Transport Phenomena of Non-Newtonian Fluids
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Lecture - 12
Equations of Change for Isothermal Systems

Welcome to the MOOCs course Transport Phenomena of Non-Newtonian Fluids, the title of this lecture is Equations of Change for Isothermal Systems. In the previous lecture we have started discussing about a few basics of transport phenomena, we have discussed transport phenomena at different scales and then we have studied a few basics of continuum hypothesis and then we have also seen a few basics of a transport by molecular mechanism.

In this particular lecture we are going to discuss derivations of equations of change for isothermal system. So, we start with derivation of equation of continuity right.

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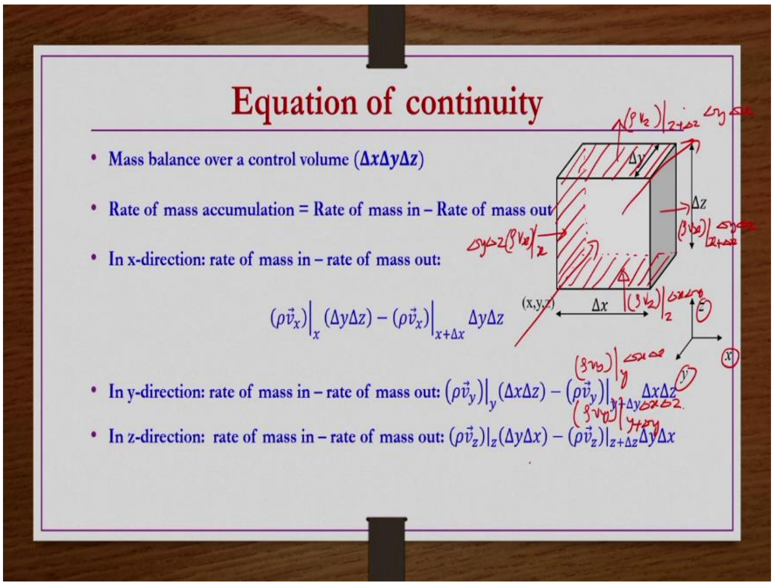

So, here what we do? We take a control volume of size Δx Δy Δz right and or which we will be doing a kind of and over which we will be doing a mass balance alright. So, this is the control volume we have selected, horizontal direction is x direction vertical direction is z direction and then another direction is the y direction, this is how coordinate system we have taken.

The size of the element control volume in the x direction is Δx in the y direction is Δy and then in the z direction it is Δz ok. Now mass balance over a control volume Δx Δy Δz we are going to do right. So, now here rate of mass accumulation should be equals to the rate of mass in minus rate of mass out. So, mass may be entering in all 3 directions and then maybe leaving all 3 direction that generalized one we are going to take now.

So, let us say mass that is entering through this shaded phases in x direction at x is equals to x so that is nothing, but $\rho v_x|_x$ multiplied by area of the phase through which it is entering. So, area of the phase through which it is entering is nothing, but Δ y Δ z alright.

Similarly the mass the rate of mass that is going out in the x direction, but at location x + Δ x that is nothing, but $\rho v_x|_{x+\Delta x}$ multiplied by the area through area of the phase through which it is living so that is the area is nothing, but Δ y Δ z.

Likewise the rate of mass that is entering in the z direction is nothing, but $\rho v_z|_z$ multiplied by the area of the phase through which it is entering so that is nothing, but Δ x Δ y. Similarly, the rate of mass that is leaving or going out in the z direction is $\rho v_z|_{z+\Delta z}$ location multiplied by the area of the phase through which it is leaving that is nothing, but Δ y Δ x.

Likewise in the y direction also the rate of mass that is getting in is $\rho v_y|_y$is equals to y multiplied by the area of the plane through which it is entering is Δ x Δ z so that is this direction here. So, now similarly the rate of mass out leaving at location y + Δ y is nothing, but $\rho v_y|_{y+\Delta y}$ multiplied by the area of the phase through which it is living that is Δ x Δ z right so, that is what we are doing.

So, now the rate of mass in the entire control volume minus rate of mass out in the entire control volume should be, you know rate of mass in the entire control volume should be addition of the mass in all 3 direction x y and z direction. Similarly rate of mass out in the entire control volume that is leaving rate of mass leaving out at x + Δ x and then y + Δ y and z + Δ z locations ok.

So, individually we do and then we combine them together so that we have the entire net rate of mass flow through the control volume. So, in the x direction if you write you have $\rho v_x|_x$ Δ y Δ z - $\rho v_x|_{x+\Delta x}$ Δ y Δ z.

Similarly, in the y direction $\rho v_y|_y$ Δ x Δ z - $\rho v_y|_{y+\Delta y}$ Δ x Δ z. Likewise in the z direction $\rho v_z|_z$ Δ y Δ x - $\rho v_z|_{z+\Delta z}$ Δ y Δ x ok.

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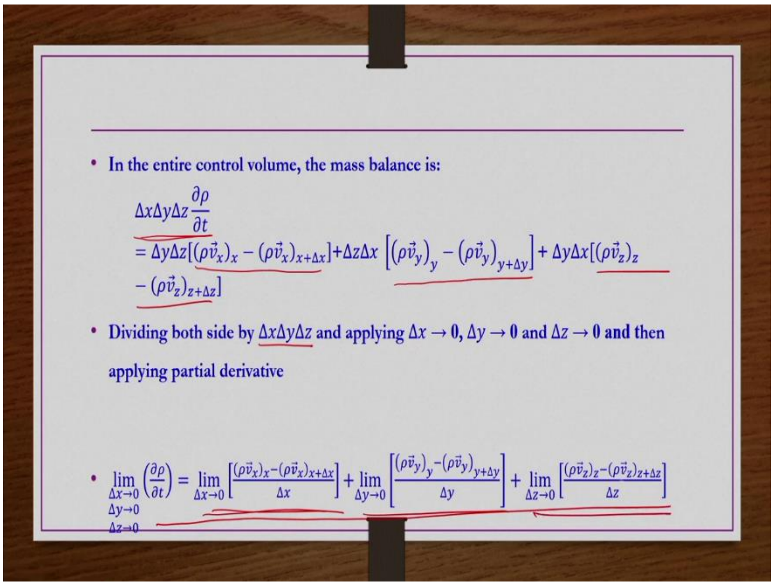

So, now in the entire control volume the mass balance is whatever the rate of accumulation is there that is nothing, but $\frac{\partial \rho}{\partial t}$ΔxΔyΔz that should be balanced by the $[(\rho\vec{v}_x)_x - (\rho\vec{v}_x)_{x+\Delta x}]$ Δ y Δ z + $\left[(\rho\vec{v}_y)_y - (\rho\vec{v}_y)_{y+\Delta y}\right]$Δ z Δ x + $[(\rho\vec{v}_z)_z - (\rho\vec{v}_z)_{z+\Delta z}]$Δ y Δ x right.

Now, this equation what we do? We divide by Δ x Δ y Δ z both sides and then applying the limiting conditions of Δ x → 0 Δ y → 0 Δ z → 0 that is the very very small size of control volume. And then applying the principles of derivatives then we get this equation after you know dividing by Δ x Δ y Δ z we get this equation after dividing by Δ x Δ y Δ z both sides.

And then applying Δ x → 0 Δ y → 0 Δ z → 0 we will be having this equation. Now this is nothing, but $-\frac{\partial}{\partial x}\rho\vec{v}_x$ and this is nothing, but $-\frac{\partial}{\partial y}\rho\vec{v}_y$ and this is nothing, but $-\frac{\partial}{\partial z}\rho\vec{v}_z$.

