Transport Phenomena of Non-Newtonian Fluids
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Lecture - 20
Laminar flow of GNFs between Parallel Plates and along Inclined Surface

Welcome to the MOOCs course Transport Phenomena of Non-Newtonian Fluids. The title of today's lecture is Laminar Flow of Generalized Newtonian Fluids between Parallel Plates and along Inclined Surfaces.

Till now what we have seen? We have seen only pipe geometry for the circular pipe geometry; infinitely long circular pipe geometry when L/D is very large and then laminar flow conditions are existing; how to obtain the volumetric flow rate, how to obtain the velocity profile, average velocity, maximum velocity, volumetric flow rate, friction factor etcetera those things we have seen for the laminar flow conditions.

When the geometry is pipe or circular cylinder, then we have also seen for the same circular cylinder or pipe geometry if the flow is under the transition or you know turbulent flow conditions; how to get the velocity profiles, how to obtain the friction factors.

If it is possible to obtain analytically how to get them, if not what are the correlations or empirical correlations existing for these friction factors etcetera. Those things we have seen even for the transition and turbulent flow conditions, but when the geometry is the pipe geometry. So now, what we do? This and then next couple of lectures we will be taking different geometries, other than the pipe geometries. Let us say in today's lecture we are going to take two geometries; one is the parallel plates.

If a fluid is confined between two infinitely long parallel plates, then if the flow is under laminar conditions how we can obtain the velocity profile for a given generalized Newtonian fluid. Generalized Newtonian fluid in the sense, any time independent non-Newtonian fluids which can reduce to the Newtonian behavior under certain limiting conditions. That we know.

So, that is Power-law fluids, Bingham fluids, etcetera, or visco-plastic fluids generalized one. So, these kind of fluids we are going to study now for the flow between parallel plates; infinite parallel plates. Another geometry that we are going to take is inclined surface.

If you have an inclined surface and then along that inclined surface if the material is flowing down because of the gravity, so then how to obtain the velocity profile for that geometry and then how to obtain the volumetric flow rate of a certain fluid flowing down the inclined surface. That is what we are going to see, and then we wind up the class which with an example problem as well.

So, the methodology is same, whatever we have done for the case of pipe flow; so that is we have to draw the geometry and then make out or list out all the assumptions or restrictions, if we are solving the problem for under laminar flow conditions. And then simplify the continuity equation, momentum equation so that you get a relation between shear stress and the pressure drop or whatever the driving force that is causing for the flow to occur, so that relation we have to obtain.

Once we have that relation for the shear stress then getting the velocity profile, volumetric flow rate is etcetera are just a kind of a mathematical simplifications. So, that is what we are going to do now, for the case of an infinite parallel plates; flow between infinite parallel plates.

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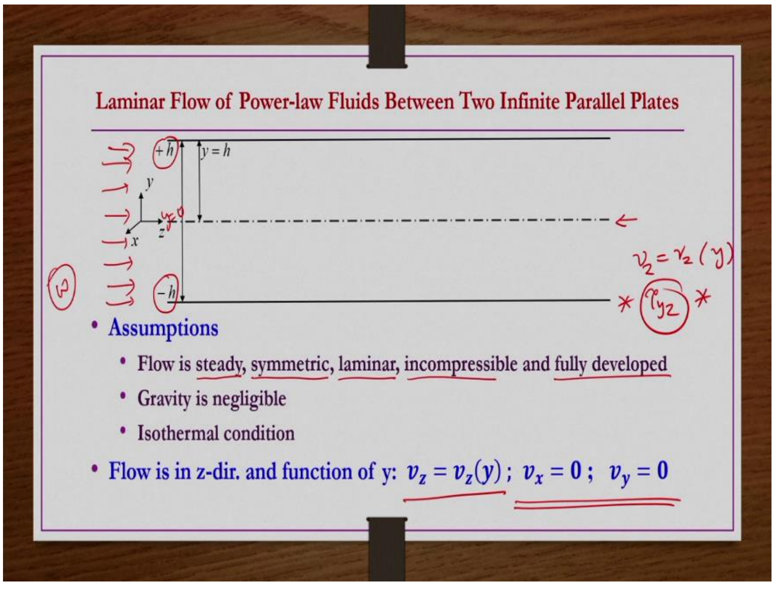

So, first we take a power-law fluid for this geometry. We are going to take Bingham plastic fluids as well. So, but whether it is power-law fluid or Bingham plastic fluids as long as the flow is the laminar symmetric one dimensional flow; then what we can have? We can have a kind of an analysis simplification. Simplification of a momentum equations up to that point it is going to be same, right.

So, because up to getting the relation for the shear stress we are not incorporating any information about the rheology of the fluids. So, till that point whether it is a power-law fluid or Bingham plastic fluids the methodology is going to be same.

So, schematically if you see you have a two infinite parallel plates which are separated at a distance 2 h. So, the center axis you know whatever is there that we are taking $y = 0$. So, the geometry we are taking $y = 0$ at the center of the two these two infinite parallel plates ok. So, that is the central axis we are taking.

So now, between these plates, these plates actually in general we in real-life application these plates are infinitely long, but the gap between these two plates is in general very small. It may not be very very small that it is negligible, that you take the viscous dissipation etcetera also into the consideration. In fact, those kinds of things would also be there.

But however, you know the gap between these two is in general very very small so that let us say in the z axis; in the z direction the length of these parallel plates if it is L, so L/h if you do its going to be very long; very large way. If you take L/h value for this kind of geometry in general you are going to have that L/ h very large value.

When you have this L/h kind of values or L/D kind of values that we have in pipe geometry, so here L/h if you have a very large value of L/h then you can say that flow is a kind of a fully developed, ok. So, this is the geometry. So now, the material whatever the fluid that we are considering here.

So, let us say now here power-law fluid that is flowing between these infinite parallel plates and then that flow is because of the pressure gradient, ok. Obviously, because of the pressure difference only the flow is taking place otherwise the, however the flow will take place because this is a horizontal plate so gravity is also not there.

So, some external force has to be there so that to flow takes place, so then that external force is because of the pressure difference that we are generating or maintaining in order to have a required flow rate of this fluid whatever we are confining between these two plates ok. So, this is geometrically this is what we are having now. So now, what we do? We have to list out the assumptions; we have to list out the assumptions right.

Before seeing the assumptions you know if it is a laminar one-dimensional flow. So, what we see? The flow is dominated in which direction? It is dominated in the z direction. So, we are going to have only v_z velocity right. And then that is, that flow is confined between y = - h to y = + h, so whatever the v_z component of velocity is there that is varying in the y direction.

So, v_z as function of y that is what we are going to have in this case whereas, the v_x and then v_y are going to be negligibly small compared to the v_z. So, then v_x v_y we can cancel out or we do not need to consider because v_z is dominating, compared to v_z magnitude wise v_x, v_y are going to be negligible we can neglect them.

And then what are these stress component are going to exist here? So, there are nine stress components any flow field if you take in general. So, out of which here in this case since the flow is in the z direction and then that flow is varying in the y direction we are going to have only τ_{yz} component of the shear stress. Rest all other we are not going to have.

Since the material is a time independent non-Newtonian fluid and there are no elasticity, so normal stresses are altogether not existing. And then out of the other τ_{xz} τ_{zx} etcetera all those components are you know negligible. So, what we are going to have? We are going to have only τ_{yz}.

So, what we have to do? We have to when we do solve this problem, the primary aim is to get what is this expression for τ_{yz}. In this case τ_{yz}, in the case of pipe flow it was τ_{rz}. If you have the other geometry like inclined surface we are going to see, so there you may be having τ_{xz} or something other geometry depending on the geometry and then coordinate system how you selected.

So, if you have that information about the shear stress and then after that everything is you know straightforward mathematical simplification to get the velocity profile, to get the you know volumetric flow rate etcetera or even friction factor, right.

So, the primary thing that we are going to; we are doing in this kind of problem solving first we are getting the expression for τ_{yz} or shear stress x. Whatever the shear stress distribution is there that expression we are going to get in general first. How we get that one? Simply by simplifying the momentum equation, right.

Simply by applying the constraints of the problem to the momentum equations generalized momentum equations, then after applying those simplifications or restrictions to these momentum equations we will be having an expression for a shear stress, upon solving that expression for shear stress you will be getting the velocity profile.

So, for this problem what are the assumptions? Flow is steady, obviously the under those limitations only we are studying and then laminar so that; and then symmetric. Laminar flow conditions we are taking, right. So, symmetric in the sense whatever the flow distribution is there between y = 0 to y = + h, the same distribution we are going to have between y = 0 to y = - h, ok.

The fluid is incompressible and then flow is fully developed, because L/h is a very large or the length wise these parallel plates are very long, the gap between these two parallel plates is very small. So, because of that one we can have a fully developed flow conditions under as a kind of valid condition, ok.

Three-dimensional way if you see, so then width of the plate in the x direction that is going to be w. So, that is going to be useful when you do the volumetric flow calculations. So that is what now, yeah.

So, now another assumption is that it is a; rather assumption reality we are having horizontal plates, so then gravity is negligible. Then isothermal conditions; there are no reactions, there are no mass transfer conditions, etcetera standard things and then flow is dominated in the z direction. So, v_z component is existing that v_z component of velocity is varying in the y direction. So, v_z is function of y that is only existing v_x v_y are going to be very small compared to the v_z. So, then we can take them as 0.

So, now, having enough understanding about the geometry and then restrictions of the problem what we can do? We can go to the simplifications of you know momentum and then, conservation of mass and conservation of a momentum equations.

