Transport Phenomena of Non-Newtonian Fluids
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Lecture - 16
Bingham Plastic Fluids Flow through Pipes

Welcome to the MOOCs course Transport Phenomena of Non-Newtonian Fluids. The title of this lecture is Bingham Plastic Fluids Flow through Pipes. We have started discussing how to obtain the velocity profile, volumetric flow rate expressions and then friction factor equation etcetera when different types of generalized Newtonian fluids are time independent non-Newtonian fluids are flowing through pipes.

So, in this process we have already seen what are the corresponding equations for the velocity profile, volumetric flow rate, friction factor etcetera when the fluid is power law type and Ellis model fluid ok. These two things for these two fluids we have already seen in the previous two lectures.

So, now we have a kind of recapitulation of those equations before going in today's lecture where we will be discussing about the flow of Bingham plastic fluids through circular pipes.

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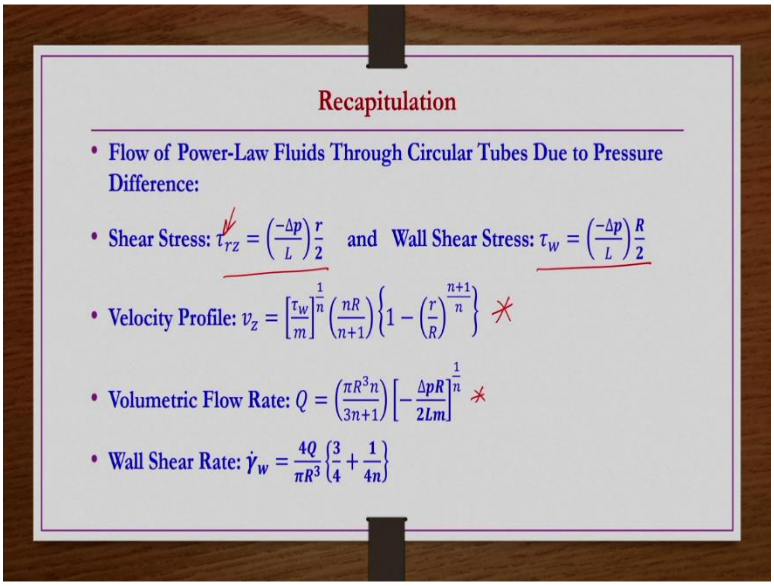

So, for the case of Power-law fluids we obtain the shear stress expression as this one $\tau_{rz} = \left(\frac{-\Delta p}{L}\right)\frac{r}{2}$ that is shear stress linearly varying with the radial coordinate r ok. If you wanted to know the maximum shear stress that is at wall. So, then you substitute r = R in this equation then you will get $\tau_w = \left(\frac{-\Delta p}{L}\right)\frac{R}{2}$.

Now, in this equation in place of τ_{rz} we substituted $m\left(\frac{-dv_z}{dr}\right)^n$ and then we simplified to get the velocity profile that after simplification and then applying the boundary condition this is what we get. And then in this equation if you substitute r = 0 you will get maximum velocity equation ok. And then volumetric flow rate we got this expression for the case of power law fluids corresponding wall shear rate we found as $\frac{4Q}{\pi R^3}\left\{\frac{3n+1}{4n}\right\}$.

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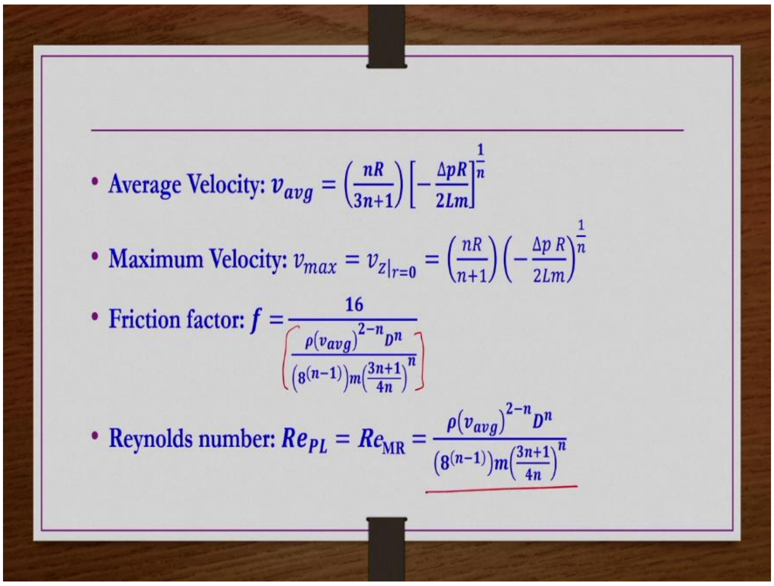

Then average velocity we obtain like this that is $\frac{Q}{\pi R^2}$ when we did we got this expression. Maximum velocity we got this expression by substituting r = 0 in v_z expression, then friction factor we got this expression we have written in a form $\frac{16}{Re}$ form. So, that analogously we can define you know what is Reynolds number for power law fluids because $\frac{16}{Re}$ is the friction factor for Newtonian fluids flowing through pipes. So, the analogously we have written here also. So, this Re_{PL} or Re_{MR} is nothing, but this particular expression.

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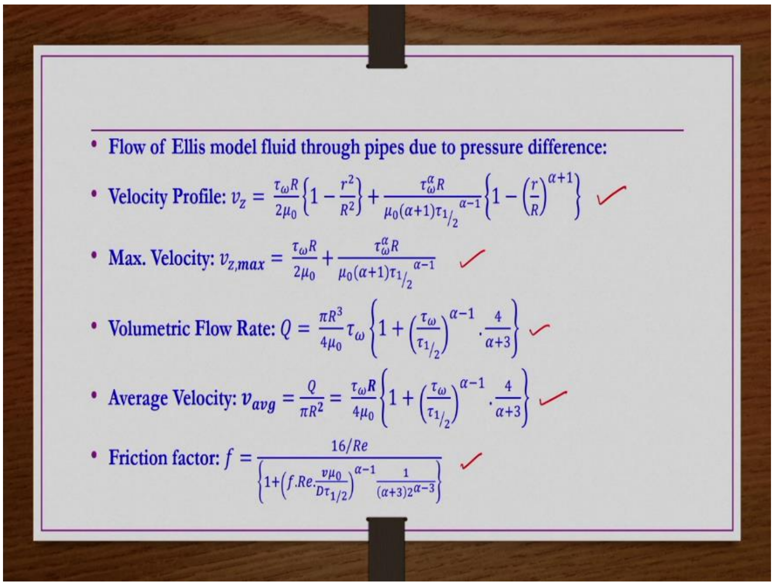

Similarly, for Ellis model fluid we obtain the velocity profile as this and then maximum velocity by substituting r = 0 in this equation we get this one. Then volumetric flow rate we got this expression, dividing this expression by πR^2 we got the average velocity this is $\frac{\tau_w R}{4\mu_0}\left\{1+\left(\frac{\tau_w}{\tau_{1/2}}\right)^{\alpha-1}.\frac{4}{\alpha+3}\right\}$ then friction factor we got this expression. And then we found that friction factor is not explicit. So, it has to be obtained by trial and error approach ok.

Now, we will be having a generalized discussion, how would be the velocity profile looking like when a viscoplastic fluid flowing through pipe. Then we apply specific type of viscoplastic fluid like Bingham plastic and Herschel-Bulkley fluids and then obtain the corresponding volumetric flow rate expression once we are getting the velocity profile expressions. So that is what we are going to see now.

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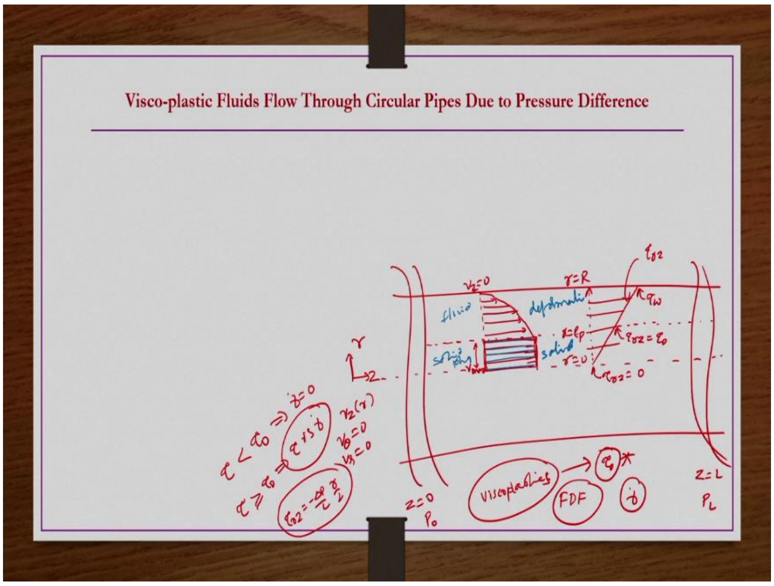

So, let us say we have the same situation where we have infinitely long cylinder ok. So, the flow is taking place because of the pressure. So, let us say if this is your center of the circular pipe. So, then if you take coordinate directions like this, this is r, this is z. So, v_z is there any and it is function of r, v_θ is 0, v_r is 0 at z = 0 P = P_0 at z = L P = P_L right. Because of this pressure difference the flow is taking place and then we have taken a region where the flow is fully developed and then laminar flow right.

