Transport Phenomena of Non-Newtonian Fluids

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Lecture - 15
Power-law and Ellis Model Fluids Flow through Pipes

Welcome to the MOOCs Course Transport Phenomena of Non-Newtonian Fluids. The title of this lecture is Power-law and Ellis Model Fluids Flow through Pipes. So, before going into the details of today's lecture what we will do? We will have a kind of recapitulation of what we have studied in the last class.

In the last class what we have discussed? We have discussed you know how to develop the velocity profile for a time independent non-Newtonian fluid or specifically in the case of a power law fluids flowing through a pipes circular pipe that is what we have seen.

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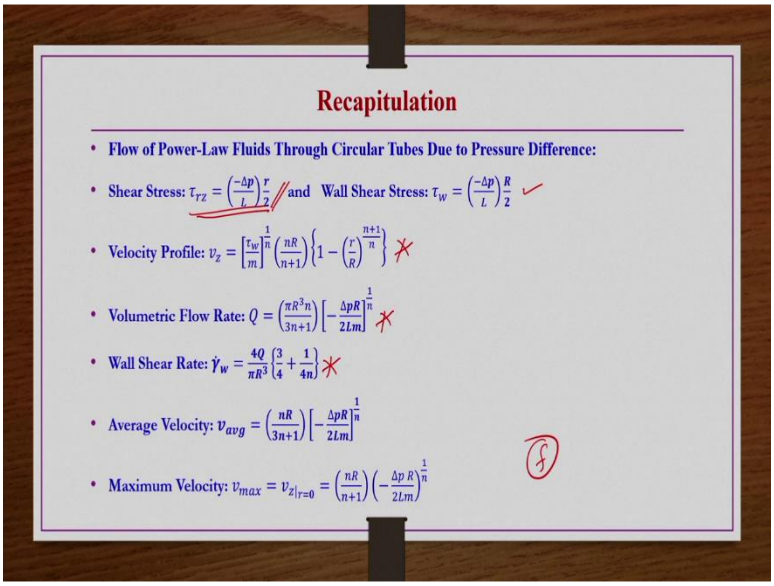

So, we have a recapitulation of that one in the previous class we have seen the flow of power law fluids through circular tubes due to pressure difference if it is taking place. So, how to obtain the velocity profile how to obtain the volumetric flow rate how to obtain the average and maximum velocity etcetera those things we have discussed right.

So, that what we get we got first term by simplifying the equations of continuity and then momentum. What we got a few information about the pressure distribution and then relation between pressure and then shear stress etcetera then we realize that now for the case of flow through circular tubes due to the pressure difference the shear stress that we got it as $\tau_{rz} = \left(\frac{-\Delta p}{L}\right)\frac{r}{2}$.

Remember till this point we did not incorporated any information regarding the rheology of the fluids right. So; that means, any fluid that is flowing through a pipe due to the pressure difference then shear stress can be related with this particular expression irrespective of the rheology of the fluid that is what we can understand. And remember all that the analysis whatever we have done in the previous class is for one dimensional motion where only v_z component of velocity is existing and that v_z is function of r right.

So, for such conditions shear stress is a linear function of pressure gradient and then the shear stress is 0 at the center and then maximum at the wall. And then between these two limits of r = 0 and then r = R shear stress linearly increases. And then wall shear stress that is the maximum shear stress that you wanted to find out you have to substitute r = R that we get this one.

Then after this in this equation what we have done we have substituted for τ_{rz} power law fluid for power law fluid $\tau_{rz} = m\left(\frac{-\,dv_z}{dr}\right)^n$ that is what we know and in that we substituted in this expression and then simplified then applied boundary conditions to get the velocity profile like this.

Once getting the velocity profile what we have done? We got the volumetric flow rate and then that is given by this expression. Then we got the wall shear rate is nothing but $\frac{4Q}{\pi R^3}\frac{3n+1}{4n}$ this is what we got. Then average and then maximum velocity expressions we got like this is what we have seen right.

But whenever there is a fluid flowing through pipe. So, then what is that important engineering property that you would prefer to measure when especially there is only momentum transfer. And then; obviously, what is the property that you would prefer to have the information about is the pressure difference or pressure gradient versus volumetric flow rate information that is what you get. Also in addition you try to have the information what is the friction factor.

This is one of the very important information required for the designing of this you know wherever this pipe information connecting pipe straight pipe etcetera are there. So, then this friction factor is very essential factor that one should have for the design as well as you know controlling the operational parameters.

So, that friction factor will try to measure for the case of power law fluids flowing through circular pipes right. Friction factor when Newtonian fluid flowing through circular pipes and then flow is under the laminar flow condition. Then what we know? Friction factor $f = \frac{16}{Re}$ that we know. So, what is that we are going to have for the power law fluids case that we are going to do now in today's class.

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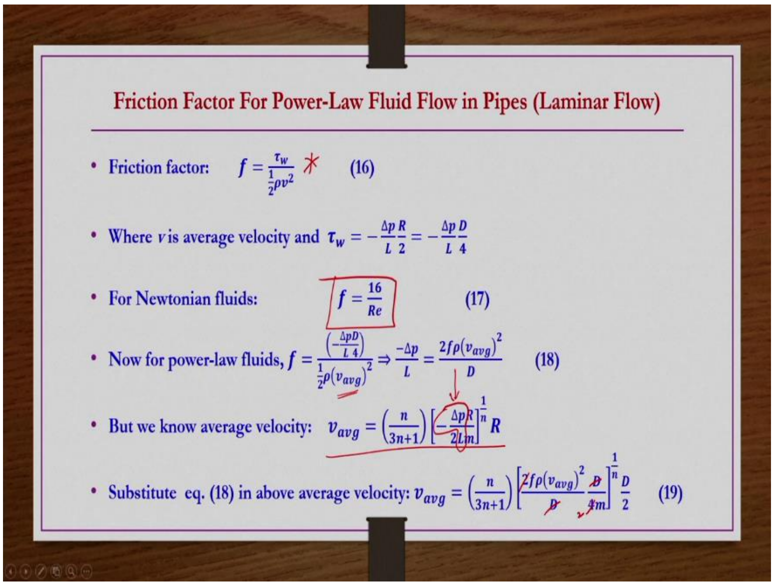

So friction factor how we can define? We can define like a dimensionalize wall shear stress that is wall shear stress non-dimensionalized using the kinetic energy $\frac{1}{2}\rho v^2$. Because this friction factor is essential when the fluid is under motion ok. So, this $f = \frac{\tau_w}{\frac{1}{2}\rho v^2}$ is the definition that we can have. So, then now this f friction factor we have to find out for the case of power law fluids.

Here this v is nothing but the average velocity and then τ_w is nothing but the wall shear stress that is $\left(\frac{-\Delta p}{L}\right)\frac{R}{2}$ or $\left(\frac{-\Delta p}{L}\right)\frac{D}{4}$. Now, for Newtonian fluids we already know that this $f =$

$\frac{16}{Re}$ that we have derived in our basic fluid mechanics course or you know basic transport phenomena course right.

Now, what is this if the fluid is non-Newtonian fluid especially if the fluid is power law fluid what is this f factor that we are going to see now. So, from this definition equation number 15 what we have $\frac{\tau_w}{\frac{1}{2}\rho v^2}$. Here in place of τ_w we can write $\left(\frac{-\Delta p}{L}\right)\frac{D}{4}$ and then divided by $\frac{1}{2}\rho {v_{avg}}^2$. This equation if you rearrange you can write $\frac{-\Delta p}{L} = \frac{2f{v_{avg}}^2}{D}$ right.

So, now if you like you know delta P that is applied pressure difference that you know D, L that is geometry of the pipe that length and then dimensions of the pipe that is length and diameter etcetera you know, density of the fluid also you know. If you know the average velocity then you can find out the friction factor right. So, in the previous lecture we obtain the average velocity for a power law fluid flowing through circular tube. So, that is nothing but this right $\frac{n}{3n+1}\left[\frac{-\Delta PR}{2Lm}\right]^{\frac{1}{n}}$ R right.

