Transport Phenomena of Non-Newtonian Fluids
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Non- Isothermal Flow of Non-Newtonian Fluids
Lecture - 25
Free Convection between Two Vertical Plates

Welcome to the MOOCs course Transport Phenomena of Non-Newtonian Fluids. Title of today's lecture is Free Convection between Two Vertical Plates. So, till now whatever the isothermal flow of non-Newtonian fluids when we have studied in different geometries primarily, we have made use of our continuity equation, momentum equation.

And then by using some mathematical procedures we try to obtain velocity profile then volumetric flow rate then friction factor etcetera that is what we have seen right. So, but if you have a non-isothermal flow of non-Newtonian fluids then, what are the additional things you may be requiring?

Obviously, when the system is at non-isothermal conditions, so then you may required more additional information, something like let us say in a given condition if the viscosity of fluid is changing is varying with respect to the temperature, so, that additional information is required which we have not considered till now right.

Sometimes let us say if there is a density variation because of the temperature changes, because of the non-isothermality of the system then so, those associated changes in the density with respect to temperature should also be encountered should also be incorporated in the system.

Then obviously, energy equation must be coming into the picture, because the system is at non-isothermal conditions. So, that is the additional thing is required. So, then sometimes we also require something like you know caloric equations of state etcetera those kind of information are also required.

So, before getting into the more details of non-isothermal flow of non-Newtonian fluids we make a list of things that are required in general in order to study this non-isothermal flow of non-Newtonian fluids.

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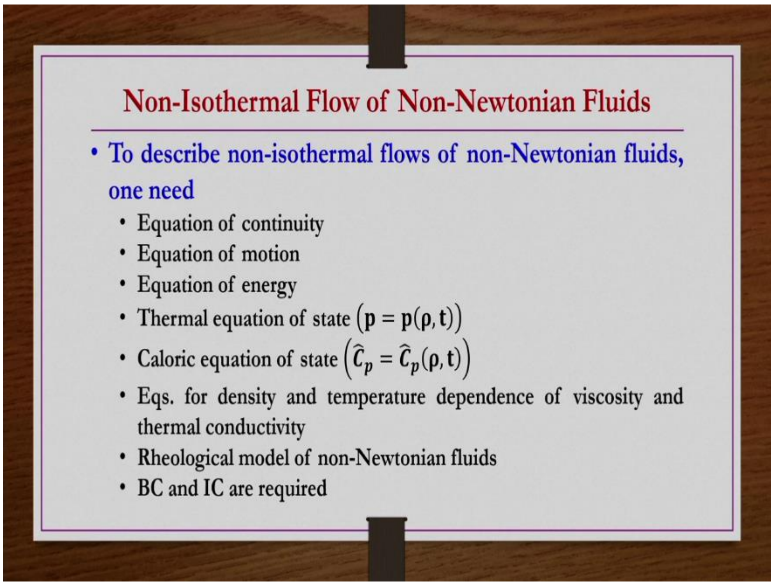

So, what are the things are required? Whether it is isothermal system or non-isothermal system in the transport phenomena definitely we need continuity equation or equation of continuity. Then equation of motion that is required, because if the non-isothermal flow we are talking about non-isothermal flow we are not just talking about the conduction where there is no flow etcetera.

We are also taking incorporation we are also taking the information of the flow, flow of non-Newtonian fluid. So, then if the flow is involved, so then equation of motion definitely would be there. Then, equation of energy because the system at non-isothermal condition. So, definitely we need to know how the energy is changing with respect to the space and time. So, that is the reason equation of energy is also required.

Then, thermal equation of state, the p function of ρ, t etcetera, then caloric equation of state, $\hat{C}_p$ function of ρ, t etcetera. Then for density and then temperature dependence viscosity and thermal conductivity, how it is dependent on temperature etcetera those information, those equations are also required.

And then obviously, the rheological model of non-Newtonian fluid is definitely is required whether the flow is isothermal or non-isothermal, if we are anticipating that material is having non-Newtonian behavior.

So, then that rheological model information must be had, so that in order to solve in order to solve these problems. Then obviously, boundary conditions and then initial conditions are also required if you are solving for the time dependence as well ok. So, now, here so majority of them are not required for all the problems in general, but many of them are definitely required.

So, what happens? Let us say if you are incorporating all the information how the velocity is changing with respect to the space in time, how the temperature is changing with respect to space in time. Similarly, pressure distribution, density distribution etcetera if you wanted to know all of these things, so definitely, all these things should be incorporated in the solver.

And then when you incorporate all this information then analytically solving such problems becomes relatively impossible. So, then we have to go for numerical solutions. Even for the isothermal systems if you do not if you have the dependence of velocity on more than one variable then it becomes very difficult.

So, now, at least in the velocity case you know in the flow cases, isothermal flow cases we can say that you know velocity is function of only y or only z those kind of thing or velocity is function of r etcetera. Those information we can say we can deduce from you know the basic problem statement, but for majority of the cases the temperature you cannot say whether it is function of only y, function of only z.

It must be explicitly mentioned or otherwise you have to you may be solving the temperature function of both y and z. Let us say flow through pipes when we have taken v_z is function of r that is what we have taken and then accordingly constraints we have listed out. And then we solve the problem for v_z as function of r right.

But the same pipe flow if the pipe and the flow is at non-isothermal conditions then temperature you cannot say it is function of r only, it will also be function of z. Even though velocity is function of r only, v_z even if v_z is function of r temperature you cannot say it is function of r only, it will also be function of z in the case of pipe flow if the system is at non-isothermal conditions ok.

So, those are the additional problems are there right. So obviously, now, what we understand? More equations are coming, more constraints are coming the solving heat transfer problems or solving non-isothermal flow problems that is where flow is also involved. It is not just you know temperature variations are there and there is no flow it is not like that.

So, when there is a flow also there and then temperature variations are there. So, that is non-isothermal flow conditions are there. The solving problems is going to be relatively difficult compared to the isothermal flow of non-Newtonian fluids what we had seen in till now ok.

However, for any problem as engineering students we can list out the constraints and then we make we can make the problems as simpler as possible. But not going far away from the reality still maintaining; not exactly the reality, but close to the reality we can solve the problems by making certain kind of assumptions as engineering students. So, that is what we are going to do in the case of this non-isothermal flow of a non-Newtonian fluids in coming few lectures.

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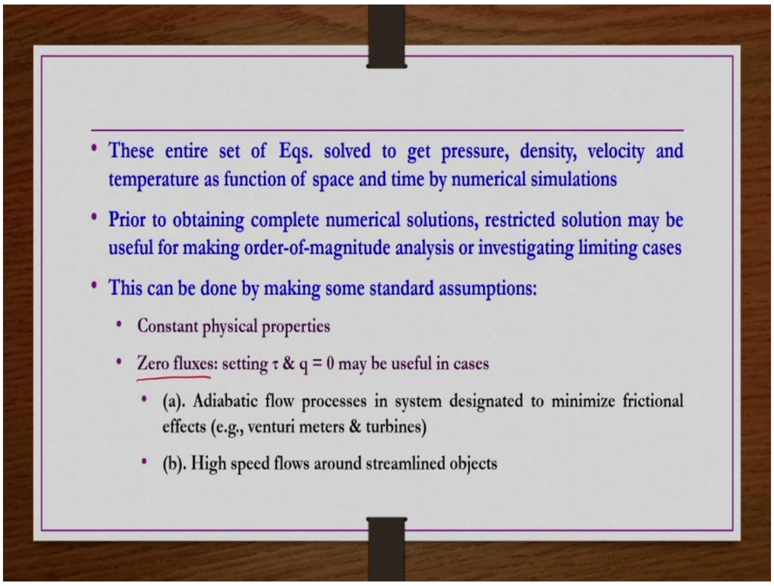

So, as I mentioned the entire set of equations whatever we listed in the previous slide if you wanted to solve, so then you have to depend on the numerical simulations. However, prior to solve the problems numerically it is a good idea to have a restricted solution.

