Multiphase Microfluidics
Dr. Raghvendra Gupta
Department of Chemical Engineering
Indian Institute of Technology, Guwahati

Lecture – 06
Taylor Flow 2

So, in this lecture we will continue the discussion about the Taylor flow. So, in the previous lecture we were talking about the Brotherton's problem.

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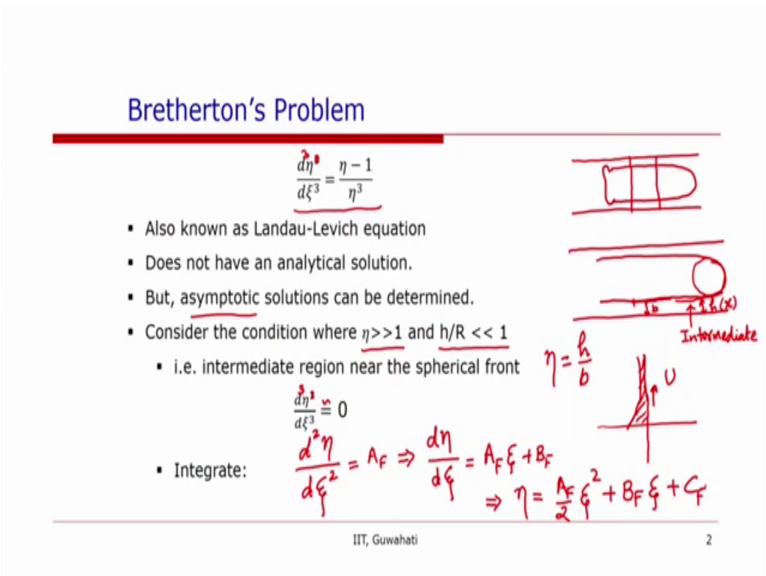

And we developed a relationship or the partial differential equation for the profile, for the bubble profile or for the bubble interface in the non-dimensional coordinate eta and xi. So, this equation we developed for the front of the bubble, but if you again go through the lecture, you will find out that there is no such assumption that stops us to use this equation at the back of the bubble. So, we can analyze the front of the bubble and back of the bubble independently using this equation. This equation is also known as Landau-Levich equation, and has been used to understand number of problems.

So, 2 problems that we understand already is one is the flow of a long Taylor bubble, when I say long; that means, that the bubbles should be long enough that it has a constant thickness film region. So, if we look at the bubble ok so, this constant thickness film region should be present for the bubble, to pass as at the long bubble. The other application of this equation can be when there is no tail of the bubble. For example, a air is passing through a capillary, which is initially filled with liquid for example, for the coating applications.

So, there also the same analysis can be used to find out the thickness of the coating. The other application which comes from Landau-Levich equation that if a plate dipped in a liquid is brought out, then the thickness of the film. So, this plate is being pulled out with a velocity U and the thickness of the liquid film that will be left behind can be analyzed by the same analysis. As you can see that this is a non-linear equation. So, it is not possible to have it is analytical solution. Nonetheless its asymptotic solutions are available and we can find or we can try to understand the nature of the solution of this equation at different regimes.

So, to start with we will consider a region; where eta is greater than 1 and h over R is less less than 1. So, to remind you we considered when we were analyzing this problem let us look at again. What we did is we consider the front of the bubble. And we had that at the the front of the bubble is spherical. And this is constant thickness film region; In the middle this is the intermediate film region; where h is a function of x. So, and the film thickness here is b; you might remember that eta is equal to h over b. So, the film thickness non dimensionalize by the film thickness in the constant film constant thickness film region.

So, we are considering the region where eta is greater than 1, but it is a still sufficiently less or sufficiently small than the channel radius. So somewhere in this region; where we have the film to be of sufficiently thickness or sufficiently thick and it is very small when you compared in the tube areas. So, in this region, if you look at the term eta minus 1 eta cube as eta will be large. So, if eta is large, then eta cube will be for the larger and then you can approximate this to be equal to 0. So, that is why one can approximate this equation as d eta cube over d xi cube is equal to 0.

So, if you integrate this equation. So, let us integrate this equation, and we will get sorry this is d cube eta over d xi cube, you might have this mistake carry it over everywhere. So, please note that, and this will be; so, we will have d 2 eta over d xi 2 is equal to a constant. Let us call that constant AF. Further if we integrate, it again we will get d eta over d xi is equal to AF xi plus BF, BF is another integration constant. And then we get eta is equal to AF by 2 xi square plus BF xi plus CF. So, that is the film profile and now we have 3 constants AF, BF and CF. And remember this is in this particular region.