So, now previously we have seen if it is a Newtonian fluid then we get the parabolic profile, if it is shear thinning fluid then we get flatter kind of profile and then if it is shear thickening fluid, then we get a sharper kind of profile that is what we have seen. However, irrespective of the type of fluid shear stress linearly increasing with increasing r that is what we have seen.

But now how about this viscoplastic fluids? Because, this viscoplastic fluids we know the one of the major important characteristic of the material is yield shear stress. This yield shear stress is the characteristic of the material ok it is available right you know. So, for a given gamma dot range if the fluid is displaying a viscoplastic behavior. So, then corresponding yield stress for that particular fluid is available from rheological data. So, that is a characteristics of the material.

And then what we know? If the applied stress τ is less than this yield stress what we have seen? There will not be any deformation there will not be any deformation that is whatever

$\frac{-dv_z}{dr}$ is there or $\dot{\gamma}$ is there that would be 0 right. But however, if applied stress is gradually increases and then it crosses τ_0 then there will be a kind of deformation and then we will be having τ versus $\dot{\gamma}$ relation.

If it is Bingham plastic then we have a linear relation; if it is Herschel Bulkley fluid then we can have you know non-linear relation after crossing this τ_0 ok. So, this is what we know, but what is this τ expression? τ expression we know if it is one dimensional laminar flow through pipes, then we know τ_{rz} is nothing, but $\left(\frac{-\Delta p}{L}\right)\frac{r}{2}$.

Because this we know it is irrespective of the nature of the fluid because till this point or till getting this expression, we did not apply the rheology of the fluid. So; that means, for this case also we are going to have a same profile $\tau_{rz} = \left(\frac{-\Delta p}{L}\right)\frac{r}{2}$ that is τ_{rz} is linearly increasing by increasing r right. So, then that we have like this expression.

So, now this is our τ_{rz} right. Now here at the wall τ_{rz} is maximum that is τ wall at the center $\tau_{rz}$is 0. So, as we moving towards the wall from the centre of the pipe from r = 0 to r = R right; that means, what happens? There would be certain location. So, now let us say here at certain level here if you measure the shear stress would be some value at this level it may be some further higher value like this you know it is gradually increasing.

So, we have to identify a point where the shear stress is equals to the yield stress. Yield stress is having some value let us say 10 Pascal or 15 Pascal whatever the yield stress it is having. So, that value you are getting when $r = r_p$ let us say that is at this location, at this location when $r = r_p$.

So, at $r = r_p$ whatever this τ_{rz} is there is equals τ_0. And then we know when the $\tau_{rz} < \tau_0$ there will not be any deformation the material would be flowing as a kind of plug it does not deform. So, it move like a solid plug, it move like a solid plug until and unless that applied stress $\tau_{rz} > \tau_0$.

So; that means, let us say between r = 0 to $r = r_p$ the material is moving with a constant velocity constant velocity let us say some constant velocity it is moving with right. So, what is that velocity that we come to know right. So, till this up to this point you know the material now is having two regions. One region where the material is moving like a plug like a solid that is between 0 to r_p why because? Because in this region of r that is from r = 0 to r = R_p the shear stress applied shear stress is less than the yield stress.

So, if the applied stress is less than the yield stress. So, then deformation will not be there. So, material will be moving like a solid without any deformation and then after crossing this r_p value applied stress would be more than the τ_0 yield stress. So, then deformation will start taking place and then as we move towards the wall what will happen?

The velocity will gradually decrease and then it reaches the 0 velocity $v_z = 0$ at this wall right $v_z = 0$ at the wall. So, and then we know at the center the velocity is maximum that is v_z is max at the center at r = 0 and then same velocity maximum velocity is maintained between these two limits of r = 0 to r = r_p right.

So, now this is the primary difference between the previous case and then this case. In the previous case where we had only viscous nature of the material there is no plastic nature in the materials. So, then material was deforming depending you know whatever the shear stress you give only small shear stress also you give start deforming and then there will be a kind of deformation.

But now here deformation will not take place until and unless the applied stress is more than the yield stress. So, that reason is this one. So; that means, from 0 to r_p from r = 0 to r_p, the material is moving like a solid plug and then from r_p to R value it is moving like a fluid deformed fluid with a deformation right.

So, now whatever the velocity that we are going to get for this kind of viscoplastic fluids we are going to derive two equations. One equation for the solid plug region, where a constant maximum velocity we have to get another region where the velocity is decreasing with increasing the r value that is deformation region.

So, between r = r_p to R it is a deformation region and then between r = 0 to r_p it is a solid plug like reason ok. So, this is deformation region and this is solid plug like region ok. So, that is what we are going to see. So, these two velocity profiles are you know it is not only one profile another one is constant value because between r = 0 to r_p it is constant and then from r = r_p to r = R it is decreasing gradually with increasing the r.

So, these two equations we are going to develop now. So, this is true for any type of viscoplastic fluid whether it is Bingham plastic or Herschel-Bulkley fluid or any other type of a viscoplastic model fluid model that we use ok include in the Casson model etcetera.

So, now for this is once you understand after this the mathematically doing simplifying the equations and getting the solutions are quite similar whatever we have done in the previous two cases of power law fluid and the Ellis model fluid. Only difference is that the velocity profile is being divided into two regions one solid plug like core region another deformation or deforming region where the velocity is decreasing gradually with increasing r.

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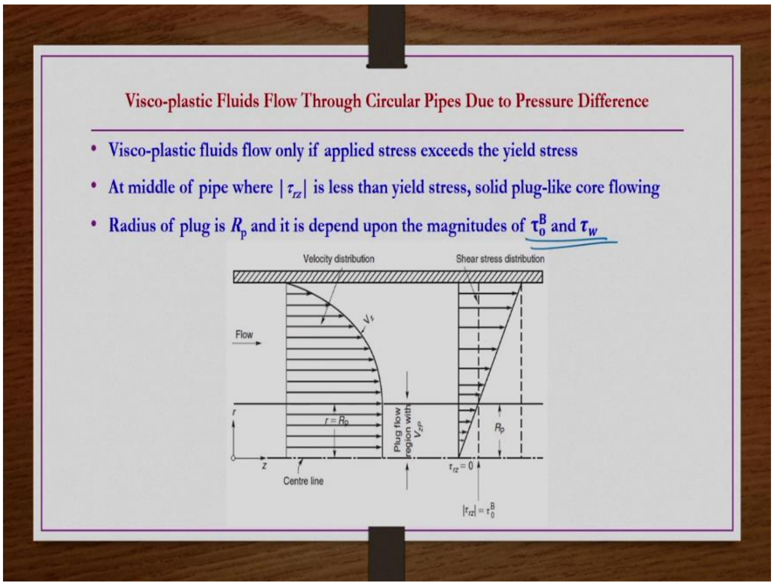

So, viscoplastic fluids through circular pipes due to pressure difference the pictorially the same thing whatever I have explained is given here ok. Viscoplastic fluids flow only for applied stress exceeds the yield stress, at middle of pipe where τ_{rz} is less than yield stress, solid plug like core flowing between r = 0 to r = R_p.

And then radius of plug is R_p and it is dependent upon the magnitude of what is the value of τ_0^B and then what is the value of τ_w wall shear stress and then characteristic yield stress of the material ok.

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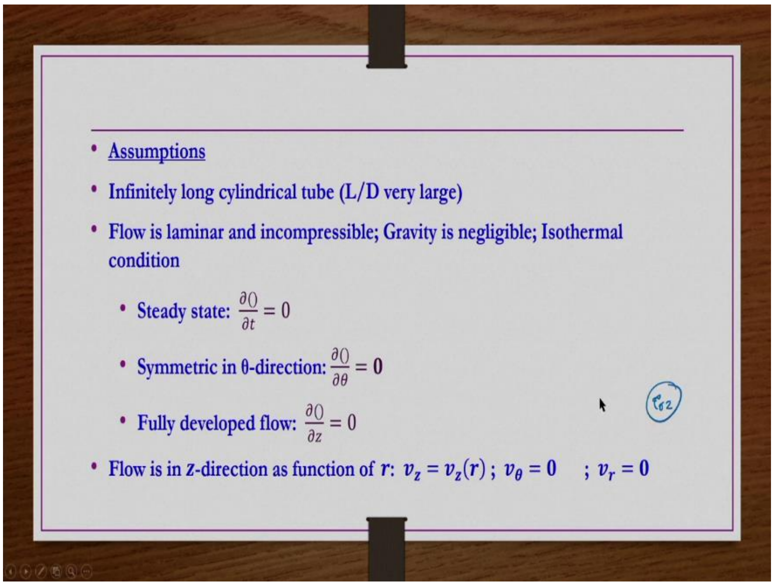

So, now assumptions for this flow also exactly same as we have done for the previous two cases. Because geometry flow nature everything is same except the rheology of the material is changing. So, we have infinitely long cylindrical tube L/D is very large so, that fully developed flow is there.

Flow is laminar and incompressible gravity is negligible isothermal conditions, steady state symmetric, fully developed flow and then only v_z component is existing which is function of r whereas v_θ v_r 0 are very very small compared to the v_z that we can neglect them. And then only component of shear stress is existing is τ_{rz} right.

So, applying these assumptions or constraints to the equations of continuity and then momentum, we get some simplified equations as we got in the previous cases. So, equation of continuity steady state.

