ok. So; however, after integration you get $T = -\frac{C_2}{r} + C_3$.

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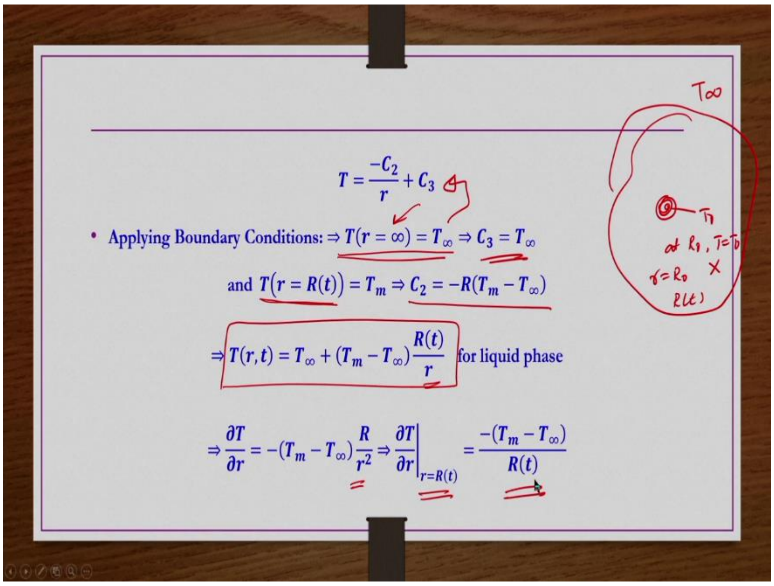

$$T = \frac{-C_2}{r} + C_3$$

- Applying Boundary Conditions: $\Rightarrow T(r=\infty) = T_\infty \Rightarrow C_3 = T_\infty$

  and $T\big(r = R(t)\big) = T_m \Rightarrow C_2 = -R(T_m - T_\infty)$

$$\Rightarrow \boxed{T(r,t) = T_\infty + (T_m - T_\infty)\frac{R(t)}{r}} \text{ for liquid phase}$$

$$\Rightarrow \frac{\partial T}{\partial r} = -(T_m - T_\infty)\frac{R}{r^2} \Rightarrow \left.\frac{\partial T}{\partial r}\right|_{r=R(t)} = \frac{-(T_m - T_\infty)}{R(t)}$$

So, now the boundary condition we have to carefully use that is what we are saying that $T = T_\infty$ at $r = r_\infty$ it is clear it is clear. So, the crystal is here that is at T_0, but far away liquid whatever is there. So, that is at T_∞. So, this is clear. So, then C_3 would be you know T_∞ if you substitute this boundary condition here. And then at $r = R_0$ $T = T_0$ this boundary condition we cannot use because the surface location is changing, right.

Initially this is at $r = R_0$, but gradually this r is changing because R is function of time. So, we cannot use that boundary condition. So, what should we use? We should use at r = some R function of time $T = T_m$, $T = T_m$ that is the melting point temperature, right.

Then you get C2 = $-R(T_m - T_\infty)$. So, now, you have both C1 C2 you substitute here. So, then T as function of r and t you get $T_\infty + (T_m - T_\infty)\frac{R(t)}{r}$. So, it is this temperature distribution is function of both space and time now and this is for the liquid phase only.

So, from here $\frac{dT}{dr}$ is nothing but $-(T_m - T_\infty)\frac{R}{r^2}$ that is what we have, but this we wanted to know this $\frac{dT}{dr}$ we wanted to know at r = R as function of time t; that means, $\frac{dT}{dr}$ at r = R (t) is nothing but $\frac{-(T_m - T_\infty)}{R(t)}$, because $\frac{R}{r^2}$ you get r $\frac{1}{r}$. So, then R (t) right.