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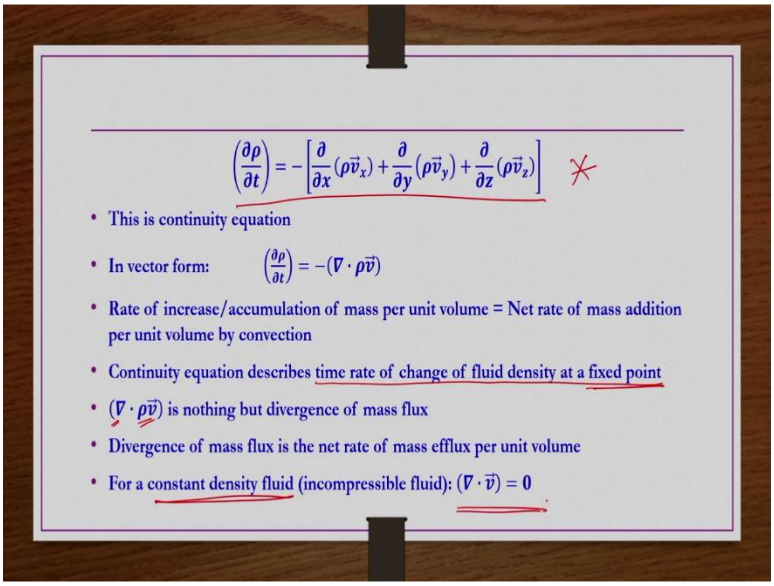

So, that is $\left(\frac{\partial \rho}{\partial t}\right) = -\left[\frac{\partial}{\partial x}(\rho \vec{v}_x) + \frac{\partial}{\partial y}(\rho \vec{v}_y) + \frac{\partial}{\partial z}(\rho \vec{v}_z)\right]$ this is what we have and this is nothing, but the continuity equation. This is the continuity equation that we are going to use extensively in the rest of the course.

In vector form the same equation we can write $\left(\frac{\partial \rho}{\partial t}\right) = -(\nabla . \rho \vec{v})$ what does it indicate rate of mass rate of increase or accumulation of mass per unit volume is nothing, but the rate net rate of mass addition per unit volume by convection. That is continuity equation describe the time rate of change of fluid density at fixed point at fixed point ok.

So, $(\nabla . \rho \vec{v})$ is nothing, but the divergence of mass flux that is ∇ is nothing, but it indicates the divergence $\rho \vec{v}$ is nothing, but the mass flux that is divergence of mass flux is the net rate of mass reflex per unit volume that is what it mean by. And then for a fluid of constant density we can write the same continuity equation as $(\nabla . \vec{v}) = 0$ right.

So, now this is about the continuity equation derivation in rectangular coordinates. What we can do? We can do the same thing in cylindrical coordinates we can do the same thing in spherical coordinates and then accordingly we can develop the equation of continuity in cylindrical and spherical coordinates as well.

Or otherwise we whatever the continuity equation that we have derived in rectangular coordinates we can apply appropriate transformations for r θ_z for cylindrical coordinates, and then r θ_ϕ for spherical coordinates and then do the simplification then we get the continuity equation in cylindrical and then spherical coordinates respectively as well.

However, we can also do the same thing for any arbitrary irregular shape of control volume as well and then we can develop the continuity equation also. But; however, in such case where the geometry is irregular the continuity equation we write in an integral form.

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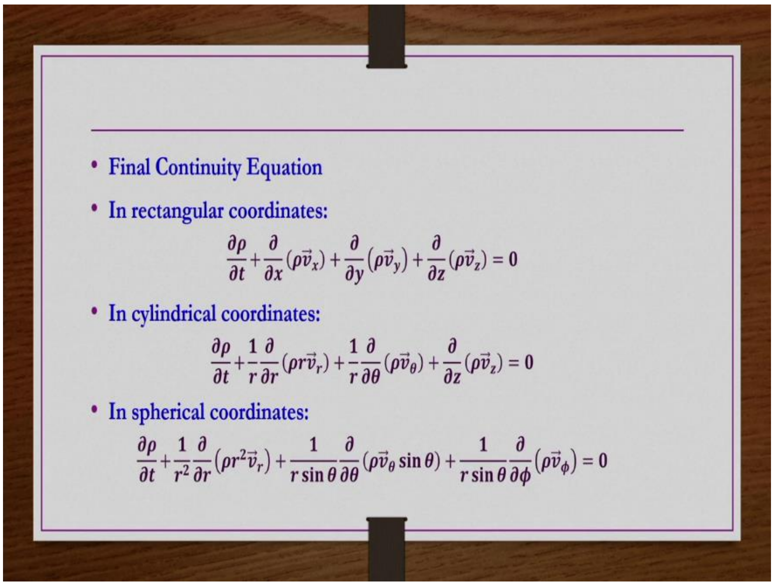

So, now we are enlisting this continuity equation in rectangular coordinates whatever we derived is nothing, but $\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho \vec{v}_x)+\frac{\partial}{\partial y}(\rho \vec{v}_y)+\frac{\partial}{\partial z}(\rho \vec{v}_z)=0$. Same thing in cylindrical coordinates if you do you get $\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r\vec{v}_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho \vec{v}_\theta)+\frac{\partial}{\partial z}(\rho \vec{v}_z)=0$.

In spherical coordinates if you do you get $\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2\vec{v}_r)+\frac{1}{rsin\theta}\frac{\partial}{\partial \theta}(\rho \vec{v}_\theta sin\theta)+\frac{1}{rsin\theta}\frac{\partial}{\partial \phi}(\rho \vec{v}_\phi)=0$ ok.

So, what is the point of going through this continuity equation is that, any flow field that you take and then you try to solve the problem whether it is theoretically whether it is just using analytical expressions analytical approaches or computational approaches whatever you do.

The final flow field solve the final flow field that you get as a solution that must satisfy the continuity equation. Then only we can say the solution is reliable if you are doing the solution in microscopic level ok.

If you are doing the transport phenomena problem at microscopic level and then after solving you get the velocity profile. If that velocity profile is satisfying the continuity equation then only you can say that the solution is reliable otherwise, you cannot say that your solution is correct that is one of the important thing that we are going to keep in mind for the rest of the course.

And then whenever we are using these equations to solve the problems then also we are going to have a kind of a recapitulation of the same information. So, the basically the solution for any flow field that you get whether it is by analytical approach or numerical approach whatever it is that solution must satisfy the continuity equation, then only we can say the solution is reliable.

Now, we see equation of motion alright so now that is momentum equation that we are going to derive right. So, whatever the momentum equation that you have seen in UG level transport phenomena primarily that you have derived for a Newtonian fluids and then the momentum equations corresponding momentum equations are nothing, but the Navier stokes equation, but now in this course.

What we do we will be deriving? A momentum equation very generalized one that can be used for any type of fluid ok. However, we apply the limiting condition of Newtonian fluids and then try to get back the Navier stokes equations as well.

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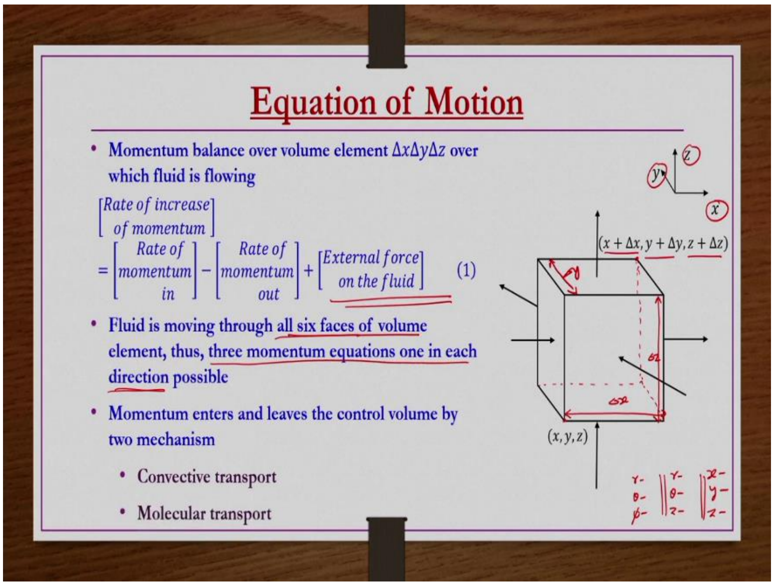

Let us say this is the control volume that we are taking. So, if it is x y z location and then this location is x + Δ x, y + Δ y, z + Δ z. The coordinate system horizontal direction we have taken x direction vertical direction we have taken z direction and the other third direction is the y direction.