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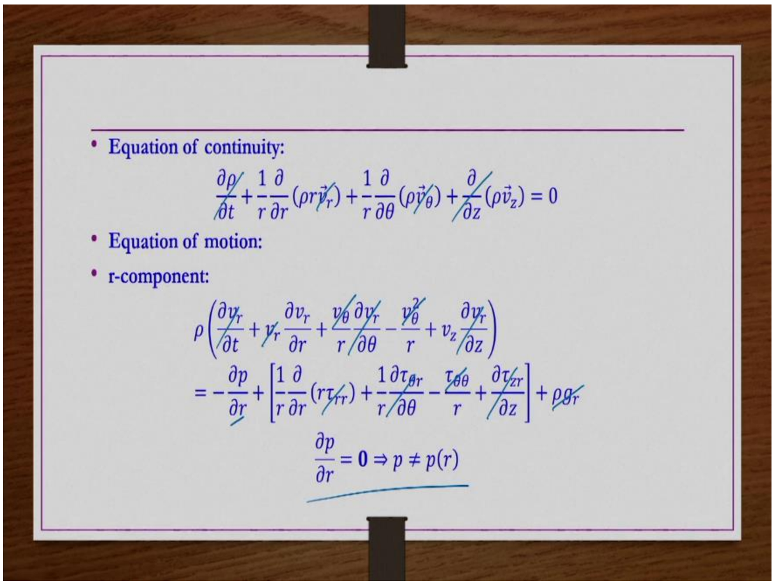

So, this term is 0 v_r is not existing, v_θ is not existing and then fully developed flow. So, $\frac{\partial}{\partial z}$ of any flow variable is 0. So, then continuity is maintained. Then equation of motion r component of equation of motion, it is given here. So, steady state this term is 0, v_r is 0, v_θ is 0 symmetry.

So, $\frac{\partial}{\partial \theta}$ of anything is 0, v_θ is 0 and then fully developed flow. So, $\frac{\partial}{\partial z}$ of any flow variable is 0, pressure we cannot say anything we do not have any generalized pressure conditions in most of the continuum flow problems, then only τ_{rz} is existing.

So, this is 0 symmetry. So, this term is 0 only τ_{rz} is exiting. So, this is also 0 and then because of the fully developed flow $\partial\tau_{rz}$ is existing or τ_{rz} is existing this item is also 0 because of the fully developed flow. That is $\frac{\partial}{\partial z}$ of any flow variable is 0 and then we are not taking gravity into the consideration. So, we have $\frac{\partial p}{\partial r} = 0$ that is $p \neq p(r)$.

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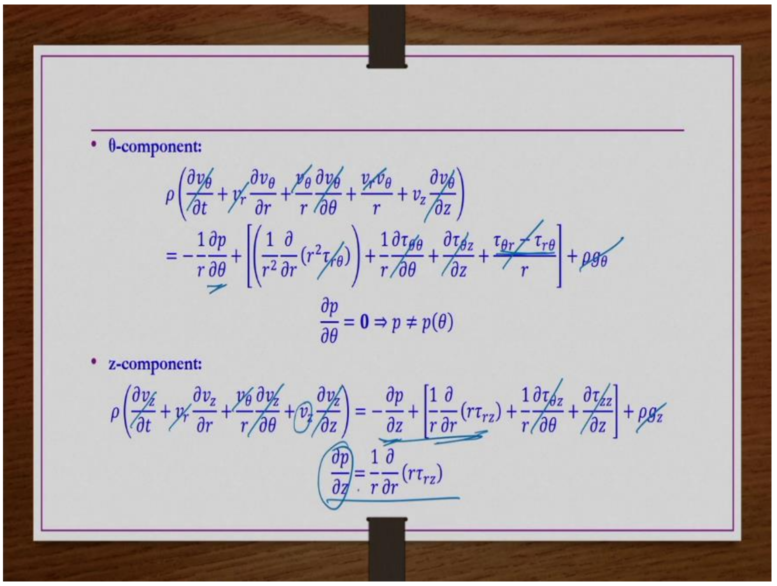

So, θ component of momentum equation given here, here also we apply the constraints. So, steady state this term is 0, v_r is not existing, v_θ is not existing symmetry. So, this term is 0, v_r v_θ both are 0 and then fully developed flow. So, $\frac{\partial}{\partial z}$ of anything is 0, pressure we cannot say we do not have any generalized boundary conditions anything.

So, only τ_{rz} is existing. So, this $\tau_{r\theta}$ is 0, because of symmetry this term is 0, because of fully developed flow this term is 0 and then these two are equals to each other is at least for the case of a simple laminar flow. So, then the difference is 0 and then there is no gravity.

So, what we get here? $\frac{\partial p}{\partial \theta} = 0$; that means, pressure is not function of θ as well. So, but in the flow problem, it is given the flow is taking place because of the pressure difference only and then the pressure difference is in z direction. So, that way we understand it is varying only z direction and then whether this variation is linear or non-linear what it is that we can understand by simplifying this z component of momentum equation that is given here.

So, steady state this term is 0, v_r is not there, v_θ is not there, symmetry this term is 0, v_z is existing, but because of the fully developed flow the other term is 0. So, left hand side all terms are negligible $\frac{\partial p}{\partial z}$ we cannot cancel out because of the fully developed flow condition. Because fully developed flow conditions is for the flow variables not for the scalers like temperature and pressure. So, $\frac{\partial p}{\partial z}$ has to be like that and then τ_{rz} is existing and then it is function of r.

So, we cannot cancel out this term also because of symmetry this term is 0 because of the fully developed flow this term is 0, gravity we are not taking into the consideration. So, then we have only these two terms. So, $\frac{\partial p}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})$ and then like previous cases here also we have seen left hand side term is only function of z whereas, the right hand side all terms are you know function of r.

So, what we can do? We can treat them individually and independently and then we can do the integration to get the required expressions.

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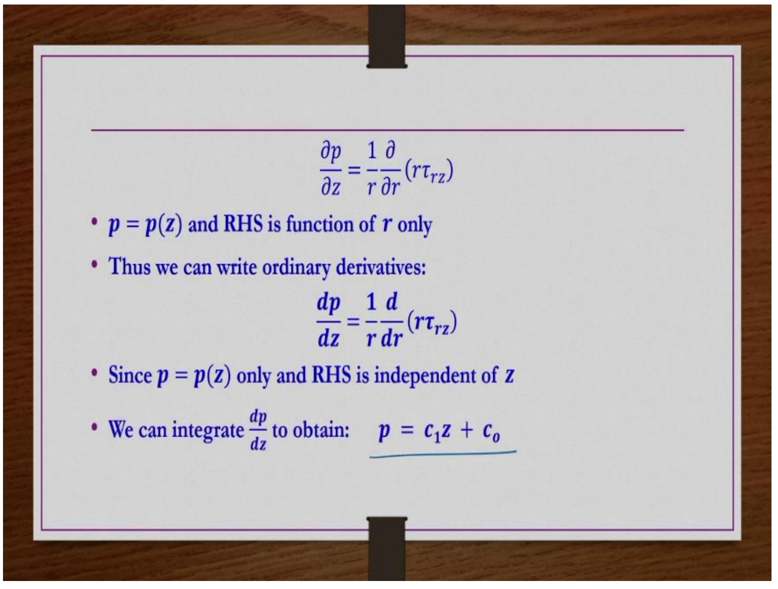

First we take $\frac{\partial p}{\partial z}$ is equals to some constant and then we integrate. So, then we get $p = c_1 z + c_0$ as expression.

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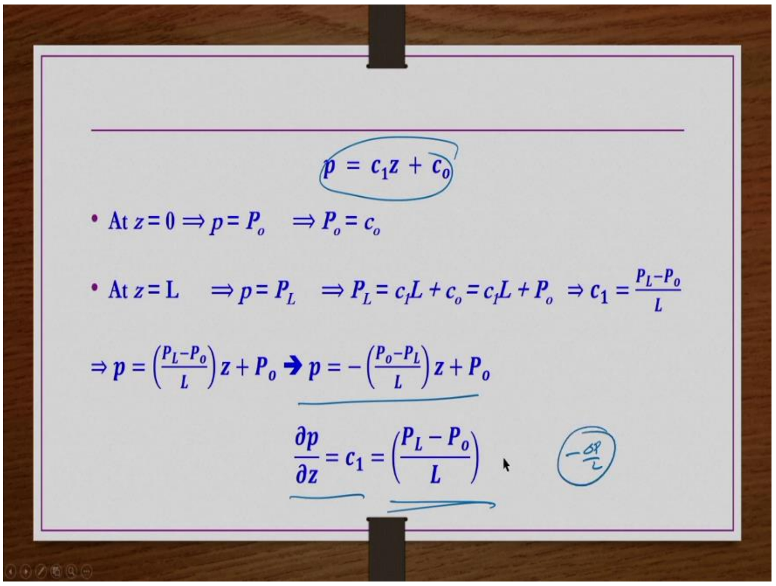

Now we apply the boundary condition at z = 0, p = P_0. So, c_0 = P_0 at z = L, p = P_L. So, c_1 we get $\left(\frac{P_L - P_0}{L}\right)$. So, when you substitute this c_1 and c_0 in this expression we get $p = -\frac{\Delta P}{L} z + P_0$ where whereas, $-\Delta P$ is nothing, but whatever this $-P_0 - P_L$ ok.

So, or we can say $\frac{\partial p}{\partial z}$ = c_1; that means, you know c_1; that means, what we can understand that pressure is dependent in the z direction, but the that is a linear dependence ok that is a linear dependence that is what we can understand. So; that means, $\frac{\partial p}{\partial z}$ we can take it as a constant, so that we can write it as $-\frac{\Delta p}{L}$.