So, now what we do here? Whatever $\frac{-\Delta p}{L}$ is there in this expression in place of $\frac{-\Delta p}{L}$. We are going to write $\frac{2f{v_{avg}}^2}{D}$ when you write it is this one this is what you get. So, this R also what I am trying to do I am trying to write it as $\frac{D}{2}$.

So, $\frac{D}{2}$ and then there is already 2. So, $\frac{D}{4m}$ and then in place of $\frac{-\Delta p}{L}\frac{2f{v_{avg}}^2}{D}$ and then this is all under whole power 1/n and then this R is also $\frac{D}{2}$ I am writing. So, this D this D is cancelled out these 2 and then 2 1's are 2 2's are. So, this is what we get.

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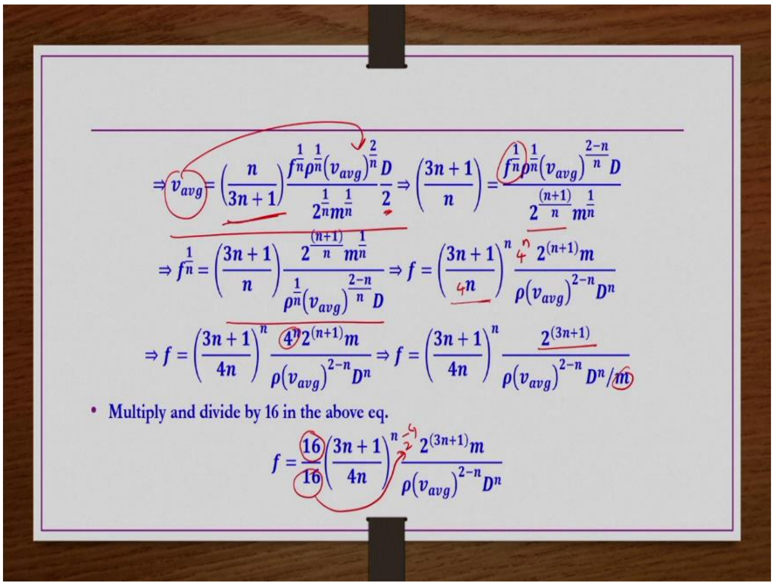

So what we do? This expression further you simplify you get this equation right. What we have done? Whatever the f ρ that whole power 1/n is there then we expanded or individually we have written like $f^{1/n}\rho^{1/n}v_{avg}{}^{2/n}$. So, that is $\frac{v_{avg}{}^{2/n}}{2^{1/n}m^{1/n}}$ we are having and then this $\frac{D}{2}$ as it is.

Now out of this what I am trying to do? I am trying to take this constant $\frac{n}{3n+1}$ to the left hand side. So, that I have $\frac{3n+1}{n}$ and then this v_{avg} I will bring in to the right hand side and then combined with this v_{avg}. So, that I have $v_{avg}{}^{\frac{2}{n}-1}$ that is $v_{avg}{}^{\frac{2-n}{n}}$.

What I am trying to do further? I am trying to join this $2^{1/n}$ and then 2, so that I can have $2^{\frac{n+1}{n}}$. Now, if I keep only f power 1/n one side and then all other terms to the other side. Then I have this term right then next step both sides I am trying to do whole power n.

So, then left side I have f that right side I have $\left(\frac{3n+1}{n}\right)^n \frac{2^{n+1}m}{\rho v_{avg}{}^{2-n}D^n}$ right. So, this $\frac{3n+1}{4n}$ is a common factor that we get in general for shear rate expressions whatever we have seen previously also when we are talking about the capillary viscometer.

So then what I am trying to do next step? I am trying to multiply this term by 4^n and then divide the term right side by 4^n. So, that I can write this f = $\left(\frac{3n+1}{4n}\right)^n$ 4^n and then remaining terms are as it is. Next step what I am trying to do? I am trying to write this 4 as 2^2. So, that 4^n I can write 2^{2n}. So, that 2^{2n} is combined with 2^{n+1} I can how 2^{3n+1} and then all other terms are same as it is and then this m I am taking to the denominator here like this right.

So now next step what I am trying to do? I am trying to multiply this expression by 16 and then divide by 16. So, that 16/16 and then all this as it is. So, now in the next step what I try to do? I keep the 16 as it is and wherever the denominator 16 is there that I will take to the numerator and then write it as 2^{-4}.

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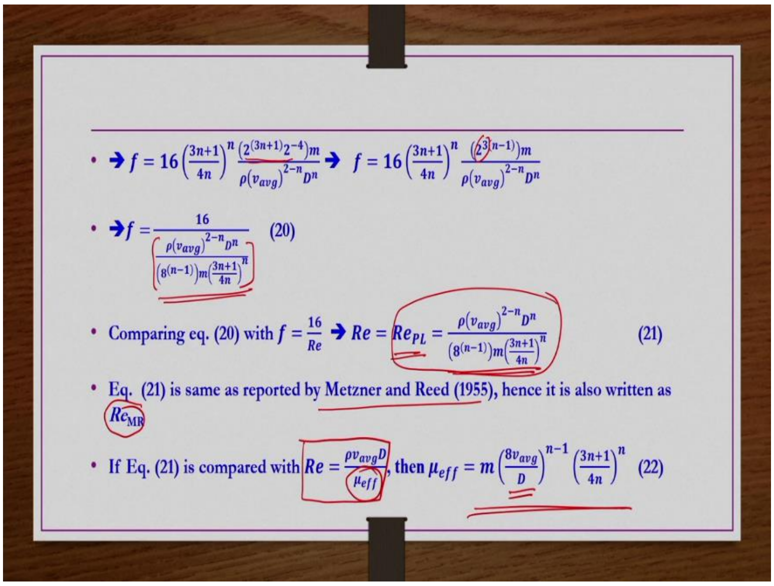

When I write it then what I get? 2^{-4} and then 16 is here as it is. So, then this term is as it is. So, this term I can write it as when you join these 2 terms I can we can write it as 2^{3n-3} that is $2^{3(n-1)}$. So, 2^3 I can write as 8. So, then what I can have? This term $\frac{16}{\frac{\rho(v_{avg})^{2-n}D^n}{\left(8^{(n-1)}\right)m\left(\frac{3n+1}{4n}\right)^n}}$.

So, now this is in a forms $f = \frac{16}{Re}$ form for Newtonian case friction factor $f = \frac{16}{Re}$ that form we have written for the case of power law fluids. So, by analogy what we can write whatever this denominator this thing is that expression that entire thing we can write it as Re for power law fluids. So, then we are writing Re_{PL}. So, Re_{PL} is nothing but $\frac{\rho(v_{avg})^{2-n}D^n}{\left(8^{(n-1)}\right)m\left(\frac{3n+1}{4n}\right)^n}$.

So, this Re_{PL} expression whatever is given is also same as given by the Metzger and Reed. So, it is also written as Re_{MR} in some books. So then what we have? If you write this Re_{MR} or Re_{PL} in the form like $\frac{\rho v_{avg} D}{\mu_{eff}}$ in terms of μ_{eff}. Because we know that for non-Newtonian fluids the viscosity changes.

So, which viscosity should you take? So, in that way if you think if you write it as in place of viscosity you simply writing viscosity because if viscosity changing Reynolds number would also be changing. So, rather writing changing viscosity with shear rate if you write effective viscosity in the definition of Reynolds number.

Then what should be that effective viscosity? That should be nothing but $m\left(\frac{8v_{avg}}{D}\right)^{n-1}\left(\frac{3n+1}{4n}\right)^{n}$ this $\frac{8v_{avg}}{D}$ is nothing but the nominal shear rate or true shear rate for the Newtonian case and then nominal or apparent share rate for the case of non-Newtonian fluids that we have already seen in the case of a capillary viscometers fine.