So, restricted solutions in the sense under certain kind of you know constraints like you know like momentum transfer we have listed or some constraints like steady, isothermal and then you know symmetric, fully developed flow etcetera those kind of constraints under those constraints we have solved problems.

Likewise, in this case of in this case of non-isothermal flows of any system also if you make some kind of constraints, so, you are going to have certain advantages like you know making order of magnitude analysis. So, and then or investigating limiting cases something like that, those things you can see.

What does it mean by? Let us say even for example, you know getting back to our viscoelastic behavior. When a same viscoelastic fluid is flowing through a packed bed of a particle of different sizes. So, then we have seen under such one conditions of particle size the viscoelasticity is not necessary to consider, that is what mean by you know this order of magnitude analysis. Such kind of analysis we can do here also in the case of non-isothermal flow.

Let us say the Reynolds number is very small right, but the Prandtl number or Peclet number is very very large. So, then we can apply something like a thermal boundary analysis and then saw simplify the problem. And then find out you know required temperature distribution etcetera, those things we can do right.

So, limiting cases also, so like you know order of magnitude limiting cases like, you will come to know whether is it really important to incorporate all the information, all the equation that are we have that we have taken etcetera. Let us say if you are taken the flow through pipes right vertical pipe.

Is it really important to have the convection effect or free convection effect? That we can, those kind of analysis we can do by having the restricted solutions, how? We go problem to problem right. And then this can be done by obviously, making some standard assumptions.

Some of the standard assumptions which are very common to the non-isothermal systems you know flux, heat flux is 0, right. Like in the momentum transfer we can say you know shear stress is zero at a given location those kind of constraints.

Another constraint is that constant physical properties. So, constant physical properties, zero fluxes may be useful in many of the cases as a standard assumptions. Some examples where we can have in general zero fluxes case are adiabatic flow processes in systems designated to minimize frictional effects like in venturi meters and turbines.

Then high speed flows around streamlined objects etcetera. So, these are some kind of examples we can have. So, that is a very generalized introduction about a non-isothermal flows what we need to have how should we proceed kind of thing right. So, now what we start?

We start with a free convection problem, free convection problem between two vertical plates right. So, here non-isothermality is there in the system, but we are not solving only temperature distribution. But that temperature distribution we utilize to find out how it is affecting the density and then that change in density how it is it causing you know convection.

And then because of that free convection how the velocity distribution is changing that is what we are going to see. Basically, we are going to find out here also the velocity distribution, but now the velocity distribution is also affected by the non-isothermality of the system because of you know free convection existing in the system ok.

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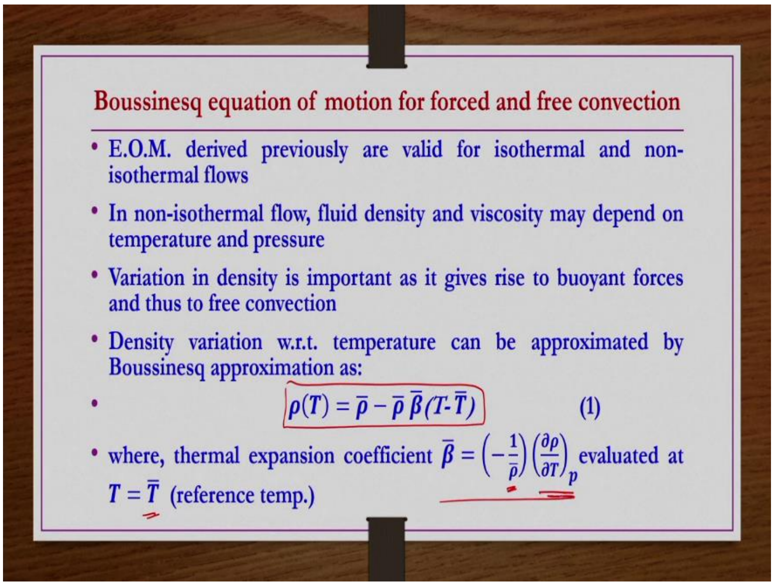

So, free convection usually you know for that purpose we use the Boussinesq's equation of motion for forced and free convection. So, that we can say under which conditions forced convection is dominating, under which conditions free convection is dominating that is what we can understand. So, since we are starting with the free convection problem, so, we see a few details about the Boussinesq's approximation.

So, equation of motion derived previously are valid for isothermal and non-isothermal flows as well that we have already mentioned while deriving the equations ok, irrespective of the nature of the flow, irrespective of the nature of the fluid generalized momentum equations we have developed. So, in non-isothermal flow fluid density and viscosity may depend on temperature and pressure.

So, is it depending both on temperature and pressure or both density and then viscosity are dependent on this temperature and pressure or only density is depending on the temperature, but not depending on pressure all these constraints of the problem one should have ok. So, variation in density is important as it gives rise to buoyant forces and thus to free convection.

Whatever the density variations are there because of the temperature differences, what happens that buoyant forces cause a change in the velocity distributions? How we is it going to change in the velocity distribution, that has to be discussed specific to problem to problem.

So, we are going to discuss one problem today. So, density variation with respect to temperature can be approximated by Boussinesq's approximation and then it is given by these things $\rho(T) = \bar{\rho} - \bar{\rho}\,\bar{\beta}(T - \bar{T})$. So, whatever the bar are there so that indicates evaluated at reference temperature $\bar{T}$ ok.

So, $\bar{\rho}$ is nothing but density at temperature $\bar{T}$ ok. And then $\bar{\beta}$ is nothing but coefficient of volume expansion at constant pressure, but temperature $\bar{T}$. So, that is $\bar{\beta} = \left(-\frac{1}{\bar{\rho}}\right)\left(\frac{\partial \rho}{\partial T}\right)_p$ right. So, this $\bar{\rho}$ is a temperature $\bar{T}$ and then this ρ how it is changing with respect to temperature at constant pressure this first derivative from here we can get it ok.

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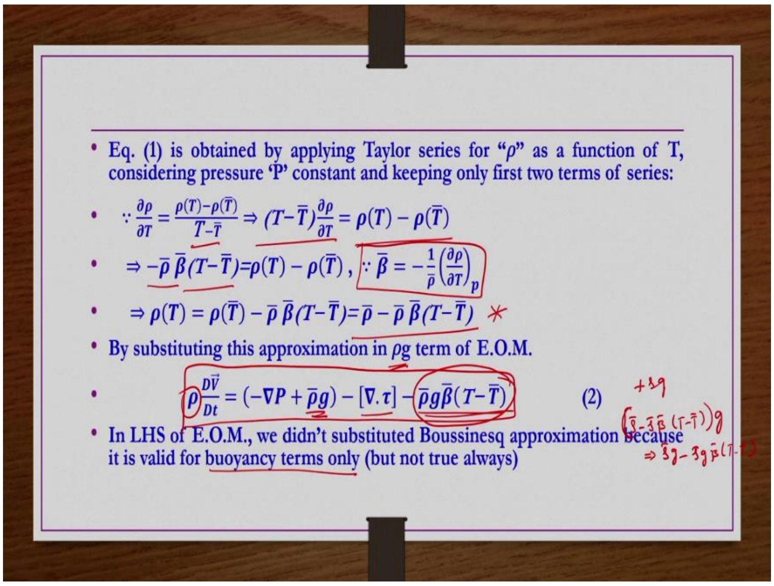

So, how do we get this equation? So, let us say density $\frac{\partial \rho}{\partial T}$ whatever is there that if you do the Taylor series expansion at constant pressure then $\frac{\partial \rho}{\partial T} = \frac{\rho(T)-\rho(\bar{T})}{T-\bar{T}}$. So that means, if $T - \bar{T}$ if you take to the left hand side $(T - \bar{T})\,\frac{\partial \rho}{\partial T} = \rho(T) - \rho(\bar{T})$, $\rho(\bar{T})$we can call it $\bar{\rho}$.

