Transport Phenomena of Non-Newtonian Fluids
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Lecture - 21
Laminar flow GNFs along Inclined Surface and Concentric Annulus

Welcome to the MOOCs course Transport Phenomena of Non-Newtonian Fluids. The title of today's lecture is Laminar flow of Generalized Newtonian Fluids along Inclined Surface and Concentric Annulus.

So, in the previous lecture also we have taken inclined surface geometry, but however, we have taken Ellis model fluid flowing down along inclined surface. So, that problem we have solved. And then we have also solved an example problem based on that analysis.

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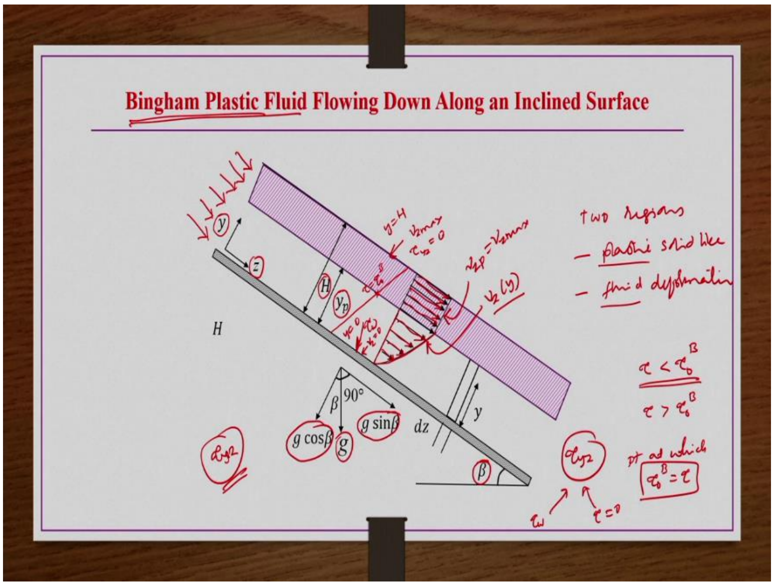

So, now what we do? We take the Bingham plastic fluid flowing down along inclined surface right. So, the geometry is same like this, so we have an inclined surface and then coordinate system we have taken this is y direction this is z direction. And then thickness of the fluid whatever is there that is H. So, the fluid is coming down on the surface like this. So, the thickness of the fluid is H. Usually these fluids are you know they flow down as a very thin layers maybe 3 mm, 4 mm or maximum 5 mm something like that ok.

And then these applications in general we find such kind of application in polymer industries right. So, now the gravities acting this direction, but the surface along with the fluid is flowing down that is it the angle β making angle β with the horizontal right. So, then obviously, what we have to do since the surface is inclined.

So, corresponding horizontal and then vertical components for this g we have to find out. So, then in the flow direction we have g sin β and then in the normal to the flow direction we have g cos β has the components of the gravity, that is fine.

So, now, next thing is that since it is Bingham plastic fluid obviously, we have a two regions of flow possible that we know. So, one is the plastic solid like region and then another one is the fluid deformation region. Or the other one is the deformation region, so where the material flows down as a fluid because the deformation is taking place.

In the region where deformation is not taking place that region material flows like a solid fluid. That we already know from the basic understanding of the Bingham plastic fluids. Why that happens? Because you know if the applied stress whatever is there if it is less than characteristic yield stress of the material. So, then the material does not deform it flows like a plug right.

Because this material viscoplastic Bingham plastic material is a viscoplastic it is having both plastic and viscous nature. So, the plastic is in general refer for the solid kind of thing here as a kind of common terminology right. So, it is having some characteristic that yield stress; one of the characteristic is the yield stress. So, that indicates below that stress region or the whenever the applied stress is below that one that material will not deform and then it flow like a solid plug ok.

So, that is the one region another one region is the when applied stress is more than this τ naught B then material will start deforming and then that material will flow like a viscous material, that we know. So, now, what we have to do? In order to find out these two regions for this geometry as well we have to find out the point at which this applied stress is becoming equal to the yield stress ok.

So, that is what we have to see. So, now, here the flow is taking place in the you know z direction. So, then obviously, v_z component is dominating and then that z direction velocities varying with respect to y ok. So, that we have to find out.

And then here only shear stress that is possible is that τ_{yz} is possible here rest all other component of this stresses are 0 or very small compared to the τ_{yz} that we can neglect right. So now, basically we have to have a basic information about the shear stress how it is varying.

So, for such kind of geometries we understand that shear stress linearly varies along the direction normal to the flow right. That is normal to the flow direction is y direction here. So, that shear stress linearly varies in y direction. So, let us say now one point the shear stress at one particular point it would be maximum shear stress another particular point it may be 0 or very small negligible 0 where the velocity is maximum, those points we have to first find out, right.

So, we cannot blindly say that you know upper region is the fluid like region or upper region is a solid like region we cannot say, from this understanding we only we can have. So, how do we get? So, we can get that one, so along on the surface on the solid surface you know the material is having zero-velocity. So, then in near to that on the surface and near to that region the gradients are going to be very high.

So, then obviously, the stress is going to be very large and then that high shear stress is going to be is known as the wall shear stress that we know right. So, but now here in this case the outer surface of the fluid right. So, the bottommost surface of the fluid which is attached to the solid surface along with which is flowing. So, at that surface y = 0. So, at y = 0 we have the maximum shear stress.

But the outer outermost layer of the fluid this is the outermost layer of the fluid. So, that is having you know exposing expose to the atmosphere or you know maybe there may be hot air is blowing etcetera in order to dry this fluid etcetera; those things may be taking place. But that is exposed to the open atmosphere. So, then obviously, it will be having the free surface. So, the maximum velocity would be there at this layer that is at y = H. At y = 0 we have τ_w and then $v_z = 0$.

So, at y = 0 we have τ_w maximum shear stress and then $v_z = 0$. Whereas, at y = H we have the maximum velocity and then shear stress τ_{yz} is 0 that is what we understand. So now, between these two point y = 0 to y = H if you draw the shear stress line linearly like this. So, it is a 0 value it is maximum value. So, then gradually what happens when you move outwards towards the outermost fuel layer fluid layer. So, then the shear stress decreases and then it becomes 0 at the outermost layer ok.

So, there would be some point; there would be some point you know at which applied shear stress $\tau = \tau_0^B$, what is that point that we do not know. So, let us call that is at y_p distance from the surface; that is at the y_p distance from the surface right. So now, towards the wall the shear stress distribution is there and then at the wall shear stress is maximum. So, then towards the wall what we have? Here we have a velocity distribution for example, like this right.

So, but after crossing when the applied stress is becoming τ naught B and then after that it is gradually decreasing as we move towards the outer surface. So, then gradually and then what happens? That means, applied stress is less than the yield stress, so then material is flowing like a plug.

So, the slanted area region whatever is shown that is also the fluid that is also a material region. So, that region the material is flowing like a solid plug and then whatever this remaining unshaded portion is there. So, that portion is the region where the material is deforming and then fluid; and then that material is flowing like a fluid viscous fluid right.

So, then obviously, so this v_{zy} we have to find out in addition to that one we have to find out what is this v_{zp} and $v_{z\,max}$ right. So, how do we find out? In order to find out either v_z or from there v_{zp} you have to find out what is τ_{yz} right, what is τ_{yz} that you have to find out.

Once you find out this τ_{yz} what will happen you can relate this one to the velocity gradient and then from the velocity gradient you can find out the velocity. So, how you can relate this one to the velocity gradient or shear rate, that is based on the nature of the fluid. So, then for the Bingham plastic fluids we have to take a $\tau_{yz} = \tau_0^B \pm \mu_B \frac{dv_z}{dy}$ so, but $\pm$ that we have to decide.

So now, as y increasing at y = 0 v_z is 0 and then as y increasing gradually velocity is increasing at $y = y_p$ to $y = H$ it is having the maximum velocity. That means, as y increasing the velocity increasing, so then we can have the + $\frac{dv_z}{dy}$ ok.

So, this is the basic understanding about the flow. So now, what we do? We simplify the continuity and momentum equation based on the constraints of the problem constraints of this problem right. So, then we can get this relation. So, before solving that or before simplifying the continuity or momentum equation we have to list out the constraints of the problem, right.

So, now what we do? We list out the assumptions are the constraints of the problem under which we are obtaining this velocity profile and then followed by volumetric flow rate etcetera.

