Multiphase Flows
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Lecture - 07
Momentum Equation through Reynolds Transport Theorem

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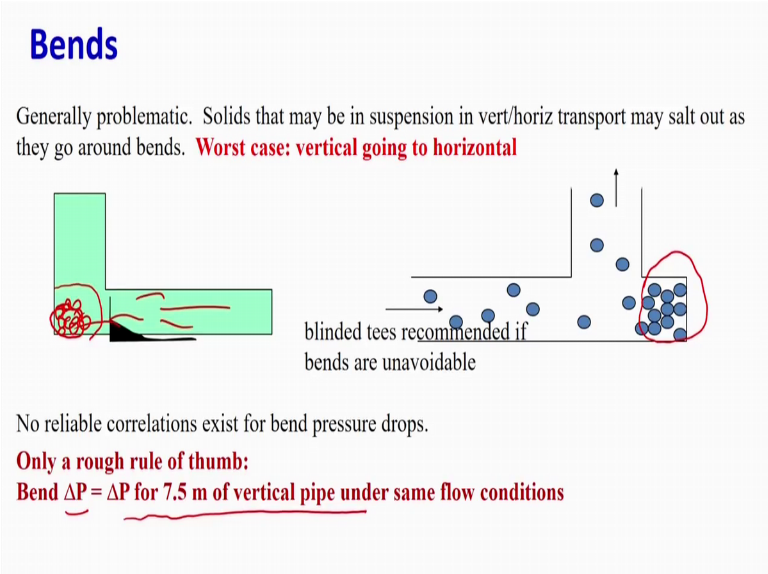

And now what we are going to see we are going to see that how to write the equations for the multi phase flow for all these conditions or all these flow regimes which we have learnt. Now, before going to that the flow equations, I would first like to briefly revise some of the para things which we have already studied maybe in the fluid mechanics, but to bring everyone at the same platform so that we can understand the equation in more detail.

I will do some revision and that revision is very critical very necessary because we are going to build up our equations based on that. Now first thing we always discussed in the multi phase flow or in any flow is that two kind of measurement or two kind of the phenomenon or two kind of parameter; what we say is the Eulerian flow and Lagrangian flow or a Eulerian measurement and Lagrangian measurement.

So, first what we are try going to understand that what is Eulerian, what is Lagrangian? So, that we should be very clear whether we are writing the equation in the Eulerian frame of domain or we are writing in the Lagrangian frame of domain, whether we are doing the measurement in Eulerian frame of domain or we are doing the measurement in Lagrangian flow up domain. So, what we are going to do we are going to discuss about the Lagrangian versus Eulerian.

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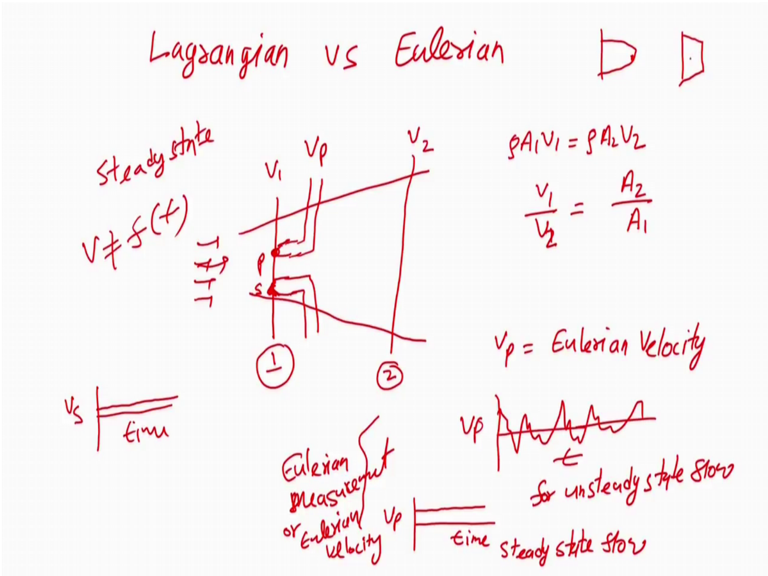

Now what is Lagrangian and what is Eulerian. Now if we; you might have studied about this in many places at the discrete places, in your mass transfer, in your fluid mechanics and we have tried to derive the equations, also we will again revise it because this is very very critical.

Now, what we are going to see is the equations, we are developing the measurement techniques, we will be developing it will be either the Eulerian measurement or it will be the Lagrangian measurement, and we have to know, we should know; how to interchange the Eulerian information to the Lagrangian information or the Lagrangian information to the Eulerian information. So, that we can use the system or we can analyze the system in more detail.

Now, to understand this; what is Eulerian, what is Lagrangian, let us assume a system in which a flow is taking place and the flow is changing with the position. Now we can easily assume any diverging or converging section and if the section will be diverging, definitely the area will change if the area will change my velocity is going to change.

So, suppose the fluid is flowing here in a diverging friction. So, what will happen? Suppose, I am taking two places here; one is section one and another one is section two. So, what will happen? Section one; suppose has a velocity V 1, section two has a velocity V 2, why it is going to happen because my area has changed we know the equation of continuity that A 1 V 1 will be equal to A 2 V 2, if the flow is incompressible or we can say that overall mass transfer rate is going to be the same.

So, overall mass transfer is going to remain same. So, rho A 1 V 1 will be equal to rho A 2 V 2 if rho is same rho; rho will be canceled out A 1 V 1 will be equal to A 2 V 2, it means I can write V 1 upon V 2 is nothing, but it will be equal to A 2 upon A 1 ok.

So, if your area will change your velocity will also change now it means if A 2 is higher than the A 1, then what will happen the V 1 will be higher than the V 2 ok. So, that is going to happen. Now what we are going to see that if suppose, I am interested in measuring the velocity at any of this point ok, say I am interested to measuring the velocity at point one at any point here in between.

Now, what will happen suppose I have a probe which can measure the velocity say pitot tube I can place the pitot tube at this place, this point and it will give me the velocity at this point. So, suppose I am putting a pitot tube here, I hope everyone knows what is pitot tube. So, I am putting a pitot tube along this point p and if I put a pitot tube I will measure the velocity at that point.

Now, if I assume that the flow is steady at steady state it means what the V is not a function of time ok. So, V will not change with the time. So, what will happen you will measure certain velocity here? Now velocity at this point and velocity at this point is going to be different why because the area is keep on changing now what you are measuring here is the velocity at a fixed place ok.

So, that is called Eulerian velocity. So, what you will measure say pitot tube is giving a velocity V p which is the I am saying the point velocity the velocity at a point p this V p is nothing, but is the Eulerian velocity, it means what when we fix our frame of reference or when we fix our position or specify our location and measure something at that location is called Eulerian set or Eulerian measurement or Eulerian quantity.

So, here we are measuring the velocity now I have fixed my position. So, that what we are going to see is the this velocity how it is changing with the time now if with the time this flow is at steady state, then dou V by dou t will be 0 and you will be measuring the same velocity throughout the domain or throughout the time, you are doing the measurement ok, if it is not efficient the flow is not a steady state then with the time, you will see that how the velocity is changing at that location.

So, suppose a flow is not steady or is the unsteady state flow then you will see at point p how the velocity is changing with the time this is time and this is V p for unsteady state flow for a steady state flow what you will see you will see this is V p and this is the time, you will see there is no change in the velocity and this is for a steady state.