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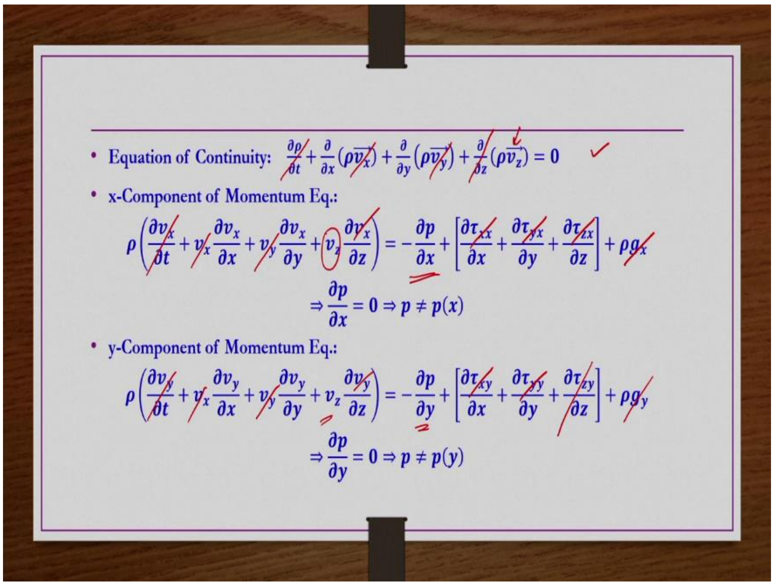

So, we are going to simplify the continuity equation right, then we are going to simplify the momentum equations. So, the geometries in Cartesian coordinates. So, the continuity equation in Cartesian coordinate is given here. So, this is the continuity equation that we have already derived in the Cartesian coordinate. So, it is an incompressible as well as steady flow.

So, then first term is 0, v_x is negligible, v_y is negligible and then flow is fully developed flow; v_z is existing, but flow is fully developed. If the flow is fully developed, that means any variations in the flow properties along the flow direction would be negligible. So, $\frac{\partial v_z}{\partial z} = 0$, so then continuity is satisfied ok.

So, what does it mean by? Whatever the assumption that we have taken or you know constraints of the problem that we have listed out they are consistent. They are consistent if we are getting the continuity equation is being satisfied, then that is what it mean by. So, if continuity equation is satisfied by applying the constraints of the problem, that means, the whatever the listed out constraints are there in the previous slide they are reliable physically acceptable.

Then momentum equation, first we see the x-component of momentum equation. So, this is the momentum equation, so now we apply the constraints. Steady problem, so first term is 0, v_x is 0, v_y is 0, v_z is existing, but v_x is 0 so then left-hand side all the terms are cancelled out. Pressure we do not know in general; pressure limitations in general any of the such kind of problems we do not know, so then we just keep it as it is, right.

So, only τ_{yz} is existing, so all these components are you know shear stress are negligible. So, then we can strike out. And then we have not taken any gravity because the geometry is horizontal geometry. So, then what we get? $\frac{\partial p}{\partial x} = 0$, that means pressure is not function of x direction; pressure is not varying in the x direction, right.

Similarly, y-component of momentum equation if you simplify. So, because of the steady state this first term is 0, v_x is 0, v_y is 0, v_z is existing, but v_y is 0 as well as $\frac{\partial}{\partial z}$ of you know velocity components are 0. So, then this term is also 0. So, here also left-hand side altogether all the terms are very small; that we can take left-hand side = 0.

Now pressure we do not know what it is, we have only one shear stress. So, then this is τ_{xy} is 0 τ_{yy} is 0. Because of the fully developed flow condition this is 0. And then the gravity is not there, so this is 0. So, then here also we get $\frac{\partial p}{\partial y} = 0$, so that means pressure is not function of y as well.

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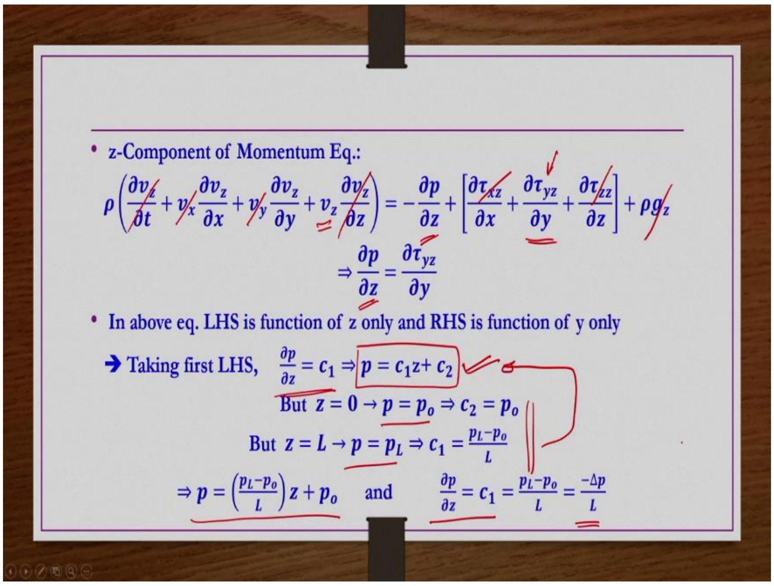

Now, z-component of momentum equation if you simplify. So, steady state term this is 0, v_x is 0, v_y is 0, v_z is existing, but it is not function of z it is function of y only. Why it is not function of z? Because of the fully developed flow. Because of the fully developed flow $\frac{\partial}{\partial z}$ of velocity component is 0.

Pressure we do not know, so let us leave it as it is. And then this τ_{xz} is not existing, τ_{yz} is existing, and then τ_{yz} is function of y right. Shear stress is 0 at the center between two parallel plates and then it is maximum at the wall. So, from center to the wall or the any of the top plate or bottom plate when you move in the y direction and that is y = 0 to y = + h or y = 0 to y = - h the shear stress is varying.

So, that is the reason this $\frac{\partial}{\partial y}$ of this one we cannot cancel out ok. And then this shear stress component is anyways 0, so then we can cancel out. And then the gravity we are not taking because of the horizontal geometry. So, then here what we get? Left-hand side again all terms are negligible that we can take left-hand side is equals to 0. Right-hand side we have two terms remaining that is $\frac{\partial p}{\partial z}$ and then $\frac{\partial \tau_{yz}}{\partial y}$. So, $\frac{\partial p}{\partial z} = \frac{\partial \tau_{yz}}{\partial y}$ this is what we have.

Now, coming to the case how to solve this equation, because left-hand side it is function of z and then y right-hand side is function of y. But however, already we have seen the pressure is not function of x and y. That means, for the when you for the right-hand side term whatever the $\frac{\partial p}{\partial z}$ is a constant one, because right-hand side term is function of y, but left-hand side term whatever the pressure is there it is not function of y. We have already seen by simplifying the x and y components of momentum equation.

So, with respect to the right-hand side term left-hand side is constant. Similarly, this τ_{yz} it is not changing in the z direction because of the fully flow conditions, right. So, it is changing only in the y direction. So, when you take the right left-hand side term; when you take the left-hand side term for the solving or integrating it, so right-hand side term is going to be constant for it. So that means, in these two terms individually we can take and then equate to some constant and then integrate them.

So, let us say that constant you can take c_1 or anything like that. So that means, $\frac{\partial p}{\partial z}$ if you take let us say constant c_1, then p is c_1 z + c_2, apply the boundary conditions. At the z = 0 let us say if you take p = p_0 then c_2 you get p_0 at z = L if you take p = p_L then you will get

$c_1 = \frac{p_L - p_0}{L}$. Straightforward simplifications by applying these two boundary conditions in this pressure distribution equation, right.

So, now if you substitute c_1 and c_2 here in this equation back, then you get $p = \left(\frac{-p_0 - p_L}{L}\right) z + p_0$ that is what you are going to get or $\frac{\partial p}{\partial z}$ is nothing but c_1 that is $\frac{-p_0 - p_L}{L}$ that is $\frac{-\Delta p}{L}$. So, there $\frac{\partial p}{\partial z}$ is nothing but a constant that is $\frac{-\Delta p}{L}$.

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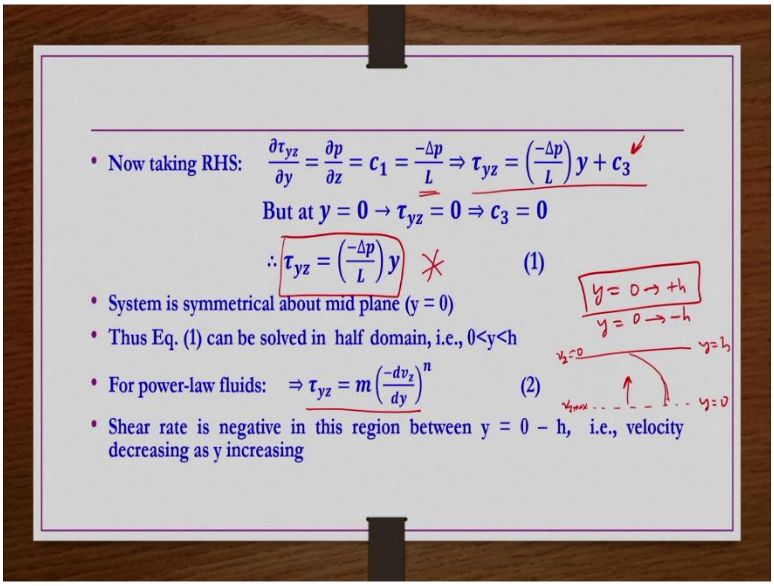

Now again τ, now whatever the $\frac{\partial p}{\partial z} = \frac{\partial \tau_{yz}}{\partial y}$ is there that equating to c_1. So now, $c_1 = \frac{-\Delta p}{L}$. That means, τ_{yz} is nothing but $\left(\frac{-\Delta p}{L}\right) y + c_3$, get it on integration.

Now, we have to apply a boundary condition in order to get this c_3. So, what is the boundary condition that we take? At y = 0, at the center of these two plates between these two infinite parallel plates at the center point, that center point we have designated as y = 0; at y = 0 shear stress is going to be 0, and then velocity is going to be maximum. So, at y = 0 $\tau_{yz} = 0$, so obviously the constant c_3 is going to be 0.

That means, $\tau_{yz} = \left(\frac{-\Delta p}{L}\right) y$. So, here also in this geometry also, when the flow is laminar and then fully developed flow then shear stress is linearly varying in the y direction; is linearly varying in the y direction, ok.

Now, till this point what we have done? We have done, simplifying of continuity and momentum equations after having enough understanding about the geometry and then confinements or constraints of the problem we got this expression, right. We have not applied any restrictions or you know assumptions about the rheology of the fluid, right.