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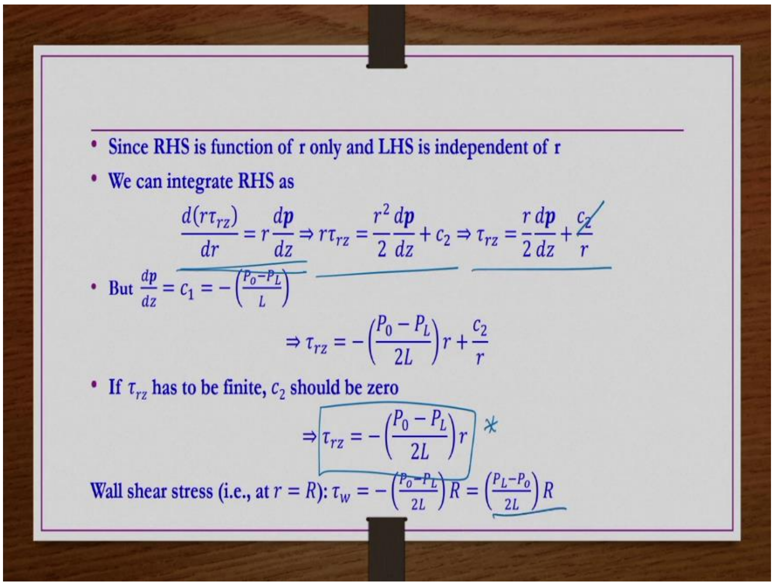

And the subsequent integration of this equation $\frac{\partial p}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})$. So, that can be written like this. So, r we have taken to the other side. So, then r$\frac{dp}{dz}$ we get right. So, then when you integrate you get this one for and then you bringing the r to the right hand side you get this expression.

Now, shear stress cannot be infinite for any value of r. So, c_2 has to be zero so; that means, shear stress you get $\left(-\frac{\Delta p}{L}\right)\frac{r}{2}$ this is what you get as a shear stress right. So, the same like you know previous case and until this point we have not incorporated anything about the rheology of the material.

So, it is valid for Newtonian, non-Newtonian fluid anything provided that the flow conditions are you know same like whatever we have listed in the one of these previous slides ok. So, wall shear stress if you wanted to find out you substitute r =R. So, then you get $\frac{-\Delta p}{L}\frac{R}{2}$.

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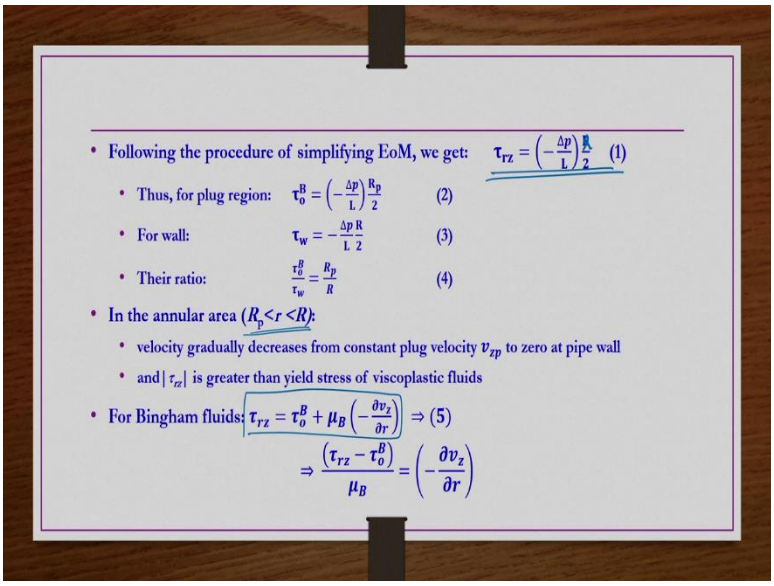

So, now this is the expression that we have. So, when r = 0 what you get? τ_r = 0 ok. So, $\tau_{rz} = \left(\frac{-\Delta p}{L}\right)\frac{r}{2}$. So, if r = 0, then $\tau_{rz} = 0$ that is at the centre shear stress is 0 when r = R then shear stress is maximum that is wall shear stress, when r = r_p then shear stress is nothing, but equals to the yield stress; that is the intermediate point or connecting point between the two flow regions, solid plug region and then deforming fluid kind of region ok.

So, that is. So, for the plug region $\tau_0^B = \left(\frac{-\Delta p}{L}\right)\frac{R_p}{2}$ for wall $\tau_w = \left(\frac{-\Delta p}{L}\right)\frac{R}{2}$ and then their ratio if you do $\frac{\tau_0^B}{\tau_w}$ is nothing, but $\frac{R_p}{R}$. This ratio will give you know fraction or portion of the cross section which is moving like a solid plug and then remaining portion moving like a viscous fluid ok.

So, now first in we have to get the velocity profile. So, first what we do? We get for the deforming region between $\frac{R_p}{R}$. Because that region is how to get the velocity profile is known to us and then it is straight forward because we have already done a few problems.

So, in the annular area or in the deforming area in the or in the region where the fluid is deforming, what is the expression for the shear stress? If it is a Bingham plastic fluid $\tau_{rz} = \tau_0^B + \mu_B\left(\frac{-dv_z}{dr}\right)$ this is what we know right.

So, now here in this case what we do? We can rearrange this equation and then get the v_z expression which is valid between R_p to R that is only deforming region ok. So, this equation now if you rearrange $\frac{\tau_{rz}-\tau_0^B}{\mu_B} = \left(\frac{-dv_z}{dr}\right)$ right.

So, in this equation in place of τ_{rz} we can write $\left(\frac{-\Delta p}{L}\right)\frac{r}{2}$ in case of τ_0^B we can write $\left(\frac{-\Delta p}{L}\right)\frac{R_p}{2}$ if required otherwise we keep it as it is and then do the simplification.

So, for τ_{rz} you have to substitute $\left(\frac{-\Delta p}{L}\right)\frac{r}{2}$ because it is varying because it is varying with r and then it has to be integrated accordingly with respect to r for the next step to get the velocity profile. For τ_0^B you do not need to substitute that equation number 2, here because it is constant anyway it is a characteristics of the material, r_p is the region up to which it is moving like solid plug ok or the point up to which shear stress is not crossing that yield stress.

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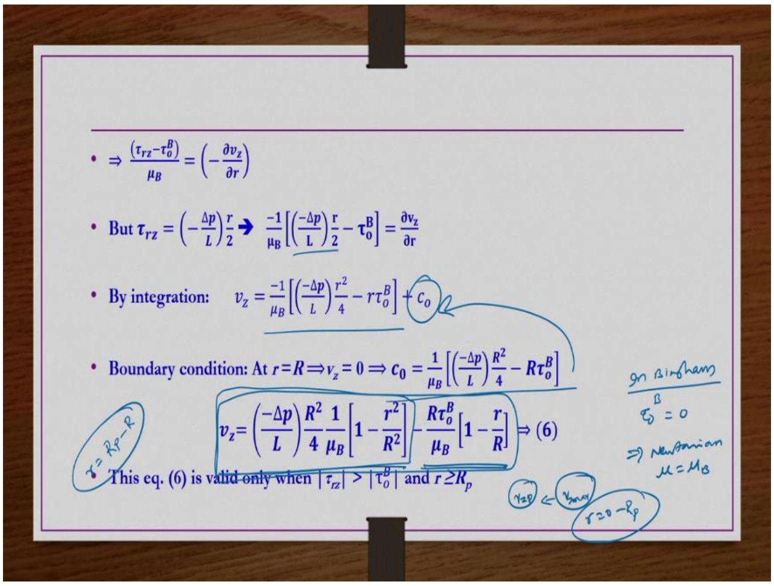

So, this expression now τ_{rz} we can substitute $\left(\frac{-\Delta p}{L}\right) - \left(\frac{-\Delta p}{L}\right)\frac{r}{2}$. So, then we have this expression right now you can integrate this equation. So, we get $\frac{-1}{\mu_B}\left[\left(\frac{-\Delta p}{L}\right)\frac{r^2}{4} - r\tau_0^B\right] + c_0$ some constant right.

So, if the r = R, then $v_z = 0$ because of the no slip condition. So, that we substitute then we get c_0 is equals to this expression. So, now, this you substitute here in place of c_0 and then you join the terms r^2 terms together and then r terms together then you get the velocity profile this one $v_z = \left(\frac{-\Delta p}{L}\right)\frac{R^2}{4}\frac{1}{\mu_B}\left[1-\frac{r^2}{R^2}\right] - \frac{R\tau_0^B}{\mu_B}\left[1-\frac{r}{R}\right]$.

Now, if you compare this one with the Newtonian case or before comparing with the Newtonian case if τ_0^B is 0 that means, what? In Bingham plastic fluids if you substitute if you substitute $\tau_0^B = 0$; that means, you get Newtonian behavior with $\mu = \mu_B$ right. So, under such conditions this term should be 0 because τ_0^B 0. So, then second term in the right hand side should be 0 right.

So, then what you have? You will be having v_z is only this part that is $\left(\frac{-\Delta p}{L}\right)\frac{R^2}{4\mu}\left[1-\frac{r^2}{R^2}\right]$ which is nothing, but parabolic profile and then which is for the Newtonian case which is for the Newtonian case ok. So, that kind of analysis we can do.