So, these all are called you Eulerian or Eulerian velocity or Eulerian velocity where the position has been fixed ok, if I change the position if I suppose, now if I am measuring instead of position p, I am measuring putting up new pitot tube which is at a position s ok, I will see another velocity V s with time if the flow is steady state this V s will be there, it can be more than V p, it can be lower than the V p, it can be same to V p depending upon the flow condition whether the flow is laminar or flow is turbulent we know, if the flow will be laminar, then what you will see you will see a parabolic profile. So, definitely if you are measuring at the center velocity will be higher if you are measuring near the wall, velocity will be lower.

If your flow profile is turbulent then what will happen that near the wall you will see a tilted profile where the velocity will not be the same, but at the central portion the velocity will almost be same everywhere in that case, if the 2 point is between this flat region your V p and V s will be the same if you are a laminar flow you are going to have a parabolic profile V p and V s will be different. So, what we will see we will see that how the velocity is changing with the location. So, if I change the location the velocity is changed for a steady state for unsteady state flow, the velocity can change even at the same location with the time for a steady state flow the velocity at the same location with the time will be this same, but if you change the location it may change ok.

So, these all are called Eulerian field or Eulerian measurement or Eulerian velocity, but the problem is our most of the conservation law whether the conservation of mass conservation of momentum conservation of energy conservation of angular momentum all those equations are written in terms of the Lagrangian field what do you mean by the Lagrangian domain it means if I am measuring the velocity of a molecule. So, what will happen if the flow is taking place in this system then the Lagrangian system if I am talking about if I do to the same diverging section?

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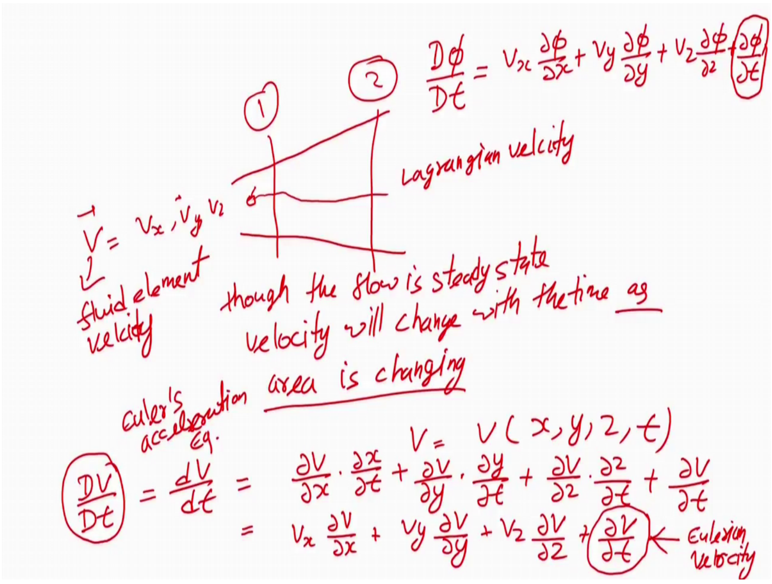

Say if one section, section two, if I am talking about a Lagrangian system then what will happen; if Lagrangian means you are sitting on a molecule of a fluid element and you are seeing, how the velocity of that fluid element is changing with the time.

It means; suppose there is a molecule of the fluid element, I am sitting on top of it, what will happen? I will see that though the flow is steady state my velocity is changing with that time ok, please remember, this though the flow is steady state and this is very critical. So, though the flow is steady state the velocity will keep on changing with the time if I am sitting on the molecule.

Why because the area is changing my velocity is going to change and this is called the Lagrangian velocity. So, what is happening my domain is not fixed and domain is changing with the time and because the domain is changing my environment is also changing with the time and that is the reason that I am seeing a different velocity field altogether I am not measuring the velocity, I am not seeing the velocity at a fixed domain I am changing the velocity with the time.

It means what is changing the velocity is changing with the time and the location of the molecule is also changing, it means here, if I am aging the velocity, this velocity will be the function of x, y, z and time t, it means if I have to write in terms of the derivative I will write it as DV by Dt it will be what it will be dou V by dou x into dou x upon dou t because position is changing with the time dou V by dou y into dou y upon dou t plus dou V upon dou z into dou z upon dou t plus dou V upon dou t, how this is changing velocity is changing with the time ok. So, if that is also happening.

So, now this is called actually the substantial derivative or material derivative which is being widely used to understand the Lagrangian velocity field that; what is the Lagrangian velocity. This Dx by Dt will be equal to what it will be the velocity if V is a vector we know that and has a 3 component V x V y V z, then what will we can write it as a vx dou V upon dou x plus vy dou V upon dou y plus vz dou V upon dou z plus dou V upon dou t.

Where V x V y and V z at V is nothing, but it is a velocity of the fluid element when it is velocity of the fluid element, this DV by Dt will be equal to the substantial derivative or the material derivative in which this is what is called the Eulerian part where we are seeing that how velocity is changing with the position and position is changing with the time in all these 3 and how the velocity is also changing with the time.

So, this part if suppose you are doing the Eulerian measurement, if I am measuring the pitot tube, what I am measuring is this part that how my velocity is changing with the time if I put a pitot tube at a particular location, I am not changing the position here in this part. So, what is happening at a fixed location, I am seeing that how the particle velocity or how the fluid velocity is changing with the time this is Eulerian velocity and this is what is the Lagrangian velocity DV quantity or the material derivative and whatever the conservation equation.

We see we all see in terms of the substantial derivative where the fluid element velocity comes into the picture or the fluid velocity also comes into the picture and that fluid velocity is being measured by how the fluid element is changing the position with the time that is nothing, but Dx upon Dt which is nothing, but the x velocity or x element of the velocity of the fluid V DV Dy by Dt or dou y by Dt which is the y element velocity of the fluid dou z by dou t which is the z element velocity of the fluid ok.

So, that is the component similarly any quantity any parameter, you can write it in this form. So, I can say that dou phi by Dt where phi is any quantity it can be written as vx dou phi by dou x plus vy dou phi by dou y plus vz dou phi by dou z plus dou phi by dou t, but this part is the Eulerian part if you do the Eulerian measurement you actually get only this while writing the conservation equation we see all this.

So, what we need to learn how to convert these two parameters and how to start the measurement in such a way that the Lagrangian information or the Eulerian information can be interchangeably used to understand about the system and this whole equation this whole equation whatever in terms of the velocity we know it is also called as Euler's acceleration equation and you have already have used in your fluid mechanics once you do the fluid mechanics you derive the Navier stoke equation, you derive the Eulerian equation and you mean this equations like if the Eulerian acceleration is 0, how the equations will be converted you do all sort of things. So, this is called Eulerian acceleration equation in which this is stilled that how the fluid element is being actually accelerating.

So, this is; what is the Lagrangian domain. What is the Eulerian domain to summarize again Eulerian domain means if your location is fixed, if your frame of reference is fixed and you are seeing the element of the fluid velocity or how the fluid velocity is changing in that element, in that fixed element is called Eulerian. So, what is happening the fluid element is keep on changing, but the location is fixed in the Lagrangian field, you are seeing that how the velocity of a fluid element is changing with the time. So, what is happening fluid element is fixed, but the location of the fluid element or the position of the fluid element is changing with the time?