So, now the size of this control volume if you take if you see in the x direction it is Δ x in the y direction it is Δ y and then in the z direction it is Δ z ok. Now, momentum balance over the volume element Δ x Δ y Δ z over which fluid is flowing we are going to take.

So, now this element whatever we have taken we have taken in a fluid element which is flowing. At a particular instant at particular location we have taken one fluid element whose volume is Δ x Δ y Δ z that, is whose size in the x direction is Δ x y direction is Δ y and then in z direction it is its size is Δ z ok.

So, now, this control volume what we have we have the momentum is entering alright momentum is entering and then leaving all 6 phases that is what we have to understand first ok. So, that is what we are going to see now each phase you know what is the momentum entering and then at each phase what is the momentum leaving out and then we do the balance.

So, the balance if you write for momentum [rate of increase of momentum] = [rate of momentum in] – [rate of momentum out] + [external force on the fluid] if at all they are acting alright.

Whatever the extra forces are acting that may be because of the gravitational field because of the reactions because of the electric field because of the magnetic field, whatever the additional terms would be there. All those terms should be coming and then be being added up in the right hand side in the balance of momentum equation ok.

So, since now we are doing for from chemical engineering point of view we take the external force because of the gravity field only we are going to take and then do the equation derivation right. If at all as mentioned already, if at all additional terms are there because of the reactions because of the electric field or because of the magnetic field because of the nuclear sources whatever is there all those terms should be added up in the right hand side if at all they are existing.

So, now, what we understand that, the fluid is moving through all 6 phases of a volume element whichever the phase you take the control volume we have taken cubic control volume. So, all throughs all 6 phases through which the fluid is moving so what we will have? We will be having three momentum equations one in each direction one in each direction we are going to have a momentum equation.

Continuity equation we have only one continuity equation if you take a rectangular coordinates, if you take cylindrical coordinates then also one equation, if you take a spherical coordinates then also only one continuity equation is possible. But, now here what happens? The momentum that is entering the fluid is moving through all 6 phases. So, then all 6 phases in a momentum may be transferring in different directions and then different direction different orientations as possible.

So, then; obviously, if you take the rectangular coordinates we will be having one momentum equation in x direction we will be having one momentum equation in y direction we will be having one momentum equation in z direction right. So, that is what we are going to have.

So, if you have geometry in cylindrical coordinates then you will be having one momentum equation in r direction you will be having one equation in θ direction you will be having one momentum equation in z direction.

Likewise if you are having you know control volume in spherical coordinate system then you will be having one momentum equation in r direction another equation in θ direction another equation in ϕ direction, so that is possible ok. So, now this momentum whether it is entering or leaving it may enter and leave either of the two transport mechanisms or both the transport mechanism depending on the situation to situation.

So, what we do? We are doing a generalized case so then what we are taking the momentum whatever is there that is entering and then leaving because of both the transport mechanism that is, molecular transport mechanism and then convective transport mechanism. So, momentum enters and leaves the control volume by two mechanisms convective transport and then molecular transport ok.

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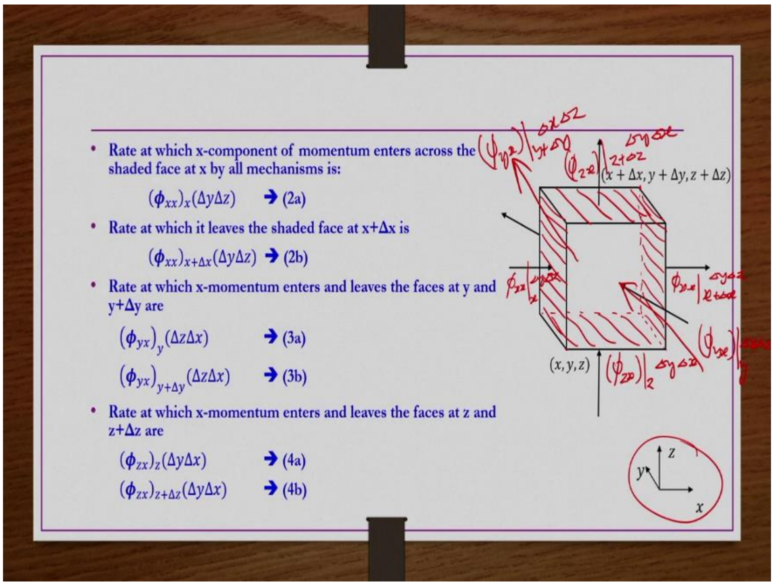

So, now we are going to do your balance the same control volume I have drawn here ok. So, this is our x y z this location is x + Δ x, y + Δ y, z + Δ z location. So, so now x momentum entering at x location what it is alright? So, entering because of the molecular transport entering because of the convective transport both, are possible.

So, then what I am writing momentum flux tensor that I am writing combined momentum flux tensor that I am writing that is nothing, but phi I am taking right. ϕ in x direction it is entering at location x multiplied by the area of the phase through which it is entering so that is nothing, but area of this phase is nothing, but Δ y Δ z.

Similarly the same x momentum leaving in x direction at location x + Δ x so that is nothing, but $\phi_{xx}|_{x+\Delta x}$ multiplied by the area of the phase through which it is leaving that is Δy Δz.

Similarly in the z direction whatever the x momentum is entering so that is ϕ_{zx}that is entering at location z and then the area of the plane through which it is entering so that is nothing, but Δ y and then Δ x, because this is Δx this is Δy and then this is nothing, but Δz ok, the directions are given like this.

Similarly, same x momentum leaving at z + Δ z location is nothing, but $\phi_{zx}|_{z+\Delta z}$ multiplied by the area of the phase through which it is living that is nothing, but again Δy Δx.

Likewise in the y direction if you take in the y direction x momentum is entering at y location is nothing, but $\phi_{yx}|_{y}$ multiplied by the plane through which it is entering that is Δ x Δ z. Similarly, x momentum leaving in the y direction is $\phi_{yx}|_{y+\Delta y}$ multiplied by the area of the phase through which it is leaving that is Δ x Δ z.

So, the overall x momentum that is entering in all 3 directions and then leaving all 3 direction can be obtained by adding all these component. So, rate at which x component of momentum enters across the shaded phase at x by all mechanism. So, this phase shaded phase we have taken is nothing, but $\phi_{xx}|_{x}$ Δ y Δ z. The rate at which it is leaving at location x + Δ x is $\phi_{xx}|_{x+\Delta x}$ Δ y Δ z.

Similarly, x momentum entering at location y and then leaving at location y + Δ y are nothing, but $\phi_{yx}|_{y}$ Δ z Δ x and $\phi_{yx}|_{y+\Delta y}$ Δ z Δ x. Likewise rate at which x momentum entering and leaving the phases at z and z + Δ z are nothing, but $\phi_{zx}|_{z}$ Δ y Δ x and $\phi_{zx}|_{z+\Delta z}$ Δ y Δ x respectively alright.

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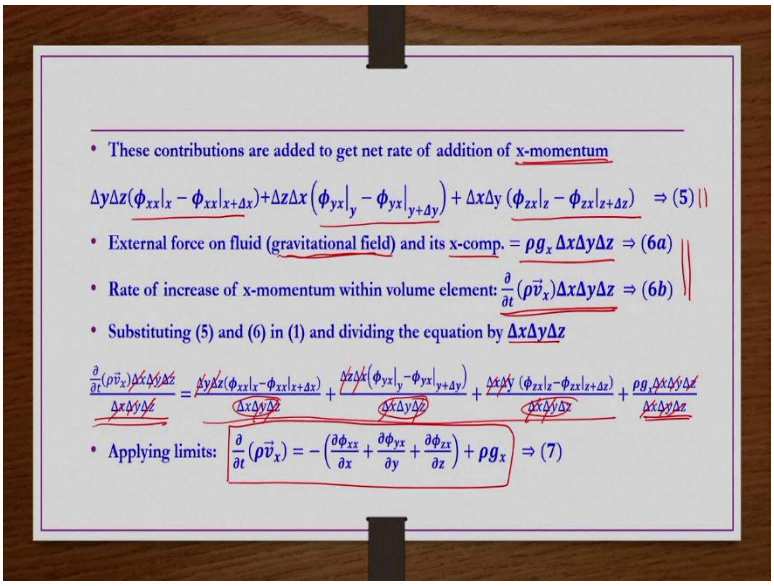

So, these contributions are added together to get net rate of addition of x momentum. So, that is nothing, but $(\phi_{xx}|_x - \phi_{xx}|_{x+\Delta x})$ Δ y Δ z + $\left(\phi_{yx}|_y - \phi_{yx}|_{y+\Delta y}\right)$ Δ z Δ x + $(\phi_{zx}|_z - \phi_{zx}|_{z+\Delta z})$Δ x Δ y alright. So, this is the net rate of addition of x momentum it's only x momentum only in the x direction ok.