So that means, up to this problem whether the fluid is power-law fluid, whether the fluid is Newtonian fluid, whether the fluid is a visco-plastic fluid, whatever it is as long as the you know you know fluid geometry fluid is generalized Newtonian fluid. So, this expression is not going to change; this expression is not going to change. So, the rheology of the fluid is coming into the picture from this point onwards, ok.

So, now this expression you can use in order to get the velocity profile. How to get that one that all now depends on the mathematical simplification, depends on the rheology of the fluid and then subsequent mathematical simplifications. So, the flow is symmetrical about mid plane that is what we have seen.

So, that is whatever the flow is taking place between y = 0 to y = + h the same thing is taking place between y = 0 to y = $-$ h So, in order to reduce mathematical simplifications what you can do? You can take the flow between y = 0 to h and then do the required simplifications ok.

So, between y = 0 to h we take for power-law fluids $\tau_{yz} = m\left(\frac{-\,dv_z}{dy}\right)^n$. Why minus we are taking? Because at y = 0 the velocity is maximum, and then at y = h the velocity is 0; v_z is 0 here v_z is maximum.

And then as y increasing; as y increasing what happens? The velocity is decreasing right. So that means, you are going to have the flow profile something like this let us say. So that means, as y increasing velocity is decreasing; that means, the velocity gradient is going to be negative. So, that is the reason here we have taken $\frac{-\,dv_z}{dy}$.

So, that is shear rate is negative in this region between y = 0 to y = h, that is velocity is decreasing as increasing that is the reason we have taken this one.

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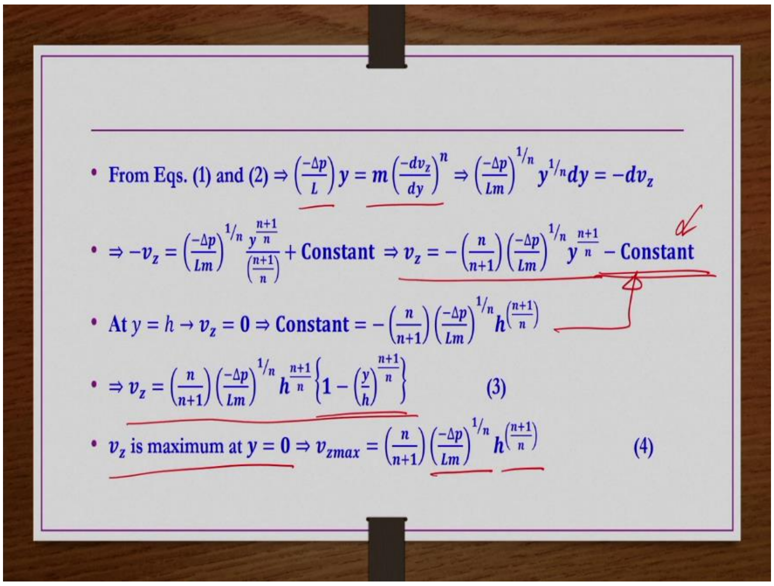

So, now this $\tau_{yz} = \left(\frac{-\Delta p}{L}\right) y$. So, $\left(\frac{-\Delta p}{L}\right) y = \tau_{yz}$, in place of τ_{yz} we have written $m\left(\frac{-\,dv_z}{dy}\right)^n$. So, now what we want? We want this velocity expression v_z expression is required. So, then v_z term we are keeping one side rest all other terms we are taking other side. So, that we have $-\,dv_z = \left(\frac{-\Delta p}{Lm}\right)^{1/n} y^{1/n} dy$. This is what we are going to have.

Now, if you integrate this equation we get $-\,v_z = \left(\frac{-\Delta p}{Lm}\right)^{\frac{1}{n}} \frac{y^{\frac{1}{n}}}{\left(\frac{n+1}{n}\right)} + Constant$. So, now, next step what we do? We, both sides multiply by minus so that we get $v_z = -\left(\frac{n}{n+1}\right)^{\frac{1}{n}} \left(\frac{-\Delta p}{Lm}\right)^{\frac{1}{n}} y^{\frac{n+1}{n}} - Constant$.

Now, this constant we have to obtain. How do we obtain? We can obtain you know = 0 $v_z = v_{max}$ that we can apply or y = + h $v_z = 0$ that also we can apply. So, what is v_{max} or $v_{z\,max}$ we do not know; so, the better is you apply a boundary condition where you have the fixed known value as a boundary value.

So, that is at y = + h we have $v_z = 0$, so then we get in this equation constant is $= -\left(\frac{n}{n+1}\right)^{\frac{1}{n}} \left(\frac{-\Delta p}{Lm}\right)^{\frac{1}{n}} (h)^{\frac{n+1}{n}}$. So now, this constant you can substitute here in this equation and then simplify then you have this $v_z = \left(\frac{n}{n+1}\right)^{\frac{1}{n}} \left(\frac{-\Delta p}{Lm}\right)^{\frac{1}{n}} \left\{1 - \left(\frac{y}{h}\right)^{\frac{n+1}{n}}\right\}$.

After substituting this constant here what we have done? From the both the terms we have taken $(h)^{\frac{n+1}{n}}$ common, so that we have $\left\{1-\left(\frac{y}{h}\right)^{\frac{n+1}{n}}\right\}$ as a within the parenthesis, right. So, this is what we get velocity profile, right.

If you wanted to know the maximum velocity in this expression if you substitute y = 0 you get the maximum velocity, because at the center line that is at y = 0 velocity is maximum. So, $v_{z\,max} = \left(\frac{n}{n+1}\right)^{\frac{1}{n}}\left(\frac{-\Delta p}{Lm}\right)^{\frac{1}{n}}(h)^{\frac{n+1}{n}}$, fine. So now, we got the velocity profile as well as the maximum velocity.

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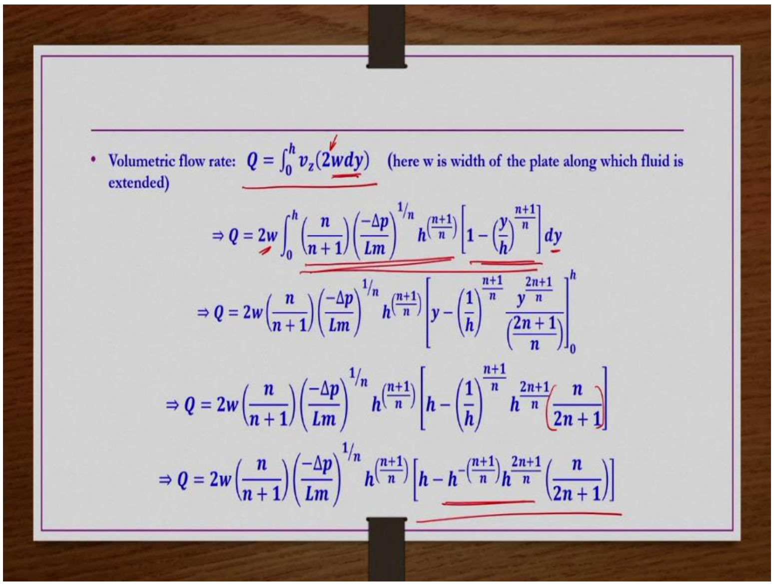

Now, we try to obtain the volumetric flow rate. Volumetric flow rate is nothing but integral $\int_0^h v_z(2wdy)$; w is nothing but width of the plate along which the fluid is extended in the x direction as I mentioned while discussing the geometry. Then y is this two is multiplied because we are the whatever the velocity profile that is there that we got between y = 0 to + h only, but the volumetric flow rate we are getting for the entire geometry. Because it is flowing in the entire geometry, so average whatever the volumetric flow rate is that is what we are going to get.

So, that is the reason since y whatever the velocity profile is there y = 0 to y = + h, same velocity profile is there between y = 0 to y = − h. So, that is the reason the velocity is multiplied by 2 here ok.

So, now v_z you can substitute here. So, v_z is this expression right; 2w is a constant so we take outside of the integration and then dy. So, if you do the integration, so this all this term before the parenthesis whatever is there that is a constant term. So, only this part you how to integrate.

So, when you do it you get y $\int dy$ is y, then from the second term $\left(\frac{1}{h}\right)^{\frac{n+1}{n}}$ is a constant that you take common. And then integration of $y^{\frac{n+1}{n}}$ is nothing but $(y)^{\frac{\frac{2n+1}{n}}{\frac{2n+1}{n}}}$ and then limits 0 to h.

So, when lower limit is 0, so both the terms should be 0. So, when you substitute the upper limit here, so we have $h - \left(\frac{1}{h}\right)^{\frac{n+1}{n}} h^{\frac{2n+1}{n}} \left(\frac{n}{2n+1}\right)$, right. So, this is what we have. So now, next step what we do? This whatever $\left(\frac{1}{h}\right)^{\frac{n+1}{n}}$ is there that I am writing $h^{-\left(\frac{n+1}{n}\right)}$ so that you know I can combine these two terms or you can directly write $h^{\frac{2n+1}{n}-\frac{n+1}{n}}$ that you can do.

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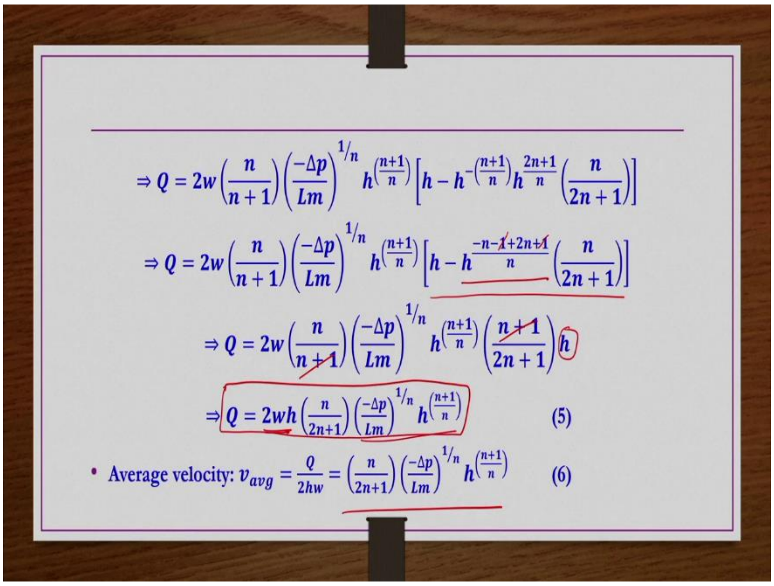

So then, when you do that one simplify it, you get it that this one. So, what you have here? h^1 you are having. So, what you can take? So, $h - h\left(\frac{n}{2n+1}\right)$ from these two terms now, you can take h common and then if you do the LCM; so, $h\left(\frac{1-n}{2n+1}\right)$ is nothing but $\frac{n+1}{2n+1}$. This is what you are having here, right.