So, that is called when the position of the fluid element changes with the time it is called Lagrangian measurement or Lagrangian velocity or Lagrangian quantity when the element is fixed observation element is fixed and the fluid element in that observation circle or in that observation frame is keep on changing and we see with the time, how the quantity is changing in that fixed frame is called Eulerian measurement or Eulerian quantity. So, these are the two quantities which we need to understand while doing most of the measurements we have, we measure in the Eulerian field domain like thermocouple like pitot tube like pressure gauge, what we do in thermocouple? We fix the thermocouple probe or the, that a particular location. So, we measure the temperature at that location.

In case of pitot tube, you fix the probe at a particular location you measure the velocity at that particular location and what you measure is the Eulerian field most of our conservation equation is written in terms of the Lagrangian field and that is why there is always a confusion between the Eulerian field measurement and Lagrangian field measurement and I hope that now it will be little bit clarified.

So, what we are going to do we are now going to see that how to interchangeably use the equations how to write the equation for the different multi phase flow domain now before that what we are going to do we are going to derive the equation and that is very common equation is called Navier stoke equation and the equation of conservation of mass

So, first we will see that how that equations are derived. So, that once for all our basics are clear and then by using that equation of momentum, we will try to see for different multi phase flow regime how that the same equation can be modified what we will try to do we will try to fix our discussion in the single direction or in the one dimensional domain you can convert that the same equation very easily in two dimensional and 3 dimensional domain; however, it will be very numerically very challenging to get the solution its almost impossible to get an analytical solution for that that is the reason that why we will be limiting our discussion for this course in A 1 dimensional domain. So, that we can solve the problem, but one can easily extend the same discussion in two dimensional and 3 dimensional domain, but then you need numerical approaches numerical methods or advanced computational methods or computational methods to solve those equations.

Second thing; what we are going to do we will be limiting our discussion most of the time in the laminar flow regime, again for the sake of simplicity of the problem. So, that we can understand the things because turbulent will be completely different ball field and if you go in the turbulent domain your number of equations will be increased your number of variable will be increased and it would be very difficult to get an analytical solution for that. So, one is interested can do that we will also try to look into the computational fluid dynamics method, how to solve the computationally the equations;

we will briefly try to see towards that, but that is not the main focus of this course, but the main focus of this course is to write the equation in one dimensional field in one dimensional domain for laminar flow equation if you get the turbulent flow; do not be worried, it is exactly same you are solving just few more equations to account the turbulent field.

We will also try to have a small discussion on the turbulent field, but most of the time we will be limiting our discussion on the laminar flow field. So, these all I am classifying or clarifying. So, that in future classes there will be no confusion and we will be keep on carrying our discussion on the laminar flow field ok, one dimensional laminar flow field that is what we are going to cover turbulent flow field, if you read the paper if you read the book some books discuss about the turbulent flow field do not get confused you can easily solve it only thing is the analytical solution will not be possible and you have to depend on the computational fluid dynamics method or advanced numerical approach methods to solve those problem people use alias people use dns people use some computational field in the finite volume method or finite difference method or computational approach to solve those equations, but we will limit ourselves in one dimensional laminar flow.

So, before going to the multi phase flow, as I said that what we are going to see is how this Lagrangian field Eulerian field is being kind of generated how to write the equations properly and how the basic notion of how to develop and how to write first the general momentum equation or general mass conservation equation. So for that; the very basic thing which has been started is called the Reynolds transport theorem ok.

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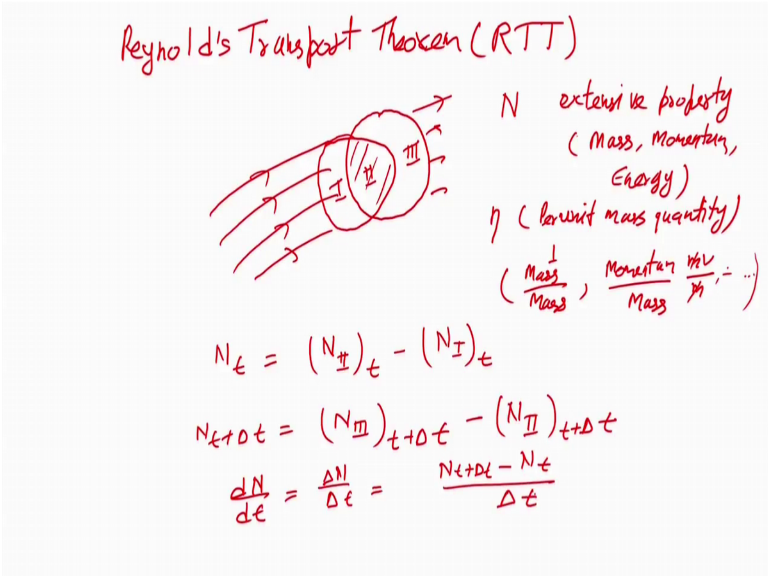

So, what we are going to do is theorem also known as widely RRT ok. So, what Reynolds transport theorem says, suppose, there is a fluid element which is moving a fluid is flowing here and if I take a small element here a small say volume which is being there and if suppose at time t equal to t plus delta t. this volume is being kept, then what will happen that you will see that it will move to a certain distance and it is the same element, I am saying that it has moved to the certain distance and what will happen you will see this is again moving.

Now, what we can do we can divide this element in 3 part; one is the part one, this is the part two which is actually the intercept and this is the part 3. So, what is happening a fluid element is there which is being placed in a moving flow and this will be swept with a velocity of the fluid and because of that swept some of the property will actually move outside of this control surface from the outer side and some of the property, from this control surface fund will move to the inside of the control volume two. So, there is a control volume and these are two; one and two is the two control surfaces which is moving in the property is moving in from control surface one and moving out from control surface two.

So, now if I have to write here; so, if I see that suppose if I am talking about any extensive property which is a property of mass and I am denoting it with a number and so, this is extensive property ok. So, this can be energy mass this can be momentum this can be energy and I am defining one more quantity eta which is in intensive property or I will say the per unit mass quantity, it means if I am defining and as a mass, then eta will be mass upon mass it means the value will be equal to 1, if I am defining the momentum it will be momentum per unit mass momentum is defined by mV divided by m it means this will be cancelled out it will be the velocity and. So, on if you define the energy and all is all. So, on

So, if you define kinetic energy t will be half mV square it will be only half V square. So, and. So, on you can define the quantity eta which is intensive property per unit mass quantity and is the extensive property which is mass energy momentum any quantity we are defining. So, we are defining a general equation. So, suppose a quantity N which is being swept and it is being from control surface one, it is entering to the control through volume two and then it is going out again through the control surface 3. So, what will happen at any time t quantity N suppose at time t the quantity N will be what the quantity N will be in what will be the quantity N in control surface two at time t minus how much is come in from control surface one at time t.

Now, at time t plus delta t if I just think about at t plus delta t how the things will happen this has been changed the location has been moved. So, what will happen this will be moved again and this quantity two will be moved and it will be seeing that how much of this quantity has moved at t plus delta t from control surface 3 and how much has come in the control surface in the control volume two t plus delta t. So, that will be at time t equal to at an time equal to t, the quantity and or the system on this N, this will be the total change in the quantity which will be coming in at t plus delta t, how much quantity will be going out that will be this quantity.