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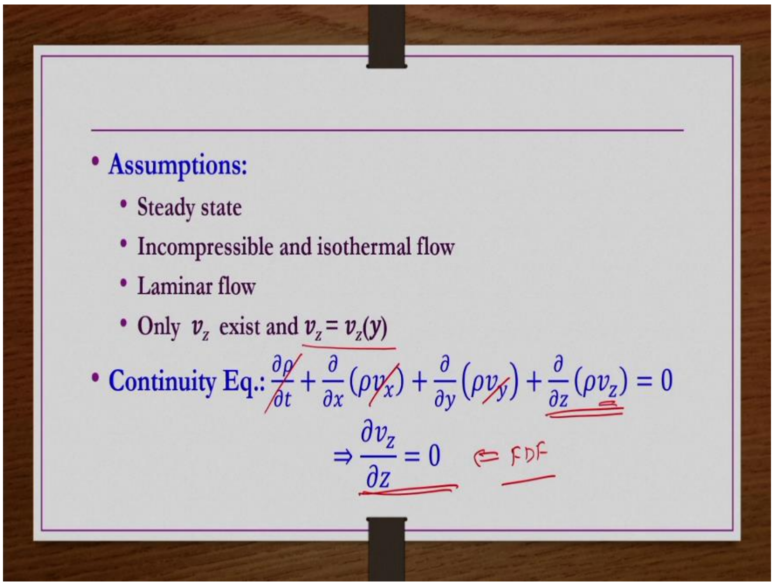

So, assumptions are the basically we are solving all these problems for steady laminar flow, incompressible fluid we are taking. And then temperature variations or the reactions mass transfer etcetera we are not taking right. And then in this geometry we have seen only v_z is existing that is function of y. Whereas, the v_x and then v_y are negligible or 0 or very small compared to the v_z that we can neglect them straightforward.

So, the continuity equation when we simplify it. So, in the Cartesian coordinates we have to take the Cartesian coordinates because of the geometry ok. So, now because of the steady state this term is 0, v_x is 0 v_y is 0, v_z is not 0 and then v_z is function of y only right. So, it is not function of z, so that way you can straight forward you can take off.

Or what we can do? We can take this one as a kind of constraint that we are getting fully developed flow. Because from the geometry we do not understand whether the flow is fully developed or not. So, such kind of constraints also if you wanted to get you can get from that only.

So, either way it is ok either you can take it as continuity satisfied or you can take this equation is given fully developed flow. So, but however, see we have taken v z is function of y, so then we can take of this one also.

So, that is what we can do otherwise we can take these $\frac{\partial v_z}{\partial z} = 0$ from the continuity equation. So, that you know we can get a one condition that fully developed flow we can take without any difficulty. So, we do not know whether we should we; for certain geometries it is in general very clear that you can take the fully developed flow, but certain geometries it may not be very clear.

So, that such kind of conditions we may get from the simplification of our continuity equation. Sometimes whether the symmetry is existing or not that also we may not clearly understand from the flow geometry. So, those kind of conditions also we may get by simplifying the continuity equation as well.

So, continuity equation is having two advantages whether, first advantage is whether the flow or the constraint that we have taken that are physically reliable or not other one is that from the limited number of constraints if at all other constraints we need to be understand. So, those things we can understand by simplifying the continuity equation.

Like you know whether the flow is symmetry or not, fully developed flow or not if you do not understand those things we can see. You can understand by simplification by simplifying the continuity equation.

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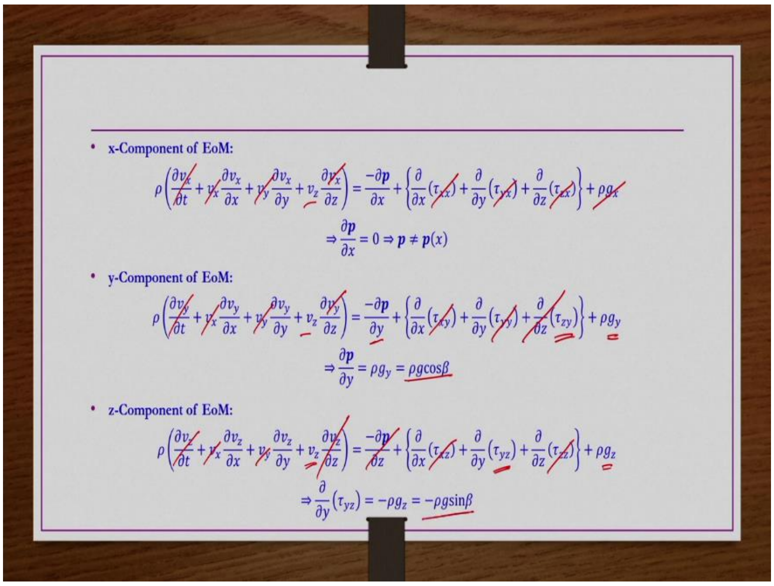

So, now x component of momentum equation here steady state this is 0, v_x is 0, v_y is 0, v_z is there, but v_x is not 0 pressure we do not now. So, just leave it as it is. This component of shear stress is not there, this is not there, this is also not there. And then gravity is not there in the x direction it is there in the y and z direction, so this is also 0. So, what we understand from here? $\frac{\partial p}{\partial x} = 0$; that means, pressure is not function of x then y component of equation if you simplify.

So, steady state this term is 0, v_x is 0, v_y is 0, v_z is not 0, but v y is 0. So, here also left hand side altogether all the terms are negligible, pressure we do not know. So, this component of shear stress is not existing, this is not existing, this is existing. But what we understand that from the fully developed flow condition $\frac{\partial v_z}{\partial z}$ is 0 that is that we got from the continuity equation or we understand that v_z is function of y only. So, then obviously, the shear stress is also going to be function of y.

So, by applying either of the constraints or either of the conditions what we can say this is 0 $\frac{\partial}{\partial z}$ of any flow variable is 0. Gravity is there in the y direction, so then what we understand here? $\frac{\partial p}{\partial y} = \rho g_y$ and then ρg_y is nothing but ρ g cos β right.

Then z component of equation of motion, so v_z is existing. So, by steady state this term is 0, v_x is not there, v_y is not there, v_z is there, but $\frac{\partial v_z}{\partial z} = 0$ from the continuity equation that we understand, right. So, the pressure cannot be there in the z direction and the flow duration because the flow is taking place because of the gravity, so this is not there. So, this component of shear stress is not there, this component of shear stress is there and then it is function of y; whereas, this component of shear stress is not there.

Then g_z gravity is there in the z direction. So, then what we understand from here? $\frac{\partial \tau_{yz}}{\partial y} = \rho g_z$ that is $-\rho g$ sin β.

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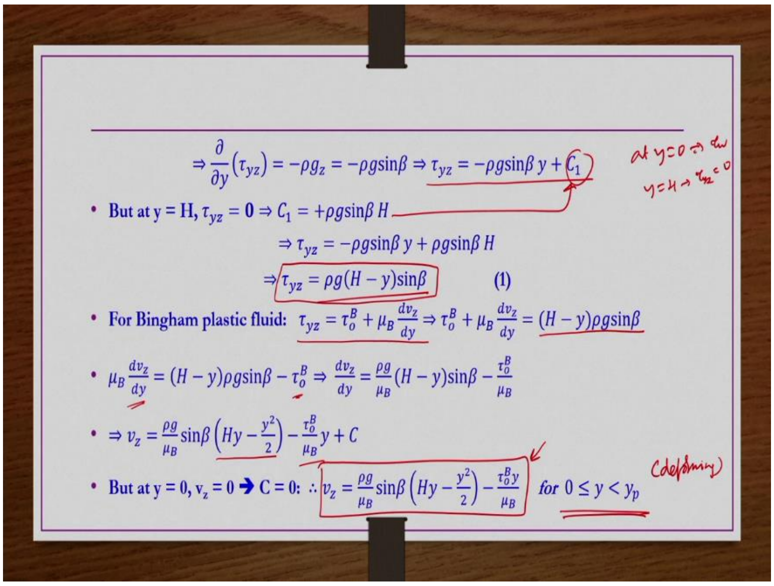

So, now this equation if you solve then you can get an expression for the shear stress as $-\rho g$ sin β y + C_1 on integration this is what you get, right. Now shear stress final expression you can get a few get these constants C_1. So, what we have? We have two boundary conditions at y = 0 then we have a this so called $\tau = \tau_w$. But at y = H we have so called τ_{yz} is nothing but 0, because the outermost layer which is at y = H that is exposed to the atmosphere ok.

So, that layer of the fluid is having the maximum velocity. So now, that if you apply here then you get C_1 constant as ρ g sin β H now if you substitute this C_1 in the above equation here then you get τ_{yz} = ρ g H - y sin β. So, till this point we have not applied any constraints of the fluid rheology.

So, till this point it is same as whatever we have seen in the previous problem Ellis fluid flowing along the inclined surface. So, till this point it is this problem also same. So now, from this point onwards the fluid rheology will come into the picture, for Bingham plastic fluids we have $\tau_{yz} = \tau_0^B + \mu_B \frac{dv_z}{dy}$.