So, this is what about the friction factor for a power law fluid flowing through circular tubes right. So, now what we do? We take a different type of fluid flowing through circular tubes, but again due to the pressure difference.

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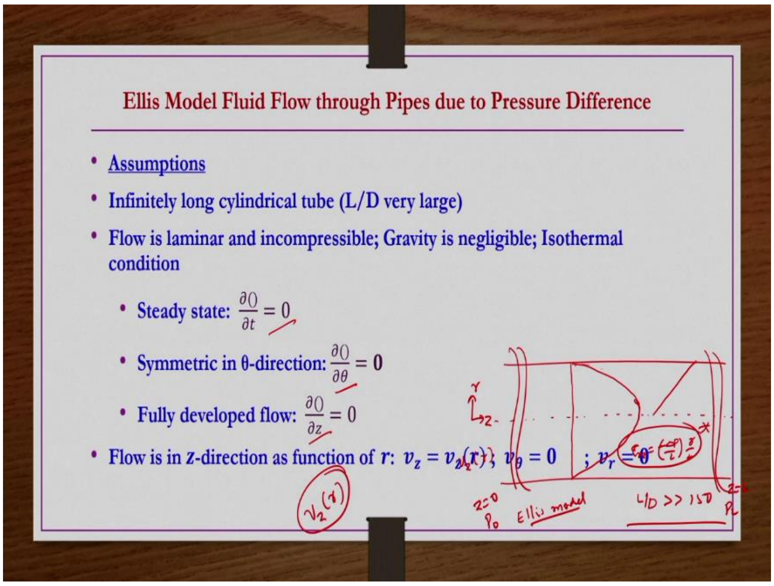

So; that means, whatever previous lecture that we have taken a circular tube infinitely long cylindrical tube right. So, that is L/D ratio is very large. So, then we have taken the only the fully developed region or we have taken L/D very large. So, that we have a fully developed flow. So, that if you have a fully developed flow the analysis would be easy for you that is in the flow direction the velocity profile will not change.

Let us say if this is your r direction if this is z direction. So, then what you have the flow direction only v_z you are having and then that is function of r. So, that velocity profile if it is Newtonian fluid then. So, if it is a Newtonian fluid the velocity profile is parabolic like this which we have already seen. We have already seen in the previous lecture and then depending on the non-Newtonian behavior or the power law index the profile changes.

It becomes flatter for shear thinning fluids and then it become very steeper for sharp kind of thing for shear thinning fluid that we have seen right and then shear stress is linearly increases with r ok. So, that is what we have seen. So, all those conditions are same here again only thing that only change that we are going to have is that in place of power law fluid.

Now we are taking Ellis model fluid right. So, the pressure at the inlet z = 0 is nothing but P_0 at the pressure at z = L is nothing but P_L and then because of this pressure difference the flow is taking place and then the all this analysis whatever we have done in the previous lecture. And then whatever we are going to do today's lecture that is only for the fully developed region it is not valid for the entry region or exit region ok. So, fully developed flow you can have when you have L/D is very large right

So, since compared to the previous lecture everything is same except the fluid. So, then what we have? We quickly go through the initial constrains assumptions etcetera simplifying of continuity equation momentum equations etcetera quickly. Because they are going to remain same they are not going to change.

The change compared to the previous lecture for the case of a power law fluid will occur only from the point where you get this expression $\tau_{rz} = \left(\frac{-\Delta p}{L}\right)\frac{r}{2}$. So, till get getting this point you are going to have exactly the same thing whatever we have done for the case of power law fluids in the previous lecture right.

So, but; however, since we are in the beginning of solving such kind of problem we will do once again ok. So how we do? We first list out all your assumptions or constraints of the problem then based on those constraints we are going to simplify the continuity and then momentum equations and then from those are simplified continuity and momentum equation you may be getting some relations using those relations you may get the velocity profile.

So, that process is to the standard process. So, that we are going to follow here also. So, assumptions we have taken infinitely long cylindrical tube that is L/D is very large and then flow is laminar and incompressible. We are not considering gravity; we are not considering non isothermal conditions flow is isothermal.

Then steady state and then symmetric in θ direction and then a fully developed flow that is what we have taken. And then in addition to that what we have we have only v_z component of velocity and then that v_z component of velocity is varying in the radial direction r direction right, v_θ v_z v_θ v_r may also be there some values.

But their magnitude wise they are very much small, they are very much small compared to the v_z and then especially in the radial direction they may be very very small ok. So, compared to the v_z v_θ v_r are negligible that is what the basic bottom line ok. So, now, these are same as previous class.

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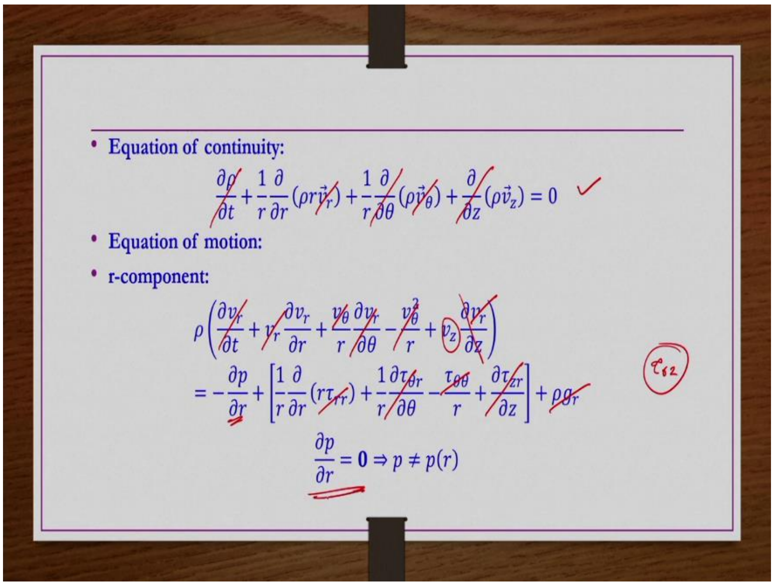

So, then we are going to apply these constraints to continuity and then momentum equation. So, continuity equation steady state. So, this is 0 and then v_r is not existing or it is 0 v_θ is not existing or it is 0; r by symmetry also this term is 0 and then since fully developed flow $\frac{\partial}{\partial z}$ any flow variable is 0. So, then continuity is satisfied. So, continuity is satisfied.

Then equation of motion r component of equation of motion is this one. So, steady state this term is 0; v_r is 0; v_θ is 0 and then because of the symmetry $\frac{\partial}{\partial \theta}$ of anything is 0 and then v_θ is 0 v_z is there, but v_z is not function of z is function of r only and then v_r is 0 and then because of fully developed flow also this term is 0.

Pressure we cannot say pressures in general we do not have any generalized boundary conditions ok. So, we just leave it as it is this term τ_{rr} would be having only v_r terms which is not there and then because of the symmetry this term is 0 this $\tau_{\theta\theta}$ also would be having only v_θ term. So, then it will be 0 and then because of the fully developed flow $\frac{\partial}{\partial z}$ of any flow variable is 0.

So, this is 0 and then gravity we are not considering also only τ_{rz} is existing in the in the in the current flow conditions whatever we have taken. So, all other terms of shear stress are 0. So, what we get here? We get $\frac{\partial p}{\partial r} = 0$; that means, pressure is not function of r.