So, rate of change of this quantity will be dN upon Dt and if I assume that this delta t is very small then I can say that delta N delta N and upon delta t. So, this will be equal to delta N will be nothing, but N t plus delta t N t plus delta t minus N t upon delta t, I can write it in this way, now if I write it N t plus delta t is what.

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$$= \frac{(N_{III})_{t+\Delta t} - (N_{II})_{t+\Delta t} - (N_{I})_{t} + (N_{II})_{t}}{\Delta t}$$

$$= \frac{(N_{II})_{t} - (N_{II})_{t+\Delta t}}{\Delta t} + \frac{(N_{III})_{t+\Delta t}}{\Delta t} - \frac{(N_{I})_{t}}{\Delta t}$$

$$= \frac{\partial}{\partial t}\int_{CV} \rho\eta \, d\forall \qquad \vec{V}$$

$$(N_{III})_{t+\Delta t} = \int_{CS} \rho\eta(\vec{V}\cdot\hat{n})\,dA \qquad = \int_{CS} \rho\eta(\vec{V}\cdot\hat{n})\,dA$$

$$= \int_{CS} \rho\eta(\vec{V}\cdot\hat{n})\,dA$$

N t plus delta t is N t 3 into t plus delta t minus; it will be N 2 t plus delta t ok, then this will be this minus again N t at two at N t, then again it will be plus N of one into t sorry this will be N 2 and this will be N one t, this will be in this case upon delta t.

So, what you can say you can say that the system final minus initial I can write it as N to t minus N 2 t plus delta t upon delta t this will be the one then I can write it N, sorry, I can write it this will be again, there is something this is minus, this is plus. Now this will be plus N 3 t plus delta t upon delta t which is moving out of the control surface known minus and one t upon delta t these are the 3 quantities.

Now, this is what this is the control volume quantity that how much is the quantity is moving out of the control volume in per unit time. So, I can write it out this quantity as dou by dou t of integral control volume integral of the same quantity rho in terms of the intrinsic property which is eta into DV ok.

So, this is how much of that quantity this quantity is moving per unit volume because section two is a control volume. So, I have multiplied with by the rho and I have just made it the in quantity N and in terms of the eta. So, it means eta is what quantity per unit mass I have multiplied kg per unit volume. So, mass mass will be cancelled out it will be carried as N capital N if you write it it in this way and then per unit volume how much is the quantity is changing. So, I can write this whole term is dou by dou t of this term.

Now, what about the surface 3; it means how much of the same quantity is going out from the surface 3 and how much of this quantity is coming in from the surface one. So, we can write it out as how much quantity is going out from the surface separately. So, I can write it out N 3 t plus delta t is equal to that how much is the quantity going out of the surface it will be nothing, but the integral surface integral, it will be the control surface integral of the quantity which is going out of the system.

Now, that will be nothing, but the eta which is the intrinsic property and the way it has been swept out from that surface now the surface velocity, suppose is V, if V is the velocity of the surface at which the surface is being swept or it means you can say the velocity of the fluid then the surface will be swept as the velocity V in the direction normal to the area. So, this we will write as a V dot N which is the unit vector in the normal direction of the dA.

So, that will be the amount of the quantity which will be swept out from the surface 3 similarly the similar amount will be swept in from the surface one into the similar amount will be swept in from the surface one it means the overall change will be written as this minus this can be written as this will be the total swept or total change will be the change at the relative change in the velocity it means V in minus V out and I am writing that as a V. So, that will be the V dot N into dA where this d V is nothing, but the change in the velocity that how much is the velocity of coming swept in and how much mass is being swept out.

So, that is the overall total change in the velocity how it is changing with the time will be written here. So, this whole quantity can be written as dN by dt.

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$$\boxed{\left.\frac{dN}{dt}\right|_{system} = \frac{\partial}{\partial t}\int_{CV} \eta \rho \, d\forall + \int_{CS} \rho \eta (\vec{V}\cdot\hat{n})\, dA}$$

Reynold's Transport Theorem

$$N = mass = m$$

$$\eta = \frac{m}{m} = 1$$

$$\left.\frac{dm}{dt}\right|_{system} = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho (\vec{V}\cdot\hat{n})\, dA$$

$$= \int_{CV} \frac{\partial \rho}{\partial t}\, d\forall + \int_{CS} \rho (\vec{V}\cdot\hat{n})\, dA$$

Dn by Dt of the system will be equal to how the quantity is changing with the time ok; with the time how the quantity is changing in the control volume dou by dou t plus in the control surface, how the quantity is being changed. So, this V dot N which is a unit vector into t now this is called the overall Reynold transport theorem. Now any transfer of the quantity can be derived through by using this Reynold transport.

So, what we have done we have done a very simplified approach we have taken we have taken a moving fluid element in which I have taken an element here now element is moving with the flow. So, what will happen some of the quantity, suppose any quantity whether it is a mass, whether it is a energy, whether it is a momentum, in its angular momentum any of the quantity will also be swept because of the fluid velocity element.

So, what will happen the swept will be because of the change in the property at the surface; how much, it is being swept in from the left surface to the control volume how much it is being swept from the control volume to the right surface and that is the total change will be what we will see the effective change in the quantity. So, effective change in the quantity with the time you will see how the quantity is changing within the control volume and how the quantity is changing from the control surface ok.

Now, if you have a different six elements all the surface element will be changing to quantity and you can have a proper equation that how any quantity is changing with the time from the control surface and from the control volume. So, this is Reynold transport theorem and you can derive all your conservation equations by using the Reynold transport theorem ok. So, what we are going to first see that how to derive the mass conservation equation by using the Reynold transport theorem. Now for mass conservation equation what will be my property my property will be mass. So, N will be what N will be equal to mass that is say written by m.

So, what will be eta? Eta will be the intrinsic property which will be per unit mass. So, eta will be equal to m upon m it means we will write it as a one. So, in the same system same Reynold transport equation I am going to replace the N and eta. So, what is going to be dm upon Dt. So, rate of change of mass of the system is equal to dou by dou t of the control volume you will write eta will be equal to one. So, this will be rho of the control volume DV dv is nothing, but the control volume integral plus you are going to have a control surface eta will be equal to 1 V dot N into dA.

Now, what we can do we can this is independent of the time. So, I can put the time inside the control volume is anywhere of the at arbitrary. So, we can write it dou rho upon dou t in the control volume V and this will be the control surface V dot N and is the direction normal to the surface dA. Now, we can use the gauss divergence theorem.

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Gauss divergence theorem

$$= \int_{CV} \frac{\partial \rho}{\partial t}\, d\forall + \int_{CV} (\nabla \cdot V\rho)\, d\forall$$

$$= \int_{CV} \left[\frac{\partial \rho}{\partial t} + (\nabla \cdot \rho V)\right] d\forall$$

$$\boxed{0 = \cancelto{0}{\frac{\partial \rho}{\partial t}} + (\nabla \cdot \rho V)}$$ Mass conservation Eq.

$$\boxed{(\nabla \cdot V) = 0}$$ Incompressible flow

So, what does theorem do the theorem changes the control surface integral to the control volume and if you apply the gauss divergence theorem here this quantity will be written as exactly same control volume dou rho upon dou t into DV plus this can be changed to the control volume integral and it will be written as del dot V and rho will be also coming in here the rho will be there here the rho will be there.