So, external force on fluid now we are taking only gravitational field only and its x component if you wanted to write that would be nothing, but ρ g_x Δ x Δ y Δ z right. If any additional terms reaction terms etcetera are also there they are also causing some momentum transport then additional terms should be there. So, but we are limiting ourselves by including only one external force term that is because of the gravitational field.

Now, rate of increase of x momentum within the volume element is nothing, but $\frac{\partial}{\partial t}(\rho\vec{v}_x)$ Δ x Δ y Δ z. So, now what we do these two equations equation 5 and 6 we will be substituting in equation number 1 which is nothing, but the overall balance equation and then we divide by Δ x Δ y Δ z both sides then we get this equation.

So, accumulation term is $\frac{\partial}{\partial t}(\rho\vec{v}_x)$ Δ x Δ y Δ z that we are dividing by Δ x Δ y Δ z and then net rate of addition of x momentum is nothing, but these 3 terms and then these 3 terms by equation number 5. So, then we are dividing by Δ x Δ y Δ z both sides.

So, then here we have and then external force if any that we have taken gravitational force. So, ρ g_x Δ x Δ y Δ z given by equation number 6 a divided by the volume of the control volume that we have taken that is Δ x Δ y Δ z.

So, now, what we can do? We can cancel out this Δ y Δ z here. So, Δ x is remaining so likewise Δ is Δ x Δ z Δ x Δ x we are cancelling out here and also right.

And then we apply the limiting conditions of Δ x →0 Δ y →0 Δ z →0 then we have this equation. That is $\frac{\partial}{\partial t}(\rho \vec{v}_x) = -\left(\frac{\partial \phi_{xx}}{\partial x} + \frac{\partial \phi_{yx}}{\partial y} + \frac{\partial \phi_{zx}}{\partial z}\right) + \rho g_x$ this is what we have ok.

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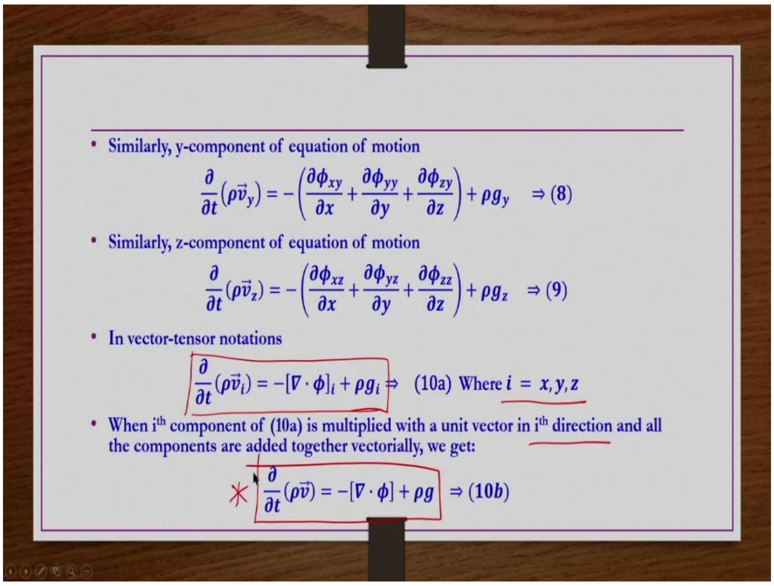

Now, similarly y component of equation of motion if you do the balance and then do the simplification you will get $\frac{\partial}{\partial t}(\rho \vec{v}_y) = -\left(\frac{\partial \phi_{xy}}{\partial x} + \frac{\partial \phi_{yy}}{\partial y} + \frac{\partial \phi_{zy}}{\partial z}\right) + \rho g_y$. And then likewise in the z direction the z component of equation of motion is nothing, but $\frac{\partial}{\partial t}(\rho \vec{v}_z) = -\left(\frac{\partial \phi_{xz}}{\partial x} + \frac{\partial \phi_{yz}}{\partial y} + \frac{\partial \phi_{zz}}{\partial z}\right) + \rho g_z$ this is what you get ok.

So, in vector tensor notations if you wanted to write this equation you can write $\frac{\partial}{\partial t}(\rho \vec{v}_i) = -[\nabla . \phi]_i + \rho g_i$ where i can be anything x, y, z. So, it is not one equation it is representation of three equation, if i = x then it will be giving one equation that is equation number 7. If i = y then you will get the equation number 8 if i = z then you get equation number 9 ok.

Now, i^{th} component of equation number 10 is multiplied with the unit vector in i^{th} direction and all the components are added together vectorially then we get this equation. A generalized equation $\frac{\partial}{\partial t}(\rho\vec{v}) = -[\nabla.\vec{v}] + \rho g$ ok, this is what we get right. This is again generalized form of momentum equation it is three equations alright whichever coordinate system you take it is it indicates three equations.

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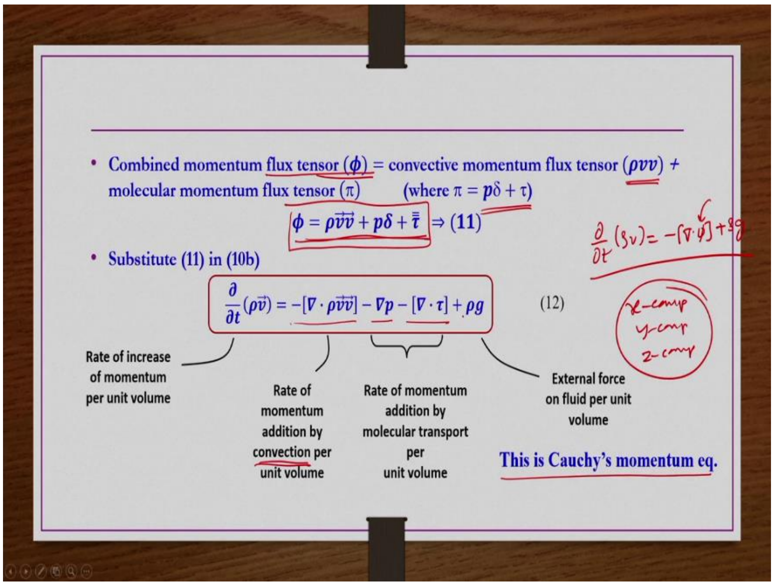

Now, what is ϕ? ϕ is nothing, but combined momentum flux tensor I as I mentioned that includes the contribution because of convective momentum flux as well as the molecular momentum flux both the terms it includes. Rather why we have written like this because ϕ is having 2 components right.

Momentum flux because of the convection whatever is there that is nothing, but $\rho v v$. And then momentum flux because of the molecular transport mechanism is nothing, but π and then that π is nothing, but p δ Δ + τ; that means, we are having ϕ is equals to this equation, $\rho\vec{v}\vec{v}$ + p δ + $\bar{\bar{\tau}}$.

So, now this we have written whatever the ϕ_{xx} if you write for individual components like convictive component and then molecular transport component like individually so then you have you will be writing three terms at each point and then in one direction.