So that means, from this $\bar{\beta}$ definition what we can have, in place of a $\frac{\partial \rho}{\partial T}$? We can have $-\bar{\rho}\ \bar{\beta}$. So, left hand side $-\bar{\rho}\ \bar{\beta}\ (T - \bar{T}) = \rho(T) - \bar{\rho}$. So that means, $\rho(T) = \bar{\rho} - \bar{\rho}\ \bar{\beta}\,(T - \bar{T})$ ok. So, this equation is giving you know how the density is varying with respect to the temperature by taking a reference temperature $\bar{T}$ ok.

By substituting this equation in ρ g term of equation of motion then what we get? This is the generalized equation of motion that we have seen in vectorial form we have written $\rho \frac{D\vec{V}}{Dt} = (-\nabla P) - [\nabla.\tau] + \rho g$ was there. In place of + ρ g now we are writing in place of ρ what we are writing?

$\bar{\rho} - \bar{\rho}\,\bar{\beta}(T - \bar{T})$ and then this entire thing is being multiplied by g. So, whatever $\bar{\rho}$ g is there from here $\bar{\rho}$ g is combined with the pressure term and then written here. And then remaining $-\bar{\rho}\,g\bar{\beta}\ (T - \bar{T})$ is written as additional term, so this term is buoyant term ok. So, in LHS of this equation here, here in place of ρ we are not writing this Boussinesq's approximation.

Because it is valid for buoyancy terms only, but; however, in some cases like you know very high flow rates there also like you know supersonic flows etcetera, in such conditions also if the velocity, in such conditions also if the density is varying you know with respect to temperature in such kind of high flows then also in the left hand side we have to do, we have to apply such kind of approximation.

But for the time being for our chemical engineering problems we are not doing it, we are doing that replacing ρ by $\bar{\rho} - \bar{\rho}\,\bar{\beta}\,(T - \bar{T})$ only in the right hand side term in place of ρ g term ok, there only we are doing it.

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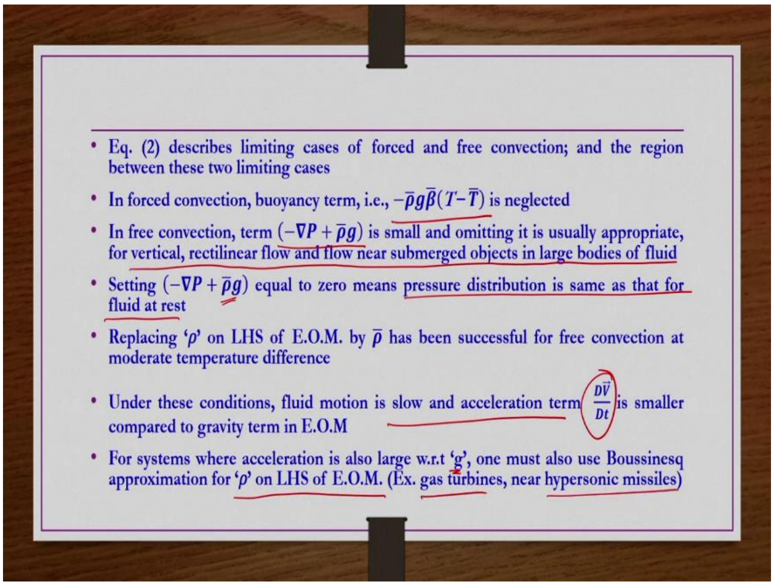

So, the previous equation whatever we have seen describes two limiting cases of forced and free convection and the region between these two limiting cases. What does it mean by? If it is a forced convention is dominated, so then last term whatever is there that can be neglected ok, in relative in relation to the or in comparison to the other term.

If it is free convection is dominating in the problem then what we can see? Whatever this $(-\nabla P + \bar{\rho}g)$ term can be neglected in comparison to the other term of a free convection buoyant term ok. So, this kind of analysis we can have.

So, in forced convection buoyancy term that is $-\bar{\rho}g\,\bar{\beta}\,(T - \bar{T})$is negligible. In free convection, term $(-\nabla P + \bar{\rho}g)$ is small and usually omitting it is not going to affect the solution much. And then in general some examples where we can omit this $(-\nabla P+\bar{\rho}g)$ despite of which the solution is not going to be affected much.

So, those examples are vertical, rectilinear flow and flow near submerged objects in large bodies of fluid etcetera. Then, setting $(-\nabla P+\bar{\rho}g)=0$ means that pressure distribution is same as that of fluid at rest. Whatever the hydrostatic pressures are there, so that is you know the pressure distribution is same to that hydrostatic pressure ok. So, that is what it mean by.

If you are setting up if you are taking m$(-\nabla P+\bar{\rho}g)=0$; that means, pressure distribution is same as that for the fluid rest fluid at rest. Because this $\bar{\rho}g$ is nothing but something like h ρ g that hydrostatic pressure etcetera those kind of terms, so that is what it mean by ok.

So, replacing ρ on LHS of equation of motion by $\bar{\rho}$ has been successful for free convection at moderate temperature differences. Under these conditions fluid motion is slow and acceleration term $\frac{D\vec{V}}{Dt}$ is smaller compared to the gravity term in equation of motion in general.

And then for systems where acceleration is also large with respect to g, one must also use Boussinesq's approximation on LHS of equation of motion, some examples like you know gas turbines, near hypersonic missiles etcetera ok. So, but for most of the chemical engineering problems we do not need to worry about you know applying the Boussinesq's approximation in the left hand side density term of equation of motion.

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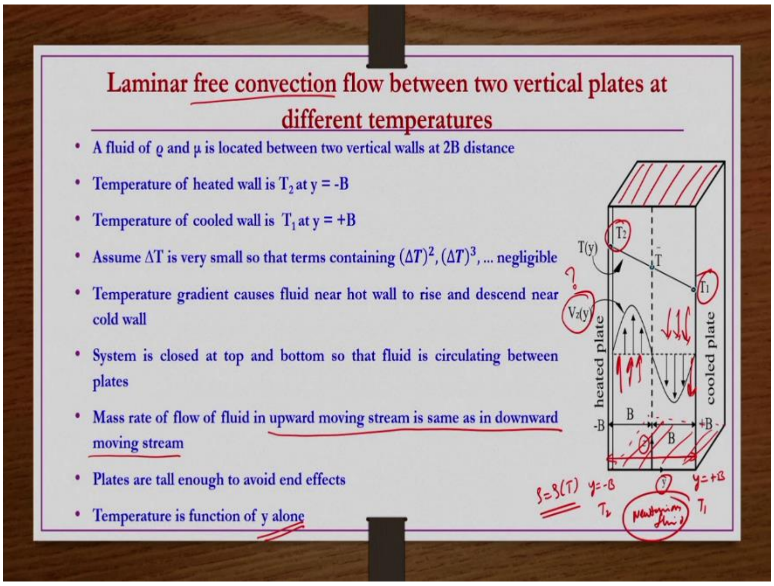

So, now coming to the problem that we are going to solve, laminar free convection flow between two vertical plates at different temperatures. So, schematically if you see we have taken two vertical plates ok, which are long enough that you can say that end effects are negligible. These two plates are separated at distance 2B ok, coordinate system is taken such a way that the y = 0 is in between these two plates, horizontal axis is y axis, vertical axis is z axis.