So, from here what we do? We write in place of τ_{yz} $(H - y)$ ρ g sin β. Then in the next step what we do? We keep this the $\frac{dv_z}{dy}$ term one side and then that whatever minus; whatever τ_0^B is there that we take to the other side. So, then minus τ_0^B. Then both sides you know we divide by μ_B, so then we have $\frac{dv_z}{dy} = \frac{\rho g}{\mu_B}(H - y)\sin\beta - \frac{\tau_0^B}{\mu_B}$.

Now if you integrate this equation what will happen? You will get a expression for the v_z. So, that $\frac{\rho g \sin\beta}{\mu_B}\left(Hy - \frac{y^2}{2}\right) - \frac{\tau_0^B y}{\mu_B} + C$ this is what you are having.

So, now at y = 0 that is the surface of the inclined surface the top layer of the inclined surface on which the first layer of the fluid is having zero-velocity because of the no slip condition. So, if you substitute y = 0 in this equation v_z would be 0 and then you can get this constant C as well as 0 right. Because all the terms are being multiplied by y, so then you get this C constant as 0.

So, then finally, you get $v_z = \frac{\rho g \sin\beta}{\mu_B}\left(Hy - \frac{y^2}{2}\right)\frac{\tau_0^B y}{\mu_B}$. And then this is valid between 0 to y_p region only deforming region; it is valid for the deforming region only ok.

So now, in this equation if you substitute $y = y_p$ then what you can have? You can have a; you can have a maximum plug velocity because it is valid up to y_p. From y_p to h it is not valid. Some y_p to h we have a constant velocity in that constant velocity you can substitute we can get by this velocity expression by substituting $y = y_p$. Because it is valid up to the y_p point and then from y_p point or not it is having constant plug velocity like a solid material. So that we do know.

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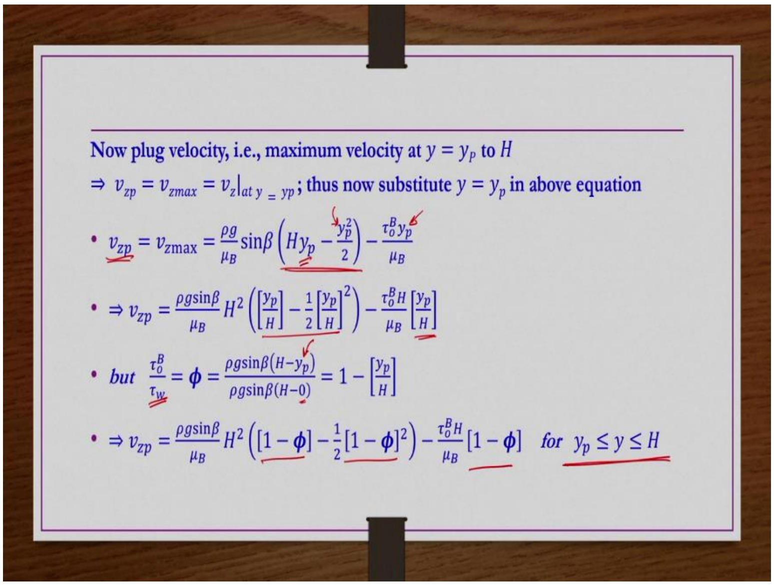

So, that v_{zp} you can get by substituting y = y_p in v_z expression that just we got. So, in the v_z here in place of y we have written y_p, here also in place of y^2 we have written y_p^2, here also here also in place of y we have written y_p and then v_z is nothing v_{zp} right. So, now, v_{zp} = $\frac{\rho g\sin\beta}{\mu_B}$.

From both the terms first from this term what we do? We take out H^2 common, so that we have $\left[\frac{y_p}{H}\right] - \frac{1}{2}\left[\frac{y_p}{H}\right]^2$. And then from this term what we are doing? We are dividing and multiplying by H, so that we have $\frac{\tau_0^B H}{\mu_B}\left(\frac{y_p}{H}\right)$. Why are we writing this in terms of $\frac{y_p}{H}$? Because this $\frac{y_p}{H}$is related to the shear stress and then yield stress or the ratio between yield stress and then shear stress that applied wall shear stress ok.

So, then that we are writing in terms of ϕ. Is it directly equal to ϕ, like in previous problems we have seen or is it $1 - \phi$ or $\phi - 1$; that we have to see. So, how do we know? For that we have to do $\frac{\tau_0^B}{\tau_w}$ we have to do, because $\frac{\tau_0^B}{\tau_w}$ is nothing but ϕ. And then τ_0^B is nothing but ρ g sin β $(H - y)$in place of y we have to write y_p.

Because at y = y_p applied shear stress is becoming equal to the yield stress that is the point which is separating the fluid; which is separating the flow region into two regions deforming region and non deforming region right. And then wall shear stress in this case we are having at y = 0; our coordinate system is like that. So, then y = 0 if you substitute here you get wall shear stress.

So, now from here $1-\frac{y_p}{H}$ you are getting as ϕ; that means, wherever $\frac{y_p}{H}$ is there we can write $1-\phi$ when you write it. So, here $1-\phi$, $(1-\phi)^2$ $1-\phi$; this is what we are having. And this is valid for plug like solid plug like region which is flowing you know from y_p to H region as a solid plug material ok.

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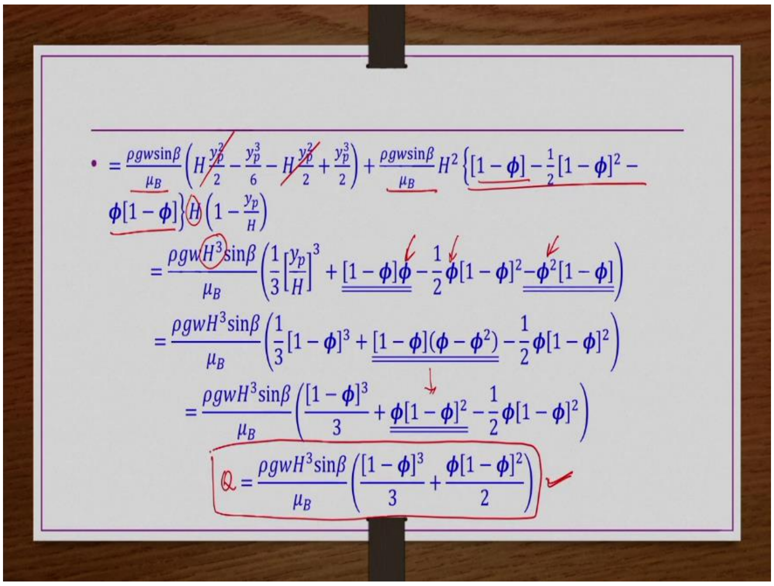

Now, volumetric flow rate we can get by integrating this v_z multiplied by w; dy w is nothing but the width of the fluid or the width of the fluid that is extended in the x direction. So, $Q = \int_0^H v_z w dy$ and then now this v_z is having two components v_z as function of y to y_p point; 0 to y_p point and then v_{zp} from y_p to H point.

So, that is the region this integration we separated out in the two parts. So, now, v_z is this one ρ g μ; $\frac{\rho g \sin\beta}{\mu_B}$ multiplied by this $1-\frac{\tau_0^B}{\mu_B}$ multiplied by y is this one. So now, this only the parts which are having y component that are kept into the integration and then remaining constant we have taken outside of the integration.

Whereas the v_{zp} is constant value, so this is all the constant v_{zp}. So, its expression is lengthy, but it is one constant value it is not varying in the y direction ok, it is one fixed value and then integration w dy, right. So, when you do the integration for this part you get $\frac{y^2}{2}$ here $\frac{y^3}{6}$ you get, because divided by 2 is there. And then here you get simply $\frac{y^2}{2}$; and then here what do you get? All these all these multiplied by simply y and then in for this case limits y_p to H.

So, other than integration what we have done? We have done in place of τ_0^B we have written ρ g $(H - y_p)$ sin β also we have written ok. So, that what we can do? $\frac{\rho g w \sin\beta}{\mu_B}$ we can take common from all the terms right. So, when we do $\frac{\rho g w \sin\beta}{\mu_B}$ you take common and then you substitute the limits.

When you substitute the limits this is what you get; from the first term and then from the second term you get or that or from the v_z point you get $\frac{\rho g w \sin\beta}{\mu_B}$ and then H^2 we are taking common. So, because all these terms are having H^2 terms are there right, it is already there H square term, so that is we are taking common.

So, the same step we have rewritten here once again, next step what we are doing? We are taking $\frac{\rho g w \sin\beta}{\mu_B}$ common from both the terms earlier we have written for the two terms separately. And then from the both the terms we are also taking H^3 as common so that within the parentheses we get these terms right.

Because here what happens? This is $-\frac{H{y_p}^2}{2}$ this is $+\frac{H{y_p}^2}{2}$ is cancelled out. So, only two terms are there $\frac{{y_p}^3}{2}$ and then this $-\frac{{y_p}^3}{6}$. So, that comes out $\frac{1}{3}{y_p}^3$ and then we are dividing by H. So, $\frac{1}{3}\left[\frac{y_p}{H}\right]^3$ we are getting from this point.