So, if τ_0^B is very small or negligible. So, then whatever the velocity profile you get, you get almost like a parabolic profile almost like if it is completely 0 then; obviously, it is Newtonian. So, then you should get the parabolic profile that you are getting by consider only first term in the right hand side of this equation right.

And now this equation is valid if it is having certain value of τ_0^B, this is valid between R_p to R value of r only right where $\tau_{rz} > \tau_0^B$ that we know ok. So, now next point is what is v_{zp} or plug like region or $v_{z\,max}$ because between r = 0 to R_p we know the material is moving with a constant maximum velocity right. So, what is that velocity if you wanted to find out in this equation you can substitute $r = R_p$.

Then simplify. So, you get the plug velocity you cannot substitute r = 0 where all to get this expression to get the required expression. Because at the center also at r = 0 also the material is having maximum plug like velocity at $r = R_p$ also it is having the same value maximum velocity value right. But you know this expression is valid only between R_p to r.

So, you cannot substitute r = 0 in this equation number 6. So, you have to substitute $r = R_p$ in this equation and then do the simplification to get the maximum plug velocity that you get like this right.

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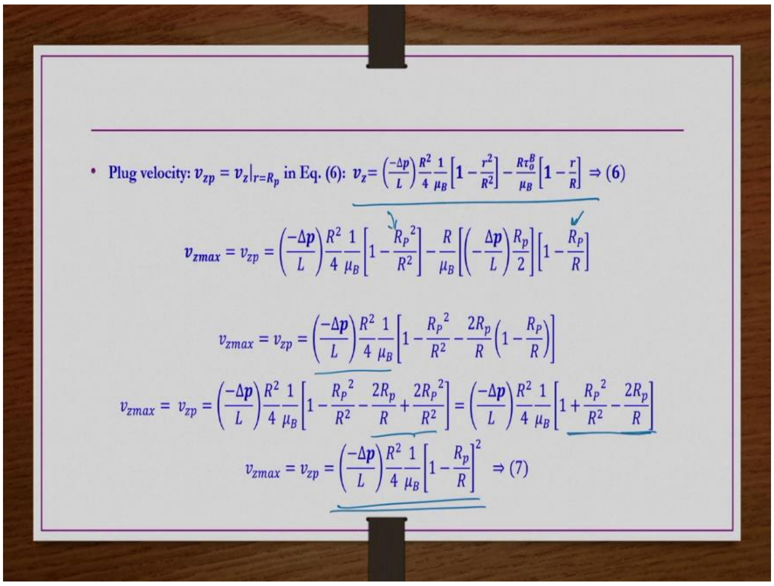

So, in this equation you substitute r = R_p here and then here. So, then you get this expression. So, further what we do? We can take these terms $\left(\frac{-\Delta p}{L}\right)\frac{R^2}{4}\frac{1}{\mu}$ we can take common and then remaining terms when we combined $\left[1-\frac{R_p^2}{R^2}\right]$. So, from here $-\frac{2R_p}{R}\left(1-\frac{R_p}{R}\right)$. So, you get. So, $-\frac{2R_p}{R}+\frac{2R_p^2}{R^2}$ you get next step right.

So, now this $\frac{2R_p^2}{R^2}$ and then this $-\frac{R_p^2}{R^2}$ when you do you get $+\frac{R_p^2}{R^2}-\frac{2R_p}{R}$. So; that means, this term we can write $\left[1-\frac{R_p}{R}\right]^2$.

So, maximum velocity or v_z or v_z plug velocity v_{zp} is given by this equation and then it is constant for a given pressure drop, for a given pressure drop for a given material viscosity plastic viscosity it is a constant value, it does not change with r ok. Because its one constant maximum value right.

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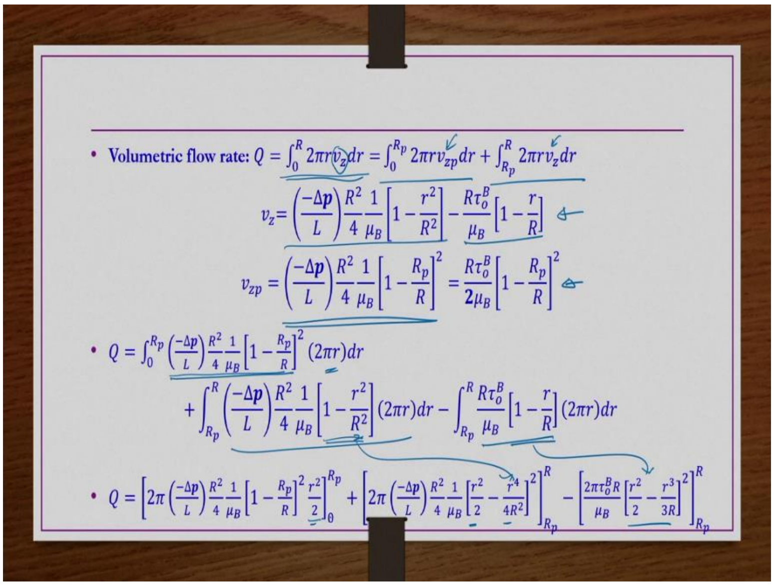

So, now what we do? Next step we find out the volumetric flow rate. So, volumetric flow rate is $\int_0^R 2\pi r v_z dr$ this is what we know now, but this v_z is having 2 fractions this v_z having 2 values one value or one expression between 0 to R_p and then another expression between R_p to R.

So, then that is we are writing correspondingly 0 to R_p we call the velocity maximum velocity maximum constant value v_{zp} and then v_z it is varying with r. So, corresponding equations we already derived in the previous slide. So, this v_z and this v_{zp}. So, v_{zp} we take this part of the equation and then we substitute here and then do the integration.

So, in this part this first term 0 to R_p $\int_0^{R_p} 2\pi r dr$ as it is and this part is v_z plus integral R_p to R then this part multiplied by $2\pi r dr$ minus this part multiplied by $2\pi r dr$ under integration. So, when you do the integration. So, this $\left(\frac{-\Delta p}{L}\right)\frac{R^2}{4}\frac{1}{\mu_B}$ all these are constant. So, then they treated as constant.

So, now you take this r you know this $\left[1-\frac{R_p}{R}\right]$ is also constant here. So, here this entire thing is constant. So, integration of r is $\frac{r^2}{2}$. So, here now r you have to bring inside the parenthesis integration of r dr is nothing, but $\frac{r^2}{2}$ here r^3 by here integration of $r^3 dr$ is $\frac{r^4}{4}$ that is what you get here again similarly you do the integration.

So, then first term $\frac{r^2}{2}$ corresponding to this one, second term $\frac{r^2}{2} - \frac{r^4}{4}$ corresponding to this one and then corresponding to this one we have $\frac{r^2}{2} - \frac{r^3}{3}$. Whereas, the constants R etcetera are kept as it is and then first term limits 0 to R_p second and third term limits R_p to r.

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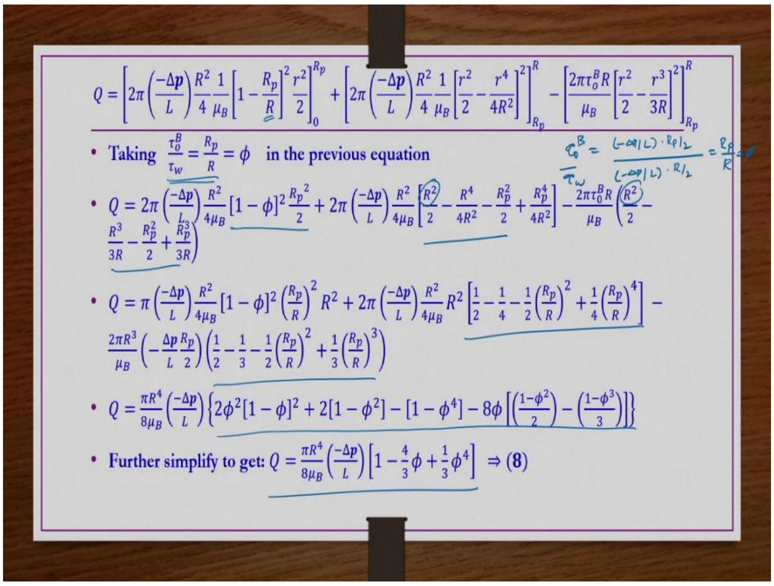

So, what we do? Now here we substitute in this case first case we substitute 0 to R_p limits, second case that is second and third terms we substitute the limits R_p to R. Then what we get? That is what we write and then further what we write? We also write $\frac{R_p}{R}$. $\frac{R_p}{R}$ is nothing but $\frac{\tau_0^B}{\tau_w}$ that we can write as ϕ. Because what is τ_0^B?

It is nothing, but $\left(\frac{-\Delta p}{L}\right)\frac{R_p}{2}$ and then what is τ_w? It is nothing, but $\left(\frac{-\Delta p}{L}\right)\frac{R}{2}$. So, that is nothing, but $\frac{R_p}{R}$ and then that $\frac{R_p}{R}$ I am just calling it v. So, we are substituting the limits and then same time we are wherever $\frac{R_p}{R}$ is there we are writing ϕ.

So, here for example, this term in place of $\frac{R_p}{R}$ if I write ϕ I get $(1-\phi)^2$ and then upper limit of this r is R_p. So, $\frac{R_p^2}{2}$. So, that is what we get. So, $(1-\phi)^2\frac{R_p^2}{2}$ you get here in the first term likewise all other terms you substitute the limits you get here.