So, one of the plate is located at y = − B another plate is located at y = + B and then gap between these two is 2B. So, the heated plate it is at y = − B and then corresponding temperature is T_2 as given here, shown here. And then cooled plate temperature is maintained at T_1 as shown here.

Now, what happens in this problem? The velocity distribution, what is this velocity distribution that we have to find out ok. Just pictorially it is shown like you know parabolic profile like that, but it may not be true ok, so, we have to derive it. So, now, in this problem we are talking about the free convection, laminar free convection flow, what happens here?

That means, free convection is taking place; that means, the variations in density are taking place because of the temperature variations. Further to mention these plates are closed both at the top and bottom, this is closed at the both at the top and bottom right.

So that, whatever the fluid is there so that is circulating because of the convection only.

That means whatever the fluid elements are there towards the heated plate their density is decreasing because of the higher temperature. And then since the fluid molecules density is decreasing the fluid molecules are raising towards the heated plate.

And then towards the cold plate the temperature of the fluid elements are less, so then density is decreasing. So, then you know, so called the fluid elements or the fluid layers towards the wall are you know having the decreasing velocity.

And then because of this temperature difference that is $T_2 - T_1$ is maintained here. So, because of that one that temperature difference is causing the density variations and then that density variation is causing buoyant forces within the system. And then because of that buoyant force this material is circulating, the fluid is circulating, you know it is raising towards the left wall and then it is falling towards the right wall.

Because left wall is heated plate and then right wall is cold plate. So, this circulation is continuously taking place ok because the both the top and bottom of the system are closed right. Now, for this case how this V_z is changing? So, if it is isothermal condition, so then we can solve the way that we have solved by simply simplifying the continuity and then momentum equations.

And then one of the momentum equation would be giving expression for τ_{yz} and then τ_{yz} for the given fluid nature if you substitute power law, Newtonian or Bingham plastic and then integrate it, apply the boundary condition, get the velocity profile that is done, but in this case we have to follow the similar approach.

But in addition to that we have we should also consider how this ρ is changing with respect to temperature. And then that change in ρ should be brought into the equation of motion in the right hand side term where ρ g term is there ok. So, whatever the listed out details of figure are given here fluid of density ρ and then viscosity μ is located between two vertical walls at 2B distance.

For the simplicity we are taking as a first problem only Newtonian fluid ok, so, that we can solve the problem easily for the time being, because it is the first problem on non-isothermal flow of fluids ok. So, temperature at heated wall is T_2 at $y = -B$, temperature of cold wall is T_1 at $y = +B$.

Assume ΔT is very small, so that terms containing ΔT^2 ΔT^3 etcetera are negligible. If ΔT itself is very small, so then their square cube terms should also be very very small, so that we can neglect them. And then temperature gradient causes fluid near hot wall to rise and descend near cold wall.

System is closed at top and bottom, so that the fluid is circulating between the plates. Mass rates of flow of fluid in upward moving stream is same as in downward as in downward moving stream.

So; that means, let us say between $y = -B$ to $y = 0$ fluid is raising. Whatever the rate it is raising the same rise, at the same rate it is falling in the region $y = 0$ to $y = +B$ ok. So, that is fluid raising rate is equals to fluid falling rate in the system.

So, plates are tall enough to avoid end effects, so that we can have you know fully developed flow assumptions and then temperature is function of y alone it is given it is given. So, we do not need to worry about this that temperature is function of y or z.

Because for the temperature, not only just for the temperature in general for the any of the scalars it is difficult to say whether it is function of only one coordinate it is not function of other coordinates like that it is very difficult to say. Since, in the problem statement it is given, so we do not need to worry.

So, the temperature variations are required or the temperature distribution we should understand. Because that information is required to substitute in Boussinesq's approximation that we are going to apply for the right hand side ρ term ok. So, what are the constraints in general we have such kind of problems?

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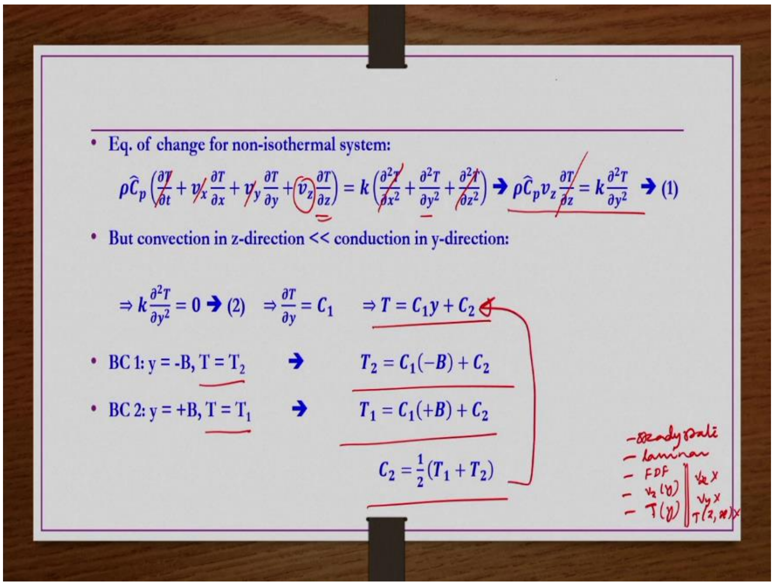

We have steady state, laminar flow it is already mentioned in the problem statement ok. And then plates are tall enough, so end effects may be avoided that is what it mentioned, so fully developed flow we can have alright.

And then v_z component is only existing and then it is function of y, temperature from schematic we do not understand whether it is function of y alone or z also. But however, luckily it is given in the problem that it is function of y only. So, need not to worry about that one.

So, these kind of assumptions you know we are having. So, v_x is 0, v_y is 0 and then T is not function of z and then x ok. So, under these constraints what we do? We first obtain the temperature distribution and then we use the temperature distribution in equation of motion to find out the velocity distribution for this free convection problem.

So, equation of change for non-isothermal system in Cartesian coordinate systems we have $\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$. This is we derived in week 3 or 4 of the course right.

So, now, here steady state, so the first term is 0, v_x is negligible, v_y is also negligible, v_z is existing, but the temperature is not function of z it is function of y only that way you can cancel out. Let us say if that information is also not given, so we retain it ok.

Then temperature is not function of x temperature is not function of z. So, we have $\frac{\partial^2 T}{\partial y^2}$ term only in the right hand side.

So, $\rho \hat{C}_p v_z \frac{\partial T}{\partial z} = k \frac{\partial^2 T}{\partial y^2}$. So now, from here this term can be cancelled out based on two constraints, one is the temperature is not function of z that is given. But from the physics of the problem the convection whatever is there that is there in the y direction right. Or the convection whatever is shown in the z direction that is very less compared to the conduction in the y direction.

So, but the, so by that constraint also we can have that convection in z direction is very small compared to conduction in y direction. So, then we can have $k \frac{\partial^2 T}{\partial y^2} = 0$. And then when you integrate you get $T = C_1 y + C_2$. We have two boundary conditions, $T = T_2$ at $y = -B$ and then $T = T_1$ at $y = +B$.