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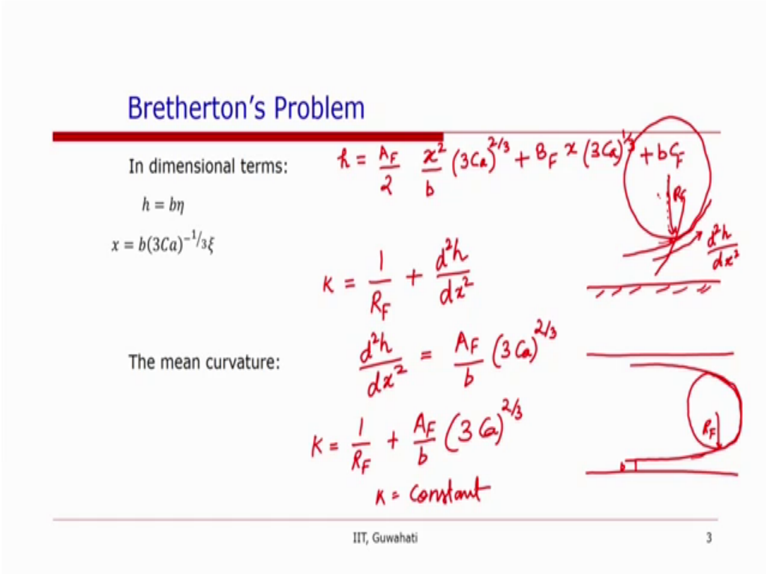

So, if we write this in the dimensional terms; so, we can write the previous equation that we have just integrated h is equal to AF by 2. So, AF by 2 and b, this multiplied by xi square over 2. So, xi square is x square over b square. So, this we can write is x square over b, and this b will go away. AF by 2 x square by 2, and xi is x over b 3 Ca power 1 by 3. So, we will have 3 Ca power 2 by 3. Plus, BF into xi into b. So, xi will be x over b. So, b will cancel out. We will have x into 3 Ca power 1 by 3 plus, b into Cr. So, that is the equation in the dimensional form and this equation is for the this sphere.

Now, the mean curvature, if we look at this region, the region that we are talking about here. This region will have 2 curvature, one along this direction; which will be given by d 2 h by dx 2. The another principal, curvature will be normal to it. So, if you are looking at the cross section. Then, this will be given by the radius of the bubble so, the bubble can the front this is at the front of the bubble. So, we can treat this at the front radius. So, this radius will be above equal to front of the bubble.

So, the mean curvature in this region will be equal to 1 over RF. If we have this is the surface, then this curvature is d plus by dh 2. And this curvature which where the sphere, of this is circular this is the radius of this circle which is RF. So, kappa is equal to or the curvature is equal to 1 over RF plus d 2 h by dx 2. And from this equation we have d 2 h by dx 2 is equal to AF over b into 3 c raised to the power 2 by 3.

So, from this we could find what is the we have integrated, we found the profile in this region, and then we saw that this the mean curvature in this region is the sum of these 2 curvatures.

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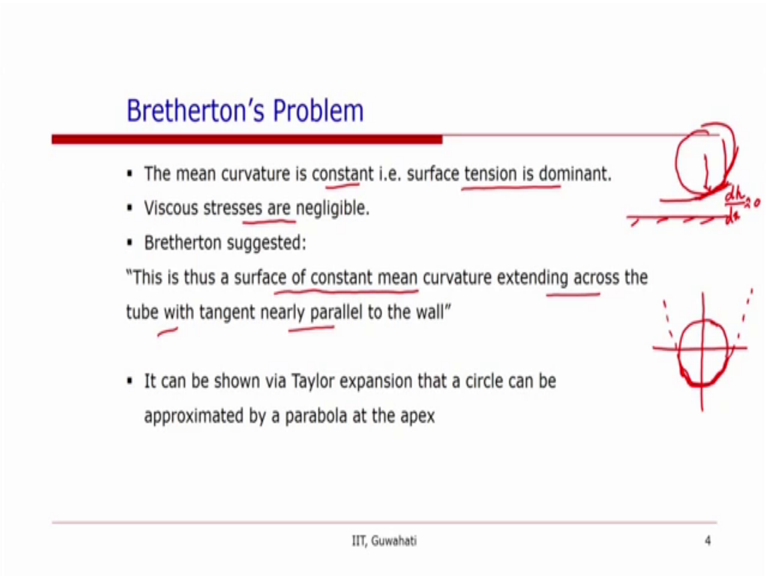

Now if we look at this region the mean curvature in this region is constant. So, in this region, if you look at the bubble; where you can fit a is fear near the bubble. So, in this region, where we are analyzing. The curvature is almost a spherical. And if you look at the mean curvature this is 1 over RF plus AF by b 3 Ca to the power 2 by 3. Where RF is the radius of this front of the bubble. And AF is a constant. B is the film thickness here; which is a constant, and capillary number is also a constant.

So, that says that kappa or the curvature is a constant; that means, the surface tension is still dominant force. Because, if the surface tension would have been changing that that change is brought about by the viscous stresses. So, in this region what we see that the curvature is constant; that means, this is dominant or surface tension in dominant in the region, in this region and the viscous stresses are still negligible. So, for this Brotherton suggested that this is a surface of constant mean curvature, which extends across the tube with tangent nearly parallel to the wall.

So, when it says that you can fit is sphere here, but this tangent is the slope of this line is almost negligible. So, you can say that dh by dx is almost 0 in this region. It can also be shown that one can fit a actually one can fit a circle, any circle can be fitted with a parabola. So, this will be fitted at the apex on the parabola. So, in this region the circle can be fitted with a parabola.