So, all the directions you will be having like you know 3 3 6 and then 18 terms you might be writing for x component of momentum equation similarly another 18 for y component of momentum equation another 18 for a z component of momentum equation. That is the reason you know writing so many terms may be confusing and then sometimes overlook errors may also lead to the wrong derivation. So, that’s the reason we have combined them all of them together as a one part.

Now, we have derived actually the equation number 10 a 10 b whatever we have written in the previous slide that is nothing, but the complete momentum equation we derived it actually. Now, what we are going to do what is this ϕ that is we are going to substitute ok. So, this we are going to do now.

So, now, in equation number 10 b we are going to substitute whatever this ϕ information is there. So, equation number 10 b is nothing, but $\frac{\partial}{\partial t}(\rho\vec{v}) = -[\nabla.\phi] + \rho g$ this is what we have. Now, in place of ϕ if you write $\rho\vec{v}\vec{v}$ + p δ + $\bar{\bar{\tau}}$ then we will be having $-[\nabla.\rho\vec{v}\vec{v}] - \nabla\mathrm{p} - [\nabla.\tau] + \rho g$ is as it is.

This equation is nothing, but Cauchy’s momentum equation and it is a generalized momentum equation. It is valid irrespective of the nature of the fluid irrespective of the nature of the flow ok, for all flow reasons for all fluids it is valid ok. So, here in this equation whatever the left hand side term is nothing, but the rate of increase of momentum per unit volume that is temporal term or accumulation term.

Then right hand side first term is nothing, but the rate of momentum addition by convection only because of the convective mechanism whatever is there that term per unit volume. Next two terms are nothing, but the rate of momentum addition by molecular transport per unit volume and then last one is nothing, but because of the external force on fluid per unit volume ok.

So, whatever the possible ways of momentum entering and leaving including one additional external force that is gravitational force all of them are included in this equation. And then again this equation is not one equation it’s the generalized equation in vectorial form we have written it is having 3 equations it is having 3 equations.

So, then it is having it includes x component of momentum equation y component of momentum equation and z component of momentum equation as well. That is if we expand this equation and then write for individual x y z component so then you will get these equations.

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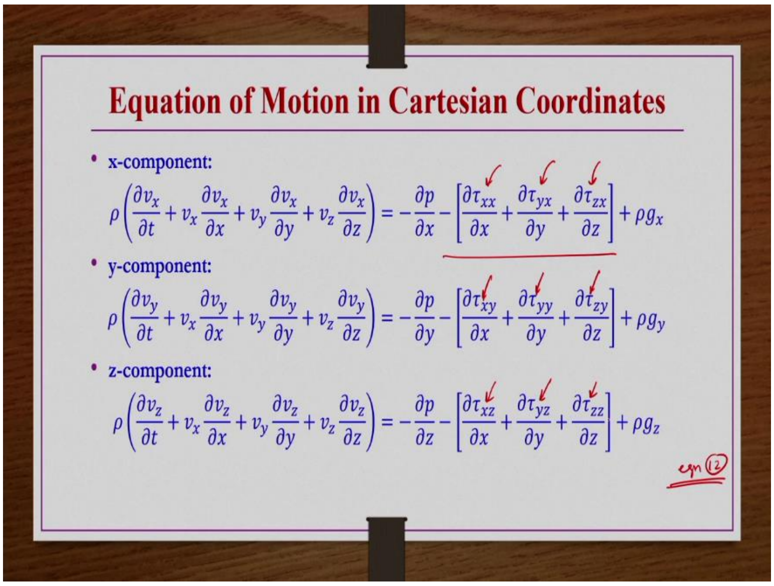

x component you get $\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial \rho}{\partial x}-\left[\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right]+\rho g_x$ right.

Now, y component if you do this again from the equation number 12 only y component for y component if you write $\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=-\frac{\partial \rho}{\partial x}-\left[\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right]+\rho g_y$.

And then likewise z component equation z component of momentum equation $\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial \rho}{\partial x}-\left[\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}\right]+\rho g_z$.

Now, these all three equations are coming from equation number 12; equation number 12 is having 3 equations and then all those three equations if you separate out you have these 3 equations right. So, one expected to remember this equation at least in Cartesian coordinates if somebody is working in the area of fluid mechanics or transport phenomena related problems; however, it is not essential right.

Because remembering or memorizing these equations in Cartesian coordinates bit slightly easier compared to the same in spherical and in cylindrical coordinates. However, it is not essential at all these equations are available in standard transport phenomena textbooks or whenever you need to solve any problem that we are going to do in the subsequent part of the course this equation would be provided ok.

Now, this equation whatever this τ_{xx} τ_{yx} τ_{zx} etcetera are there that if you substitute for the Newtonian fluid then you will get here $-\frac{\partial p}{\partial x} + \mu \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}$ you will be getting that is nothing, but the Navier stokes equation.

That is what we are going to see what are these components for a Newton's law of viscosity that we should have an information. So, they are also available in standard textbooks you know τ_{xx} τ_{yx} τ_{zx} are you know normal stress normal are nothing, but the normal viscous stresses for the Newtonian fluids, but they are extra stresses for non Newtonian fluids ok so what are they that we have to know.

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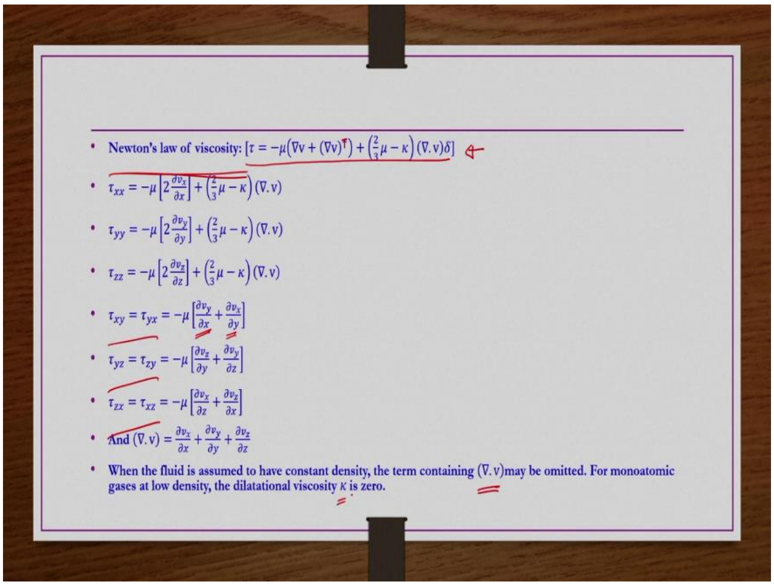

So, let us say if you have a Newton's law of viscosity then you have this τ expression like this, $\left[\tau = -\mu(\nabla v + (\nabla v)^T) + \left(\frac{2}{3}\mu - \varkappa\right)(\nabla . v)\delta\right]$ right. If you expand it is not 1 equation it is again it is having 9 components together generalized one it is having in a generalized form it is written so we can write all the 9 components individually also.

So, we write then τ_{xx} is given by this τ_{yy} is given by this τ_{zz} is given by this and then τ_{xy} = τ_{yx} = one so this is nothing, but your ∇v this is nothing, but $(\nabla v)^T$. Then similarly τ_{yz} is equals to τ_{zy} is equals to this one and then τ_z is equals to τ_{xz} is equals to this one. So, these are equal we can say only for a Newton's law viscosity when it is valid ok.

And then whatever $\nabla.v$ in the end is nothing, but $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$ ok. So, when the fluid is assumed to have a constant density the term containing $\nabla.v$ may be omitted because $\nabla.v = 0$ if other constant viscosity system that we have already seen from the continuity equation. And then for monatomic gases at low density the dilation of viscosity $\varkappa$ is 0 in general ok.