So, we have these two equations, two unknowns C_1, C_2, so we can find out them. So, now, if you add them together what will happen? $C_2 = 1/2\ T_1 + T_2$. Now this C_2 if you substitute here then you can get C_1.

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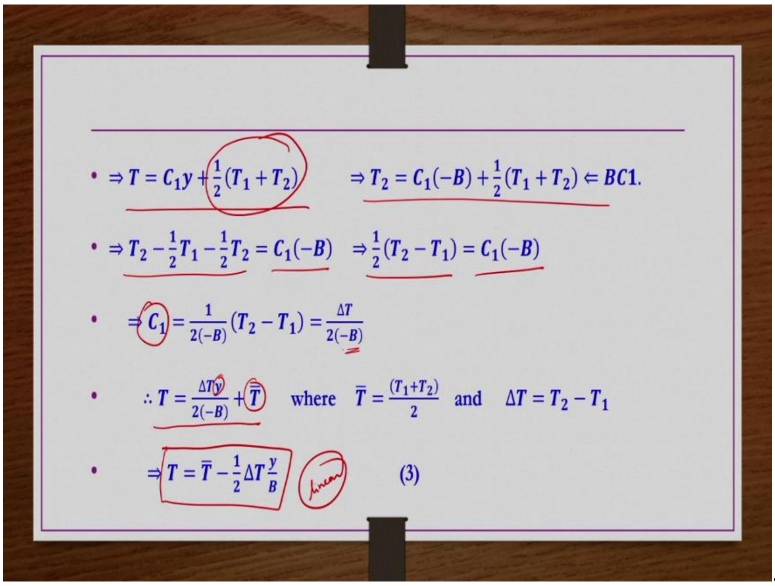

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So, $T = C_1\ y + 1/2\ (T_1 + T_2)$. Now, this first boundary condition we have at $y = (-B)$ $T = T_2$. So, $T_2 = C_1\ (-B) + C_2$, C_2 is $1/2\ (T_1 + T_2)$. So, from here if you do the simplification right, so then, what we will have?

Left hand side we have all these temperature terms, so right hand side $C_1 - B$ we are keeping right. So, then this T_2 – 1/2 T_2 is + 1/2T_2, so + 1/2T_2 −1/2T_1 is nothing but C_1 $(-B)$.

So, from here C_1 = (T_1 + T_2)/2 $(-B)$right. So, T_2 −T_1 we are writing ΔT right, I am not writing this one ΔT/2$(-B)$. So, let us keep it as it is ok there is a reason for simplification later on. So, C_1 = ΔT/2$(-B)$, 2 multiplied by $(-B)$.

Now, we have both C_1 and then C_2, this is nothing but C_2. This C_1, C_2 if you substitute in this equation T = C_1 y + C_2 then we get this equation T = C_1. C_1 is nothing but ΔT by 2 multiplied by $(-B)$, C_1 y and then y + C_2 is nothing but (T_1 + T_2)/2 that we are calling $\bar{T}$ average temperature ok.

So, this is what we are having, so; that means, T we can write T = $\bar{T}$ –(1/2) Δ T (y/B). So, what we understand here? T is a linear function of y that is what we understand. Before solving the problem from the problem schematic we do not have any information whether it is linear or non-linear etcetera.

So now, by simplifying this, an equation of energy and then applying the boundary conditions we understand the temperature is a linear function of y fine. So, temperature distribution is known. So now, what we do? We simplify the equations of motion alright, different components of equation of motion.

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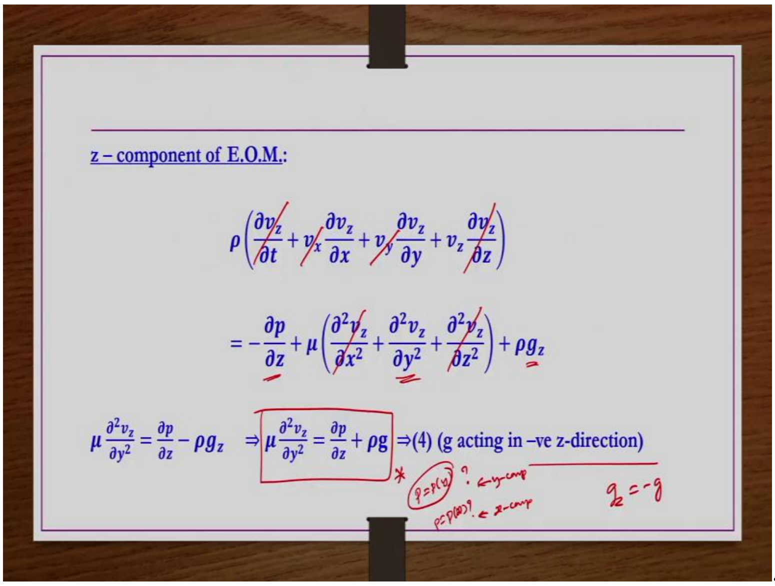

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So, start with the z component, z component equation of motion in Cartesian coordinate is this one. So, what we have here? Steady state this term is 0, v_x is 0, v_y is 0 and then v_z is not function of z it is function of y only or by fully developed flow also $\frac{\partial v_z}{\partial z} = 0$. Because plates are long enough, so that end effects are negligible that is given in the problem statement.

The pressure in general we do not know anything, so keep it as it is. And then this you know v_z is not function of x, it is not function of z, so it is function of y. So, then we have to retain this thing. And then in the z direction, z direction is vertical z direction z coordinate is in the vertical direction, so then g_z is there ok.

So, what we get? $\mu \frac{\partial^2 v_z}{\partial y^2} = \frac{\partial p}{\partial z} - \rho g_z$. Now, g is acting in the negative z direction, so g_z is nothing but $-$g. So, we have this equation, $\mu \frac{\partial^2 v_z}{\partial y^2} = \frac{\partial p}{\partial z} + \rho g$. So, let us keep it as it is, not solving this problem this equation as of now because this pressure is function of z that is what we are understanding from this equation.

Now, if you integrate this equation, so you should know whether it is function of y or not, if it is function of y also then integration become complicated. So, we should understand whether the pressure is a function of y or not especially before solving this differential equation for v_z right. So, if you wanted to know this one what you have to do? You have to simplify the y component of equation of motion, so that is what we are going to do.

Similarly, if it is pressure is function of x or not that if you wanted to know you have to simplify x component of equation of motion. So, if we find that p is not function of y, so, then this integration will become easy and then we can straightforward do the integration by assuming the right hand side terms are constant with respect to y.

But that also we cannot do directly because now here ρ is function of T and then T function of y. So, that also we have to substitute here as per the Boussinesq's approximation. So, those steps we are going to do subsequently.