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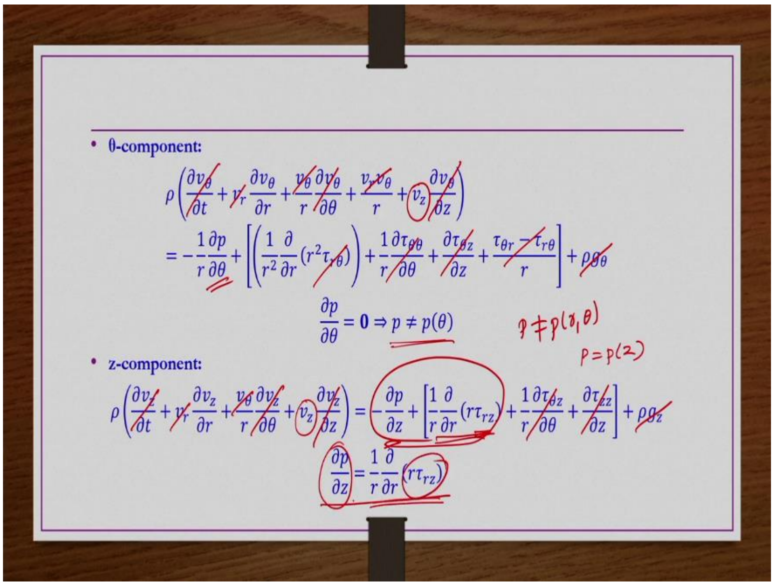

Similarly, θ component of momentum equation if you simplify steady state. So, this term is 0 v_r is not existing v_θ is not existing and then because of this symmetry this term is also

0. So, v_r v_θ both of them are 0 v_z is not 0, but because of the fully developed flow $\frac{\partial}{\partial z}$ of any flow variable is 0 fully developed flow in this case flow is in z direction.

So, $\frac{\partial}{\partial z}$ of any flow variable is 0 that is only for the flow variables not for you know scale as like pressure and temperature right. So, the symmetry boundary conditions also in general only for you know flow conditions only not for the scale as temperature profile pressure profile we cannot generalized ok. So, let us keep it as it is.

So, $\tau_{r\theta}$ is having v_r v_θ terms or this τ_r shear stress $\tau_{r\theta}$ component is not existing because of symmetry this term is 0, because of the fully developed flow $\frac{\partial}{\partial z}$ of any flow variable is 0. So, this is 0 and then for laminar flow these 2 are equal to each other. So, that is 0.

So, gravity finally, we are not taking anyway then what we get here again we get $\frac{\partial p}{\partial \theta} = 0$ that means, p is also not function of θ. So; that means, till now what we understand p is not function of r and θ it is so, but the flow problem is taking place because of you know pressure difference.

So, that what we can understand now from here itself without solving the problem further we can realize that p is function of z. What function? Is it a linear non-linear that we do not know that we can understand by simplifying z component of momentum equation. So, here steady state this is 0 v_r is 0; v_θ is 0 symmetry. So, this term is this term is also 0 v_z is the existing, but it is fully developed flow.

So, $\frac{\partial}{\partial z}$ of v_z is 0 pressure you cannot say anything any generalized conditions we cannot have. So, $\frac{\partial p}{\partial z}$ we cannot cancel out. So, only shear stress is existing τ_{rz}. So, which is again function of r because in the radial direction only the shear stress variations are there because the flow direction we are taking the z direction. So, only τ_{rz} shear stress component is existing. So, we cannot cancel out this term and then because of the symmetry this term is 0.

Because of the fully developed flow this term is 0 and we are not taking any gravity in this problem. So, what we have? We have only these 2 terms; that means, $\frac{\partial p}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})$ that is what we are having. Now, what we see here? So, left hand side $\frac{\partial p}{\partial z}$ is there.

So, already we realized by simplifying r and θ component equations what we understand that pressure is not function of r and θ. So, whereas, the right hand side what we have its everything function of r right. So what we can say? This $\frac{\partial p}{\partial z}$ is independent of whatever this thing is there in the right hand side ok and then similarly the τ_{rz} is function of r because v_z component should be this.

So, the v_z is function of r only. So, it is not function of z. So, whatever the right hand side terms are there τ_{rz} etcetera they are independent of z. So, then what we say? For the right hand side term whatever the left hand side terms are you know constants they are not going to affect it because left hand side term is only function of z and then right hand side terms are only function of r.

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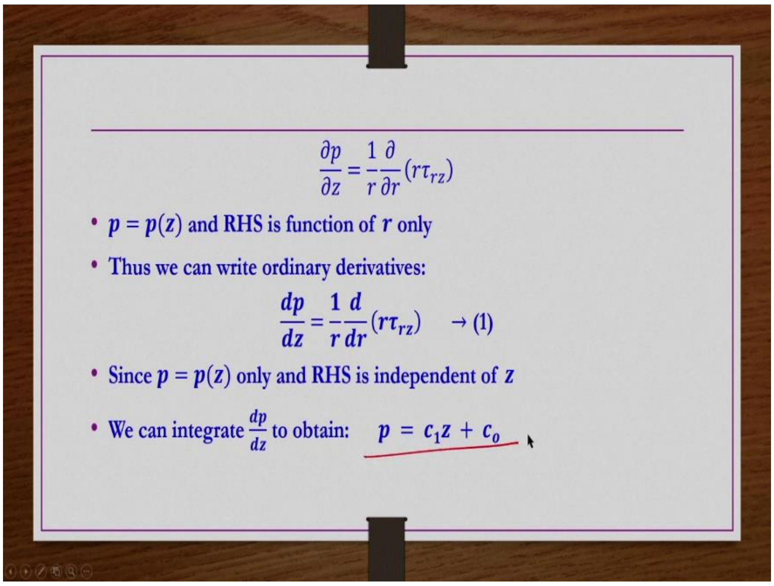

So, what we can do? We can individually integrate them when we integrate we get $p = c_1 z + c_0$.

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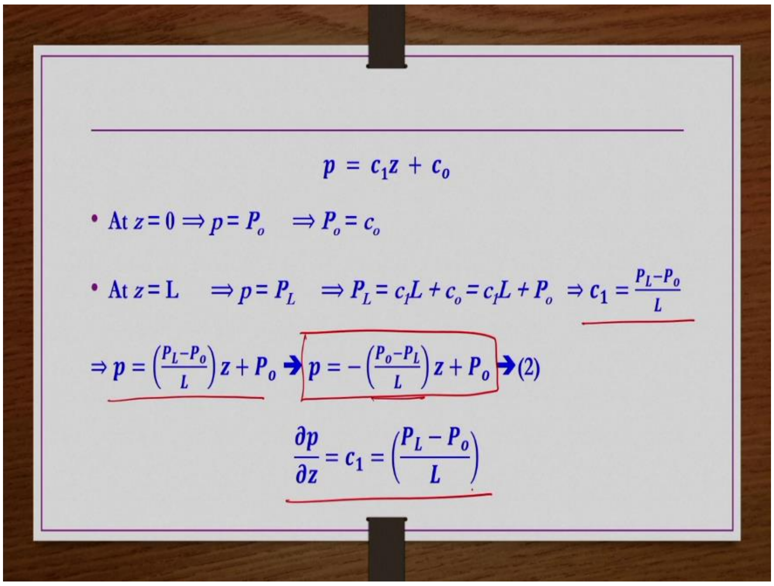

So, what are the boundary conditions? At z = 0 p = p_0. So, c_0 should be P_0 and at z =L p is = P_L. So, that $P_L = c_1 L + c_0$ then c_1 we get $\frac{P_L - P_0}{L}$; that means, p = c_1 is $\frac{P_L - P_0}{L}$ + z + P_0. So, the same thing we can write $P_L - P_0$ we can write $\frac{-P_0 - P_L}{L}$. So, that p =$\frac{-\Delta p}{L} + z + P_0$.