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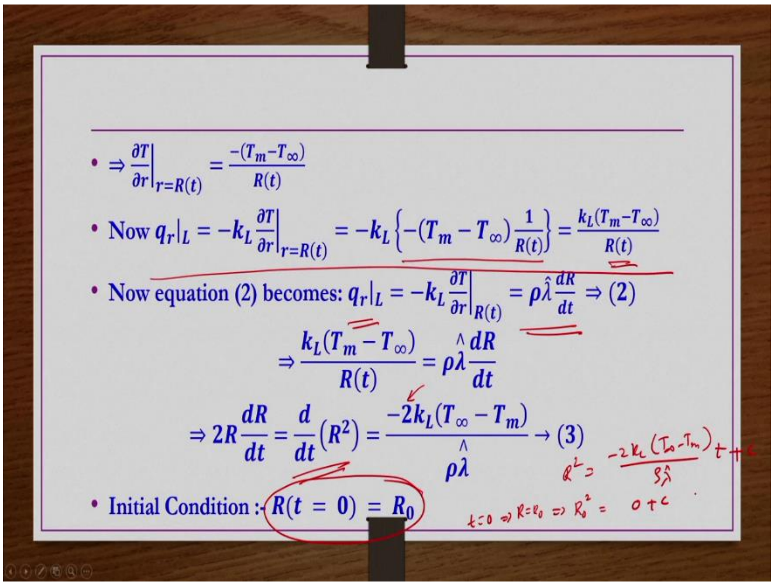

- $\Rightarrow \left.\frac{\partial T}{\partial r}\right|_{r=R(t)} = \frac{-(T_m - T_\infty)}{R(t)}$
- Now $q_r|_L = -k_L \left.\frac{\partial T}{\partial r}\right|_{r=R(t)} = -k_L\left\{-(T_m - T_\infty)\frac{1}{R(t)}\right\} = \frac{k_L(T_m - T_\infty)}{R(t)}$
- Now equation (2) becomes: $q_r|_L = -k_L \left.\frac{\partial T}{\partial r}\right|_{R(t)} = \rho\hat{\lambda}\frac{dR}{dt} \Rightarrow (2)$

$$\Rightarrow \frac{k_L(T_m - T_\infty)}{R(t)} = \rho\hat{\lambda}\frac{dR}{dt}$$

$$\Rightarrow 2R\frac{dR}{dt} = \frac{d}{dt}(R^2) = \frac{-2k_L(T_\infty - T_m)}{\rho\hat{\lambda}} \rightarrow (3)$$

$$R^2 = \frac{-2k_L(T_\infty - T_m)}{\rho\hat{\lambda}} t + c$$

- Initial Condition :- $R(t = 0) = R_0$

$t=0 \Rightarrow R=R_0 \Rightarrow R_0^2 = 0 + c$

So, this you can substitute in the simplified equation $q_r|_L$ = whatever ρ $\hat{\lambda}\frac{dR}{dt}$ in this equation we can substitute. So, $q_r|_L$ is nothing but $-k_L\frac{dT}{dr}$ at r = R (t). So, that is $-k_L\frac{dT}{dr}$ just now we got this one that is $-(T_m - T_\infty)\frac{1}{R(t)}$. So, $k_L(T_m - T_\infty)\frac{1}{R(t)}$ is nothing but $q_r|_L$. So, that we can substitute here and then equate it to the ρ $\hat{\lambda}\frac{dR}{dt}$. So, then we have this equation, right.

Now, we know this r function of time and then $\frac{dR}{dt}$ is also there. So, then we integrate and then get that relation. So, again here also what we are doing we are writing 2R $\frac{dR}{dt} = \frac{d}{dt}R^2$ by multiplying either side with 2. So, 2 is here. So, $\frac{d}{dt}R^2$ is nothing but $-\frac{2k_L(T_\infty - T_m)}{\rho\,\hat{\lambda}}$ ok. So, if you take the initial condition at t = 0 R = R0 that is given.

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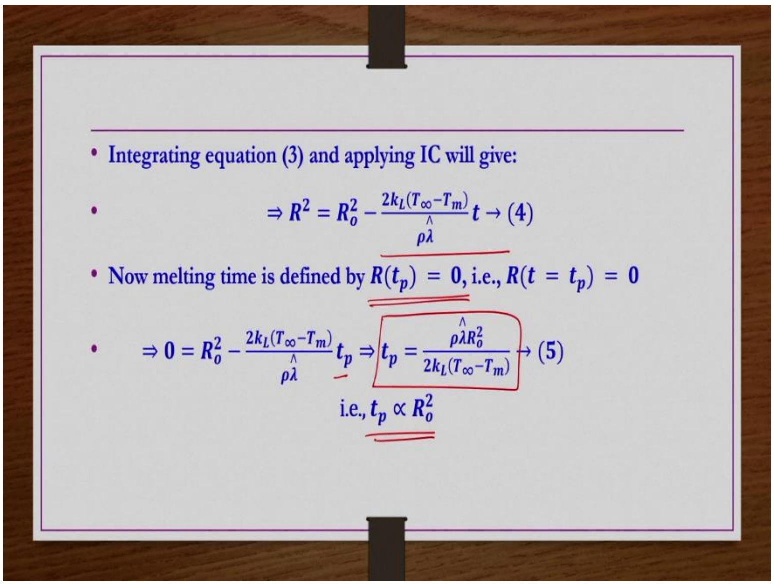