In the next step what you do you take R^2 common from this second and third terms then what you have? You have these terms in the parenthesis because you get $\left(\frac{R_p}{R}\right)^2$ or $\left(\frac{R_p}{R}\right)^3$ those kind of terms you get. So, in the next step not only writing $\phi = \frac{R_p}{R}$ we also take common $\frac{\pi r^4}{8\mu_B}\left(\frac{-\Delta p}{L}\right)$. So, then remaining terms all the remaining terms have been written in the parenthesis like this.

Now, what you can do? You can combine the terms which are having similar let us say this $\frac{(1-\phi)^2}{2}$ these kind of terms are there. So, then you can combine them and then simplify them simple algebraic rearrangement will give you this expression; $Q = \frac{\pi R^4}{8\mu_B}\left(\frac{-\Delta p}{L}\right)\left[1 - \frac{4}{3}\phi + \frac{1}{3}\phi^4\right]$. So, this is the volumetric flow rate.

So, we got the plug velocity, we got the velocity in the deforming region and then we got the volumetric flow rate as well. So, now if you divide this volumetric flow rate by cross section area of the pipe we will get the average velocity. So, that we are doing now here.

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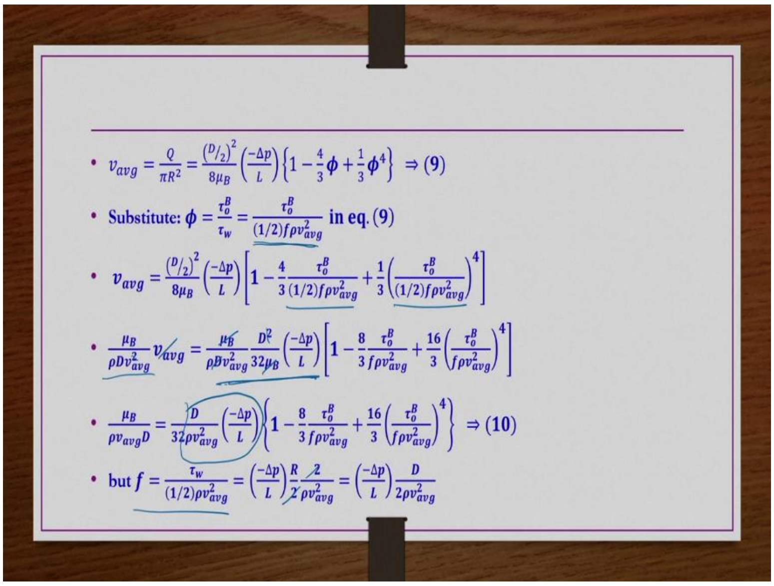

So, then you get this expression right. So, next what we do? We substitute in this equation $\phi = \frac{\tau_0^B}{\tau_w}$ because now we are going to derive expression for friction factor right. So, ϕ is nothing, but $\frac{\tau_0^B}{\tau_w}$. So, that τ_0^B we are keeping as it is and then relation between friction factor and then wall shear stress is nothing, but $\tau_w = \frac{f\rho {v_{avg}}^2}{2}$ that we are writing here.

So, in this equation whenever phi is there we are going write this one when you write you have this expression in place of ϕ you have this one, in place of ϕ^4 we have this expression right.

So, again here whatever the r^2 is there. So, that r square term we are writing $\frac{D}{2}$ we are writing $\frac{D}{2}$. So, because everything we wanted to write in terms of $D\rho v_{avg}$ and μ_B etcetera because so, that we can define the Reynolds number etcetera if required in the subsequent step.

So, now in this equation after substituting $\phi = \tau_0^B$ or $\frac{2\tau_0^B}{f\rho {v_{avg}}^2}$. Then what you do? Both sides you multiply by $\frac{\mu_B}{\rho D {v_{avg}}^2}$. So, that you know this v_{avg} and then this square can be cancelled out and left hand side we can have $\frac{1}{Re_B}$ expression.

So, likewise that we will be doing in the right hand side also, but simplification we will be doing further right. Because here you know rather writing so, many times we further simplify it. So, from here what we can see this D and then square of this one this μ_B this μ_B cancelled out. So, then you have $\frac{D}{32\rho {v_{avg}}^2}\left(\frac{-\Delta p}{L}\right)$ right.

So, now from the definition of friction factor we have $f = \frac{\tau_w}{\frac{1}{2}\rho {v_{avg}}^2}$ and then τ_w is nothing, but $\left(\frac{-\Delta p}{L}\right)\frac{R}{2}$. So, that should be multiplied by $\frac{2}{\rho {v_{avg}}^2}$. So, this 2 2 cancelled out then $\left(\frac{-\Delta p}{L}\right)$ in place of r we can write $\frac{D}{2\rho {v_{avg}}^2}$. Why are we writing this one? Because here we have $\frac{D}{\rho {v_{avg}}^2}\left(\frac{-\Delta p}{L}\right)$ is there.

So, in place of this one what I can write? $\frac{f}{16}$ because $f = \left(\frac{-\Delta p}{L}\right)\frac{D}{\rho {v_{avg}}^2}$. So, then here 32 is there; so, $\frac{32}{2} = 16$. So, in place of this term I can write $\frac{f}{16}$ Re_B that is what I can do or $\frac{f}{16}$ I can write for that moment. So, in place of this one I can write $\frac{f}{16}$. So, that I am going to do in the next step.

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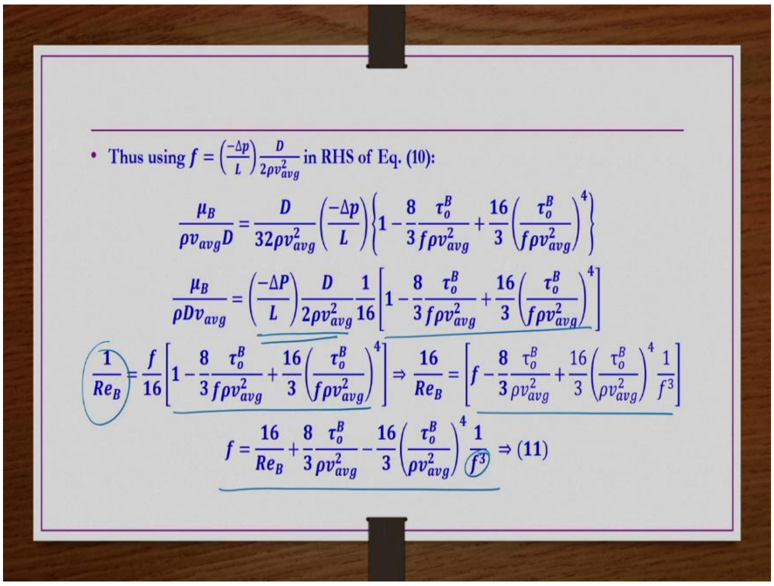

So, this is nothing, but your f. So, you have $\frac{f}{16}$ multiplied by whatever the terms are there as it is right and then left side left hand side term $\frac{\mu_B}{\rho D v_{avg}}$ is nothing, but $\frac{1}{Re_B}$. So, next step I am what I am doing? I am just taking this whatever 16 is there that to the other side and then bringing this f within the parenthesis. So, then I have this term here.

So, if I keep one f one side and then take the other terms to the other side then we can have this expression for the friction factor. So, we are going to further simplify this equation, but this itself we can take it as a kind of final expression right. So, what we can see here? The friction factor for the case of Bingham plastic fluid is not explicit right. It includes some kind of trial and error approach because in the right hand side also we are having f terms ok.

So, even if you know the geometrical information of the pipe like L D etcetera even if you know the material characteristics like μ_B ρ τ_0^B etcetera then also you cannot calculate the friction factor directly because in the right hand side also there is f term ok. So, it is a trial and error best approach should be followed. So, that this equation is satisfied when we substitute all the values of τ_0^B ρ v_{avg} μ_B etcetera right.

So, next what we do? We try to write this equation in a similar form like $\frac{16}{Re}$ form. So, that we can have a kind of analogous to Newtonian form what correction factor are we going to have in the case of Bingham plastic fluid that is what we can have.

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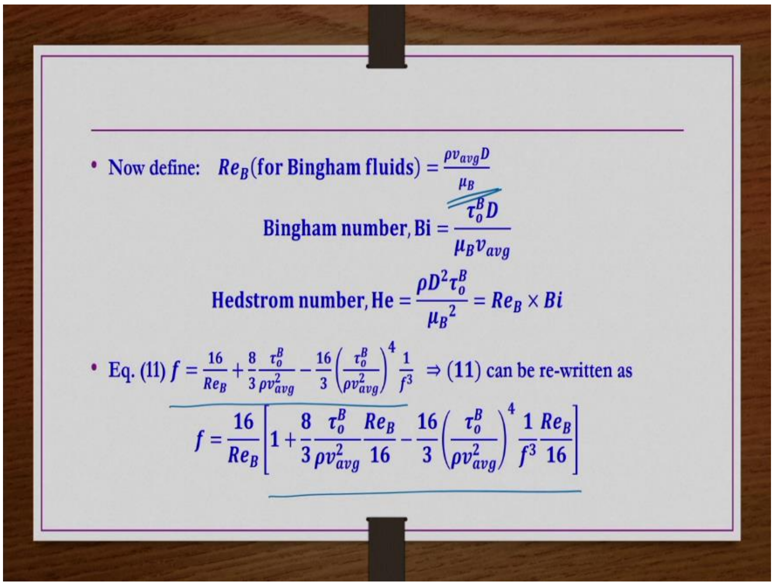

So, before that what we do? We write dimensionless numbers Bingham fluids $\rho\ Re_B$ is nothing, but $\frac{\rho D v_{avg}}{\mu_B}$ and then Bingham number B_i is nothing but $\frac{\tau_0^B D}{\mu_B v_{avg}}$. Bingham number Bi $= \frac{\tau_0^B D}{\mu_B v_{avg}}$ this is what we have.