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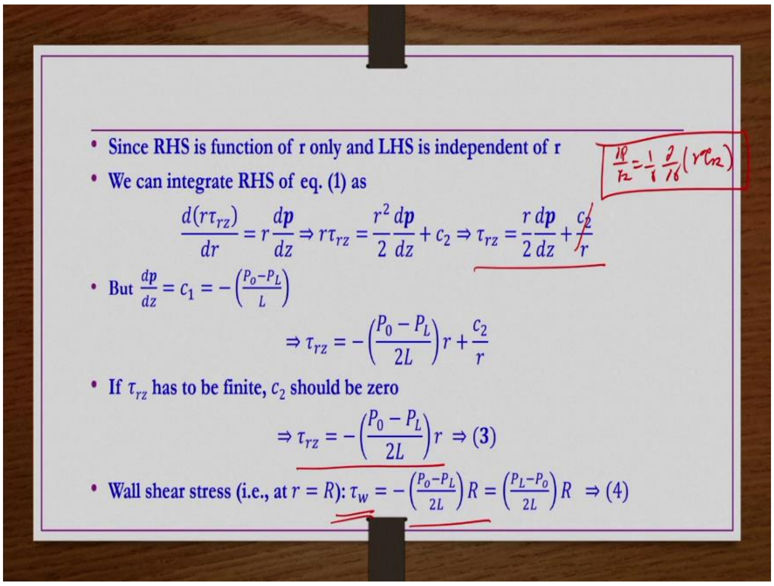

So, $\frac{\partial p}{\partial z} = c_1 = \frac{-\Delta p}{L}$ that is what we can write. Now what we can do? Whatever the $\frac{\partial p}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})$ equation was this by simplifying the z component of momentum equation. So, we treat $\frac{\partial p}{\partial z}$ as a constant for integrating the right hand side term. So, then what we do r we take to the other side. So, then r $\frac{dp}{dz} = \frac{d}{dr} r\tau_{rz}$.

So, now if you integrate with respect to r. So, then what we have $r\tau_{rz} = \frac{r^2}{2}\frac{dp}{dz} + c_2$ then; that means, $\tau_{rz} = \frac{r}{2}\frac{dp}{dz} + \frac{c_2}{r}$ right. And then shear stress cannot be infinite at any location between any value of r between 0 to R.

So, but if you substitute r = 0; here this τ_{rz} becoming infinite, but it is not possible. So, then c_2 has to be 0 so; that means, τ_{rz} is nothing but $\left(\frac{-\Delta p}{L}\right)\frac{r}{2}$ this is what we get right. So, if you know you need to know the shear stress at wall you have to substitute r = L r = R you have to substitute then we get $\tau_w = \left(\frac{-\Delta p}{L}\right)\frac{R}{2}$.

So what we do next? We try to obtain velocity profile for Ellis model fluid. Remember till this point before this slide you know we did not brought any information about the rheology of the fluid. So, till the point $\tau_{rz} = \left(\frac{-\Delta p}{L}\right)\frac{r}{2}$ that expression derivation you know the problem is same if the flow is taking place in circular tube under fully developed flow conditions right.

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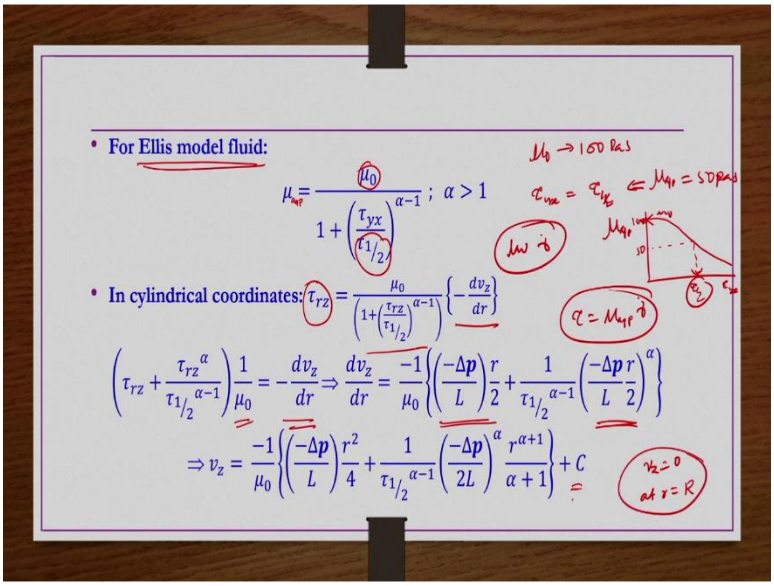

So, now after this point we are going to bring in the effect of the fluid rheology. So, for Ellis model fluid apparent viscosity is nothing but this is apparent viscosity. At the beginning itself, we have a you know made a you know generalization that if it is a non-Newtonian fluid then whether you specifically write apparent viscosity or not it is apparent viscosity because viscosity changes with shear rate ok.

So, now this apparent viscosity is nothing but $\frac{\mu_0}{1+\left(\frac{\tau_{yx}}{\tau_{1/2}}\right)^{\alpha-1}}$ α is something like same like a power law behavior index for the case of power law fluid. But only thing that in the power law fluids power law behavior index n < 1 for the shear thinning fluids whereas, in the case of Ellis model fluid this α > 1 ok.

So, that is a different, but the nature wise they are same ok. And this μ_0 is nothing but 0 shear viscosity this $\tau_{1/2}$ is nothing but you know let us say if your μ_0 is 100 pascal second. So, what is the shear stress at which this viscosity is becoming 50 pascal seconds that shear stress we call it as $\tau_{1/2}$ ok.

So, this is this we already know so; that means, you know if you plot this apparent viscosity μ_{app} versus τ_{yx}. Then this usually you know high it is large at smaller shear stress and then gradually decreases something like this. So, this is nothing but your μ_0 right. So, this let us say if it is 100 at what shear stress it is becoming 50 then μ_0 is decreasing to $1/2$ of its value.

So, that value is known as the corresponding shear stress value is known as the $\tau_{1/2}$. And then we know we have already studied this Ellis model fluid is important when the deviation from the power law fluid is more at low shear rate range. At low shear rate range if the deviation from the power law behavior is more important.

So, then it is better to use the Ellis model fluid for reliable information ok and then such cases occur in general in most of the dairy products. So, the same equation if you write for you know if you write in cylindrical coordinates then you have $\frac{\mu_0}{1+\left(\frac{\tau_{rz}}{\tau_{1/2}}\right)^{\alpha-1}}$.

So, then τ_{rz} τ is nothing but apparent viscosity multiplied by the shear rate right. Now shear rate is $\frac{-dv_z}{dr}$ and then in case only τ_{rz} shear stress component is existing. So, $\tau_{rz} = \frac{\mu_0}{1+\left(\frac{\tau_{rz}}{\tau_{1/2}}\right)^{\alpha-1}}\left\{\frac{-dv_z}{dr}\right\}$.

So now what we do? We keep $\frac{-dv_z}{dr}$ one side. So, then remaining terms to the other side what we have $\tau_{rz} + \frac{\tau_{rz}{}^{\alpha}}{\tau_{1/2}{}^{\alpha-1}}$ and then this entire multiplied $\frac{1}{\mu_0}$. So, this minus if you take to the right hand side $-\frac{1}{\mu_0}$ and then in case of τ_{rz} you can write $\left(\frac{-\Delta p}{L}\right)\frac{r}{2}$ because that is just now we derived. So, when you write you get this expression right.

So, now if you integrate this equation $v_z = -\frac{1}{\mu_0}\left(\frac{-\Delta p}{L}\right)\frac{r^2}{2}$ and then 1/2 is also already there. So, $\frac{r^2}{4} + \frac{1}{\tau_{1/2}{}^{\alpha-1}}\left(\frac{-\Delta p}{2L}\right)^{\alpha}$ and then integration of r^{α} is nothing but $\frac{r^{\alpha+1}}{\alpha+1}$and then integration constant c.

So, now that c constant if you wanted to find out the velocity v_z is 0 at wall that is at r = R because of the no slip velocity is 0. So, that you substitute here in this equation. So, then you get this c constant and that c constant you back substitute here you will get this expression.