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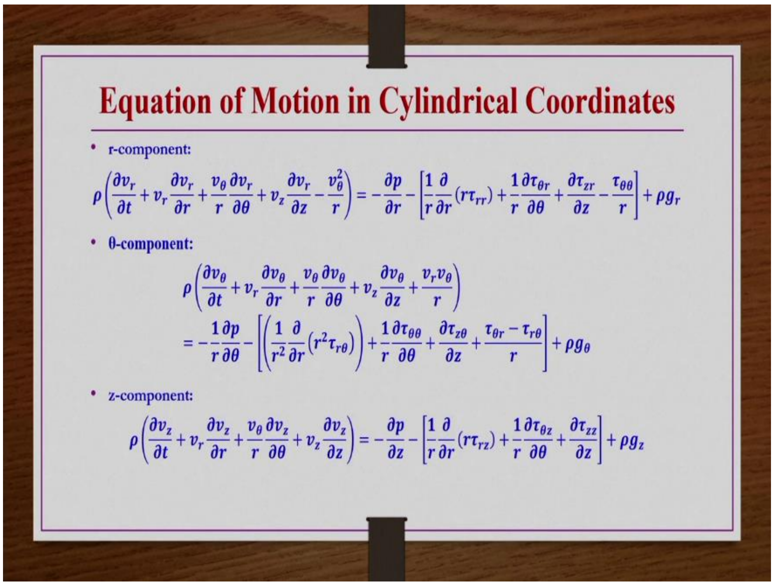

Then the same momentum equations let us say if you do the derivation in cylindrical coordinate systems are whatever the momentum equation just now we derived in rectangular coordinates. And then we apply transformation to get the cylindrical coordinate equation you will get this following equation r component of momentum equation is given this one θ component of momentum equation is given by this one and then z component of momentum equation is given by this equation ok.

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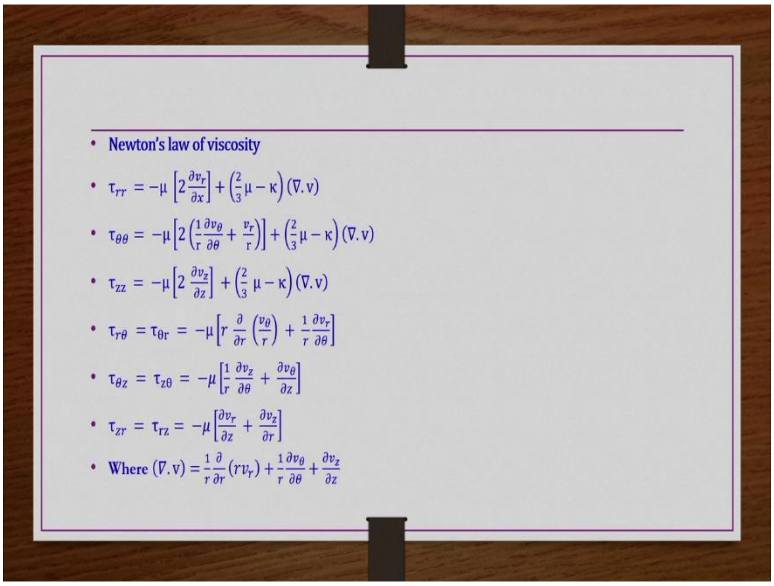

Here also in cylindrical coordinates Newton's law of viscosity if you write you get all those 9 terms written like this ok. And then likewise if you derive these momentum equations in spherical coordinate system are whatever the momentum equation just now we derived in Cartesian coordinate system you apply, a corresponding transformation to get the spherical coordinate equations.

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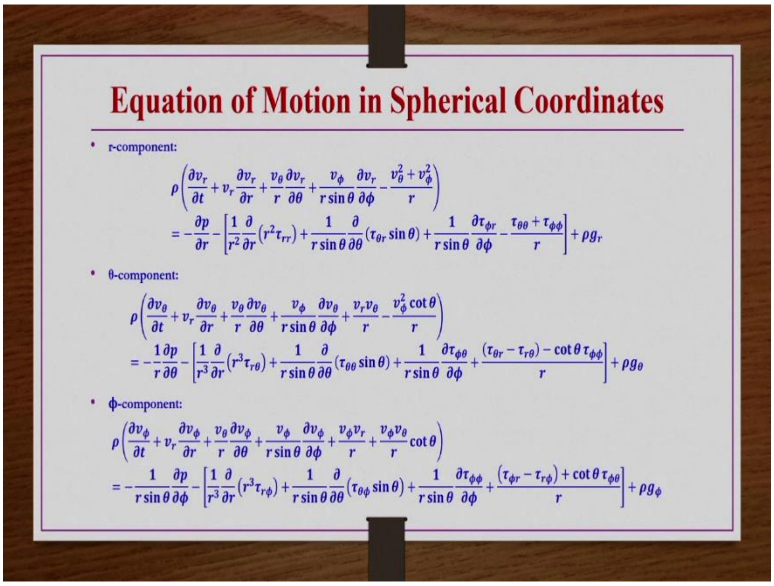

Then you will have equations of motion in spherical coordinates like this r component of momentum equation θ component of momentum equation and then ϕ component of momentum equation alright.

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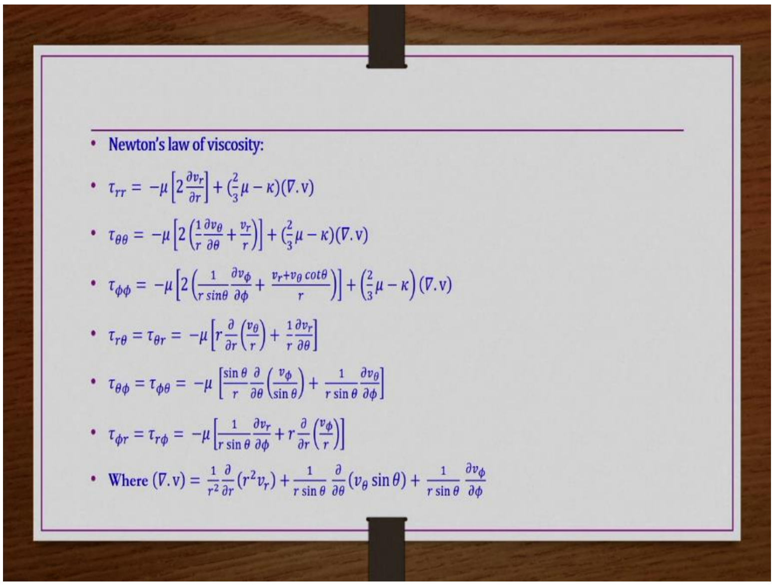

So, likewise in spherical coordinates also Newton's law of viscosity if you apply then these all these components τ_{rr} $\tau_{\phi\phi}$ $\tau_{\theta\theta}$ etcetera you get by this equation right. All these equations you do not need to memorize they are available in standard transport phenomena textbooks also they will be provided whenever you need to solve any problem in general.

So, that is about the momentum equations ok how to derive the momentum equation that is what we have seen ok. Now, what we are going to see is that different types of derivatives in general used in transport phenomena, like you know partial derivative total derivative substantial derivative etcetera what are they that is also we know; however, we are going to have a kind of recapitulation.

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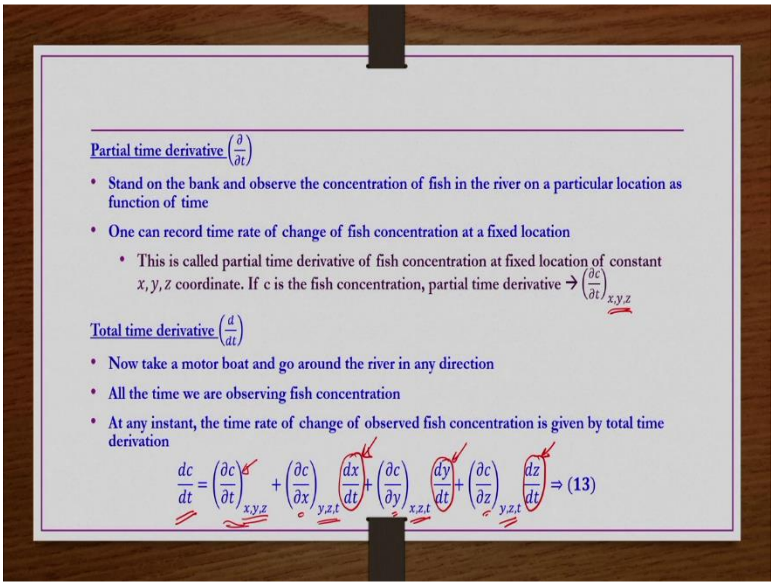

Partial time derivative $\left(\frac{\partial}{\partial t}\right)$, so it is something like you know we take as a kind of fish concentration or number of fishes you know in a lake. So, you are sitting on the bank or in a river we are doing let us say in a river if you wanted to find out the fish concentration right.