So now, this h we combined with 2w. So, 2wh we will be having here. And then this n + 1 this n + 1 is cancelled out, so then remaining term $\frac{n}{2n+1}$ we are having, the remaining two terms are as it is. So, this is the expression for the volumetric flow rate for a power-law fluid flowing between two infinitely long parallel plates.

Then average velocity if you wanted to get; if you divide this $\frac{Q}{2wh}$ you will be getting average velocity. That is average velocity is nothing but $\frac{Q}{2wh}$, so that is $\frac{n}{2n+1} - \left(\frac{-\Delta p}{Lm}\right)^{\frac{1}{n}} h^{\frac{n+1}{n}}$. This is what we are going to have, right.

So, now what we do? The same geometry, but we are going to have a different fluid we are going to take a Bingham plastic fluid, right

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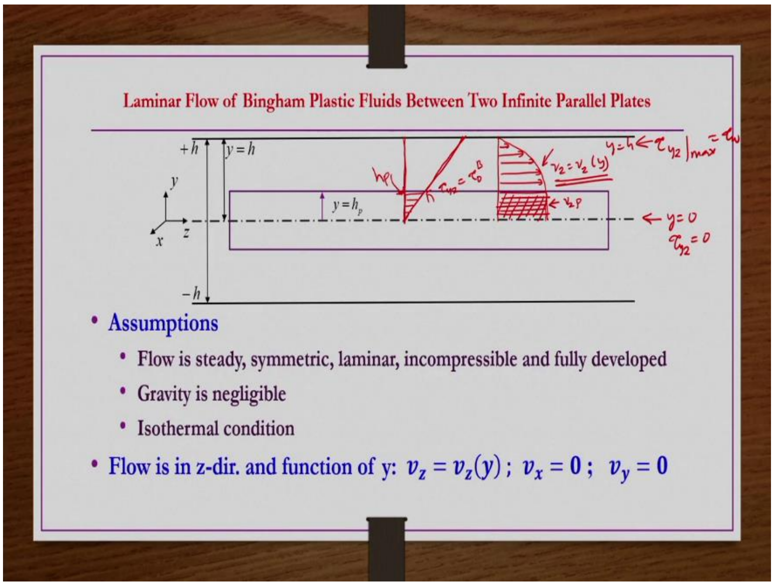

So now, we have a Bingham plastic fluid. So, what we know that Bingham plastic fluids? They do not flow until and unless the applied stress is more than the characteristic yield stress of the material that we know. So now, at y = 0; this is y = 0, the same geometry we are taking, same coordinate system everything.

At y = 0 what we have? We have the $\tau_{yz} = 0$, right. At y = h; at y = h we are going to have a maximum wall shear stress. In these two between these two points we know that the shear stress is linearly increasing. That is what we have already seen, so that we know for the power-law fluid. So, that it is increasing like this.

So now, what happens? So, let us say up to certain point. So, at y = 0 velocity is going to be maximum and then at y = h it is 0, right. So, now shear stress is 0 here, so gradually it increases; gradually it increases. There would be a point where; there would be a point where τ_{yz} whatever is there that will become τ_0^B or characteristic yield stress of the material. So, that point we do not know what it is where location we do not know. So, let us call this point is h_p, right.

Because now, it is a Bingham plastic material the velocity profile if you try to obtain for this one. So, up to this y = h_p point, the point where up to which the material is flowing as a kind of a constant velocity plug; it is having a constant plug velocity, right.

So, that is velocity profile is going to be something like this. And then within the limit of y = 0 to y = h_p it is going to a constant maximum plug velocity. After that only deformation starts, and then when the deformation is starting as we move towards the wall towards the top plate the velocity is decreasing because of a no-slip boundary condition at the wall, right.

So, now what we have here? We have a two velocity profiles rather two velocity profile, one velocity profile which is function of a y and then another velocity profile or constant velocity v_{zp} which is not a function of y which is constant; which is constant between y = 0 to y =h_p, right. But from y = h_p to y = h deformation is taking place, within that deformation region velocity is function of y. That is what we are going to see now right.

So now, like in the pipe flow conditions we have done here also in this geometry also we are going to develop the velocity profile for a two regions. One is the non-deforming solid plug kind of region where the velocity is constant that is v_{zp} or $v_{z\,max}$. Another region where the deformation is taking place, where the material is flowing like a viscous fluid not like a plastic solid; so, within that viscous fluid region or where the deformation is taking place the velocity is function of y.

So, the division of the floor geometry in two parts is the only difference here, because that is the characteristic nature of the viscoplastic fluids. Viscoplastic fluids does not flow as long as the applied stress is more than the characteristic yield stress of the material.

Yield stress of the material is the characteristic of the material; like density, viscosity, etcetera that characteristic of the material we are having in general. So, this yield stress also is a characteristic of the material for the viscoplastic material.

So now, which we have to identify the location at which the, at which the applied stress is equal becomes equal to the yield stress of the material, right. So, that location we are calling h_p, because we know that this τ_{yz} is linearly varying in the y direction. We already seen in the power-law fluid case. So, that does not change with the fluid nature, only velocity profile is going to change with the fluid nature or rheology of the fluid ok.

So, between 0 to h_p we are having constant plug like region where the material is flowing like a plastic solid material with a constant plug velocity. And then $y = h_p$ to $y = + h$ we are going to have a kind of deforming region, where the material is flowing like a viscous liquid. Because, in that region the applied stress is more than the characteristic yield stress of the material, right.

So, for that reason velocity is going to be a function of y. So, these two things we similar to kind of pipe flow things whatever we have taken, but we are now geometry is different so we have to define like this here again.

So, now this is the basic understanding of the Bingham plastic fluid flow between two infinite parallel plates. Next is whatever the assumptions constraints and then simplification of a continuity momentum equation they are going to be same, so then we can have exactly same thing like in the power-law case.

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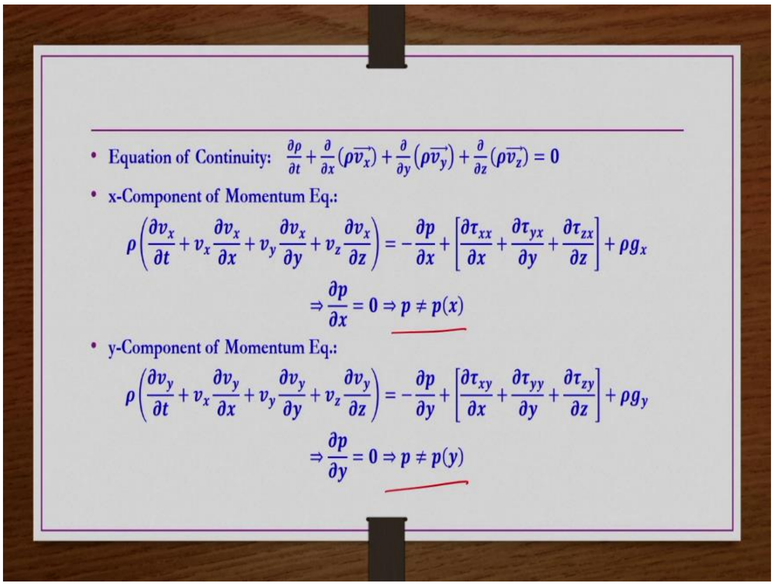

So, these are these things are same constraints, simplification of continuity equation, simplification of momentum equation, whether it is x-component or y-component whichever component of momentum equation their simplification is same. So, we are going to get the same outcome of simplification of this momentum equation because of the constraints are same, only the rheology of the fluid is change. So, that is coming into the picture only after having the shear stress expression.

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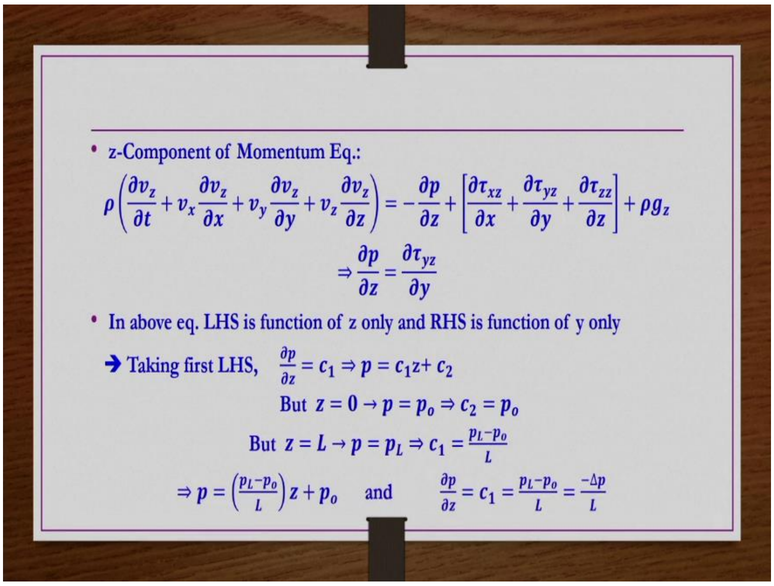

So, the simplification of z component of momentum equation is also going to be same exactly like a power-law case, because the geometry is same constraints are same; flow constraints are same. So, the pressure distribution whatever we got for the power-law case the same thing is valid here also.