- Integrating equation (3) and applying IC will give:
- $\Rightarrow R^2 = R_o^2 - \frac{2k_L(T_\infty - T_m)}{\rho\hat{\lambda}} t \rightarrow (4)$
- Now melting time is defined by $R(t_p) = 0$, i.e., $R(t = t_p) = 0$
- $\Rightarrow 0 = R_o^2 - \frac{2k_L(T_\infty - T_m)}{\rho\hat{\lambda}} t_p \Rightarrow \boxed{t_p = \frac{\rho\hat{\lambda} R_o^2}{2k_L(T_\infty - T_m)}} \rightarrow (5)$

i.e., $t_p \propto R_o^2$

So, now then you get this equation. So, that how you get? So, let us say if you integrate this equation you get $R^2 = -\frac{2k_L(T_\infty - T_m)}{\rho\,\hat{\lambda}} t$. So, when t = 0, R = R_0 so that means, $R_0^2 =$ then thus + c the constant integration constant would be there. So, that should be 0 + c. So, the integration constant is nothing, but R_0^2.

So, that if you substitute here and in place of integration constant you get $R^2 = R_0^2 -$ whatever $\frac{2k_L(T_\infty - T_m)}{\rho\,\hat{\lambda}} t$ as it is, right. So, now, melting time is defined as R as function of t_p which is when it becomes 0; that means, what is the time required when R becomes 0 when R becomes 0; that means, the crystal has completely melted the crystal has completely melted and then there is only one liquid phase there is no crystal phase at all.

So, in this equation you substitute t = t_p and then take = 0 then you get $0 = R_0^2 - \frac{2k_L(T_\infty - T_m)}{\rho\,\hat{\lambda}} t_p$; that means, $t_p = \frac{\rho\,\hat{\lambda}\,R_0^2}{2k_L(T_\infty - T_m)}$. So, t_p is proportional to R_0^2; that means, if you double the size of you know crystal then you time required for the complete melting of that crystal would be increased by would be increasing by 4 folds.

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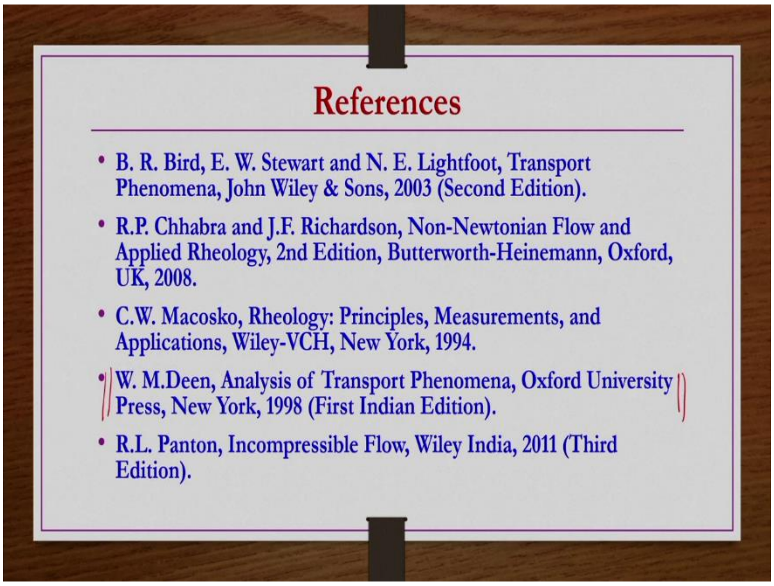

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So, the references for this lecture are provided here, but both of these problems you can find out from this book as exercise problems.

Thank you.