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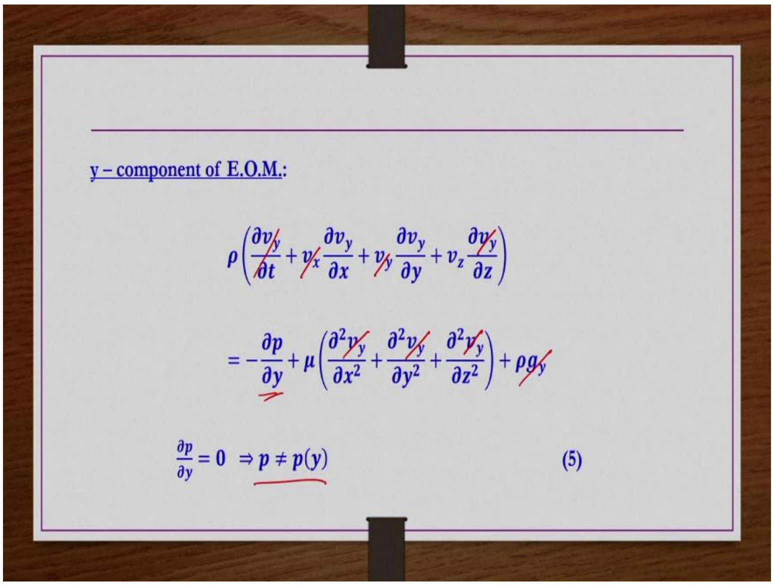

So, y component of equation of motion is given here. So, steady state this term is 0, v_x is not there, v_y is not there, v_y is not there. So, left hand side all the terms are negligible, pressure we do not know, v_y is 0. So, all these three terms are 0 and then in the horizontal direction gravity is not there, y direction is horizontal direction.

So, what we understand? $\frac{\partial p}{\partial y} = 0$; that means, pressure is not function of y. So, at least we are not worried about $\frac{\partial p}{\partial z}$ term in equation number 4 you know to integrate it and then get the velocity profile ok.

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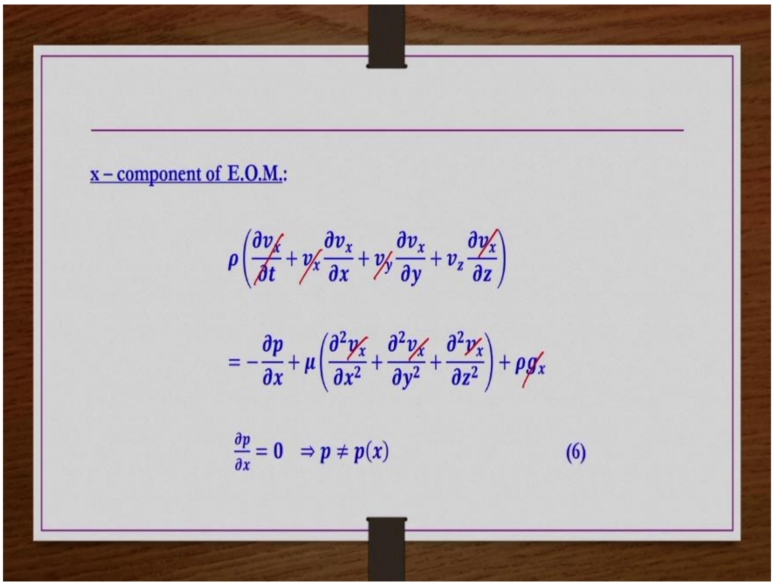

So, now next is x component of equation of motion that is given here. So, we have steady state problem, so first term is 0, v_x is 0, v_y is 0 and then v_x is 0. So, here also left hand side terms all terms are negligible. And then v_x is 0 and then in the, and then gravity is only in the z direction. So, g_x is also 0, so here also we get $\frac{\partial p}{\partial x}=0$; that means, pressure is not function of x.

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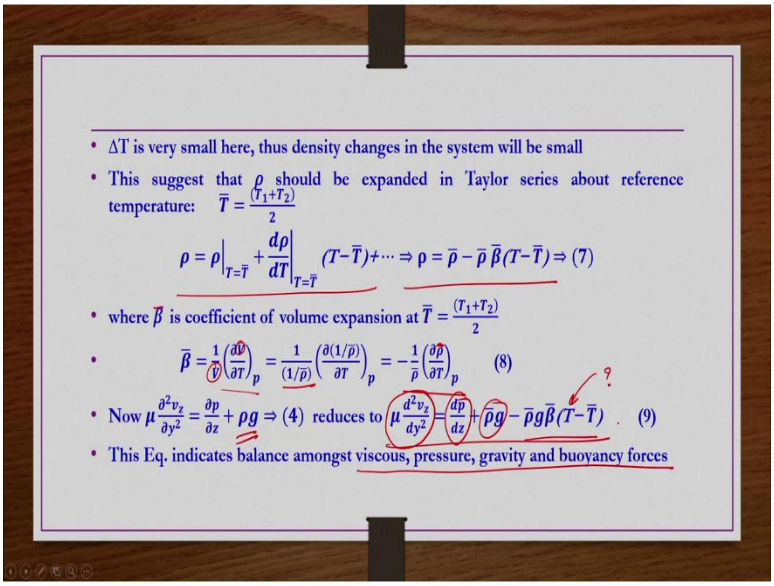

Now, what we understand? ΔT is very small here that is what we assume. So, that ΔT^2, ΔT^3 cube we can neglect thus density changes in the system will be so small. And then this suggests that ρ should be expanded in Taylor series about reference temperature $\bar{T}$ which is the average temperature of T_1, T_2.

Then according to this Taylor series expansion ρ = $\rho|_{T=\bar{T}}$ +$\frac{dp}{dT}|_{T=\bar{T}}(T-\bar{T})$ that is what we have. And then so on so, ΔT^2 term, ΔT^3 term should be there, so we are not considering it. So, this we can write ρ at $\bar{T}$ is nothing but $\bar{\rho}$.

So, this $\frac{dp}{dT}$ we already understand that it is $-\bar{\rho}\bar{\beta}$ that is what we understand from the definition of a volume expansion coefficient or coefficient of volume expansion for a given fluid at constant pressure right. So, that is we are evaluating at $\bar{T}$ temperature right.

So, β is coefficient of volume expansion at $\bar{T}$, so then $\bar{\beta}$ we can have. So, this $\bar{\beta}$ definition of coefficient of volume expansion is nothing but $\frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$. So, V we are writing a $\frac{1}{\bar{\rho}}$ because this V whatever is there this is for the volume and then this is at the reference temperature $\bar{T}$.

So, in place of V we can write $\frac{1}{\rho}$ and then that also at reference temperature ρ bar. So, that is what we have done and then we can then we can differentiate this one. So, we get $-\frac{1}{\bar{\rho}}\left(\frac{\partial p}{\partial T}\right)_p$.

This is not, there is no bar here there should not be bar, because $\bar{\rho}$ is a constant value, $\left(\frac{\partial p}{\partial T}\right)_p$. So, this if you substitute in equation number 4 in place of ρ g, ρ should be replaced by $\bar{\rho}$ $-\bar{\rho}\bar{\beta}$ (T $-\bar{T}$), so then this equation we get. Now, this equation we have to integrate to get the velocity distribution. So, before that we should also substitute what is this T that also we do subsequently ok.

So, now here this equation gives a balance amongst different forces this force indicate viscous force, this term gives the pressure distribution pressure force. And then this term gives the hydrostatic pressure terms and then this gives the buoyant force. So, balance amongst viscous, pressure, gravity and buoyant forces are given by this particular equation.