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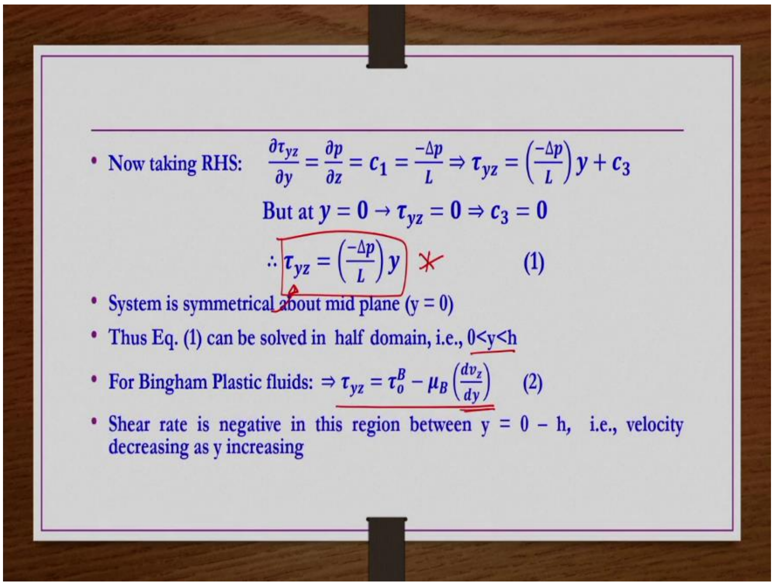

So, shear stress also same thing is valid here. So, exactly same thing we get. Up to this point did we incorporate rheology of the fluid? No, not at all. So, up to this point whether it is power-law fluid or Bingham plastic fluid it is going to be same, ok. So, now, the fluid rheology will come in the picture through this τ_{yz} part ok.

So, we have seen the symmetric about mid plane y = 0. So, between y = 0 to y = h whatever the flow is taking place distribution whatever the distribution that we are going to get, the same distribution we are going to we are going to get between y = 0 to y = − h, that is towards the bottom plate.

So, what we do? We solve only in the half domain so that to reduce mathematical listings; simplification steps we can reduce. So, then $\tau_{yz} = \tau_0^B + \mu_B \left(\frac{-dv_z}{dy}\right)$. That minus I have written before in the μ_B. Why $\frac{-dv_z}{dy}$? Because the same reason here also from y = h_p to y = h as we are increasing the y the velocity is decreasing.

From maximum v_{zp} or plug velocity it is going to 0 velocity at the top plate y is equal to which is located at y = h. So, from y = 0 to y = h when we are increasing we are increasing the y value the velocity is decreasing, that means shear rate is going to be negative. So, that is the reason we have minus here.

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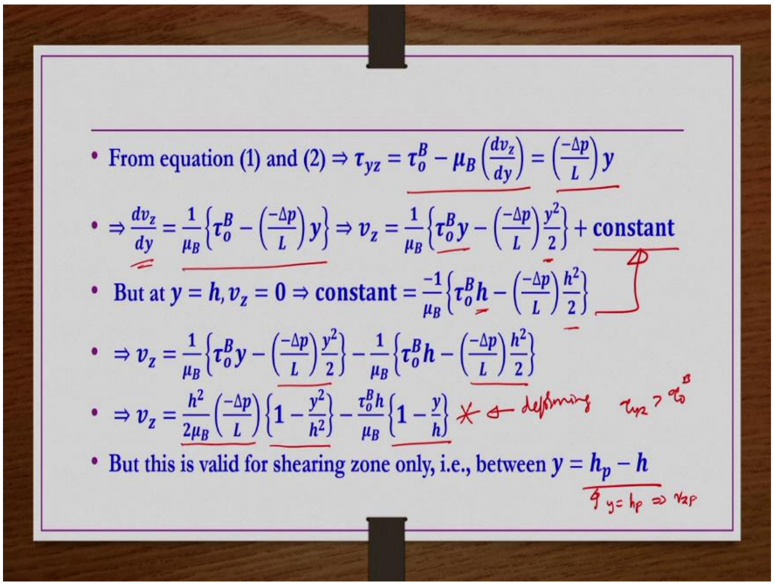

So, now here again in place of τ_{yz} if you substitute $\left(\frac{-\Delta p}{L}\right)y$ and then that you equate to $\tau_0^B - \mu_B\left(\frac{dv_z}{dy}\right)$ right. Then we wanted to get the velocity profile. So, $\frac{dv_z}{dy}$ we are keeping one side rest all terms we have taken to the other side, right. Next what we are doing? We are integrating it. So, here τ_0^B y here $\frac{y^2}{2}$ for the multiplication of $\left(\frac{-\Delta p}{L}\right)$ we get by integration and then constant, right.

So, now the boundary condition that we apply that is again same boundary condition at y = + h v_z is 0 because of the no slip boundary condition and then constant you get $\frac{-1}{\mu_B}\left\{\tau_0^B h - \left(\frac{-\Delta p}{L}\right)\frac{h^2}{2}\right\}$. Now this we substitute here.

So, and then after substituting what we are doing? We are combining the terms 1; this is the $\frac{1}{\mu_B}\left\{\tau_0^B y - \left(\frac{-\Delta p}{L}\right)\frac{y^2}{2}\right\}$ as it is, in place of constant $\frac{-1}{\mu_B}\left\{\tau_0^B h - \left(\frac{-\Delta p}{L}\right)\frac{h^2}{2}\right\}$ we are having.

So, now next what we do? We take or combine the terms which are having $\frac{-\Delta p}{L}$ terms and then τ_0^B terms. So, when you combine these terms $\frac{-\Delta p}{L}$ terms you what you can have? $\left(\frac{-\Delta p}{L}\right)\frac{h^2}{2}\,\mu_B$ you can take common. So, when you do that one remaining terms would be $1-\frac{y^2}{h^2}$. Similarly, from the remaining two terms if you take $\frac{\tau_0^B h}{\mu_B}$ common you can have $1-\frac{y}{h}$ as a multiplication.

And then this is this velocity profile is only for the deforming region or where the region or the region where applied shear stress is greater than the characteristic yield stress. So, that deformation is taking place and then material is flowing like a viscous fluid, right. So, that is between h_p to h only this equation is valid, right.

So, but if you wanted to know the plug velocity what you can do? In this equation you can substitute $y = h_p$, because this equation is valid between h_p to h you can substitute $y = h_p$ to get v_{zp}, right. So, that is what we getting we are doing that one that is that will give you the maximum velocity or the plug velocity. So now, plug velocity or maximum velocity is at $y = h_p$.

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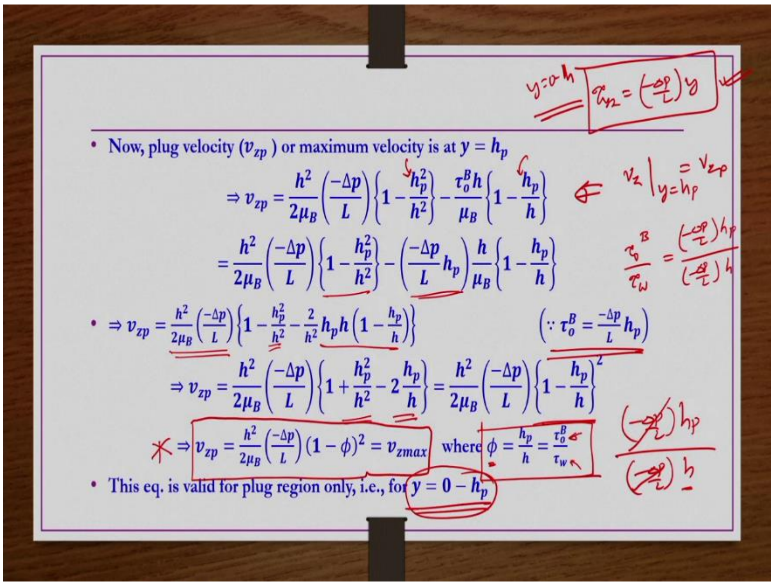

So, in the above equation that is in this equation wherever y value was there. So, we substituted h_p in the previous equation. In the v_z expression, in the v_z expression wherever y was there we are substituting h_p then we get v_{zp}. So, this is what we have.

So, now what we have further simplification of this equation we can do by taking this τ_0^B as $\left(\frac{-\Delta p}{L}\right) h_p$, because $\frac{\tau_0^B}{\tau_w}$ if you do what you get? $\tau_0^B = \left(\frac{-\Delta p}{L}\right) y$ in place of y you have the h_p, and then τ_w is nothing but $\left(\frac{-\Delta p}{L}\right) h$, at y = h. This is what we have.

So that means, in place of τ_0^B you can write $\left(\frac{-\Delta p}{L}\right) h_p$ that is what you can write. Because this τ_0 whatever the τ_{yz} expression was there $\left(\frac{-\Delta p}{L}\right) y$ this expression is valid between y = 0 to h y h.

So, in when you substitute y = h_p that will become equal to the characteristic yield stress of the material. So, in place of τ_0^B we are writing $\left(\frac{-\Delta p}{L}\right) h_p$, ok. So, that we have because of this relation, right. So now, what we do why because this entire equation v_{zp} equation we are we wanted to write in terms of the pressure that is the only thing, otherwise this also we can after simplify and then take it as a final expression, right.

So, now here what we can do? From these two terms we can take $\frac{h^2}{2\mu_B}\left(\frac{-\Delta p}{L}\right)$ as a common term. So, then we have $1 - \frac{h_p^2}{h^2}$ this is as it is. Remaining term $-2h_p - \frac{2}{h^2} h_p h \left(1 - \frac{h_p}{h}\right)$ you are having.

So now, what we do? We expand this term, when we expand this term what we will have; we will have like this is $-2\frac{h_p}{h}$ we are having and then this is $+2\frac{h_p^2}{h^2}$. So, there it is $-\frac{h_p^2}{h^2}$ is there, so then $+\frac{h_p^2}{h^2}$ we are getting. So, this you can write $\left(1 - \frac{h_p}{h}\right)^2$.

So, that is v_{zp} we are going to get this expression our $v_{z\ max}$ is this one that is $\frac{h^2}{2\mu_B}\left(\frac{-\Delta p}{L}\right)(1-\phi)^2$ ϕ is nothing but $\frac{h_p}{h}$ which is nothing but $\frac{\tau_0^B}{\tau_w}$; which is not known a priory. For a given flow geometry flow conditions it is not known.