So, you are sitting on the lake and then you select or choose a position or a fixed coordinate system you choose within the river. You fixed a location that is nothing, but you are taking the fixed coordinate system spatially fixed location and then with respect to time you are measuring how many number of fishes are crossing that particular location and then you are measuring that concentration with respect to time.

So, that if you write you get that is nothing, but your partial time derivative where you are fixing the location and then variations you are taking only with respect to time. That is stand on the bank and observe the concentration of fish in the river on a particular location any fixed location of x y z as function of time you measure that concentration. So, that is nothing, but you know partial time derivative where one can regard time rate of change of fish concentration at fixed location.

And this is called partial time derivative of fish concentration at fixed location at fixed x y z of constant x y z coordinate and then if c is the fish concentration then partial time derivative of fish concentration is nothing, but $\left(\frac{\partial c}{\partial t}\right)$ at fixed x y z coordinates. Location you are fixed, you are fixing the location and then you are not in the river right you are sitting on the bank and then you are fixing a location in the river.

And then you are trying to measure the rate of time rate of change of fish concentration at particular location and then with respect to time that concentration you are fixed a fish or fish concentration you are measuring so that is nothing, but $\left(\frac{\partial c}{\partial t}\right)$ ok.

Next is the total time derivative, now what you do? You understand; obviously, the fish concentration in the river entire river is not going to be you know $\left(\frac{\partial c}{\partial t}\right)$ at constant x y z it will not give the true picture it will give the reliable picture at fixed location only, but if you change the location the fish concentration may be very different right.

If you go to some other location the fish concentration maybe even further different compared to the second location or first location, likewise if you take n number of locations in the fish you understand the concentration of the fish is going to be very different compared to the fixed location we are wherever you are taken.

That means, if you wanted to represent the fish concentration in the river by partial time derivative you going to get a very wrong value that cannot give a true picture.

So, now, what you do? Next time you take a boat and go around the river and then go around the river and then at different locations with different with respect to time you try to measure the concentrations. So, now you are in the river, but you are in you are on a boat and then on the boat you are taking around the rivers and different locations you are selecting.

And then a different location with respect to time as well as with respect to the space you are trying to find out the concentration of the fish. Because you know that whatever the fish concentration that you representing by partial time derivative is highly inaccurate so that is the reason you are doing this one.

So, when you do this thing whatever that at any instant that time rate of change of observed fish concentration that you get by total time derivative that is nothing, but $\left(\frac{dc}{dt}\right)$ that is nothing, but $\left(\frac{dc}{dt}\right) = \left(\frac{\partial c}{\partial t}\right)_{x,y,z} + \left(\frac{\partial c}{\partial x}\right)_{y,z,t}\left(\frac{dx}{dt}\right) + \left(\frac{\partial c}{\partial y}\right)_{x,z,t}\left(\frac{dy}{dt}\right) + \left(\frac{\partial c}{\partial z}\right)_{y,z,t}\left(\frac{dz}{dt}\right)$right.

So, now, fish concentration with respect to time by keeping coordinates one all x, y, z coordinates that is already included partial time derivative. In addition to that one fish concentration at different x, y, z locations are also reported by taking you know corresponding other space and then time variables as constant. And then the multiplication factors $\frac{dx}{dt}$ $\frac{dy}{dt}$ $\frac{dz}{dt}$ are coming for all these three terms.

And these terms are nothing, but $\frac{dx}{dt}$ $\frac{dy}{dt}$ $\frac{dz}{dt}$ terms are nothing, but the x component of boat velocity y component of boat velocity and z component of boat velocity at any location at any fixed location at any fixed time that you are taking ok. So, the components of boat velocity they are not you know river velocity.

Obviously, the boat velocity is now you realize that it is very much different compared to the river velocity right. River velocity at different locations if you measure the velocity fluid stream velocity you may have the different values right different values you will; obviously, have. You will not be you are the river velocity would not be the same as the boat velocity or boat velocity is not boat is not moving at the same velocity as the river stream velocity.

At different location river stream is having different velocity, but boat velocity you are controlling mechanically you are controlling it, you can have higher or lower speed compared to the river stream velocity whatever you want right.

So; obviously, this total time derivative now you realize it is better than the partial time derivative, but still it is not very much reliable, but still it is not very much reliable because whatever this velocity components are there. So, coming into the picture in this total derivative these velocity components are not of the river stream at different locations, but of the boat right.

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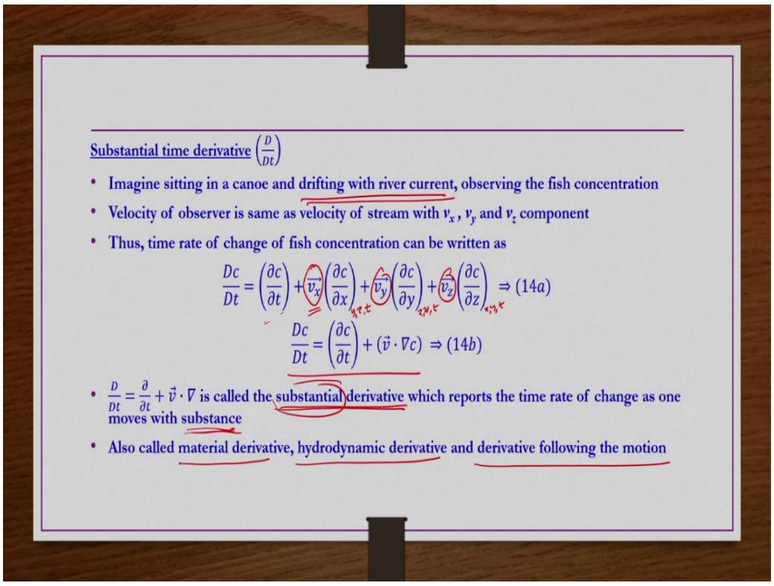

So, then that is the reason what you do? You make further improvement by defining substantial time derivative where you are imagining sitting in a canoe and drifting with river current right and then observing the fish concentration right. Velocity of observer is now same as the velocity of stream with v_x, v_y, v_z components.

Because now you are sitting in a canoe and then drifting with river currents you are not changing the velocity you know rather you are drifting with the river current velocity stream whatever the fluid streams there in the river. So, that velocity at different location that is itself you are taking. So; that means, the velocity of observer is now same as the velocity of the river stream and then that river stream having v_x, v_y, v_z components at each location; at each location ok.

Thus time rate of change of fish concentration can be written as $\frac{Dc}{Dt} = \frac{\partial c}{\partial t}$ the second partial derivative is coming into the picture $+\vec{v}_x\left(\frac{\partial c}{\partial x}\right) + \vec{v}_y\left(\frac{\partial c}{\partial y}\right) + \vec{v}_z\left(\frac{\partial c}{\partial z}\right)$ alright. So, now, here when you measure $\frac{\partial c}{\partial x}$; obviously, y, z and t are constant. And then here also $\frac{\partial c}{\partial y}$ when you measure so z, x and t are constant. And then here also when you measure $\frac{\partial c}{\partial z}$ you keep x, y and t as constants right.

So, this v_x, v_y, v_z are nothing, but the components of the velocity of the stream. So, the same equation 14a if you write in vectorial form you have $\frac{Dc}{Dt} = \frac{\partial c}{\partial t} + \vec{v}.\nabla c$.

That means, whatever $\frac{D}{Dt}$ that is nothing, but $\frac{\partial}{Dt} + \vec{v}.\nabla$ is there that is called a substantial time derivative, because it also moves with the substance. Now, here the moving substance in the case of river example is nothing, but you know the velocity of the river streams right.