And then at this point what happens? It is already H^2 is there and this H is there. So, that we are multiplying as H^3, so there is no problem. So, whatever this parenthesis term is there that is a constant, so as it is right ok.

And then whatever $1 - \frac{y_p}{H}$ is there, so that we are writing as a ϕ, we are writing as ϕ. And then that ϕ is being multiplied by this $1 - \phi$ that ϕ term is being multiplied by all these three terms. So, that is the region here we have extra ϕ, here also we have extra ϕ, here also we have in place ϕ we are getting ϕ^2 right. So, this is all simplification step, so that you know what we do now? We combine these two terms underline terms so that we have $1-\phi$ multiplied by $\phi-\phi^2$, right.

Next step what we do from this $\phi-\phi^2$ if you take ϕ common, so you have $1-\phi$. So, ϕ multiplied by $(1-\phi\,)^2$ you get from these two steps whereas, the remaining steps we are keeping as it is. So, now, we have a ϕ multiplied by $(1-\phi\,)^2$ and then $-\frac{\phi}{2}$ multiplied by $(1-\phi\,)^2$. So, then you get $+\frac{1}{2}\phi$ multiplied by $(1-\phi\,)^2$. So, that is your; that is nothing but this value right.

So, the final expression for the volumetric flow rate for the case of Bingham plastic fluid flowing down an inclined surface making β angle with the horizontal axis. So, then the volumetric flow rate you can have this expression, right.

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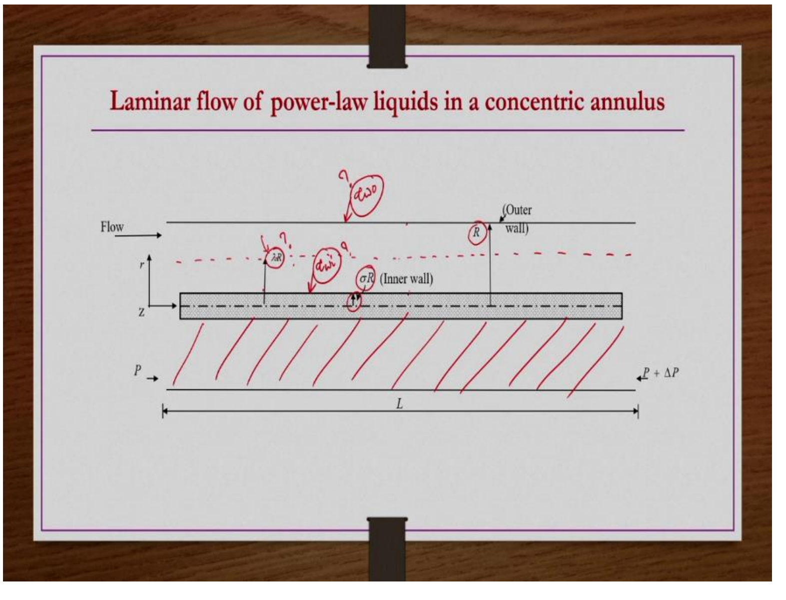

So, now we take another geometry; the another geometry that we are taking is concentric annulus ok; laminar flow of power law liquids in concentric annulus. So, this is the geometry, what we have? We have two cylinders, so one cylinder outer cylinder is having the radius R, another inner cylinder having the radius σ R right. So, these between these two cylinder there is a gap, annulus gap is there within that annulus gap what happened, the fluid is flowing the fluid is flowing.

Now, what we say? We cannot say that the velocity profile is going to be like a you know parabolic or linear directly from here, why? Because now there are two walls right. So, earlier what we have taken whether it is pipe flow or you know inclined surface or between infinite parallel plates there is a central axis which we are taking as a y = 0 and then that central axis is not the solid wall solid.

Now, here you know the flow is taking place between two solid walls right. So, then at this solid wall let us say you have the shear stress τ_{w_i} we cannot say its 0 right, and then at this surface you have τ_{w_0} let us say. So, in the previous problems till now whatever we have studied? We have seen only one wall shear stress cases whichever is the geometry we have taken.

Now here there are two wall shear stresses are there between that two we are taking, we are having the fluid flowing down flowing. Because between these two walls the fluid is flowing and then two walls are having two different wall shear stresses right. So, we cannot say one is 0 another one is maximum. So, then between these two y values the material is you know are the shear stresses you know having linear profile that we cannot say now right.

So, now, that is a one problem another problem is that the velocity also, now the velocity distribution if you wanted to have for this case let us say. So, how do you have? Like you know let us say it is a 0 velocity here, it is a 0 velocity here, can we say like this parabolic will be there or can we say you know like this or can we say the flattened flow like this, we do not know we cannot so; we are not sure. Why? Because either of these I mean whatever it is parabolic or anything we know the location of the maximum velocity in general for the previous geometries. But in this geometry we do not know at which location the maximum velocity is existing.

So, that is the problem; that is the difference compared to the other geometries that we have taken whether pipe flow or the concentric or parallel plates whatever we have taken. So, compared to those geometries here the problem is we do not know at which location we are having the maximum velocity, because that also is required. If you know that region, so that region you can take that is or that point you can take it as the point at which the shear stress is 0 and then accordingly you can do some analysis.

So, now here what happened? So, we are assuming λ R is the position from the central axis. So, because this the radial direction we are taking vertically and then z direction we are taking horizontal right. So, at certain value of R = λ R, what happens? The velocity is having maximum velocity that is the point that point we do not know, we are assuming let us say at R = λ R maximum velocity is existing.

So, now here this τ_{w_i} and then τ_{w_0}they are wall shear stress and they are not equal to each other, they are not equal to each other they are different from each other. So obviously, it is not going to be existing at the middle point, if they are equal to each other then we can say somehow it may be at the middle point between these two regions. So, τ_{w_i} is very much different from the τ_{w_0} right.

So, now what happens? The velocity profile here towards the wall maybe you may be having like this right, and then towards the outer wall you may be having like this. So, this part let us say v_{z_0}this part let us say v_{z_i}, so that is possible right.

So, symmetry between the within the annulus region symmetry is not possible that is what I mean to say, because we do not have a maximum velocity location at the center, we cannot say that one. Because the two points or the two regions two walls between which the flow is taking place those two walls the shear stress are different shear stress wall it is having right.

So now, how to find out that velocity? So, we first if you want to find out the velocity this you have to find out, this v_z v_{z_0} both of them you have to find out. And then before that what is this λ R that location also you have to find out. If you do not know that location there is no you know proper solution for the reliable solution you cannot get it ok right. So, that is what we are going to do.

We are going to follow the same approach that we have been following, but the constraints are slightly different here slightly complicated. So, within these constraints we are going to; we are going to solve the problem right. So, this whatever v_{z_i} v_{z_0} that I have shown that is a just for a understanding only I have shown, it is not that this kind of profile we are getting; we may get something like this also.

May be possible that depends on the values of $\tau_{w_i}\tau_{w_0}$ in and all that. So, what is that we do not know. But however, since we are taking the laminar flow so such complicated flows flow geometries, flow distributions may not be there right. So, now what we do? We write the constraints of the problem right then simplify the continuity and then momentum equation, so that we have some expression for the shear stress. And then we applied either of the constraints $\tau_{w_i}\tau_{w_0}$ in order to find out the λ R and all that we are going to do here.

Remember, what are the known? R is known, σR is known here and then ΔP is known through which because of which the flow is taking place through this annulus. Only three things we know we do not know what is τ_{w_0} we cannot calculate, τ_{w_i} also we do not know.

What is this λ R also we do not know. We take a generalized case that these are $\tau_{w_i}\tau_{w_0}$ are very different from each other, so that we can have a generalized solution. If you take both of them are equal, so then you can take you can get the solution for only one case of λ R which is maybe between these two values of R and σ R right.

So, if you take a different values of τ_{w_i} τ_{w_0}, so then you can solve a generalized solution; we can obtain a generalized solution which may be valued for any λ R value ok.

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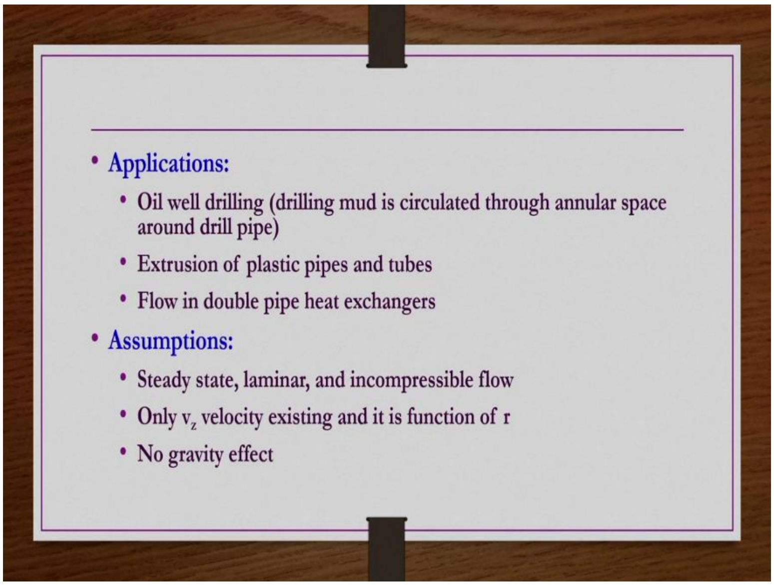

So, applications first we see for this geometry where we in general have such kind of a geometries.