So, because of that reasons most of the problems associated with the viscoplastic fluids we have done trial and error approach kind of thing that we applied. Like Hank's model or Slatter model for the case of pipe geometry we have applied we have seen you know trial and error kind of approach is required or some kind of analytical expressions are required.

Anyway, we think let us not worry about that one. So, ϕ is nothing but the ratio between the characteristic yield stress divided by the maximum wall shear stress, right. τ_0^B is nothing but $\left(\frac{-\Delta p}{L}\right) h_p$ and then τ_w is nothing but $\left(\frac{-\Delta p}{L}\right) h$. That is whatever this expression $\tau_{yz} = \left(\frac{-\Delta p}{L}\right) y$. So, in place of y if you substitute h_p you get τ_0^B, in place of y if you substitute h you will get the wall shear stress.

So, that is $\frac{h_p}{h}$ is nothing but the $\frac{\tau_0^B}{\tau_w}$ that we call we are calling as ϕ. And this is valid between obviously 0 to h_p range that is the non-deforming solid plug like material whatever is moving like a solid plug like as plastic plug. So, that material whatever that plug like velocity is there. So, that is given by this velocity and then this is valid only between 0 to h_p.

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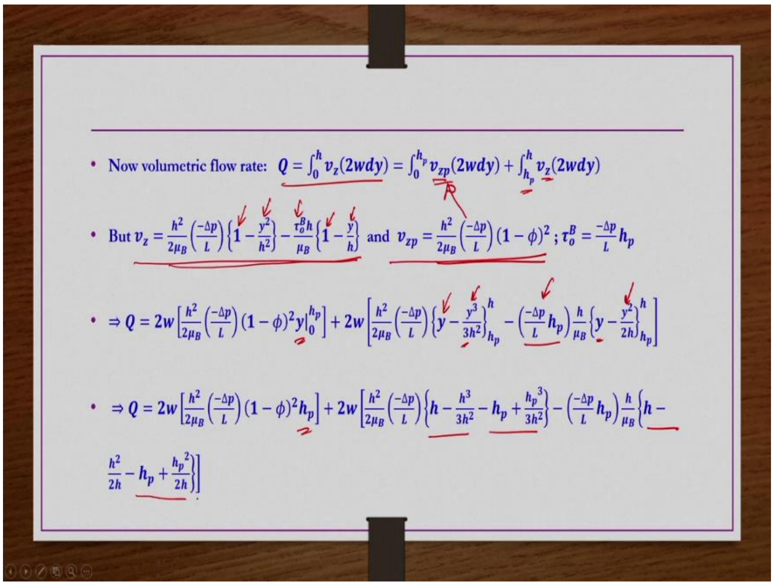

So, volumetric flow rate Q $= \int_0^h v_z(2wdy)$ that is what we have, right from the geometry. But now v_z is having, v_{zp} between y = 0 to h_p and then v_z from between y = h_p to h. So, this v_{zp} v_z; we already got in the previous slides like this. So now, we substitute them here and then integrate.

So, when you substitute v_{zp} this one here and then integrate. So, this entire thing is constant because v_{zp} is a constant velocity, it is not changing with y direction, 2 and w are also constant. So, this entire thing multiplied by y you are getting after integration and then limits are 0 to h_p. The remaining terms 2w and then this entire term when you integrate it. So, here for the integration you are going to get y and then here for integration $\frac{y^3}{3}$ you are going to get so that is here.

So, then again here for integration on integration you get y and then here on integration of this y term you get $\frac{y^2}{2}$; that is here. And then this τ_0^B we are writing $\left(\frac{-\Delta p}{L}\right) h_p$ as we have done previously for the obtain for obtaining the plug velocity, right. So, that all the terms now are in $\left(\frac{-\Delta p}{L}\right)$term L form ok.

Now, so, this limits we substitute, so h_p we are getting here. So now, here upper limit h you substitute, so you get these two terms when lower limit h_p you substitute then you get these two terms. And then last term also when you substitute upper limit you get $h - \frac{h^2}{2h}$; lower limit h_p if you substitute here. So, you get $-h_p + \frac{h_p^2}{2h}$.

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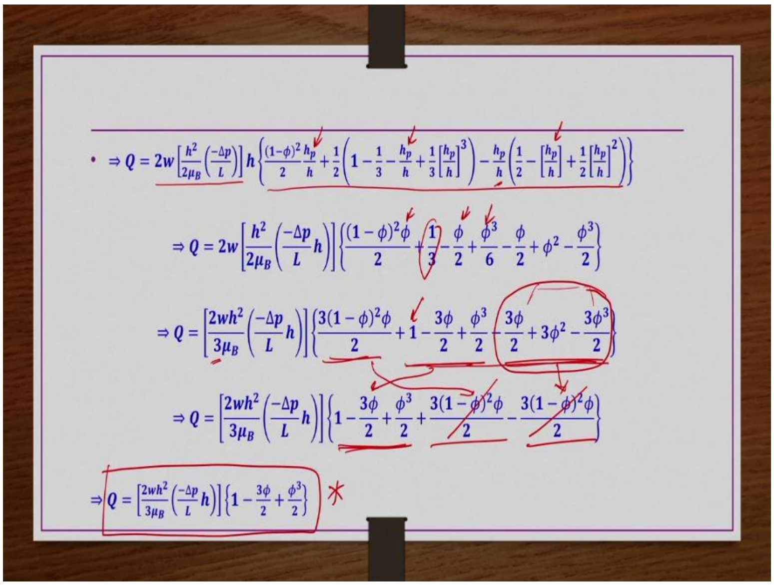

So now, from all these terms what we are trying to do we are trying to take $\frac{2wh^2}{2\mu_B}\left(\frac{-\Delta p}{L}\right)$; h as a common, so then we are getting all these terms from the remaining three terms, right.

Now, further we what you do? You combine the terms or wherever this $\frac{h_p}{h}$ is there; wherever this $\frac{h_p}{h}$ is there by taking that h common you are getting this $\frac{h_p}{h}$ terms everywhere in all the three times, right. So, in place of $\frac{h_p}{h}$ we are writing ϕ, here also we are writing ϕ, here also $\frac{h_p}{h^3}$ $\left(\frac{h_p}{h}\right)^3$ we are writing ϕ^3, right.

And then here also $\frac{h_p}{h}$ we are writing $\frac{\phi}{2}$ and then this is also $\frac{h_p}{h}$ is ϕ, so ϕ x ϕ = ϕ^2. So, like that here also $\frac{h_p}{h}$ is ϕ^2 and then this another $\frac{h_p}{h}$ we are multiplying, so $\frac{\phi^3}{2}$ that is what we are having ok.

Next step what we are going to do? We are dividing by 3 and then multiplying by 3 so that we can have some simplification, ok. Then, so except this term we have one and then all other terms are being multiplied by 3, here also it is multiplied by 3 so 3/3 1 we are writing here.

So, in the next step 1 we have written here $\frac{-3\phi}{2}$ we have written $\frac{\phi^3}{2}$ we have written here, right. So, these three terms we have written here as a first three terms and then whatever these $\frac{3(1-\phi)^2\phi}{2}$ term is that we have written here, right. So, remaining three terms whatever are there these three terms we can write them as a; if you take 3/2 common we can write them as $(1-\phi)^2$ by I mean whole square multiplied by ϕ you can write. These remaining three terms.

We can write them as a $\frac{3(1-\phi)^2\phi}{2}$ that is what we can write. So, that this $+\frac{3(1-\phi)^2\phi}{2}$and then $-\frac{3(1-\phi)^2\phi}{2}$ can be canceled out, right. So, that we have Q is equal to this expression; only first three terms would be there. So, this is the expression for the volumetric flow rate, right.

So, whether it is pipe geometry or parallel plate geometry whatever the methodology how to get the velocity profile, how to get the volumetric flow rate is same as long as the flow is the laminar one-dimensional flow without any difficulty we can do. The same methodology we have to follow. So now, before winding today's lecture quickly we take another geometry; inclined surface.

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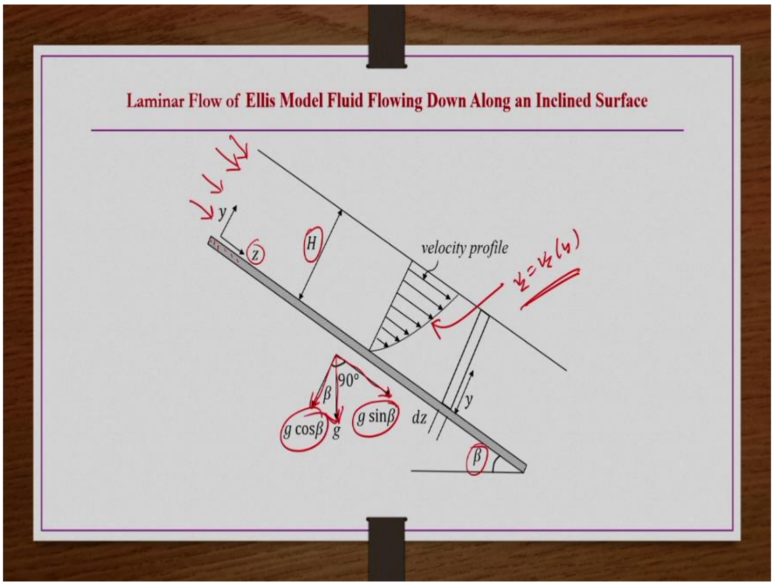

So, let us say we have an inclined surface like this. So, this is the inclined surface right. So, which is inclined to the horizontal axis at an angle β. So, there is a material is coming in and then flowing down here because of the gravity, right. The thickness of the material that is flowing down is h ok. Inclined surface on which the material is flowing; that inclined surface flow direction we are taking z the other direction we are taking y.

So, here also v_z is function of y compared to the v_z other v_x v_y are going to be 0 right. So, what is this velocity profile as function of y we are we have to get it. So now, here the gravity it is not either completely horizontal it is not either completely vertical, so it is in an inclined surface; inclined at β angle.