So, this del dot V or it will be del dot rho V here we have done this. So, this will be rho will also be here there will be a rho here ok. So, that will be the control volume. So, this will be the control surface here. So, because you are changing in the intrinsic property here the rho will be there it will be there rho. So, it will be rho V into n. So, that is the whole Reynold transport theorem. So, here the rho also will come here and you will see that what is del dot V rho now if you do that I can write it under both the control volume integral. So, del rho upon del t plus del dot rho V and this will be d v.

Now, this is control volume because we are using an arbitrary control volume system we can do that integral this will be Dx Dy dz, we will integrate it, it will come at a delta x delta y delta z and we can simplify it as a del rho upon del t plus del dot rho V will be equal to 0 and this is nothing, but the mass conservation equation .

If the flow is incompressible then del rho by del t will be 0 rho will come out and del dot V will come to 0 for incompressible flow ok. So, you can derive that how the mass conservation equation can be derived from the Reynolds transport theorem again going back to this eta how it rho comes because we are changing the property and to the intrinsic property and that is why the rho has been multiplied here again ok. So, that because you are changing the N to the eta that is why I forget here rho I introduce it and please be there and I hope now there will be no confusion why the rho comes here.

So, what you have to do you have to just replace it if you replace it you will get and just use that gauss divergence theorem which actually changes the surface integral to the control volume integral gauss divergence theorem and it will be nothing, but del dot rho v. So, by using that you can derive the mass conservation equation general mass conservation equation you can simplify it for the incompressible flow where del rho by del t will be 0 and then you can write it as del dot V will be equal to 0 because rho will be constant it will come out and that you can take it as a 0. So, right hand side. So, you will get the del dot V equal to 0 which is the incompressible mass conservation equation for the incompressible fluid or velocity continuity equation.

So, this can be derived similarly one can derive the momentum equation Navier stokes equation and for Navier stoke equation derivation what you need to do you have to take the momentum.

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Momentum $\quad N = mV$

$\eta = V$

$$\left.\frac{dN}{dt}\right|_{system} = \frac{\partial}{\partial t}\int_{CV} \rho \eta \, d\forall + \int_{CS} \rho \eta (\vec{V}\cdot\hat{n})\, dA$$

$$\frac{d}{dt}(mV) = \frac{\partial}{\partial t}\int_{CV} \rho V \, d\forall + \int_{CS} \rho V (\vec{V}\cdot\hat{n})\, dA$$

$$= \int_{CV} \frac{\partial}{\partial t}(\rho V)\, d\forall + \int_{CS} \rho V(\vec{V}\cdot\hat{n})\, dA$$

$$= \int_{CV} \frac{\partial}{\partial t}(\rho V)\, d\vec{\forall} + \int_{CV} (\nabla\cdot \rho \vec{V}\vec{V})\, d\forall$$

It means the system property, N will be equal to mV and eta will be equal to V. So, what we will do we will say dN upon Dt Reynold transport theorem of the system will be equal to control volume dou by dou t of the control volume of rho eta d z plus control surface rho eta V this is eta dot N into dA.

Now, we will replace it for the momentum it means this will be d upon Dt of mV will be equal to dou upon dou t of integral rho eta is equal to V into DV plus control system rho V into V dot N into dA.

Now, again what I will do because this is the control volume this dou by dou t I can take it inside. So, I can write it control volume dou by dou t of rho z into DV where V is nothing, but the control volume integral plus control surface rho V into V dot N into dA. So, again I can put that gauss divergence theorem and if I will put the gauss divergence theorem this will be control volume dou by dou t of rho V into DV and this will be again converted to the control volume and del dot rho V c DV ok, again the gauss divergence theorem the control surface to the control volume now again what we can do.

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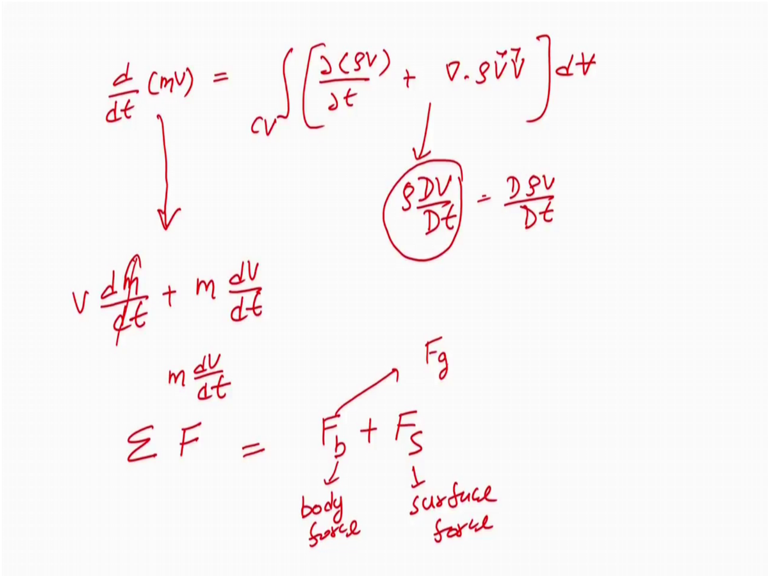

We can say that d upon Dt of mg is nothing, but this will be equal to control volume integral of dou rho V upon dou t plus del dot rho vv of the control volume DV.

Now, this is the right hand term term whatever we have seen here and this will be nothing, but capital DV upon Dt that will be the substantial derivative of the velocity or you can say that it will be the material derivative this whole term the right hand term we can write it as rho this this term we can write it as rho DV upon Dt if rho is constant if rho is not constant we can write it dou rho V upon dou t ok.

So, what we are going to do if rho is rho is incompressible we are writing going to write it it is this way d rho DV upon Dt. Now DV upon Dt now what is the left hand side term left hand side term if you will see I can write it d m upon Dt into V plus m DV upon Dt the left hand side term this term.

Now, we know that rate of change of the mass this is V equal to 0. So, what will happen you will see m DV upon Dt which is nothing, but the force. So, left hand side term is nothing, but the summation of all the forces which is acting on the element will be equal to whatever the material derivative is DV upon Dt of the control volume interior. So, now, on any element whatever the element we have thought any element what are the total forces acting the total forces acting can be written in terms of the body forces plus the surface forces Fg plus Fs. Now, let us simplify that what will be the body force or instead of g, I will write it though let it right Fb which is the body force to the surface force ok.

So, Fb is the body force Fs is the surface force now what is the body force which will be acting in absence of any other force it will be nothing, but Fg which will be the gravitational force which will be acting on the element now what are the surface forces. So, let us assume a simplified case and I am assuming the cuboid.