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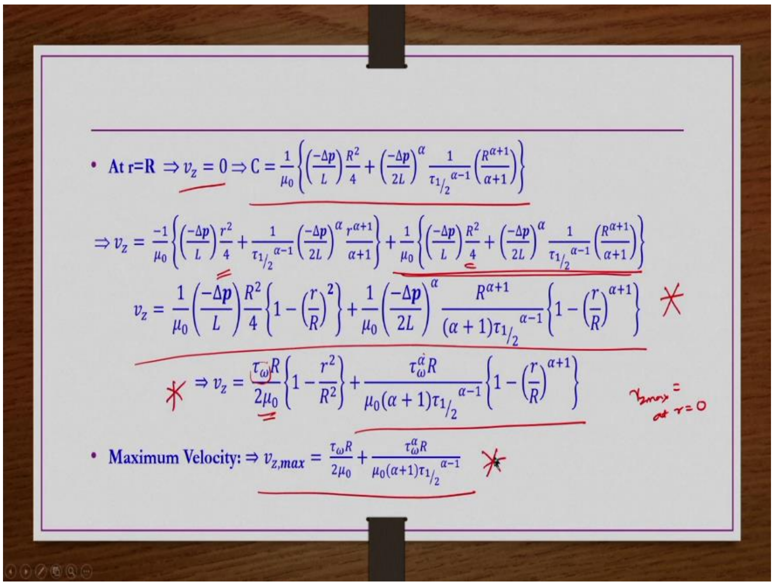

So, this c constant is this one when you substitute r = R in the previous equation then v_z will become 0 and then corresponding c is this equation. And then that c we are substituting in this v_z expression $v_z = -\frac{1}{\mu_0}\left(\frac{-\Delta p}{L}\right)\frac{r^2}{4} + \frac{1}{\tau_{1/2}{}^{\alpha-1}}\left(\frac{-\Delta p}{2L}\right)^{\alpha}\frac{r^{\alpha+1}}{\alpha+1} + c$ in place of c this is coming out.

So what we do? r powers whatever the similarities are there those terms we combine together that is r^2 terms we combine together and then $r^{\alpha+1}$ terms we combine together then we get v_z is equals to this expression this is nothing but the velocity profile right.

So, now this same expression what we can write $\left(\frac{-\Delta p}{L}\right)\frac{R}{2}$ you can write it as τ_w; so τ_w. So, 1 r is remaining r and then divided by in place of 4 we 2 is combined with τ_w so, 2 is remaining. So, $\frac{r}{2\mu_0}\left(1-\frac{r^2}{R^2}\right)$. Similarly, here also if you do this is what we get wherever $\left(\frac{-\Delta p}{L}\right)\frac{R}{2}$ is there we are writing τ_w here also. So, this is the velocity profile you get right.

Now, if you wanted to find out the maximum velocity v_{zmax}. What you do? You can get by substituting r = 0 at the center the velocity is going to maximum. So, when you substitute r = 0 then $v_z = v_{zmax}$ and then that is given by this expression. So, we got the velocity profile we also got the maximum velocity now.

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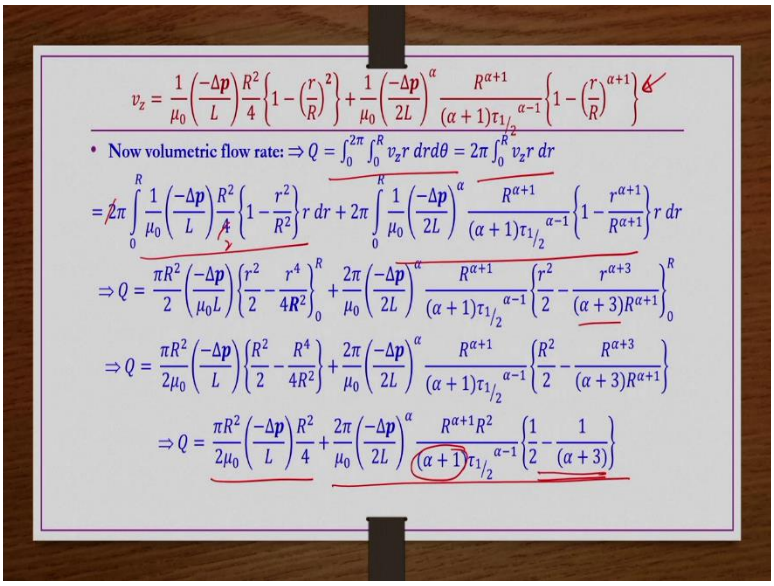

Next what we try to do? We try to get the volumetric flow rate. So, volumetric flow rate is $Q = \int_0^{2\pi}\int_0^R v_z r dr d\theta$. So, volumetric flow rate $Q = \int_0^{2\pi}\int_0^R v_z r dr d\theta$, but in θ direction we have the symmetric flow.

So then what we can have? $2\pi\int_0^R v_z r dr$ and then this v_z is nothing but this expression just now we got it. I have written the expression before converting that expression v_z expression in terms of τ_w ok. So, that v_z we are going to substitute here. So, then we have this expression.

Now what we can do? We can bring this r within the parenthesis here and then here also and then do the integration. So, then we have this expression. So, here for the first case 2 1's are 2 2's are and rest $\frac{r^2}{2} - \frac{r^4}{4R^2}$. And then here again $\frac{r^2}{2} - r^{\alpha+2}$ is there integration of $r^{\alpha+2}$ will be $\frac{r^{\alpha+3}}{\alpha+3}$will be having right.

Now this lower limit is anyway 0. So, then you do not need to worry I am substituting it because all the terms are having r value r's ok. So, now upper limit when you substitute R then you get this one. So, if you do further simplification then what you get Q is equals to this expression.

In the next step what we can do? You can do LCM of this one. So, then you get $\frac{\alpha+3-2}{2(\alpha+3)}$. So, that is $\alpha + 1$ you get. So, that $\alpha + 1$ and this $\alpha + 1$ will be cancelled out right.

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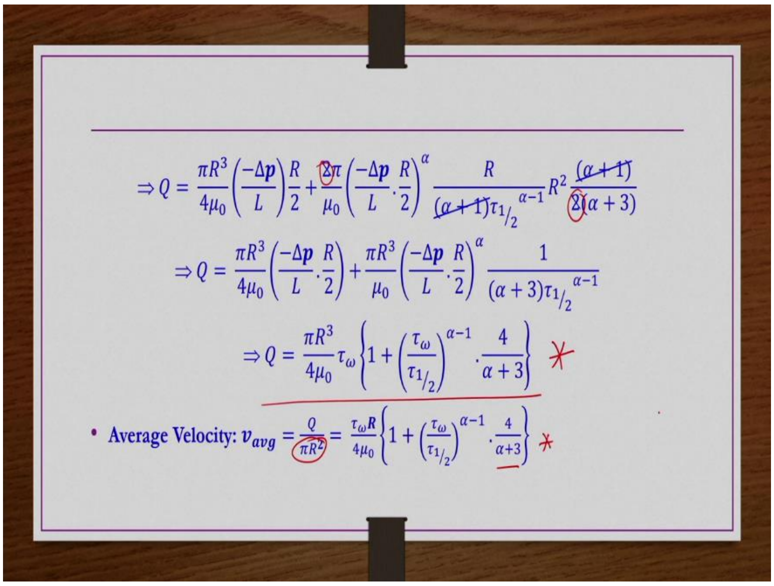

Then you have this term right this $\alpha + 1$ that $\alpha + 1$ is cancelled out and divided by $2(\alpha + 3)$ is this. So, that 2 and then this 2 π 2 we can cancelled out ok. So, this is what we get this expression. Now, again in place of $\left(\frac{-\Delta p}{L}\right)\frac{R}{2}$ you can write τ_w right when you write it you get this expression $Q = \frac{\pi R^3}{4\mu_0}\tau_w\left\{1 + \left(\frac{\tau_w}{\tau_{1/2}}\right)^{\alpha-1} \cdot \frac{4}{\alpha+3}\right\}$ this is what you have right.