So, then whatever the two components are there; so if it is g is acting in this direction, if you draw the tangent so this direction we are going to have g sin β and then this direction we are going to g cos β, right. So, now what we do? We do the same simplifications right; we write the constraints of the problem right.

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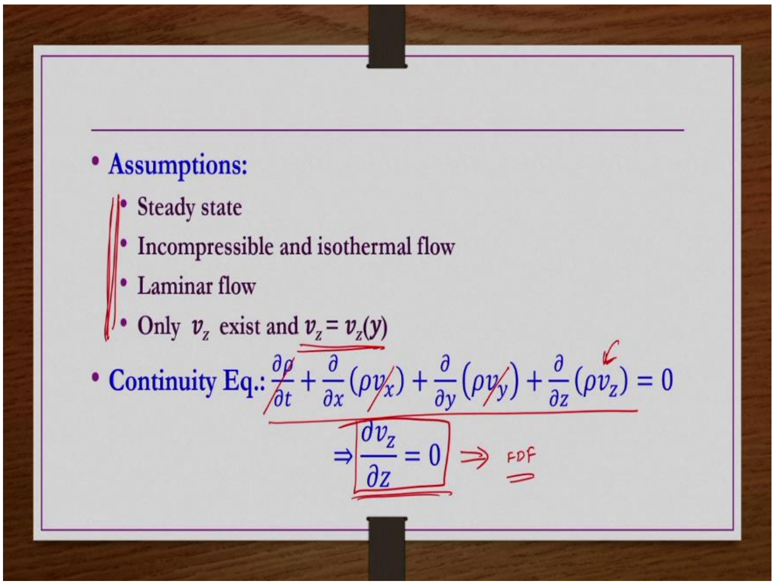

We are taking the steady state, incompressible and isothermal flow conditions, laminar flow only v_z is existing and then v_z is function of y. This material whatever is flowing down it is very thin actually. The thickness of the material is very thin, within that thin layer of the material how the velocity profile is varying? So, for such thin layer of the material that is flowing down. So, it is only one component is going to be predominant other components are going to be 0.

That is in the vertical direction or you know in the x direction we are not going to have any velocity. So, v_z is only existing v_x v_y are going to be 0. So, here also Cartesian coordinates are applicable. So, then continuity equation if you apply here. So, what we have? Steady state so this is 0, v_x is 0 v_y is 0 right; v_z is not 0, but whether the material is under fully developed flow conditions or not that also we do not know. That is the reason $\frac{\partial}{\partial z}(\rho v_z)$ we cannot cancel out.

So, what we get from simplifying the equation? $\frac{\partial v_z}{\partial z} = 0$. So, what does it mean? For this flow geometry when you apply these constraints you get another you know constraint that is the flow is fully developed. Fully developed flow constraint that you can understand through the continuity equation.

Remember, as we have been discussing that the importance of continuity equation; simplifying the continuity equation is that whether the constraints that we are taken they are consistent or not that is what we can understand. Otherwise, if some boundary conditions or some kind of limiting conditions we are not or some kind of constraints we are not very sure from the problem geometry then such kind of constraints we can get by simplifying the continuity equations.

So, that is what we are doing here, that is what we get here. We are not sure from the geometry whether the flow is fully developed or not, but from the continuity equation or by simplifying the continuity equation what we understand $\frac{\partial v_z}{\partial z} = 0$. That means the flow is fully developed in the z direction.

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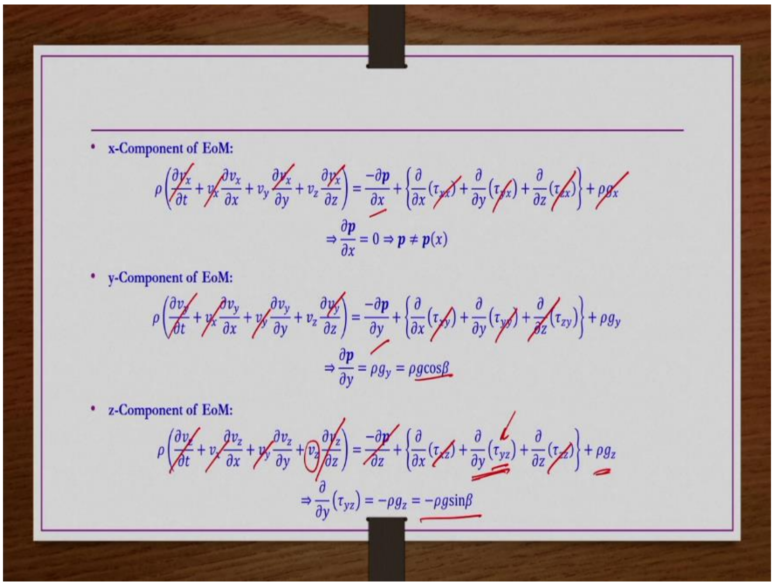

Similarly, now x-component of momentum equation if you simplify. So, v_x is 0. So, then all the terms in the left-hand side are having v_x terms because it is x-component of momentum equation, so all of them are 0. Pressure we do not know. And then only τ_{yz} is existing. So, this τ_y τ τ expressions are 0. And then gravity in the x direction whatever the gravity we are taking that is not there in the x direction it is there only in the y and z directions, ok.

So, what we understand from here? $\frac{\partial p}{\partial x} = 0$; that means, pressure is not function of x. Similarly, y-component of momentum equation if you simplify, then steady state this term is 0 and then v_x is 0 v_y is 0 v_z is not 0, but v_y is 0. So, that is 0. Pressure we do not know anything.

This only τ_{yz} is existing, but this is not existing τ_{xy} τ_{yy} are not existing; τ_{yz} or τ_{zy} is existing, but because of the fully developed flow condition that we get from the continuity equation this term is also 0. ρg_y in the y direction gravity is nothing but g cos β. So, what we get? $\frac{\partial p}{\partial y} = \rho g_y$ from the y-component of our momentum equation that is nothing but ρ g cos β.

Similarly, z component of momentum equation you know, steady state this is 0 v_x is 0 v_y is 0 v_z is not 0, but from the continuity equation we understand that $\frac{\partial v_z}{\partial z} = 0$. So, this is 0. So, next pressure, the flow is taking place only because of the gravity. So, then whatever the pressure distribution in the z direction is there that also we can take it as 0.

And then τ_{xz} is not existing τ_{zz} is not existing, τ_{yz} is existing τ_{yz} is existing and then it is function of y also because whatever the thickness or the fluid is there that is expanding in the y directions. So, from y = 0 to y = + h the shear stress is varying. So, that component you cannot cancel out. And then gravity in the z direction is existing. That means, $\frac{\partial \tau_{yz}}{\partial y} = -\rho g_z$; that means, $-\rho$ g sin β.

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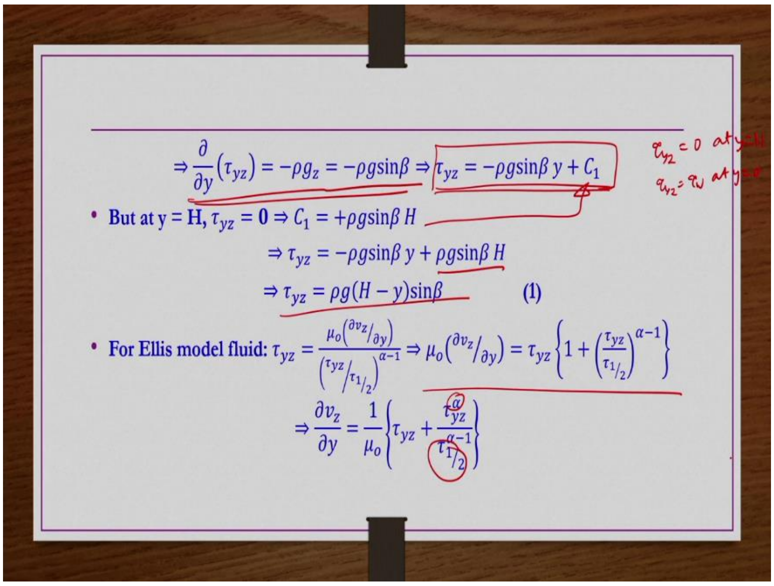

So, once we have this τ_{yz} expression rest everything is a simplification to get the required velocity profile ok. So, from this z component of momentum equation simplification whatever $\frac{\partial \tau_{yz}}{\partial y} = -\rho g_z$ we got from there we can get the τ_{yz} expression as; $\tau_{yz} = -\rho$ g sin β y +C_1 because g_z is nothing but the sin β ok.

So, what we get? $\tau_{yz} = -\rho$ g sin β y + C_1. This C_1 constant you can get by applying the shear stress boundary conditions. τ_{yz} = 0 at upper surface at y = + h; upper surface of the fluid layer we are calling you know y = h. And then τ_{yz} = maximum of tau wall at y = 0.

Because now, unlike the previous geometry or pipe flow conditions the fluid is you know at the surface at the surface inclined surface the material is flowing down. So, at the inclined surface what we have? The maximum walls shear stress we are having. We are going to have the maximum wall stress wall shear stress at the along the inclined surface.

And then this, the reason the material is you know flowing down. So, the upper layer of the fluid whatever is there that is open to the atmosphere. So, then that y = h is going to have a 0 shear stress. So, that boundary condition we apply. That is y = h τ_y = 0, so $C_1 = \rho$ g sin β h. Then if you substitute that one here in this equation you get $\tau_{yz} = \rho$ g h - y sin β.

So, $\tau_{yz} = \rho$ g h - y sin β once we have this expression. So, you can apply next step rheology of the fluid to get the velocity distribution. So, for this geometry what we are doing? We are taking Ellis model fluid. Ellis model fluid τ_{yz} is this one, this we already know.

So, this expression we rearrange so that you know we get mu naught $\mu_0 \frac{\partial v_z}{\partial y}$ one side another terms are other side. So, then what we have? $\frac{\partial v_z}{\partial y} = \frac{1}{\mu_0}\left\{\tau_{yz} + \frac{\tau_{yz}^{\alpha}}{\tau_{1/2}^{\alpha-1}}\right\}$. So, this α and then $\tau_{1/2}$ we already know this characteristics of you know Ellis model fluid. So now, what we do?