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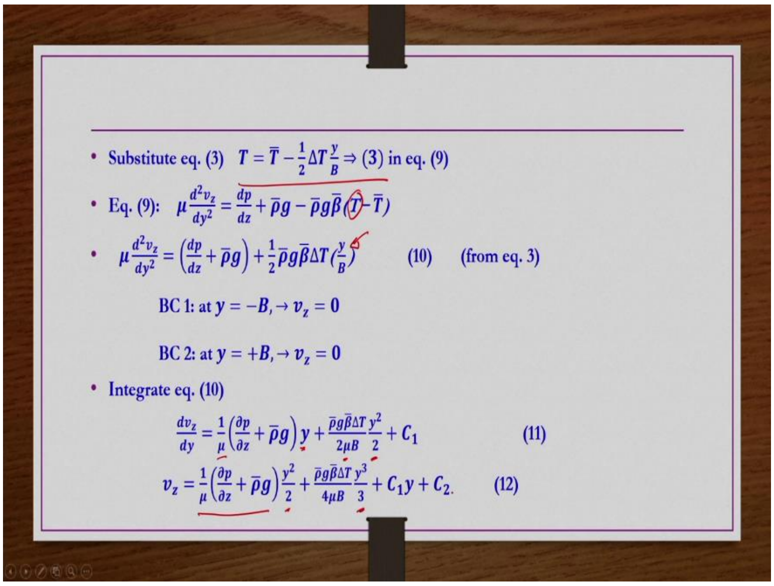

Now, we have already derived temperature by simplifying the energy equation that we got T = $\bar{T}$ − (1/2)Δ T (y/B). This is our equation number 3 in previous slides, this equation number 3 we are substituting our just derived equation number 9.

So, this is equation number 9 in place of T we are going to use this equation now. So, when you use it, so what you get? Right hand side term T −$\bar{T}$ you can write −(1/2)ΔT (y/B).

So, this term would be + 1/2$\bar{\rho}g\bar{\beta}$ ΔT (y/B). So, now, in the right hand side except this y all other terms are independent of y. So, we can integrate without any difficulty right. Before integrating what we do? We take this μ to the right hand side. So, we need boundary conditions at y at both the walls because of the no slip conditions we have the velocity 0 right.

Whether it is y = −B or y = + B the velocity is 0 because of the no slip conditions right ok. So, this is in this equation when you integrate you will get two constants. So, we have two boundary conditions, so there should not be any difficulty in obtaining the constants also. On integrating this equation we get this particular term that before integrating first I have taken mu to the right hand side.

So, it is coming under the denominator term right then after first derivative I what I get? Here first term is multiplied by y and then second term y is there. So, $y^2/2$ + one constant, once again integrating what I get? v_z is equals to this term is as it is $\frac{1}{\mu}\left(\frac{\partial p}{\partial z}+\bar{\rho}g\right)$ and then y integration of y is $y^2/2$. Now, here integration of y^2 is $y^3/3$ and then C_1 y + C_2.

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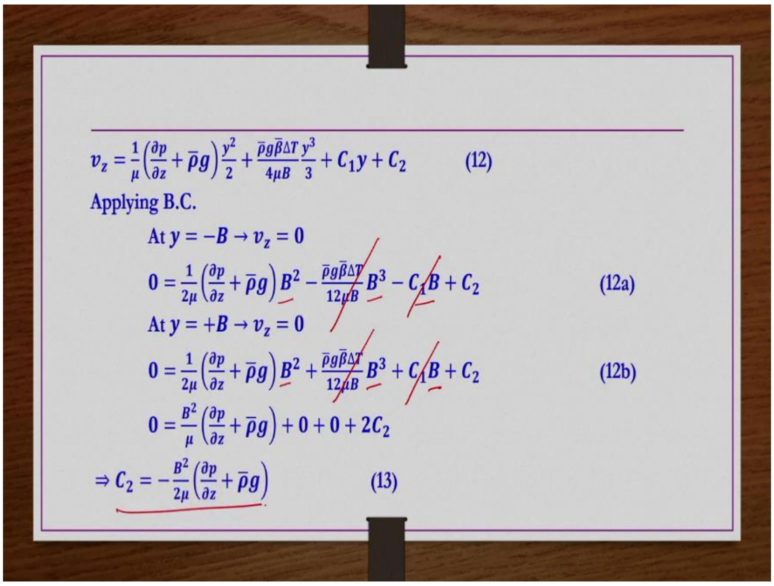

So, this is the equation, we have rewritten once again ok. Now, applying the boundary condition at y = −B v_z = 0. So, wherever y is there we are writing −B. So, B^2 here and then $-B^3$ here and then −B here. Another boundary condition at y = + B also v_z = 0. So, in equation number 12 wherever y is there we can substitute + B.

So, + B^2, + B^3, + B here. Now, these two equations if you add together what will happen? So, this term this term are same, so we can cancel out this term this term are same, but different signs, so we can cancel out. So, C_2, 2 C_2 and then 2 times of this term that is $\frac{B^2}{\mu}\left(\frac{\partial p}{\partial z}+\bar{\rho}g\right)$ right. So, from here C_2 you get this expression, this C_2 we are going to substitute here to get the C_1.

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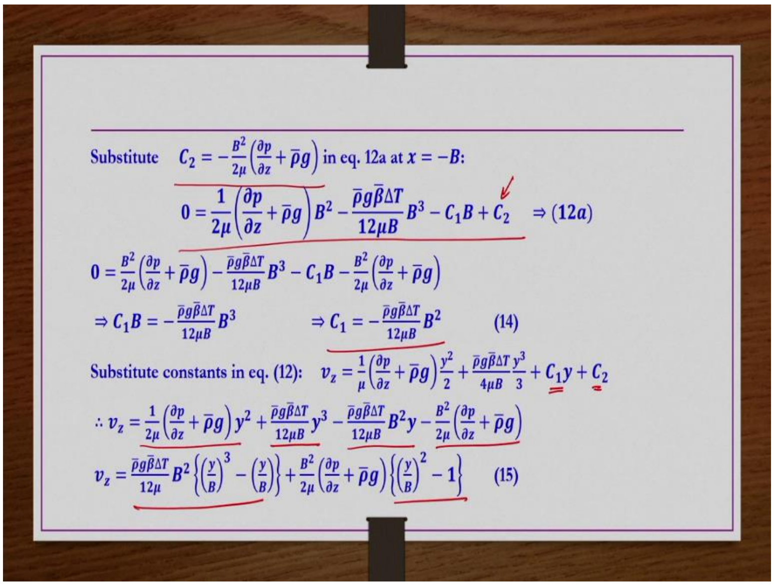

So, in that equation number 12 a, this is 12 a in place of C_2 we are writing this C_2 constant that we just obtained. Then simplify it then you will get C_1 is equals to this constant. Now, this both C_1, C_2 you substitute in equation number 12, so then you have this expression.

Next step what you do? From this term and then from this term what you do? You try to take B^2 common, what you try to do? B square if you take common $\frac{\bar{\rho}\, g\, \bar{\beta}\, \Delta T}{12\mu B} B^2 \left\{\left(\frac{y}{B}\right)^3 - \left(\frac{y}{B}\right)\right\}$ you will get.

And then from these two terms $\frac{B^2}{2\mu}\left(\frac{\partial p}{\partial z} + \bar{\rho}\, g\right)$ if you take common, then you get $\left(\frac{y}{B}\right)^2 - 1$ as the multiplication factor. So, this is the velocity profile ok. So, now, this velocity profile $\bar{\rho}$ that is density is density at reference temperature $\bar{T}$ is known, $\bar{\beta}$ for a given system might be available or must be given.

ΔT is known, viscosity is known, width of or gap between two plates is known everything is known except this $\frac{\partial p}{\partial z}$. If you know this $\frac{\partial p}{\partial z}$ then you can know the velocity profile easily. So, now still this is the final velocity distribution equation, so, we do not know what is $\frac{\partial p}{\partial z}$. So, we cannot use it for our final applications, so now we try to obtain what it is.