One major application we see in oil well drilling mud circulated through the annulus space around the drill pipe in general. Then extrusion of plastic, pipes and tubes in polymeric industries. And then flow in double pipe heat exchangers in general that is true. Any of the chemical processing industries we find double pipe heat exchangers, there also we have the flow through annulus geometry.

So, then assumptions steady state, laminar, incompressible flow standard assumptions we are having. And then only v_z is existing and it is function of r, here also no gravity effect. Remember that annulus space whatever we said that is cylindrical annulus ok. So obviously, continuity and then momentum equations we have to solve in cylindrical coordinates.

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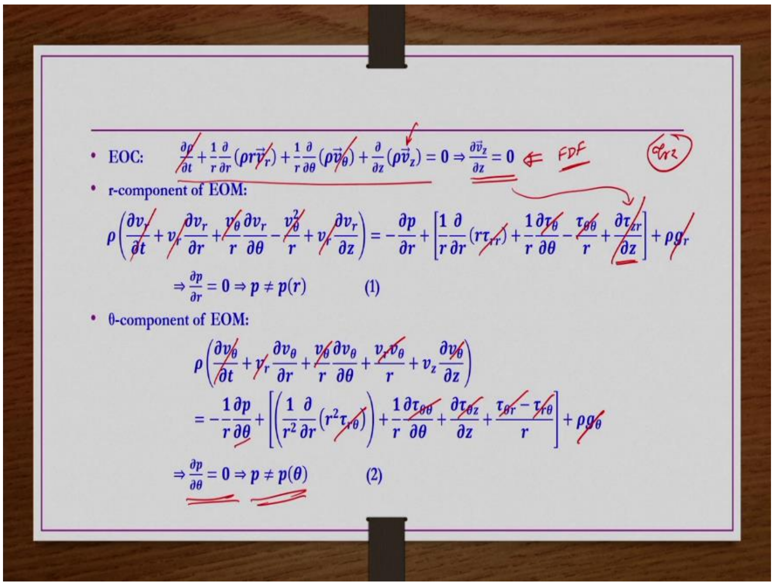

So, continuity equation if you have in cylindrical coordinates that is this one right. So, steady state this term is 0, v_r is not existing, v θ is not existing, v_z is existing and it is function of r only that way also this term is canceled out. But however, we can understand this one other way $\frac{\partial v_z}{\partial z} = 0$, because if $\frac{\partial v_z}{\partial z}$ is existing, so then we can say fully developed flow conditions are existing.

If we have these conditions this may be; if you have this condition this may be useful for solving the momentum equation subsequently, so that way we can take it ok. So, r component of momentum equation in is given here. So, steady state term this is 0, v_r is not existing, v_θ is not existing, v_θ is not existing, v_r is not existing.

So, left hand side all terms, altogether all the terms are 0. Right hand side pressure we do not know anything, so only τ_{rz} shear stress component is existing other shear stress components are not existing. So, this is 0, this is 0, this is 0, τ_{rz} is there. But because of the, so this term is 0 because of a fully developed flow which we understand by simplifying the continuity equation.

Because from the flow geometry we do not understand whether the flow is really fully developed or not, so that we understand here. So, if you have the fully developed flow $\frac{\partial r}{\partial z}$ any derivative in the flow direction of any flow property is 0 only for the vector. So, that is this entire term is 0. Since, horizontal concentric cylinders are there, so then there is no gravity term.

So, what we understand? $\frac{\partial p}{\partial r} = 0$; that means, pressure is not function of r that is what we understand from simplification of r component of equation of motion. So, then θ component of equation of motion if you right here this is what it.

So now, here steady state. So, this term is 0, v_r is not there, v_θ is not there, v_r v_θ both are not existing, v_θ is not existing. So, again left hand side all terms altogether are 0, pressure we do not know anything in general. So, next is this component of shear stress is not existing, this is also not existing, this is also not existing, these two identically equal, so they for laminar flow. So, that way also 0 otherwise, individually also these two stress components are not existing for this term, for this geometry ok.

Gravity is not there because of the horizontal geometry. And then, what we understand now here? $\frac{\partial p}{\partial \theta} = 0$. That means, here in this case also the pressure is not function of θ. So, what we understand? Pressure is not function of both r and θ it is function of z only as we seen from the schematic also.

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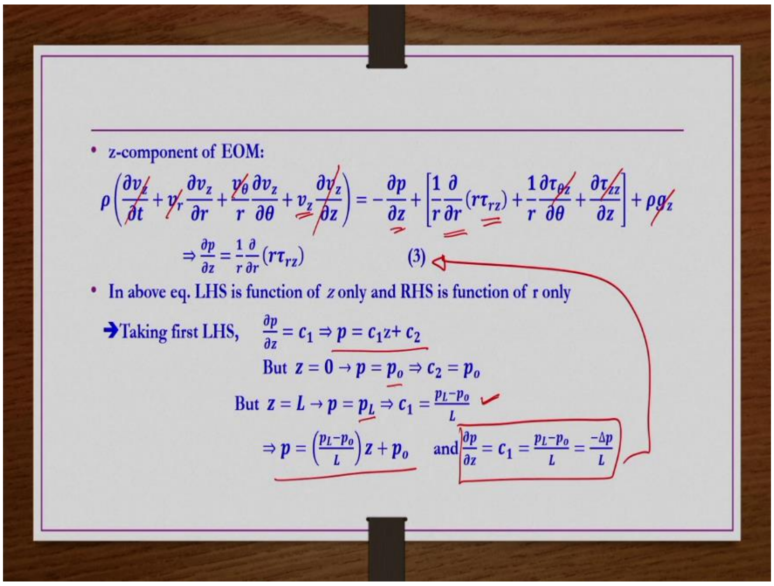

So, z component of momentum equation written here. So, because of the steady state this term is 0, v_r is 0, v_θ is 0, v_z is existing but v_z is function of r only and then from the continuity equation we understand $\frac{\partial v_z}{\partial z}$ is 0 right. Pressure we do not know, so let us keep it as it is.

τ_{rz} component of shear stress is existing and it is function of r. So, then we cannot cancel out this one $\tau_{\theta z}$ is 0, τ_{zz} is also 0 and then there is no gravity because of horizontal geometry. So, only two terms are remaining in the right hand side. So, that we can write $\frac{\partial p}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})$.

So, now here if you wanted to solve this equation you have to understand the dependency independent dependent variable. So now, left hand side terms it is function of z only it is not function of r whereas, the right hand side term it is function of r only it is not function of z. So, then what we can say? We can individually integrate to some constants right, so that it will be is we can integrate them right.

So, now taking only LHS that is $\frac{\partial p}{\partial z} = c_1$ constant c_1. So, then we get p = c_1 z + c_2 at z = 0 let us say pressure is p_0 then c_2 is p_0. At z = L that is the length of the cylinder if the pressure is p_L then substitute this one here and then apply $c_2 = p_0$. And then simplify you get $c_1 = \frac{p_L - p_0}{L}$ or $\left(\frac{-\Delta p}{L}\right)$ you can write.

So, pressure distribution you get this one by substituting c_1 c_2 in this equation. And then $\frac{\partial p}{\partial z}$ that is c_1 you get it at $\left(\frac{-\Delta p}{L}\right)$ which is nothing but Δp is nothing but $p_0 - p_L$. So, $\left(\frac{-\Delta p}{L}\right)$ that is what we are having.

So, now this you this you can make use in equation number 3 by taking RHS side term $= \left(\frac{-\Delta p}{L}\right)$ right. So that you do know here.