So, the substantial derivative reports the time rate of change as one moves with the substance that is the reason it is called as substantial derivative because it moves with the time. It is also known as the material derivative hydrodynamic derivative because it also includes the hydrodynamics of the flowing system and then a derivative following the motion that is also the other terms are given different terminologies are there.

But the terminology substantial time derivative is famously used. So, that is about the partial total and then substantial time derivatives alright. Now, we are going to write the so, called equations of continuity and then momentum that we have derived in a few slides before those equations we are going to write by using substantial derivatives.

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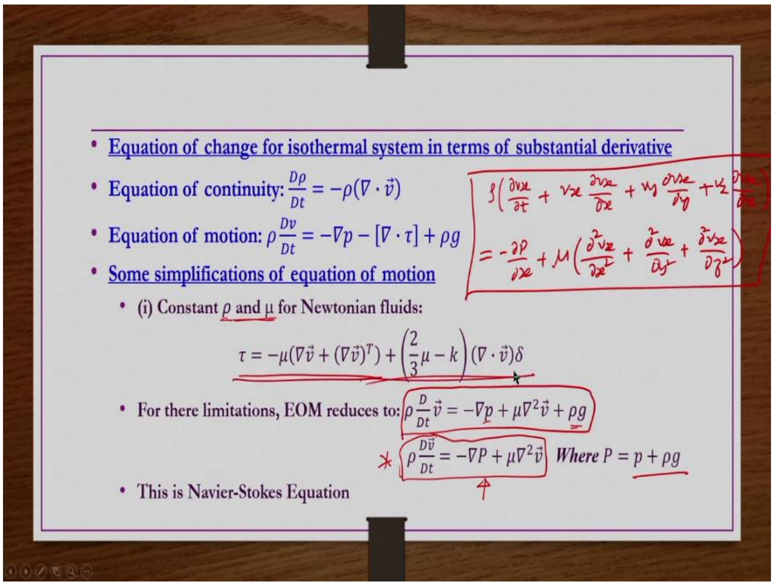

So, that if you write equation of continuity you can write $\frac{D\rho}{Dt} = -\rho(\nabla.\vec{v})$. Whereas, the equation of motion you can write $\rho\frac{Dv}{Dt} = -\nabla p - \nabla.\tau + \rho g$ you can write ok.

So, now, some simplifications of equation of motion that we are going to see; first one we take constant density and then viscosity system for Newtonian fluids, that if you do that is

Newton's law of viscosity that you applied that is in place of τ_{xx} you write $-\mu\frac{\partial \vec{v}_x}{\partial x}$ all those terms and then simplify, then you will get this equation.

$\rho\frac{D}{Dt}\vec{v} = -\nabla p + \mu\nabla^2\vec{v} + \rho g$, let us say if you combine this p and then ρg, $\rho\frac{D\vec{v}}{Dt} = -\nabla P + \mu\nabla^2\vec{v}$ where P is nothing, but p +ρ g. This equation is nothing, but your Navier stokes equation.

Let us say x component of this Navier stokes equation if you write you will get $\rho\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z} = -\frac{\partial \rho}{\partial x} + \mu\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}$.

This is what x component of momentum equation for Newtonian fluids which is also known as the Navier stokes equation x component similarly y component and z component also we can write. This you are getting simply in place of tau you are using this expression and then simplifying it.

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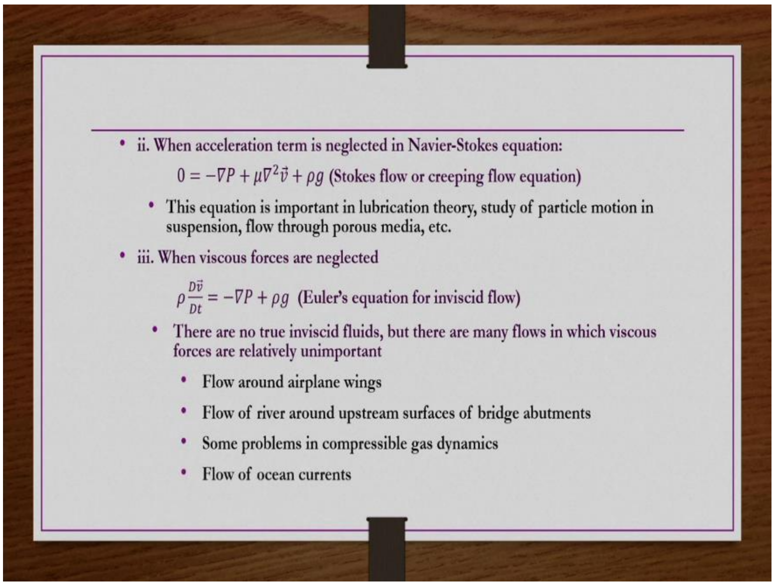

Now, when acceleration term is neglected in Navier stokes equation that is convective terms if you take off then that is whatever the transport is there that transport is there only because of the molecular transport. Then we have the equation $0 = -\nabla P + \mu\nabla^2\vec{v} + \rho g$, which is also known as the stokes flow or creeping flow equation because when there is no convection term all together left hand side terms can be taken as a 0.

This equation is important in some examples like lubrication theory study of particle motion and suspensions flow through porous media etcetera. Now, another simplification or limitation of our momentum equation if you take, where the viscous forces are neglected, that is inviscid flows. Then we have $\rho\frac{Dv}{Dt} = -\nabla P + \rho g$ here whatever the viscous terms $\mu\nabla^2\vec{v}$ is there that term is absent.

So, this is known as the Euler's equation for inviscid flow. In fact, there are no true in viscid flows in general, but there are many flows in which viscous forces are relatively unimportant or maybe very small compared to the convection and in other terms. So, that those viscous forces can be neglected in many cases some of them are flow around flow around airplane wings, flow of river around upstream surfaces of bridge, some problems in compressible gas dynamics and then flow of ocean currents etcetera.

So, this is what the recapitulation that we have in this class recapitulation because all these equations are known to you in your UG level transport phenomena course, maybe these equations are known only for the case of Newtonian fluids case. Now, in this lecture we have derived them for a generalized case irrespective of the nature of the fluid irrespective of the nature of the flow whether laminar or turbulent Newtonian, non-Newtonian no restrictions very generalized Cauchy's momentum equations we have derived here ok.

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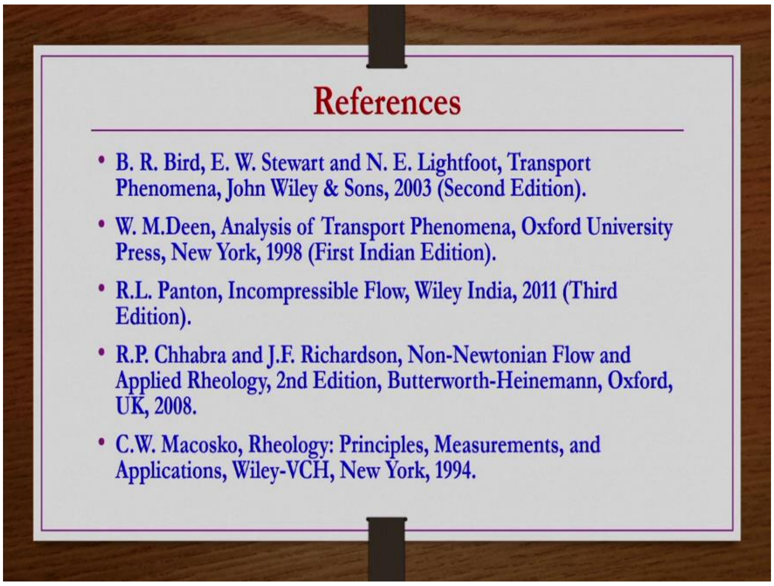

The references all these derivations and then simplifications you can find there find out in this book transport phenomena by bird Stewart and Lightfoot. However, similar information may also be found in the Analysis of Transport Phenomena by Deen are in the book Incompressible Flow by Panton and then these two are the other reference books which may be useful.

Thank you.