So, now when you multiply these two Re_B and then this Bingham number B_i, then again we will get another dimensionless number because both of them are dimensionless. So, that we call Hedstrom number and then when you do you get $\frac{\rho D^2 \tau_0^B}{\mu_B{}^2}$ you get.

So, we make use of these numbers in that expression whatever we got this equation number 11 this expression we got right. So, first we do $\frac{16}{Re_B}$ taking common from the right hand side. So, then we have this term.

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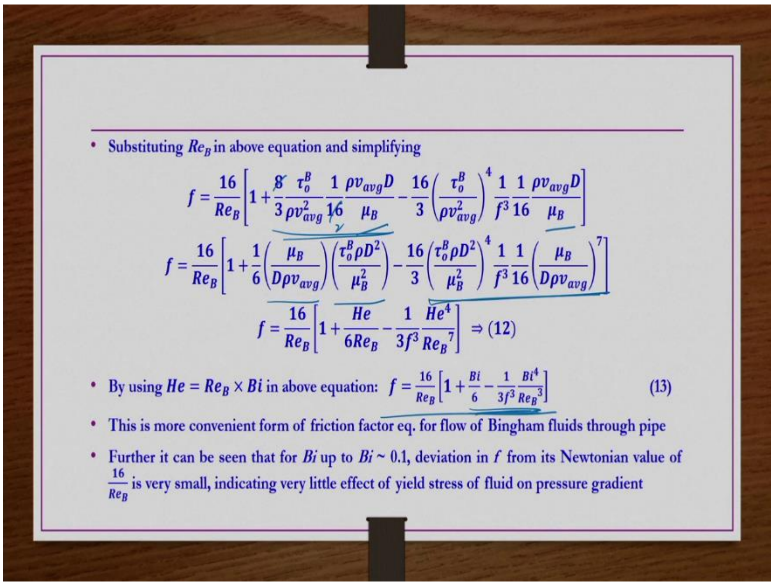

In the next step what we do? We write $Re_B = \frac{\rho v_{avg} D}{\mu_B}$ here and then here also, then what you do? We combined these terms such a way that we have $\frac{1}{6}$, here because 8 1's are 8 1's are 8 2's are and then 3 2's are 6 we get here.

So, $\frac{\mu_B}{D\rho v_{avg}}$ one terms and then remaining terms when you have from this part then you get $\frac{\tau_0^B \rho D^2}{{\mu_B}^2}$ which is nothing, but Hedstrom number same thing we do for the last term in the right hand side. So, then you get this one so; that means, this is nothing, but Hedstrom number this is nothing, but inverse of the Reynolds number.

So, you get $f = \frac{16}{Re_B}\left[1 + \frac{He}{6Re_B} - \frac{1}{3f^3}\frac{He^4}{{Re_B}^7}\right]$ right. The same expression in place of Hedstrom number if you write $Re_B x Bi$ and then simplify you get $f = \frac{16}{Re_B}\left[1 + \frac{Bi}{6} - \frac{1}{3f^3}\frac{Bi^4}{{Re_B}^3}\right]$ right.

So, what we can say this Bi is nothing, but or Bingham number is nothing, but dimensionless yield stress dimensionless yield stress of that particular material Bingham plastic material ok.

So, if the yield stress is small. So, then Bingham number will also be small. So, then let us say if Bingham number is less than 0.1 or 1 something like that. So, then altogether the second term onwards second third term you can neglect. So, then what you get? The behavior or friction factor would be close to the case of the Newtonian fluid ok.

So, let us say Bingham number is 0.1 or something like that you can happily ignore the second and third term in the right hand side of this equation number 13 to get the friction factor and then that friction factor would be very much close to the corresponding Newtonian value of having viscosity same as μ_B viscosity or whatever the yield stress is there that is having very negligible effect in the flow phenomena ok. So, that is what we can understand. So, now before concluding today's lecture we will take an example problem.

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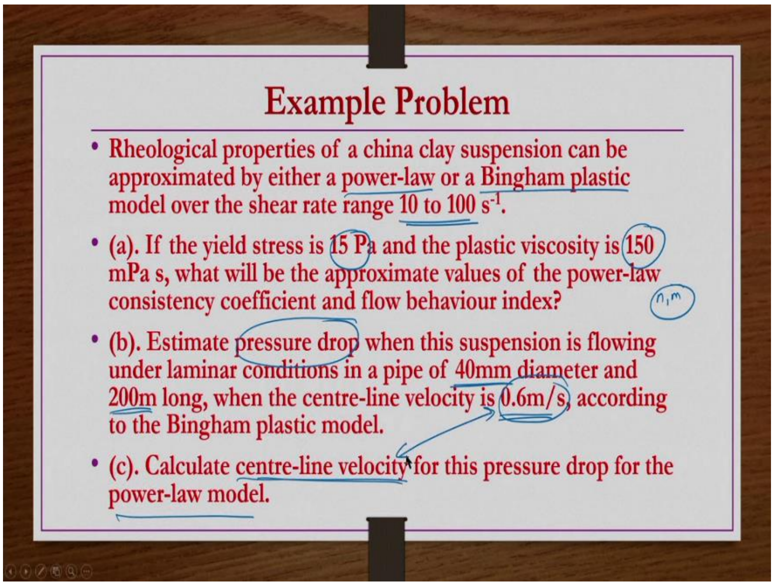

So, rheological properties of a material is given and then it has been found that either power law nature or Bingham plastic model can be you know used to represent that particular materials rheology when shear rate is between 10 to 100 second inverse. The rheological behaviour is such a way that you can use either power law model or Bingham model for that fluid if shear rate is 10 to 100 second inverse.

So, then if the yield stress is 15 pascals the plastic viscosity 150 milipascals second. What will be the approximate values of the power law consistence index and then power law consistency coefficient? That is n and m this part of the problem I think we have done already in one of the lecture in the first or second week ok. So, anyway, but this is having continuation of the problem.

So, estimate the pressure drop when the suspension is flowing under laminar condition in a pipe of D = 40 mm and then l is equals to 200 meters when the center line velocity is 0.6

meter per second. Because the same material whatever the rheological information obtained from the rheometer, it shows that you can use a either power law or Bingham plastic behavior for that data.

So, when you apply. So, then what are the corresponding center line velocity for the Bingham plastic model is given right. So, then from that you have to find out the pressure drop $-\Delta p$ you have to find out ok. So, v_{zp} expression we just derived. So, in that v_{zp} expression v_{zp} that is maximum velocity 0.6 meter per second is given.

So, in the right hand side you have $\left(\frac{-\Delta p}{L}\right)\frac{R^2}{4}\frac{1}{\mu_B}\left[1-\frac{R_p}{R}\right]^2 - \frac{R\tau_0^B}{\mu_B}\left[1-\frac{r}{R}\right]$ that term is there. So, then you substitute all those values you get the required $-\Delta p$, but that will also be required you know trial and error approach though it is a constant value because R_p value we do not know it is not given ok. So, that we do.

Then after this let us say for this pressure drop whatever the pressure drop you obtain in the b part of the problem and then same pressure drop is applied for the power law model how much would be the center line velocity? Are you going to get the same point six meter per second or you get into higher get the higher maximum velocity or lower maximum velocity compared to the Bingham plastic model if the same pressure drop is applied. So, that is what we are going to see now ok.

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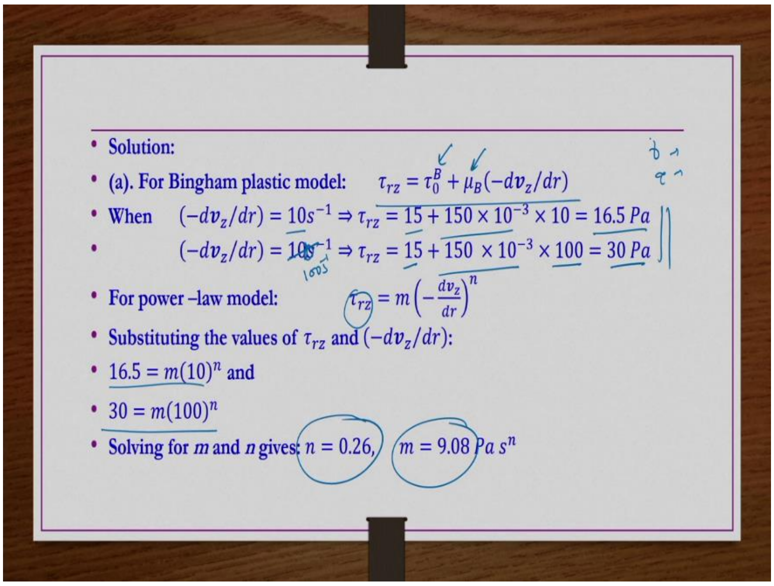

So, first part for Bingham plastic model this is the expression that we know right. Now here when $\dot{\gamma}$ or $\left(\frac{-\,dv_z}{dr}\right) = 10$ what is τ_{rz} you have to find out. Similarly, when $\dot{\gamma}$ or $\left(\frac{-\,dv_z}{dr}\right)$ =100 what is τ_{rz} you have to find out because τ_0^B and then μ_B are given.