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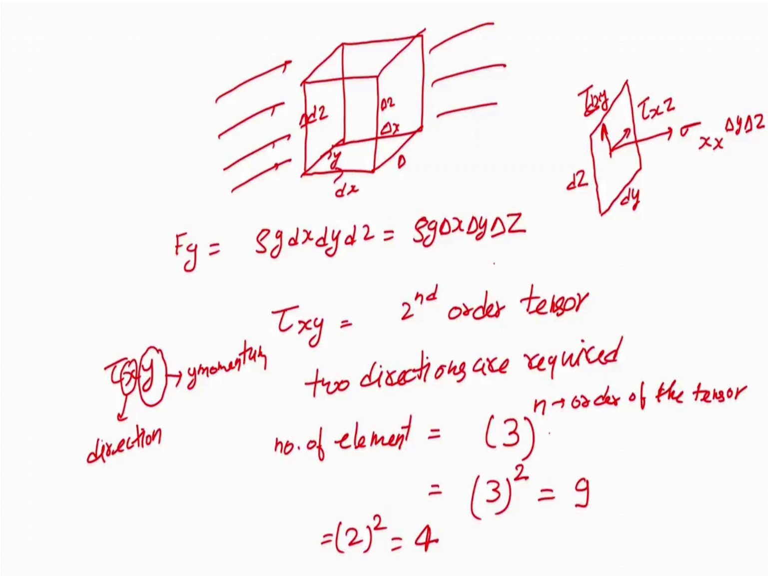

So, let us assume it is being kept inside a moving fluid element and this is the direction x this is the direction y this is the direction z ok.

So, what is the body force acting on this that Fg it will be nothing, but it will be rho into g into Dx, Dy Dz is this is the Dx Dy and Dz a very small element, I am choosing Dx, Dy, Dz or delta x delta y delta z then this will be equal to actually rho g delta x delta y delta z, if Dx divided is it is very small infinitesimally small then we can simply write it in terms of the delta x delta y delta z. So, that will be nothing, but the body force which will be acting on it on the element.

Now, what are the surface forces now if you see that the surface forces the surface forces will acting on all the surfaces let us assume a sink one surface the surface which is in the direction of state x the surface which is in the direction of x this is Dz this is Dy and this is in the direction of x or I can write it delta x delta y and delta z delta y and this is in the direction.

So, what are the forces which will be acting one will be acting which will be the normal to this surface and that force is being given by sigma. So, I will write it as sigma at x which is the force acting normal to the surface, then there will be shear forces will be acting now the shear forces is what which is tau and the tau notation whatever we are going to follow it is tau is what; it is a second order tensor.

What do you mean by second order tensor? It means we need two direction to specify distance and number of component will be what how many number of component it will be the number of dimensional space it means I will write it as. So, tau xy is what two, it is a second order tensor it means you require two directions to specify the second order tensor and the number of element will be equal to how much is the number of space you are using. So, suppose if you are using in 3 dimensional space. So, number of the space will be 3 and raise to the power N where N is nothing, but order of the tensor tensor ok.

So, it means if you have a 3 dimensional space order of tensor is two. So, you will have nine component of the shear stress if you have A 2 dimensional domain and order is 2. So, you will have 4 direct 4 component ok. So, tau will have 4 component and the notation which we are going to follow is tau xy suppose if I am writing say tau xy, it means what I am talking about I am talking about this is y momentum in x direction ok, it means tau xy will be y direction momentum, it means the velocity will be in the y direction y directional velocity it means vy component and effect of the vy component in the x direction. So, that is the way we are going to distribute it.

Now, we are talking about in the x direction. So, there will be sigma xx which is the normal stress component sigma xx, ok, sigma xx will be acting now that in terms of the force if you have to write you have to multiply it with the area and area will be what which will be the perpendicular to it. So, in terms of the force the area perpendicular to the surface this will be delta y delta z ok. So, that will be the area perpendicular. So, how much it will be this, it will be coming into the surface x now there will be the force which will be in the x direction again which will be effect of the y momentum.

So, it will be tau xy which will be acting on it and tau xy suppose the direction will be somewhere here that will be tau xy say tau xy and tau zx the direction will be somewhere

here that will be tau xz actually which will be the z momentum in the x direction y momentum in the x direction these are the forces which will be acting in the x direction

Similarly, the forces will be acting in the y direction the forces will be acting in the z direction, but I am just writing in one directional forces which is in the x direction. So, what I can say I can say that total force which is acting in the direction of x that will be written as sigma xx that would be multiplied by delta y delta z

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$$(\sigma_{xx}\ \Delta y \Delta z) \quad (\tau_{xy})\ \Delta x \Delta z \quad (\tau_{xz})\ \Delta y \Delta x$$

$$F = \left(\sigma_{xx}\big|_x - \sigma_{xx}\big|_{x+\Delta x}\right)\Delta y \Delta z + \left(\tau_{xy}\big|_y - \tau_{xy}\big|_{y+\Delta y}\right)\Delta x \Delta z + \left(\tau_{xz}\big|_z - \tau_{xz}\big|_{z+\Delta z}\right)\Delta y \Delta x + \rho g\, \Delta x \Delta y \Delta z$$

$$\frac{F}{\Delta x \Delta y \Delta z} = \frac{\sigma_{xx}\big|_x - \sigma_{xx}\big|_{x+\Delta x}}{\Delta x} + \frac{\tau_{xy}\big|_y - \tau_{xy}\big|_{y+\Delta y}}{\Delta y} + \frac{\tau_{xz}\big|_z - \tau_{xz}\big|_{z+\Delta z}}{\Delta z} + \rho g$$

Then it will be the force which will be acting in the again x direction which will be the y momentum. So, x into y now that is the force which is the y directional force which acting in the x direction. So, what will be the surface which will be acting here you have to multiply with that surface and that surface will be what this is the y momentum force it will be delta x into delta z similarly it will be tau xz that will be multiplied by delta y into delta x.

Similar. So, these are the forces which are acting on the x side of the surface similar forces will be acting on the other side. So, this is on this side of the surface whatever we have said. So, this side of the surface similarly the forces will be the similar forces will be acting on this side. So, total net change in the x direction force will be you have to just subtract it if you subtract it. You will write it as sigma xx minus at say x minus sigma xx at x plus delta x you multiply it by delta y delta z, the total forces surface forces I am adding it, it will be tau xy at surface y minus tau xy at surface y plus delta y and this will be multiplied by delta x delta z plus tau xz at xz minus tau xz at z j plus delta z upon multiply by delta y delta x plus that is the total force.

So, total force this is the surface forces plus the body force body force is nothing, but the Fg Fg can be written as Fg can be written as as we have defined it is nothing, but Fg is nothing, but rho into g delta x delta y delta z now you can divide it by delta y delta z delta x delta z that will be the F and you just divide it by the delta x delta y delta z, this will be nothing, but you can write it sigma xx minus sigma xx this is at x, this is as x plus delta x upon delta x plus tau xy at y minus tau xy at y plus delta y upon delta y plus tau x z at z minus tau x z at z plus delta z upon delta j plus rho g.

So, you can write it as by using if delta x delta y delta z are very small infinitesimally small.

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$$= -\frac{\partial \sigma_{xx}}{\partial x} - \frac{\partial \tau_{xy}}{\partial y} - \frac{\partial \tau_{xz}}{\partial z} + \rho g$$

$$= -\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) + \rho g_x$$

$$\frac{\int_{cv} \rho \frac{DV}{Dt}\, d\forall}{\delta x\, \delta y\, \delta z} = -\left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right] + \rho g_x$$

$$\rho \frac{DV}{Dt} = -\left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right] + \rho g_x$$

$$= -\left[\frac{\partial (P + \tau_{xx})}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \partial \tau_{z}\right.$$

You can write it in terms of the differential equations, we can write as d sigma xx, I will write it in terms of the dou upon dou x minus dou tau xy upon dou y minus tau x z upon dou z plus rho g ok. So, whole minus will come out it will be dou sigma xx upon dou x plus dou tau xy upon tho y plus dou tau zz upon dou z now because we are writing it in terms of the x direction gravity also I will write it it in the x direction and instead of g I will write it as rho gx.