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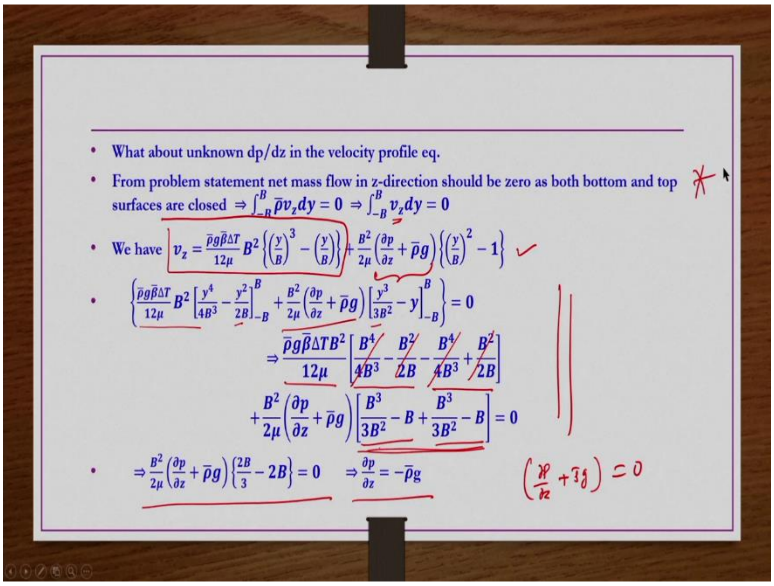

From the problem statement it is given the rate at which fluid is raising the same rate it is falling. That means, if you take for the both the limits of y = −B to y = + B integral of $\bar{\rho}$ v_z dy should be 0 right. And then $\bar{\rho}$ is constant, so that also you can take to the right hand side. So, integral −B to + B v_z dy = 0, if you do then you can get some kind of information about this $\frac{\partial p}{\partial z}$.

So, v_z just now we derived this equation, now this equation we substitute here and then integrate it. When you integrate here the first term these are the constant. In place of y^3 integration of y^3 is $y^4/4$ integration of y is $y^2/2$ and then second term this is constant.

So, integration of y^2 is $y^3/3$ and then integration of constant 1 is y. Limits −B to + B if you substitute first term, $\frac{\bar{\rho}\, g\, \bar{\beta}\, \Delta T}{12\mu B} B^2$ is as it is. Then when you substitute these limits you get upper limit these two terms, lower limits these two terms, what you can understand? These two terms are same, but the opposite signs, similarly these two terms are also same but opposite sign, so we can cancel out.

The second term $\frac{B^2}{2\mu}\left(\frac{\partial p}{\partial z} + \bar{\rho}\, g\right)$ and then this is the upper limit after substitution this is the lower limit after substitution right. So now, altogether in these two terms first term is anyway 0, because whatever the terms that are being multiplied, so that is all 0.

So, first term is 0, second term whatever this summation comes out we can write simply take it to the right hand side and we can write this way. So, what we can get? $\frac{\partial p}{\partial z} = \bar{\rho}$ g; that means, $\frac{\partial p}{\partial z} + \bar{\rho}$ g $= 0$ 0.

So that means, in this equation the second term is 0. So, in the velocity profile we need only this first term, because $\frac{\partial p}{\partial z} + \bar{\rho}$ g $= 0$ that we understand from this statement of the problem.

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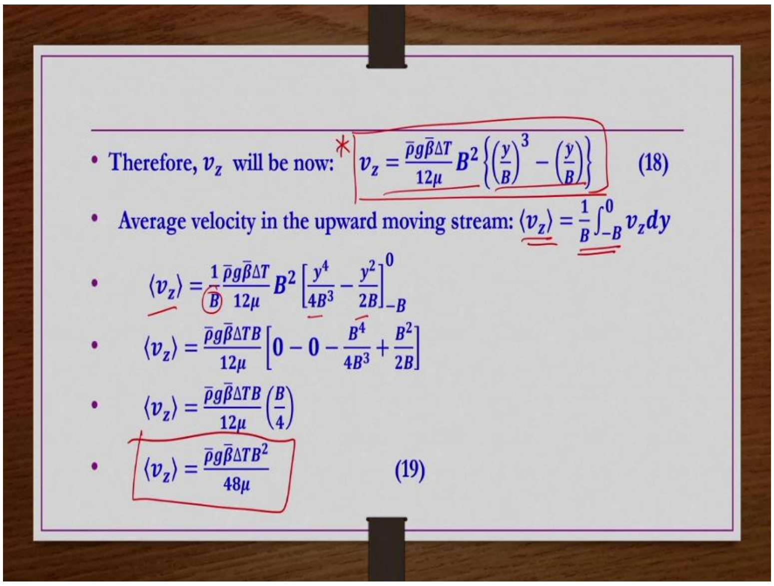

So, therefore, v_z will be only the first part, first term of equation number 12 that we have derived previously. Only first term of velocity distribution should be taken under consideration, because second term is having $\frac{\partial p}{\partial z} + \bar{\rho}$ g which is 0. So, second term of that velocity distribution equation is 0. So, we have only first term in the equation, so then velocity profile is this one.

So, before winding the class if you wanted to have the average velocity in the upward moving stream, so average velocity only in the upward moving stream. Upward moving stream is between y = $-$B to y = 0. So, $\frac{1}{B} \int_{-B}^{0} v_z dy$ if you do you get the average velocity in the upward moving stream.

So, when you do it this $v_{z\ avg}$ is equals to this all term divided by B and then integration of these terms. So, integration of y^3 is $y^4/4$ and then integration of y is, and then integration of y is $y^2/2$ limits $-B$ to 0. When you substitute and then simplify you get $v_{z\ avg}$ is $\frac{\bar{\rho}\, g\, \bar{\beta}\, \Delta T\, B^2}{48\mu}$ ok.

This is how we have to solve non-isothermal flow of any fluid whether Newtonian or non-Newtonian. So, we have taken simple Newtonian fluid in this today's lecture. So, we will be taking different non-Newtonian fluids and then more complicated geometries in the coming lectures.

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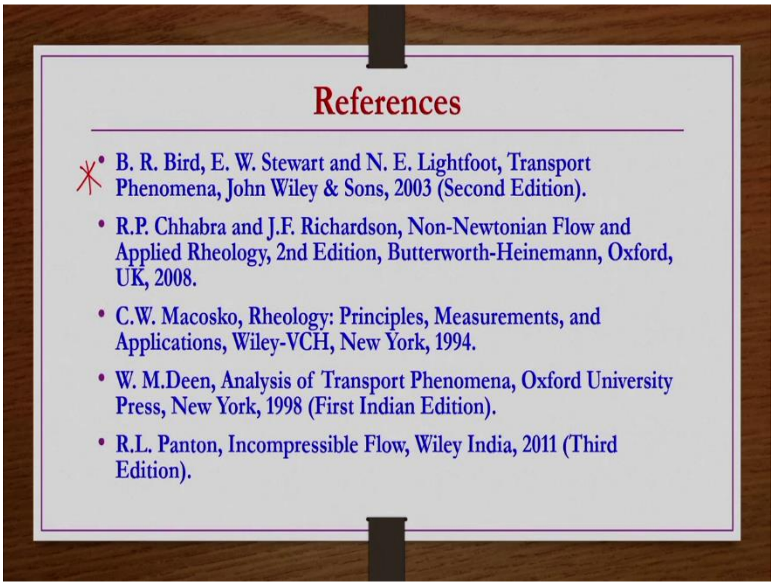

Reference: these details have been taken by the standard book Transport Phenomena by Bird, Stewart and then Lightfoot, other useful references are provided here.

Thank you.