So, that when you do when $\dot{\gamma}$ is 10 second inverse $\tau_{rz} = \tau_0^B$ is 15; μ_B is 150 milli pascal second and then $\left(\frac{-\,dv_z}{dr}\right) = 10$. So, 16.5 you are getting. Similarly, when this $\left(\frac{-\,dv_z}{dr}\right) = 100$ second inverse. So, then $\tau_{rz} =$ 15 + 150 milli pascal second into or multiplied by 100 you get 30 pascal second right.

So, corresponding to two limits of gamma dot what are the two limits of τ we know. So, for these two limits what we do now? We apply power law model. For the power law model now what happens here τ_{rz} is known now two limits of τ_{rz} are known from this calculations when we applying the Bingham plastic model. Because for the same data you can apply either Bingham plastic or power law model that is given in the problem statement.

So, when you apply this one when $\dot{\gamma}$ is 10 second inverse what is τ_{rz}? 16.5. So, 16 point 16.5 is equals to m multiplied by 10 power n and then when $\dot{\gamma}$ or $\left(\frac{-\,dv_z}{dr}\right)$ is 100 second inverse what is τ_{rz}? It is 30. So, 30 is equals to m multiplied by 100^n. So, now you have two equations and then two unknown m and n when you solve you get n is equals to 0.26 and then m is equals to 9.08 pascal second power n. So, first part is done.

Second part of the question. Finding the pressure drop $-\Delta p$ if v_{zp} or $v_{z\,max}$ is 0.6 meter per second in the case of Bingham model is applied.

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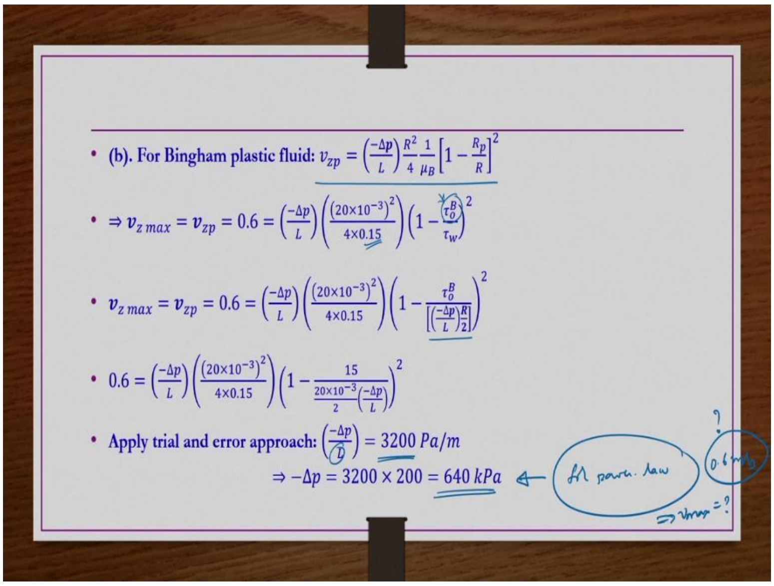

So, for the Bingham model v_{zp} we just found this expression right. So, now here v_{zp} is given as 0.6 $-\Delta p$ is not given we have to find out and then r is given or d is 40 mm. So, r is 20 mm that is given μ_B is given as 150 milli pascal second. So, that is 0.15 pascal second and then $\frac{R_p}{R}$ is nothing, but $\frac{\tau_0^B}{\tau_w}$ we just have seen in the derivation right. So, that we are writing.

Then τ_0^B is given right it is given in the material as a kind of material characteristic in the problem. τ_w we can write it as $\left(\frac{-\Delta p}{L}\right)\frac{R}{2}$ remaining we are keeping as it is in this step right. So, next step what we can have? In place of r here you can write 20 mm or 20 multiplied by 10 power minus 3 and then write the keep the remaining terms as it is.

So, now you apply the trial and error approach and then get this $\left(\frac{-\Delta p}{L}\right)$ you get 3200 pascals per meter approximately L is also given. So, L is given as 200 meters. So, then if you multiply this number by 200 you get 640 kilopascals. So, second problem is done second part of the problem is done.

Now, third part is if this pressure drop is applied for the case of power law nature. Let us if you apply power law model for that rheological data and then you provide this much of pressure drop how much is $v_{z\,max}$ or v_{max}? Is it going to be same like 0.6 meter per second which is for the case of Bingham plastic fluid or is it going to be higher or lower that has that we have to see.

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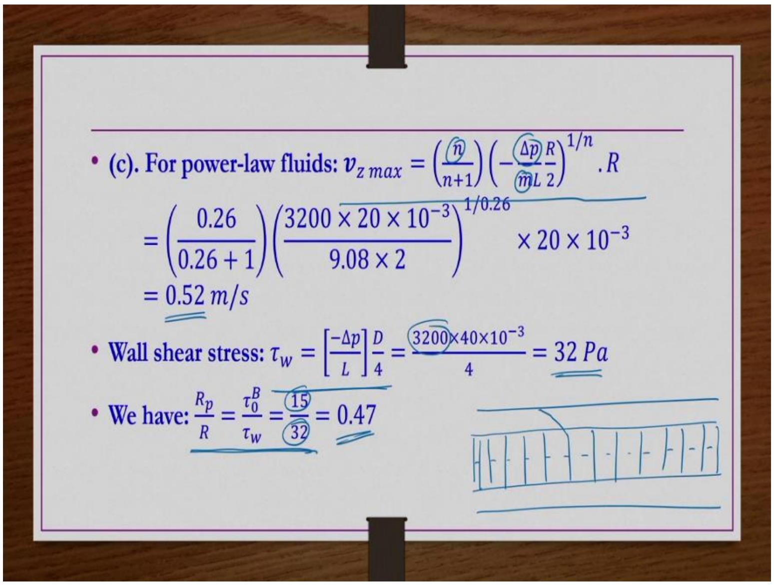

In one of the previous lectures what we have seen? For power law fluids $v_{z\,max}$ is nothing, but this expression. So, just now $-\Delta p$ we found n and m values also we have already got in the part A of the problem right l is given, r is given. So, everything you substitute here and then substitute and then do the simplification you get $v_{z\,max}$ = 0.52 meter per second.

So, what does it mean? When a material rheology for this problem when that particular whatever the china clay material is given, if that is rheology is such a way that either if you can apply power law or Bingham plastic fluids it is not going to be you know causing too much problem to your engineering solutions or process whatever.

Then when you apply the Bingham plastic model you are going to get the maximum higher center line velocity or higher maximum velocity which is 0.6 meter per second, but if you change the model rheological model to power law behavior you are getting lower maximum velocity that is 0.52 meter per second ok; this kind of analysis we can do by solving this problem.

Further corresponding wall shear stress which is not known. So, now, for the corresponding $\left(\frac{-\Delta p}{L}\right)$ of 3200 you find out τ_w that comes out to be 32 pascals. So, which is very small value in general fine.

So, $\frac{R_p}{R}$ then you get $\frac{R_p}{R}$ is nothing, but $\frac{\tau_0^B}{\tau_w}$. τ_0^B is given as 15 τ_w just now you got corresponding to $\left(\frac{-\Delta p}{L}\right)$of 3200 you got τ_w as 32. So, 15/32 is 0.47 that is approximately half of the cross section is flowing like a plug this $\frac{R_p}{R}$; $\frac{R_p}{R}$fraction whatever is there you know you know what is it is? It is that is that much of fraction.

Let us say if this is your pipe this center entire pipe and this is this center of the pipe. So, and then this is almost like you know half of the pipeline dimension. So, in this region the material is flowing like a solid plug and then after this it is moving like this decreasing like this. So, almost half of the cross section half of the pipe portion is moving like a solid plug with a maximum velocity of 0.6 meter per second that is what we can understand from this $\frac{R_p}{R}$ value.

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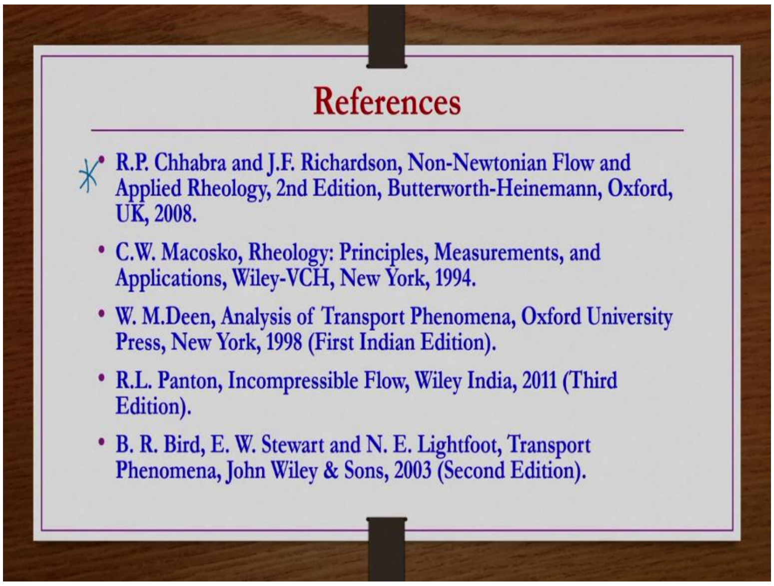

Reference the entire lecture is prepared from this excellent book by Chhabra and Richardson. Other useful references are provided here.

Thank you.