So, if volumetric flow rate is known average velocity can be find out by dividing the volumetric flow rate by cross section area of the π. So, $\frac{\pi R^2 Q}{\pi R^2}$ if you do you get $\frac{\tau_w R}{4\mu_0}\left\{1 + \left(\frac{\tau_w}{\tau_{1/2}}\right)^{\alpha-1} \cdot \frac{4}{\alpha+3}\right\}$ this is what the average velocity.

So, now we got the volumetric flow rate average velocity as well in addition to the velocity distribution and then maximum velocity. So, everything we all almost everything we got except the friction factor. So, we try to do some simplification to get the friction factor as well.

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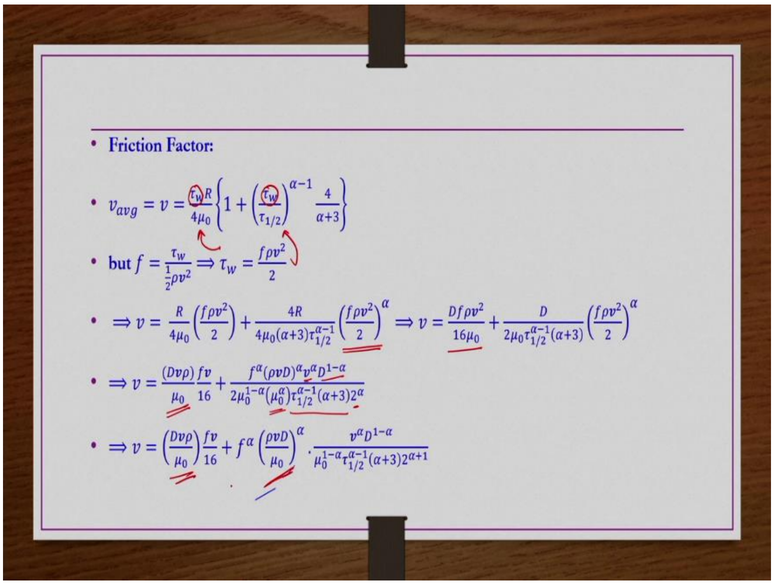

So, friction factor we know that $f = \frac{\tau_w}{\frac{1}{2}\rho v^2}$. So, now, $\tau_w = \frac{f}{2}\rho v^2$. So then what we do? In place of τ_w in the average velocity expression we are going to write $\frac{f}{2}\rho v^2$ from here. Then we get $v = \frac{R}{4\mu_0}\tau_w$ that is $\frac{R}{4\mu_0}\left(\frac{f\rho v^2}{2}\right) + \frac{4R}{4\mu_0(\alpha+3)\tau_{1/2}^{\alpha-1}}(\tau_w)^{\alpha-1}\tau_w$ that is τ_w^{α} and then τ_w is nothing but $\left(\frac{f\rho v^2}{2}\right)^{\alpha}$ is as it is right.

So now next step what I am trying to do? I am trying to write these expression R also I am writing in terms of D/2. So, that we can define Reynolds number. So, then we have here first term in the right hand side $\frac{Dv\rho}{\mu_0}\frac{fv}{16}$ + second term $f^{\alpha}\rho^{\alpha}v^{2\alpha}$ etcetera are there.

So then how I am writing? ρ^{α}, v^{α}, D^{α} and then remaining v^{α} whatever is there. So, that I am writing separately because here what you have you have only 1 D. So, D^{α} you have written here. So, $D^{1-\alpha}$ I am writing here and then divided by 2 ${\mu_0}^{1-\alpha}$ I am writing and then whatever the μ_0 is there I am writing ${\mu_0}^{1-\alpha}$ and then ${\mu_0}^{\alpha}$ remaining constants are as it is right.

So, this 2^{α} is as it is right. So, next step what I am trying to write? $\frac{Dv\rho}{\mu_0}$ within parenthesis here and then in the second terms also $\left(\frac{Dv\rho}{\mu_0}\right)^{\alpha}$ here in the as a parenthesis within the parenthesis I am writing. So, that I can write $\frac{Dv\rho}{\mu_0}$ I can write it as Re that I can write.

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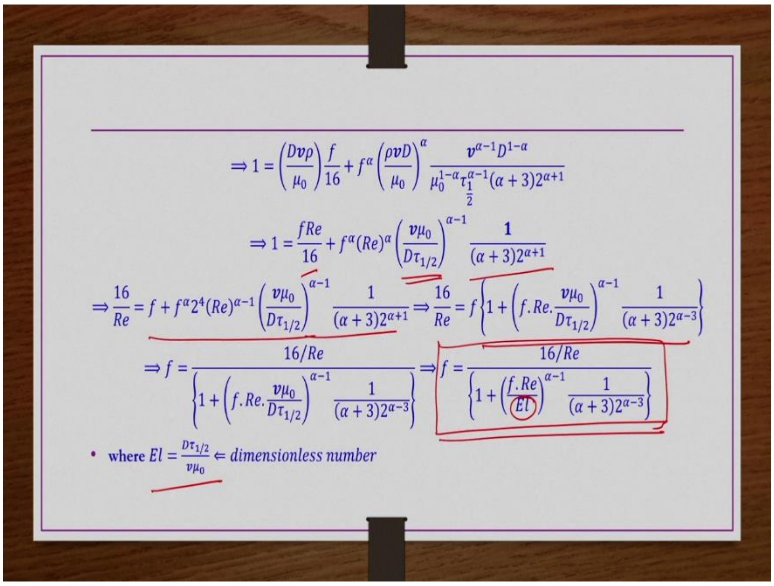

So, first term $\frac{fRe}{16}$ you I can have second term $f^{\alpha}\ Re^{\alpha}$ and then remaining terms whatever are there $v^{\alpha-1}D^{1-\alpha}{\mu_0}^{1-\alpha}\ {\tau_{1/2}}^{\alpha-1}$ I can write it as $\left(\frac{v\mu_0}{D\tau_{1/2}}\right)^{\alpha-1}$. And then remaining constants $\frac{1}{(\alpha+3)2^{\alpha+1}}$ right.

Next step what we are going to do? We are going to multiply $\frac{16}{Re}$ both sides because just wanted to write this also in the Newtonian case like in the Newtonian case f =$\frac{16}{Re}$ form is there. So, that way we are trying to write. So, that is the reason I am writing I am multiplying this equation both sides by $\frac{16}{Re}$. So, right hand side first term is f and then second term $\frac{16}{Re}$; 16 I am writing as 2^4 and then $\frac{Re^{\alpha}}{Re}$ I am writing $Re^{\alpha-1}$ this all terms as it is.

In the next step you know what I am trying to do? I am taking f common from the 2 terms in the right hand side. So, then I have this expression. So, that is f = $\frac{16}{Re}$ divided by this particular expression. Now here what I can write? $\frac{D\tau_{1/2}}{v\mu_0}$ I am writing some dimensionless number El. So, then this is what we have.

So, what you understand here you know like you know Newtonian or power law fluids the friction factor is not explicit it is not explicit. So, you cannot get directly by substituting Reynolds number etcetera or power law index power law behavior index etcetera ok. So, it is a trial and error basis ok that is what we understand here. So, this El is nothing but $\frac{D\tau_{1/2}}{v\mu_0}$ this is what we have.