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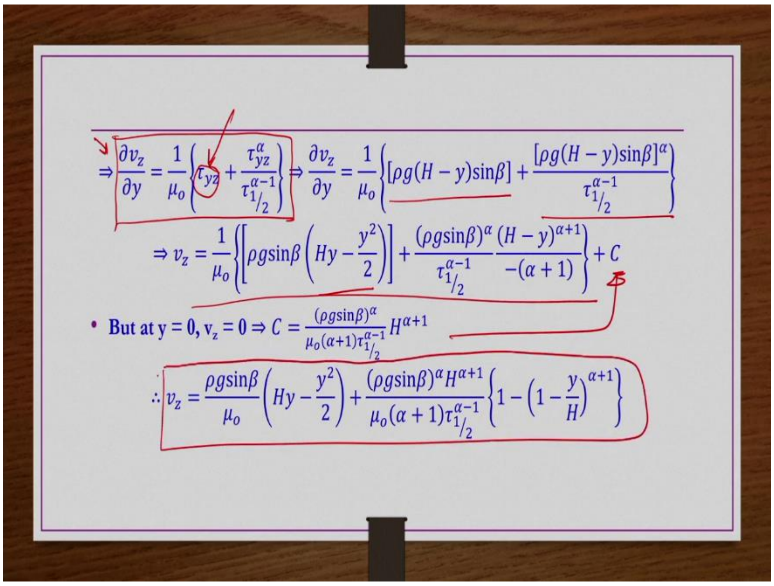

In this expression $\frac{\partial v_z}{\partial y} = \frac{1}{\mu_0}\left\{\tau_{yz} + \frac{\tau_{yz}^{\alpha}}{\tau_{1/2}^{\alpha-1}}\right\}$. This is coming only from the rheology of the material, whatever the rheological expression for the Ellis model fluid is there only from that only we got.

Now, for our problem what is τ_z; what for our this flow problem what is τ_{yz} that we already got that is nothing but ρg h - y sin β that we are going to substitute. So, when you substitute here this one. Now when you integrate this one, here $\left(Hy - \frac{y^2}{2}\right)$ you get here you get $\frac{(H-y)^{\alpha+1}}{-(\alpha+1)}$ you get; $-(\alpha + 1)$ you get here. Rest all other terms are constants, right.

So now, apply the boundary condition to get the constant C. So, at the wall that is it y =0 v_z is 0. So, then this constant is nothing but this one. In this equation, if you substitute y = 0 then v_z would be 0 left-hand side would be 0, so then constant you get this one. So, this constant you substitute here and then simplify then you get velocity profile like this. So, simple simplification only, just rearrangement I am writing directly here.

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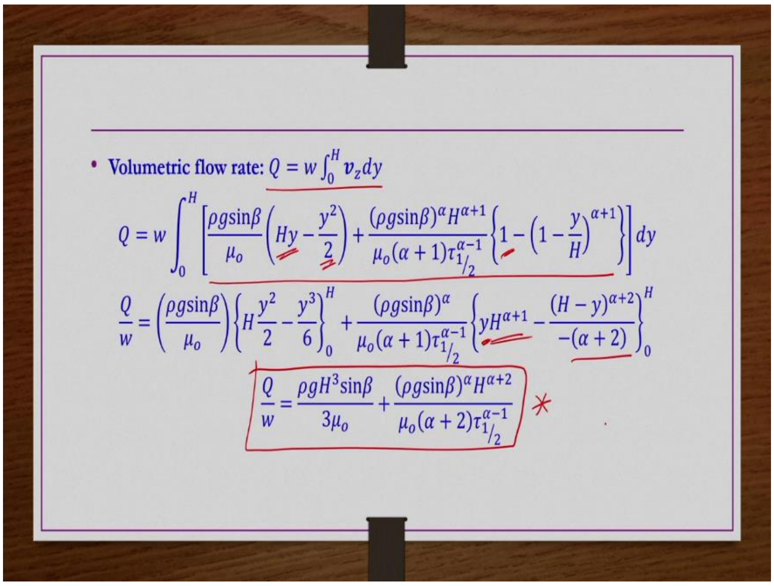

So, the volumetric flow rate Q = $\int_0^H v_z w dy$; w is nothing but the width of the fluid in the x direction that we have seen in the geometry. So, this v_z expression we got this one already. And then if you integrate this one you get here for this place $\frac{y^2}{2}$, and then for this point you get $\frac{y^3}{3}$ on integration. And then here this part you get y.

And then here what you do, before integration what you do here this term? You do the LCM, so that you can have $\frac{H-y}{H^{\alpha+1}}$. So, $(H-y)^{\alpha+1}$ if you do the integration you have $\frac{(H-y)^{\alpha+2}}{(\alpha+2)}$, right. Then whatever that $H^{\alpha+1}$ is there so that you have taken common here. So, that is what we are getting this one right.

So now, the limits 0 to H also if you substitute and then simplify, so you get this term. Straightforward simple simplification that you can do it. So, this is the volumetric flow rate Q/w = given by this expression if Ellis model fluid is flowing down along the incline surface.

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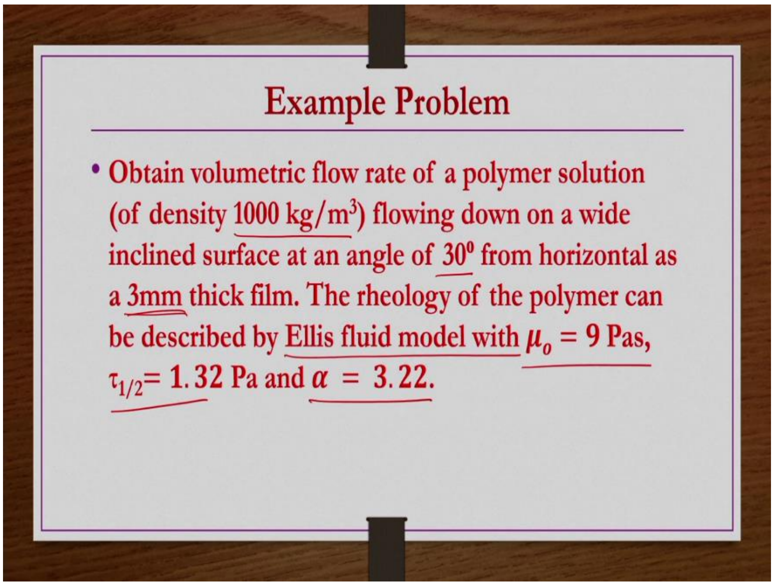

So, finally, example problem. So, the material of density 1000 kg per meter cube is flowing down on a wide inclined surface at angle β = 30°. The fluid thickness as I said; the fluid is in general very thin so 3 mm thickness only there. The rheology is defined by the Ellis model fluid, so that we have the μ_0, $\tau_{1/2}$ and then α are given, right. So, we have to find out the volumetric flow rate.

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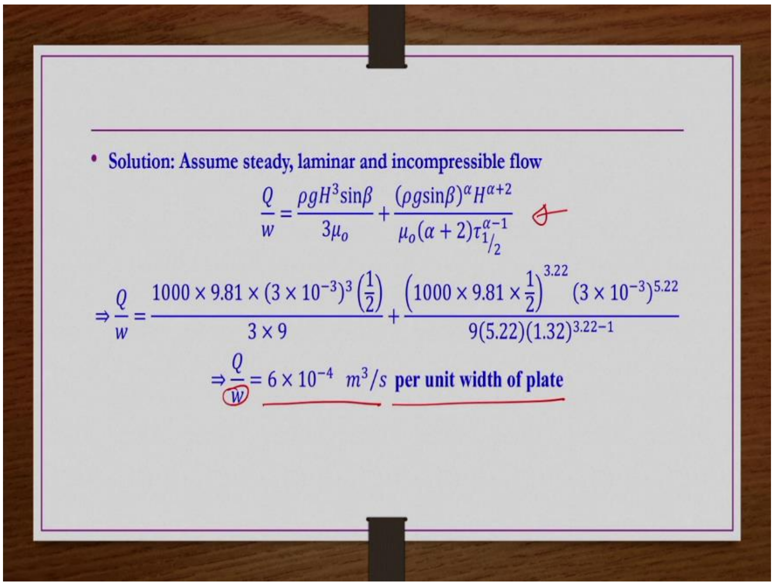

So, we just developed Q/w = this expression. So, in this expression in the right-hand side everything is known; ρ, g, H, β, μ_0, α, $\tau_{1/2}$ everything is known. So, when you substitute and then simplify you get 6 x 10^{-4} meter cube per second per unit width of the plate. So, that is the volumetric flow rate, ok.

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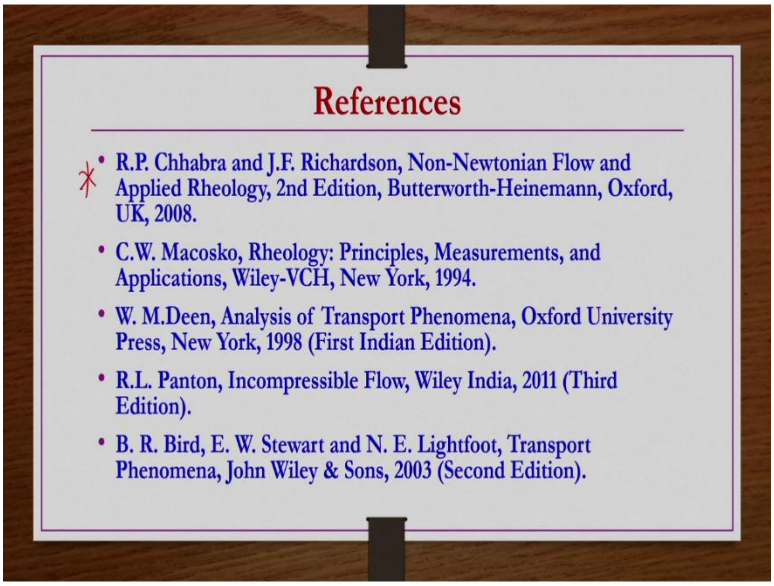

References: The entire lecture is prepared from this excellent book by Chhabra and Richardson, other reference books are provided here.

Thank you.