So, what we have done total force which is acting in the x direction similarly one can write it in for the y direction for the z direction and that will be the total summation of the force and that is the force which we are writing is integral control volume of d of V upon Dt, I am writing it for incompressible flow of dou z now this divided by the total volume, there the total volume has been divided. Now if you do that here delta V delta xy this is delta x delta y delta z is here and that you are writing as minus of dou sigma xx upon dou x plus dou tau xy upon dou y plus dou tau z z upon dou z plus rho gx.

Now, if you do that if this is infinitesimally small DV can be written as Dx Dy Dz if you integrate it it will be delta x delta y delta z that can be cancelled out and you can write it as rho DV upon Dt is equal to minus of dou sigma xx upon dou x plus dou tau xy upon dou y plus dou tau zz upon dou z plus rho gx. Now the normal stress term we know that the normal stress is the combination of the pressure plus the stress in the normal direction. So, I can write it as minus sigma t plus tau xx ok, upon dou x plus tau xy upon dou y plus dou tau zz tau xz sorry.

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$$= -\frac{\partial \sigma_{xx}}{\partial x} - \frac{\partial \tau_{xy}}{\partial y} - \frac{\partial \tau_{xz}}{\partial z} + \rho g$$

$$= -\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) + \rho g_x$$

$$\frac{\int_{CV} \rho \frac{DV}{Dt}\, d\forall}{\Delta x \Delta y \Delta z} = -\left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right] + \rho g_x$$

$$\rho \frac{DV}{Dt} = -\left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right] + \rho g_x$$

$$= -\left[\frac{\partial (P + \tau_{xx})}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right] + \rho g_x$$

You know this is xz everywhere sorry for this tau xz upon dou z plus rho gx.

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$$\rho \frac{DV}{Dt} = -\frac{\partial p}{\partial x} - \left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right] + \rho g_x$$

$$\boxed{\rho \frac{DV}{Dt} = -\frac{\partial p}{\partial x} - \left[\nabla \cdot \tau\right]_x + \rho g_x}$$

Momentum eq.

$$\boxed{\rho \frac{DV}{Dt} = -\nabla p - \left[\nabla \cdot \tau\right] + \rho g}$$

$$\boxed{\rho \frac{DV}{Dt} = -\nabla p + \mu \nabla^2 V + \rho g}$$

Navier Stokes Equation

So, you can write it as rho DV upon Dt will be equal to minus dou p upon dou x minus dou tau x x upon dou x plus dou tau x y upon dou y plus dou tau xz upon dou z plus rho gx and this term can written as minus dou p upon dou x minus del dot tau plus rho gx and this is the x component of the tau. So, this is rho Dz upon Dt.

Now, the whole Navier stoke equation can be written as dou V upon Dt minus dou p minus del dot tau plus rho g and this is nothing, but it is the momentum equation. Now if we write it in terms of mu also constant then this whole equation can be written as rho DV upon Dt will be minus dou p plus mu del two z because tau is equal to nothing, but minus mu dou V. So, delta 2 V plus rho g and this is called Navier stokes equation

So, this has been derived from the RRT by doing the property balance and property is nothing, but the momentum. So, that is the general equation which is called momentum equation or general momentum equation and from momentum equation if you assume rho is constant mu is constant you can come to the Navier stoke equation which will be in this form.

So, this is the 3 dimensional Navier stokes equation has been derived from the basic Reynold transport theory where you assuming a control volume in which some property and the properties momentum which is coming in which is being there and then is going out from a control volume. So, how much property is coming into the control volume how much property is going outside of the control volume the net change is this or written it in this form.

Now, we are going to use this equation in terms of now this is DV by Dt again if you open this DV by Dt that will go in terms of the Eulerian as well as the Lagrangian measurement variables. Now what we are going to do we are going to use this equation we are going to see that how the equation will be modified in case of multi phase flow.

But before doing that what I am going to do I am going to simplify the equation in the one dimensional form this is the 3 dimensional equation if you want you can get the solution for this, but you again need a numerical approach or computational fluid dynamics method to solve this equation in the 3 dimensional domain now what I am going to do I am going to do simplify the equation in the one dimensional form.

Now, if I have to simplify the equation in the one dimensional form what I am going to do.

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$$\rho \frac{DV}{Dt} = -\frac{\partial p}{\partial x} - \frac{\partial \tau}{\partial x} + \rho g$$

steady state flow

$$A \rho V_x \frac{dV_x}{dx} = -\frac{\partial p}{\partial x}A - \frac{\partial \tau}{\partial x}A + \rho g_x A$$

$$\dot{m} \frac{dV_x}{dx} = -A\frac{\partial p}{\partial x} - A\frac{\partial \tau}{\partial x} + \rho A g_x$$

G

$$\dot{m} \frac{dV_x}{dx} = -A\frac{\partial p}{\partial x} - P\,\partial\tau + \rho A g_x$$

$$\frac{\partial p}{\partial x} = -\frac{\dot{m}}{A}\frac{dV_x}{dx} - \frac{P}{A}d\tau + \rho g_x$$

$$\frac{dp}{dx} = \frac{\dot{m}}{A}\frac{dV_x}{dx} + \frac{P}{A}d\tau - \rho g_x$$

This rho DV by Dt will be equal to minus dou p by dou x minus I will write force in terms of the tau. So, this will be dou say tau upon dou x I am writing it in the one dimensional plus rho g let us assume that this is the my equation now what I am going to do I am writing it in terms of the one dimensional again I am going to write it in terms of one dimensional. So, say in the direction of x I am writing. So, dou zx into DV x upon Dt or Dx assuming that the flow is steady state vx upon vt term I am neglecting the steady state term.

So, the steady state flow. So, the flow is steady state it means dou vx upon dou t we are neglecting this is the one dimension. So, rho dou vx upon to vx upon dou x will be equal to minus dou p upon dou x minus dou tau upon dou x plus rho gx I am writing it in x direction ok.

Now, what I am going to do I am going to multiply with the area which is the cross sectional area everywhere I can do that into it now if I do that area multiplication this is what this is nothing, but the mass flow rate and I am going to denote it with m naught ok. So, this is how much of the solid is flowing in the x almost the fluid is flowing in the x direction m naught d V x upon Dx area into dou p upon dou x minus now area into gx.

Now, A into Dx if I try to reduce it it will be in terms of the perimeter if I am assuming that this x is very small element and I am just doing the changes in a very small element, I can just cancel it out I can write it minus a dou p upon dou x minus area upon Dx it will be perimeter which is p into dou tau A gx ok. So, into m naught into DV x upon Dx many books write it in terms of the g in a standard or multi phase flow books this some book also write it in terms of the g. So, do not get confused, if you see the g symbol it is just nothing, but rho V into x.

Now, we can simplify it what we want again I am going to write it in terms of the dou p by dou x because that is what we want to calculate. So, I will write it as minus m naught into dvx upon Dx divided by area this will be minus p upon a into d tau plus rho into gx. So, that will be what is the total pressure drop you are going to see and that total pressure drop is going to be the function of all this ok.

Now, what if you will see here what we are going to do we are going to say this is as dou p by Dx and because the pressure drop is going to take place it is going to be negative and instead of writing final minus initial I am going to write initial final it means everywhere I will take the minus sign out.