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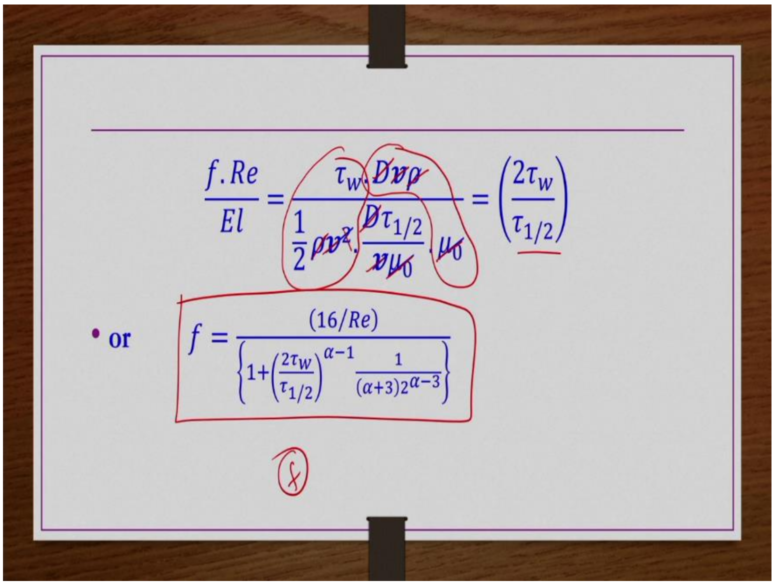

So, $\frac{f.Re}{El}$ if you further try to substitute and then try to write in terms of τ_w something like that f is nothing but $\frac{\tau_w}{\frac{1}{2}\rho v^2}$ and then Re is nothing but $\frac{Dv\rho}{\mu_0}$ and then El is nothing but $\frac{D\tau_{1/2}}{v\mu_0}$. So, now here what we can see? This v^2 and then this v this v is cancelled out this D this D is cancelled out ρ and ρ cancelled out.

So, what you have? $\frac{2\tau_w}{\tau_{1/2}}$ because this μ_0 this μ_0 is also cancelled out. So, $\frac{f.Re}{El}$ we can write in the previous equation as $\frac{2\tau_w}{\tau_{1/2}}$. So, then f we can also write like this right. Anyway τ_w again we cannot say that now equation for fiction now again we cannot say that the friction factor is explicit.

Because now we brought right side in form of τ_w etcetera no because τ_w is again related to f it is just you know writing in a simplified form only we are writing this one ok. So, still the f is not explicit it is based on the trial and error approach only you can get the solution.

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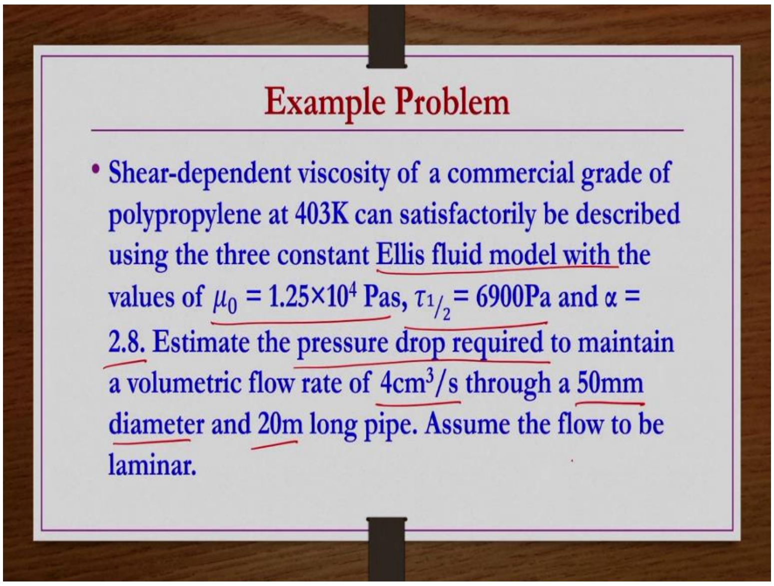

So, now before concluding today's lecture we will have example problem. So, one fluid rheology is expressed by Ellis model fluid with μ_0 is equals to this value $\tau_{1/2}$ this value and then α 2.8. What we are asking? The pressure drop required to maintain a volumetric flow rate of 4 centimeter cube per second through 50 mm diameter and then 20 meter long pipe; L is given, D is given, Q is given, and then it was told that flow assume the flow to be laminar right.

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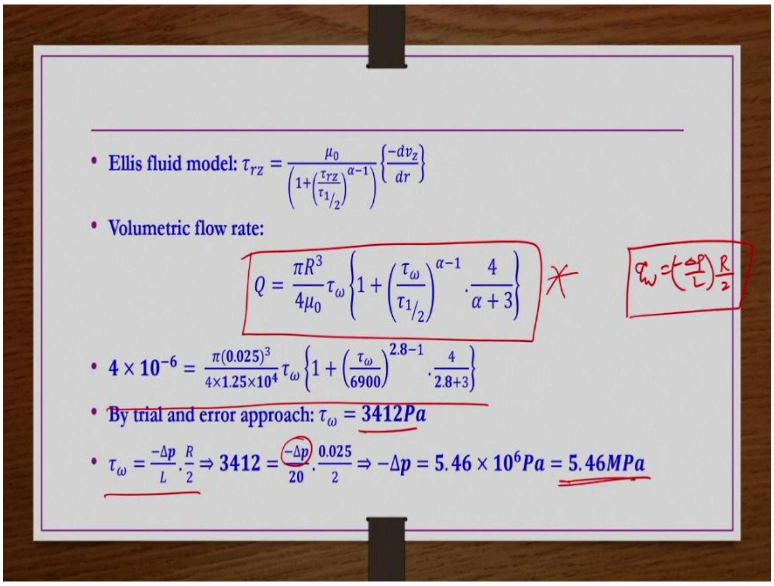

So, everything is given except $\frac{-\Delta p}{L}$ in the case of you know you know volumetric flow rate in the case of volumetric flow rate expression. Now here you know it is in terms of τ_w is again $\left(\frac{-\Delta p}{L}\right)\frac{R}{2}$ or $\frac{D}{4}$. So, this equation except the τ_w everything is given.

So, you substitute all those numbers here and then get the τ_w information and then from there τ_w information used this equation to get the $-\Delta p$ value ok. So, but solving this equation when you substitute all these values you will not get straight forward this will also include trial and error approach when you do this one you get τ_w = 3412 pascal's right.

And then when you apply $\tau_w = \left(\frac{-\Delta p}{L}\right)\frac{r}{2}$ expression here and then substitute for τ_w r and then L etcetera here in this equation what you get remaining minus delta p you can get it as 5.4 mega pascal's ok. In the next class what we try to do? We will be trying to obtain the similar expression if the fluid is viscoplastic fluid that is what we are going to do in the next class.

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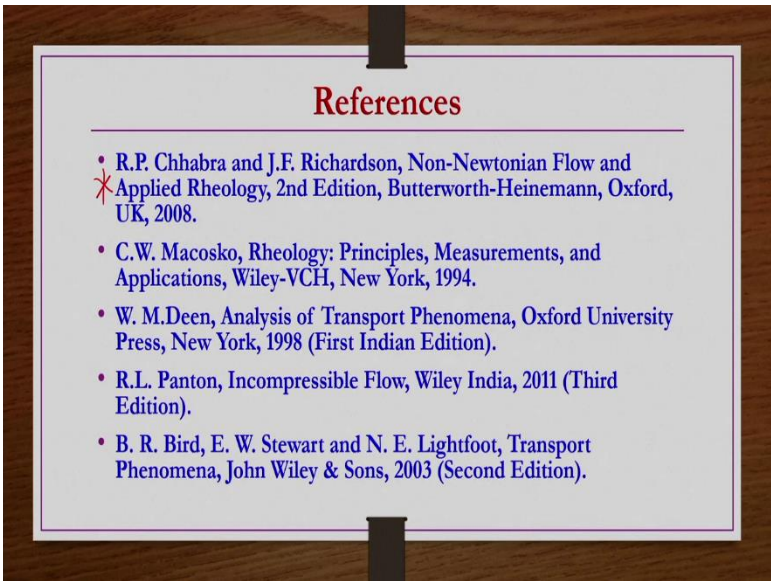

The references for this lecture the entire lecture is prepared from this reference book. However, other reference books are also given which may be useful.

Thank you.