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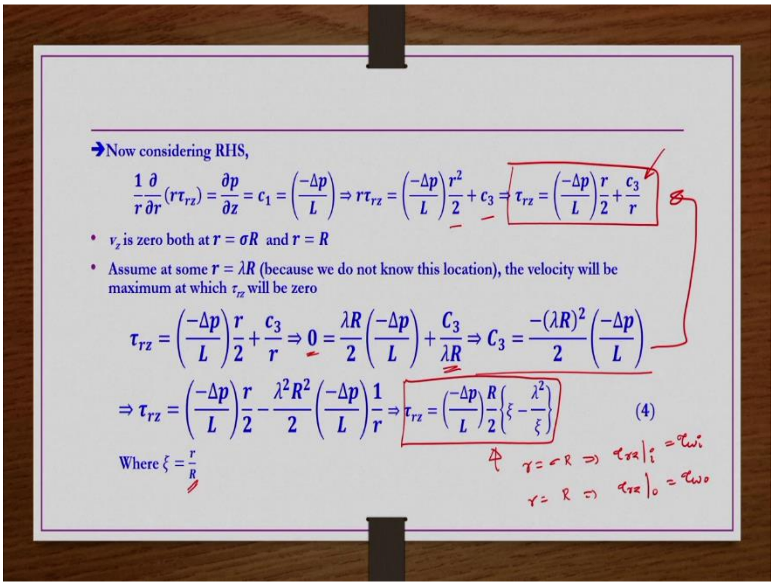

So, RHS $= \frac{\partial p}{\partial z}$ which is nothing but c_1 and then that is nothing but $\left(\frac{-\Delta p}{L}\right)$. So, now you take this r on the left hand side to the other side and then integrate then you get $r\tau_{rz} = \left(\frac{-\Delta p}{L}\right)\frac{r^2}{2} + c_3$. Then whatever the another r is there in the left hand side that also if you bring it to the right hand side you will get $r\tau_{rz} = \left(\frac{-\Delta p}{L}\right)\frac{r^2}{2} + \frac{c_3}{r}$ right.

Now this c_3 we have find out we have to find out c_3 by applying the boundary condition for the shear stress. Shear stress is having some wall shear stress at r = σ R that is inner wall. And then it is having some other wall shear stress value at r = R that is outer wall.

So, we cannot use those two boundary conditions. But we are having maximum velocity at certain location r = λ R that we are assuming; we are assuming we do not know the location, but later on we find it out ok. So, at r = λ R we are having you know maximum velocity. So, that at that point at r = λ R τ_{rz} is going to be 0.

So, in this equation if you substitute r = λ R then τ_{rz} would be 0. Because at r = λ R is the location at which we are assuming the velocity is maximum we do not know we are going to find out anyway right. So, now this now you simplify you get this equation c_3 like. This c_3 you substitute here and then rearrange the terms. So, that you get $\tau_{rz} = \left(\frac{-\Delta p}{L}\right)\frac{R}{2}\left\{\xi - \frac{\lambda^2}{\xi}\right\}$. Where ξ is nothing but $\frac{r}{R}$, $\frac{r}{R}$ we are writing as ξ right.

Now we have this τ_{rz} expression right, τ_{rz} expression; now in this expression if you substitute r = σ R then you will get τ_{rz} at inner wall that is τ_{w_i}. And then if you substitute r = R then you get τ_{rz} at outer wall which is nothing but τ_{w_0}.

Both of them are having certain values they are not equals to each other. They may be equal to each other for given pressure drop and all that, right. So, but they are you know mostly not equal to each other in most of the cases that is the reason we are solving this problem right.

So, now that is what about the shear stress, now we are having the equation for the shear stress. So, our next step is that we have to obtain the velocity profile by having the equation for shear stress as per the rheology of the fluid. So, we are taking power law fluid.

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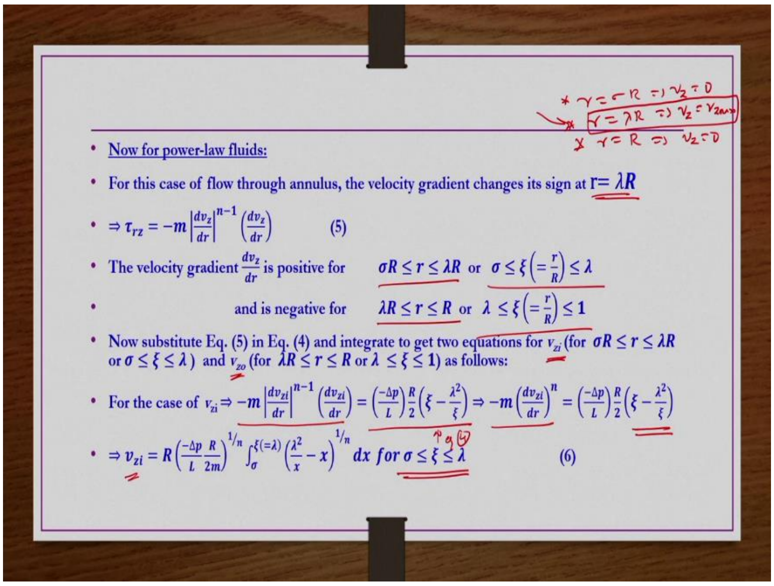

So, as per the geometry what we have? From r = σ R to r = λ R the velocity gradient you know it is a positive. Because at r = σ R we are having $v_z = 0$, at r = λ R $v_z = v_{z\,max}$ right.

Because of the you know this is the no slip condition this is the location at which maximum velocity is there.

So, as r is increasing from σ R to λ R velocity is increasing. So, towards the inner wall whatever the velocity profile is there. So, that is having the positive gradients; that is having the positive gradients. So, then you know that we have to take right, but at r = R what happens? You know again v_z is 0.

So, now this because at the upper wall outer wall again because of the no slip it is a; it is 0. So, from 0 to some maximum velocity it is increasing with y, but after that maximum velocity is again decreasing as y increasing right. So, then the velocity gradient is changing its say at r = λ R that is location and the v at which v_z is maximum right.

So, this is required to understand, because now though it is one power law fluids within single one power law fluid the flow is having two regions, two different velocity profiles are possible. Now in this case because of the two different wall shear stress values are having ok. So, that individually we have to solve or we have to solve them simultaneously so that this condition is maintained at which the velocity gradient is changing the sign that has to be maintained.

So, the solving of the equation to get the velocity distribution should be such way that both the velocity distribution towards the inner wall and towards the outer wall they should be satisfied or should become equal to each other at r = λ R and then that value should be maximum. So, for the power law fluids we have taking, so $\tau_{rz} = -m\frac{dv_z}{dr}\left|\frac{dv_z}{dr}\right|^{n-1}$.

So, velocity gradient is positive in this region σ R to λ R region and then negative λ R to R region. So, in terms of a ξ if you write these are the two regions. So, first what we do? Now v_{zi} we are calling it towards the inner wall whatever the velocity that is σ R to λ R region. And then v_{z_0} we are calling the outer wall towards the outer value; towards the outer wall that is r = λ R to r = R right.

So, now for the case of v_{z_i}, so velocity gradient is positive. So, then we can have minus this modulus we can remove directly as it is. So, then we have $\left(-\frac{dv_z}{dr}\right)^n$ and then that is equals to nothing but $\left(\frac{-\Delta p}{L}\right)\frac{r}{2}\left\{\xi - \frac{\lambda^2}{\xi}\right\}$ from the shear stress expression. This is what shear stress we got from equation number 4; equation number 4 this is what we have right.

So, now what we do this m before taking m to the other side, so this minus we take to the other side, so that this term we can write the when you take this minus to the other side this term we can write $\frac{\lambda^2}{\xi} - \xi$. And then after that this m also we take to the right hand side and then this n also we have to we will take to the right hand side. So, that we have $\left(\frac{-\Delta p}{L}\frac{R}{2m}\right)^{1/n} \int_{\sigma}^{\xi(=\lambda)} \left(\frac{\lambda^2}{x} - x\right)^{1/n} dx$ this is v_{z_i}.

We are not solving it we are writing where x is the dummy variable similar to ξ here right and then this is valid for this region only; ξ = σ to λ right.

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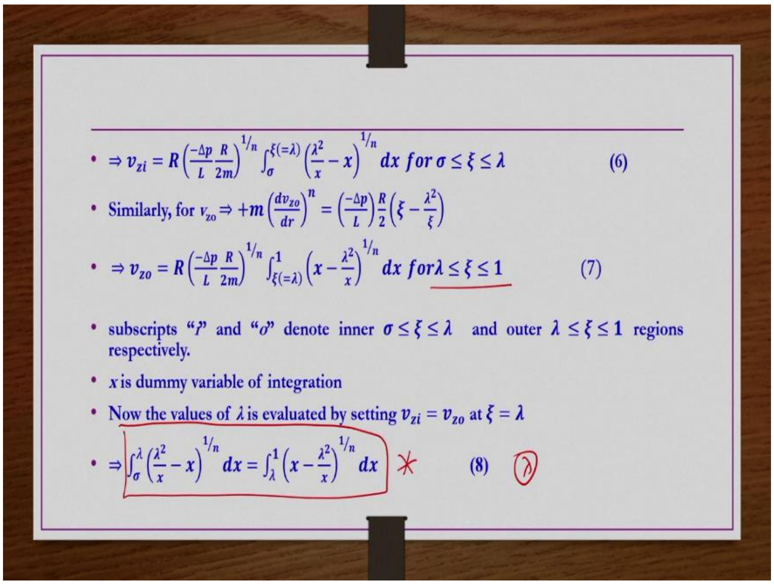

Now, similarly for the outer region v_{z_0} is nothing but $+\left(\frac{dv_z}{dr}\right)^n$. And then that should be equals to the τ_{rz} that is nothing but $\left(\frac{-\Delta p}{L}\right)\frac{R}{2}\left\{\xi - \frac{\lambda^2}{\xi}\right\}$. So, now, this m you take to the right hand side and then this n also you take to the right hand side. So, then right hand side term all the terms should be getting whole power 1/n.