So, this will be DV x upon Dx plus p upon a d tau minus rho g x because this p is now initial minus finest a final minus initial initial minus final. So, it will be in this form, we can write it now if we write it; it in this form what we can say that the total delta p across the pipeline in the single phase flow is going to be the function of 3 things one is the frictional parameter second one is the acceleration field because of the acceleration because of the frictional and because of the gravitational.

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$$\frac{dP}{dx} = \underbrace{\frac{\dot{m}}{A}\frac{dV_x}{dx}}_{\text{due to Acceleration}} + \underbrace{\frac{P_d \tau_w}{A}}_{\text{due to friction}} - \underbrace{\rho g_x}_{\text{gravity forces}}$$

$$\left(\frac{dP}{dx}\right)_F = \frac{P}{A}\tau_w = \frac{4\pi D}{\pi D^2}\tau_w = \frac{4\tau_w}{D}$$

$$\left(\frac{dP}{dx}\right)_A = \frac{\dot{m}}{A}\frac{dV_z}{dx}$$

$$\left(\frac{dP}{dx}\right)_g = \rho g_x$$

$$\tau_w = \underbrace{f}_{\text{fanning friction factor}} \rho \frac{u^2}{2}$$

So, dp by dx can be written as m naught upon a into dvx upon dx plus p upon a tau w minus rho gx.

So, this is acceleration this is due to friction or viscous forces and this is due to gravitational forces or gravity forces. So, the dp by dx in any pipeline is function of these 3 things whatever we have discussed for the limiting conveying in the previous classes that this is the function of acceleration fluid acceleration it is the function of the friction of the fluid it is a function of the gravity.

Now, if you have a multi phase flow what you are going to have this is the phase one you will have the delta p because in the overall delta p in a pipeline because of the acceleration of one phase acceleration of other phase or second phase friction because of the phase one friction because of phase two gravity force because of phase one gravity force because of phase two in single phase flow this is going to be the function this.

So, I can say dp upon dx of frictional component is written as p upon a tau w now d tau w you can write or tau w you can write if the change is very small and p upon a in a cylindrical pipe if we are thinking about then the perimeter is nothing, but pi by d area is pi by 4 d square to tau w or you can write it equal to 4 tau w upon d in single direction one dimensional.

Then dp upon dx due to acceleration is nothing, but m naught upon a into dvx upon dx that is the Lagrangian acceleration term how it is changing with the velocity is changing with the position x then dp upon dx which is the gravitational forces is equal to rho g x

So, this is the combination of all these 3 now even in the single phase flow if you want to find the pressure drop the problem is it is forced to measure the Lagrangian velocity you should know how the V is changing with the x second and most important is how the tau is being defined and the tau is being defined tau w is defined in terms of the fanning friction factor it is F rho u square by 2, where F is fanning friction factor and we know that finding the fanning friction factor itself is a difficult job and you have to go to the moody chart or some correlation which is being developed empirically developed correlation to find the F value for laminar flow it is relatively simpler.

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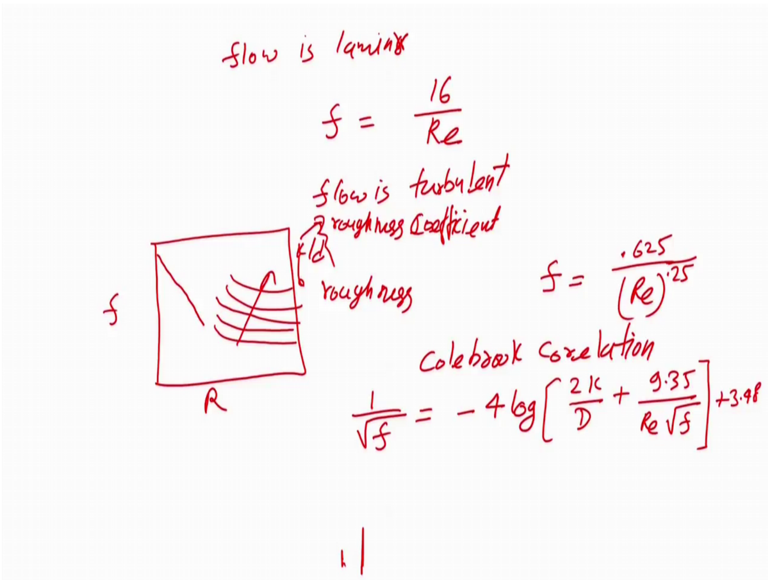

So, if the flow is laminar F is nothing, but 16 by re and if flow is turbulent then what you need to do there are several empirical based equation you have to either use that or you have to use the moody chart and where the F is being plotted whether in Reynold number, this is for the laminar flow and this is for the different turbulent flow and this is for different k upon d where k is nothing, but the roughness factor coefficient or roughness value this k upon d is called roughness coefficient and k is roughness ok.

So, the simplest equation for the turbulent flow via the pipe is smooth is equal to it is point 6.25 upon re raised to the power 0.25. This is the way F has been defined, if the roughness is placed the roll. There is different correlation is there which can be used to find that what will be the value of destruction factor and the one form of the Cole book. The correlation for the fanning friction term is one upon under root F is equal to minus 4 log of twice ki upon d plus 9.35 upon re under root F plus 3.18. This is the one form of the correlation. There are different cause both correlations have a form of this correlation available which is valid for the different in Reynold number there are other correlations also available to find the F values.

So, what I am trying to say that even for the single phase flow calculating the delta p is a challenge. And that is mainly because of the acceleration term and because of the frictional term and the frictional term the problem comes with the friction factor value that how to calculate the suitable friction factor value for the one dimensional flow in the single phase flow and different correlations are available, if the flow is laminar finding the values is relatively simpler because you have a straightforward correlation F equal to upon re.

If the flow is turbulent; then, if there is a roughness parameter, then, the becomes the solution becomes iterative. And if you see that if you have to solve this equation; then, you have to do the iteration to find the value of F for the different Reynold number or for the same little number. How the value of F will change you have to do the iterative solution to get the F value or you have to move to the moody chart, you have to see that how the winner in all number F is changing and what is your roughness parameter and depending upon the roughness value, you can find the k upon d for a particular k upon d for a particular in all number you can find the F value.

So, it is still a challenge in doing in the single phase flow the challenges will get double will increase at least. So, I will not double I will say once the multi phase flow comes into the picture why because now the problem is even to find that what is the area. So, first we have to find what is the area, what is the velocity inside of the fluid because now it is not filled with that single phase it is filled with both the phases available a particular fraction in the pipeline is being occupied by that particular phase you have to find that fraction, you have to calculate the velocity inside you have to calculate the Reynold number inside and based on that Reynold number you have to calculate the F value and then you can calculate the delta p in the pipeline.

So, what we are going to do from the next class is we are going to calculate the use the regimes. First we will focus on the gas liquid flow we will take the different regime. Homogeneous regime, Separated regime, Annular regime. All those regimes we will take we will see how this equations need to be modified in those regimes to calculate the delta p.

We will also try to see some of the empirically developed correlation like Lockhart Martinelli correlation which is widely used in all the industries to find the delta p in a pipeline. So, you will understand those systems and we will see that in each regime; how the equation will get modified what are the challenges we have to see in the each regime and then we will try to do some assignment problem on each regime to understand to have more clarity on it.