Then if you integrate v_{z_0} you get R $\left(\frac{-\Delta p}{L}\frac{R}{2m}\right)^{1/n} \int_{\xi(=\lambda)}^{1} \left(x - \frac{\lambda^2}{x}\right)^{1/n} dx$. So, x is a dummy variable and this is now valid between ξ = λ to 1 right.

So, now the solution of this; why we are not doing the complete integration? Because we have to understand or we have to find out what is this λ; λ we do not know right. And then at λ this both the two values at ξ = λ both v_{z_i} and then v_{z_0} or v_{z_0} = each other right. Then we have this expression.

So, now, this condition has to be maintained such a way that you know for a given λ value; for a given λ value this condition has to be maintained. Then only that whatever the λ value is there that correct that is correct, otherwise it is not correct.

Or this integration we have to solve simultaneously, so that you get that λ value which is analytically not possible to do. So, numerically one has to do. That is the region we have not done the integration also. In addition to this region that $v_{z_i} = v_{z_0}$ should be maintain at ξ = λ. So, that is the region. Another region that we cannot integrate this one analytically. So, we have to do the numerical integration for this.

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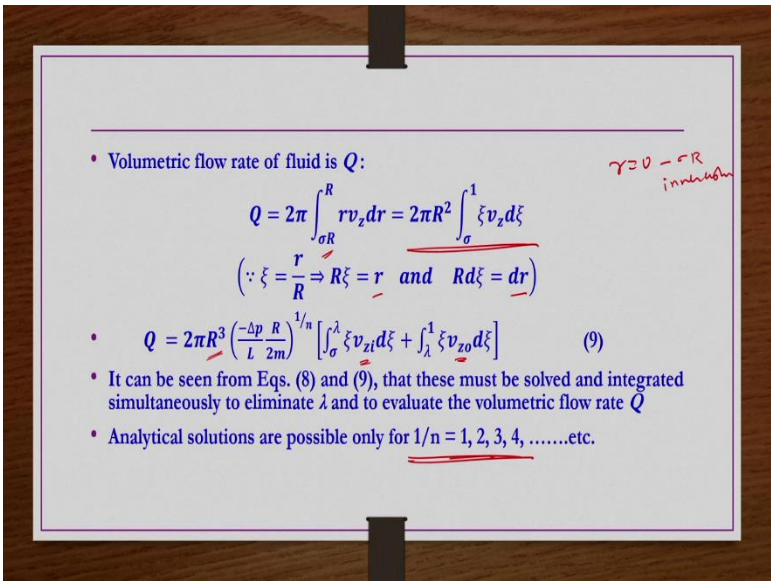

So, now volumetric flow rate then $Q = 2\pi \int_{\sigma R}^{R} r v_z dr$ it is not 0 to R because 0 to R = 0 to σ R is an inner wall inner cylinder. So, where there is no flow, so that is the region σ R to R is the limiting condition.

So, the in terms of ξ if you write it would be σ to 1 integral integration limits should be σ to 1 because we have ξ = R/R. So, in places r we can write R ξ in place of a dr we can write R d ξ. So, when we write here this one you get it right.

So, now $2\pi R^2$ then multiplied by $R\left(\frac{-\Delta p}{L}\frac{R}{2m}\right)^{1/n}$. So, R^2 R multiplied, so R^3 we are getting. And then $\int_{\sigma}^{\lambda} \xi v_{z_i} d\xi + \int_{\lambda}^{1} \xi v_{z_0} d\xi$. This is what we have, right.

Since v_{z_i} v_{z_0} are not known, so we cannot integrate this one as well, we have to depend on the numerical integration. So, analytical solution however, are possible if 1/n values are you know you know integers like 1, 2, 3, 4 etcetera.

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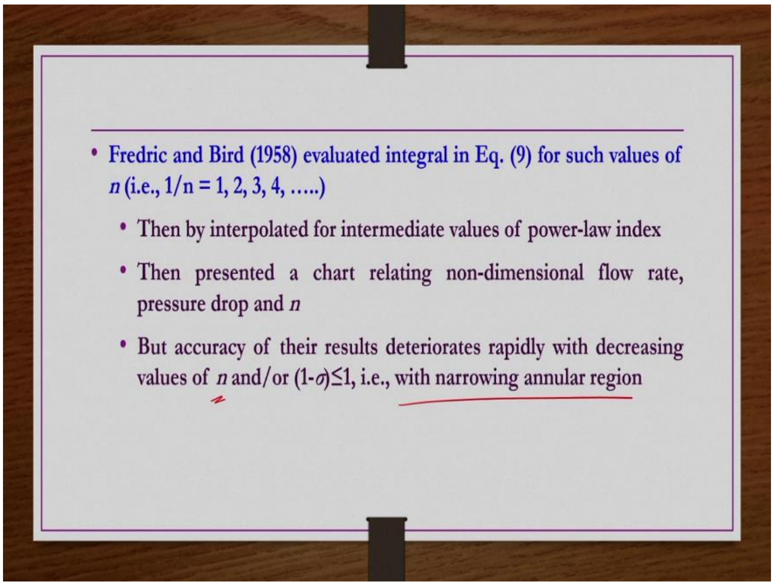

So, for that reason people have solved numerically, so some of them we will see. Fredric and Bird evaluated that integral previous equation for such values of 1 by n like 1, 2, 3, 4. And then they have interpolated for the other values of n; other values of 1/n so that wide spectrum of n values can be covered.

Then they presented a chart relating non-dimensional flow rate pressure drop and n value. But accuracy of their results deteriorate rapidly with decreasing values of n and or the gap annular region is becoming narrow either of the case the accuracy is less compare to the experimental results.

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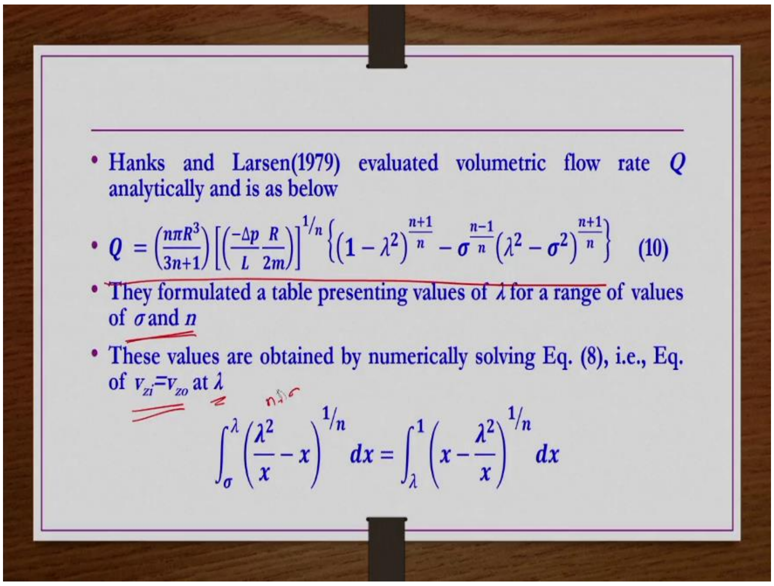

So that, and then Hanks and Larson evaluated volumetric flow rate analytically and is given by this expression, we are not going to derive this equation. So, this is the analytically the Hanks and Larson have obtained volumetric flow rate for this geometry, for the power law fluids right.

But however, in this case also λ is known not known. So, that λ as function of σ and n they obtained that they obtained by numerically. So, it is not completely analytical, so some part is done by numerical part. So, numerically they obtained you know λ values by taking $v_{z_i} = v_{z_0}$ or by maintaining $v_{z_i} = v_{z_0}$ conditions for different n and σ values.

For different n and σ values they have maintained $v_{z_i} = v_{z_0}$ are you know by maintaining that one they found the λ values right.

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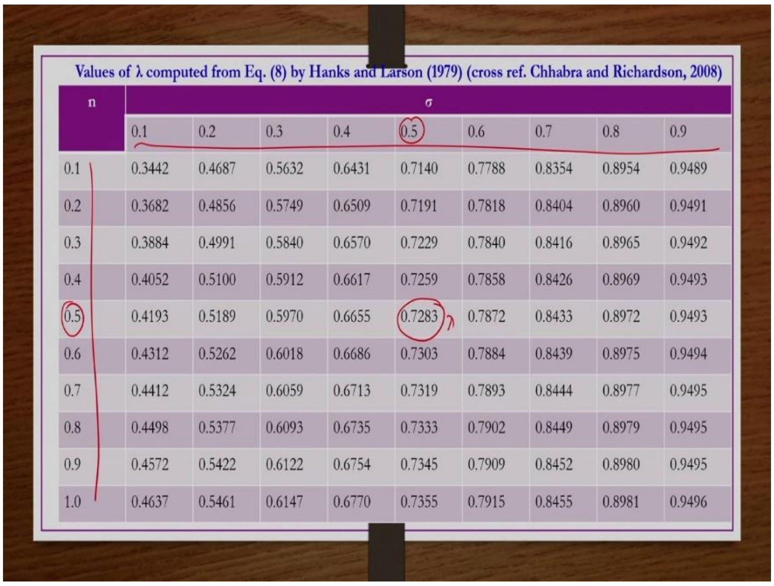

Values of λ computed from Eq. (8) by Hanks and Larson (1979) (cross ref. Chhabra and Richardson, 2008)

n	σ								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	0.3442	0.4687	0.5632	0.6431	0.7140	0.7788	0.8354	0.8954	0.9489
0.2	0.3682	0.4856	0.5749	0.6509	0.7191	0.7818	0.8404	0.8960	0.9491
0.3	0.3884	0.4991	0.5840	0.6570	0.7229	0.7840	0.8416	0.8965	0.9492
0.4	0.4052	0.5100	0.5912	0.6617	0.7259	0.7858	0.8426	0.8969	0.9493
0.5	0.4193	0.5189	0.5970	0.6655	0.7283	0.7872	0.8433	0.8972	0.9493
0.6	0.4312	0.5262	0.6018	0.6686	0.7303	0.7884	0.8439	0.8975	0.9494
0.7	0.4412	0.5324	0.6059	0.6713	0.7319	0.7893	0.8444	0.8977	0.9495
0.8	0.4498	0.5377	0.6093	0.6735	0.7333	0.7902	0.8449	0.8979	0.9495
0.9	0.4572	0.5422	0.6122	0.6754	0.7345	0.7909	0.8452	0.8980	0.9495
1.0	0.4637	0.5461	0.6147	0.6770	0.7355	0.7915	0.8455	0.8981	0.9496

So, that λ values are tabulated here right. For different n values and then different σ values given here you know the corresponding λ values are given here right. Let us say if n = 0.5 and then σ is also 0.5. So, the λ value is 0.7283 right.

So, now this is the laminar flow through concentric annulus right. So, if you have complicated geometry, so then analytical solution completely is not possible we have to depends to some extent on numerical integrations etcetera, for example this problem.

So, now, before winding up today's class what we are going to do? We are going to take an example problem.

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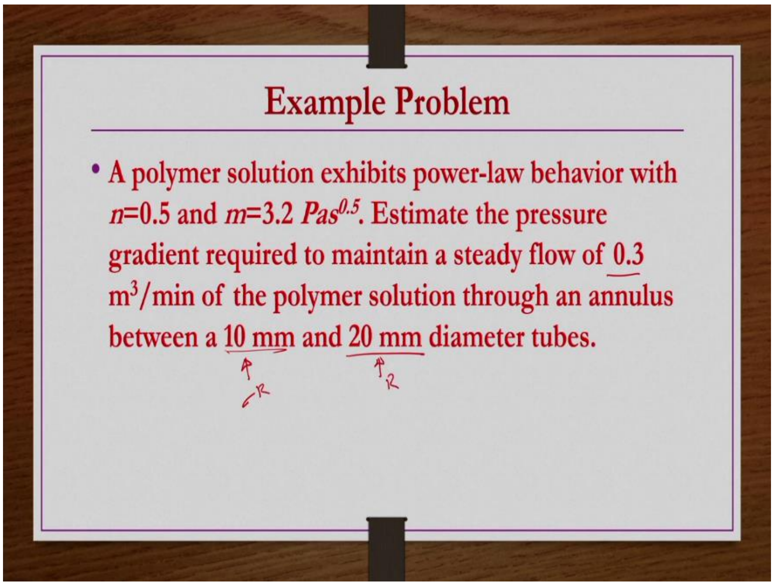

So, a polymer solution exhibits power-law behavior with n = 0.5 m is equals to 3.2 pascal second power n. The estimate we have to estimate the pressure gradient required to maintain steady flow of 0.3 meter cube per minute, so Q expression is given. So, the annulus between 10 mm and 20 mm diameter tubes, so this is σ R this is R ok

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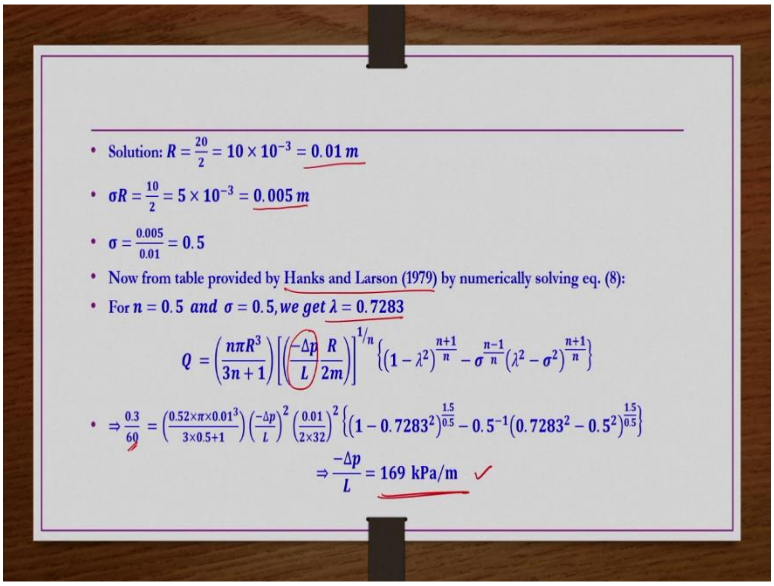

So, now we see the solution diameter of the outer wall or outer tube is given that is 20 mm, so its radius would be 10 mm, so that is 0.01 meters. Similarly diameter of inner tube is given as 10 mm, so its radius would be a 5 mm. So, σ R would be 0.005 m because diameter of inner tube is given as 10 mm. So, its radius would be 5 mm in terms of meter it is 5 into 10^{-3} meters.

So, now if you do σ R/R you will get the σ value that is 0.005 by 0.1 is nothing but 0.5 right. And then n is given 0.5 now σ we got 0.5, so both we got 0.5 0.5. So, from the tables that has been given by Hanks and Larson that is for different σ values and n values λ corresponding λ values are given. So, that λ if n is 0.5 and then σ is 0.5 from that table λ is 0.7283. Just previous couple of slides before we have that table right.

Then their volumetric flow rate expression is given by this. In this equation now everything is known including the λ except the $\left(\frac{-\Delta p}{L}\right)$. So, that you can find out by substituting, Q is also given as a 0.3 meter cube per minute. So, divided by 60 we have, so that we have meter cube per second.

So, right hand side you substitute all the values for R, n, m etcetera, λ etcetera, σ etcetera then only thing unknown is $\left(\frac{-\Delta p}{L}\right)$. So, that you get 169 kilo pascal per minute; upon the simplification you can get this values ok.

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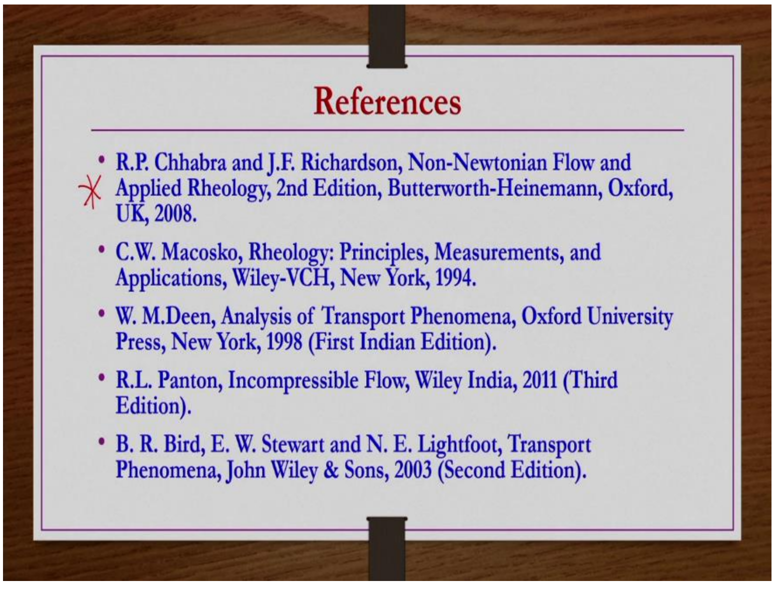

So, the references: The entire lecture is prepared from this reference book by Chhabra and Richardson, other references are provided here.

Thank you.