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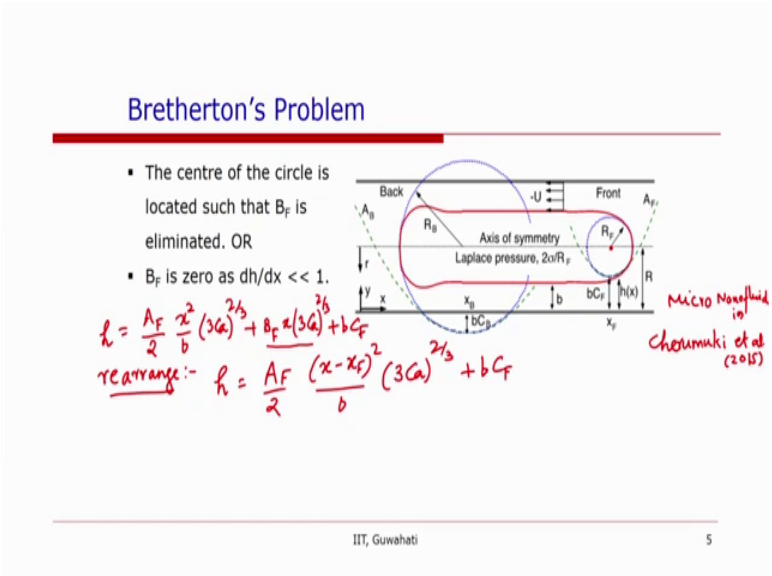

So, the equation that we had for h, we had h is equal to AF by 2 x square over b 3 Ca raised to the power 2 by 3, plus BF x 3 Ca raised to the power 2 by 3, plus bCF. We can rearrange this in such a manner, that we have h is equal to AF by 2 x minus xF squared or b 3 Ca raised to the power 2 by 3 plus bCF.

So, what I have tried to do here, that we change or we select xF or the; is such that, the xF has been chosen in such a manner that BF is eliminated. So, or that is what I have suggested, here the center of the circle is located such that BF is eliminated, or in this region if you look at a circle can be approximated by a parabola, and then you can look at a publication my cherumuki et al and micro nanofluidic. 2015, where they have described this how this parabola can be fitted. But nonetheless, if we rearrange this equation in such a manner; then we have eliminated the constant BF, taking into account the fact that as dh by dx this will touch in this the circle is about to touch of the slope is negligible here.

So, the equation can be rearranged in this manner.

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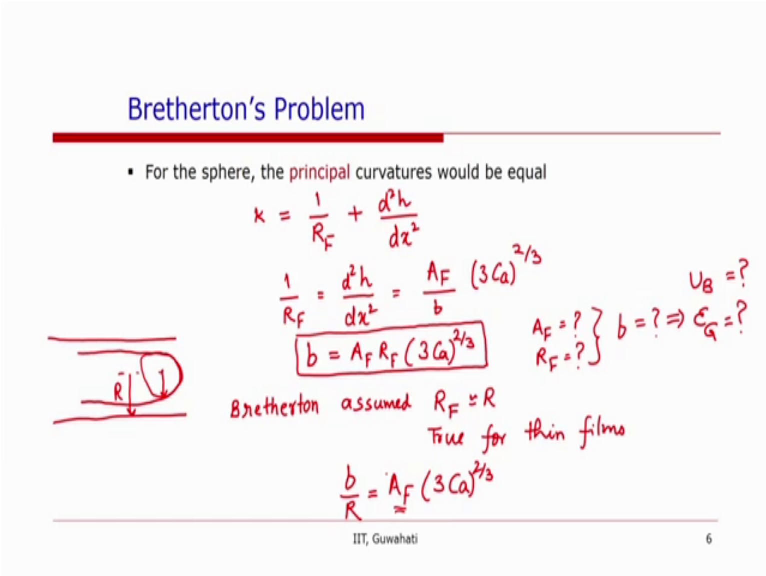

Now, in this place or as we said that there are 2 principal curvature, we had written the curvature 1 over RF the first principal curvature. And the other principal curvature was d 2 h over dx 2. Now if this is a sphere, then for a sphere, the 2-curvature; curvature in this direction, and curvature in this direction the 2-principal curvature will be equal. So, for a sphere the 2-principal curvature are going to be equal. So, 1 over R as will be equal to d 2 h over dx 2, which is equal to AF over b 3 Ca raised to the power 2 by 3. That gives us b is equal to AF RF 3 Ca raised to the power 2 by 3.

So, if we know the constants AF, and the radius of the front RF. Then we can approximate this if we know AF and RF, then we can find out what is the film thickness. And the film thickness from the film thickness we can further find out the UTP, not UTP by the knowledge of UTP, we can find out UB bubble velocity and the void fraction. So, Brotherton's in which analysis he assumed that RF; that is, the radius of the sphere, that is fitted here this is approximately equal to the channel radius R which is true for thin films. So, that will reduce that b by R is equal to AF into 3 capillary number to the power 2 by 3.

So, if one can find out AF, then one know what is the film thickness this make things simpler.

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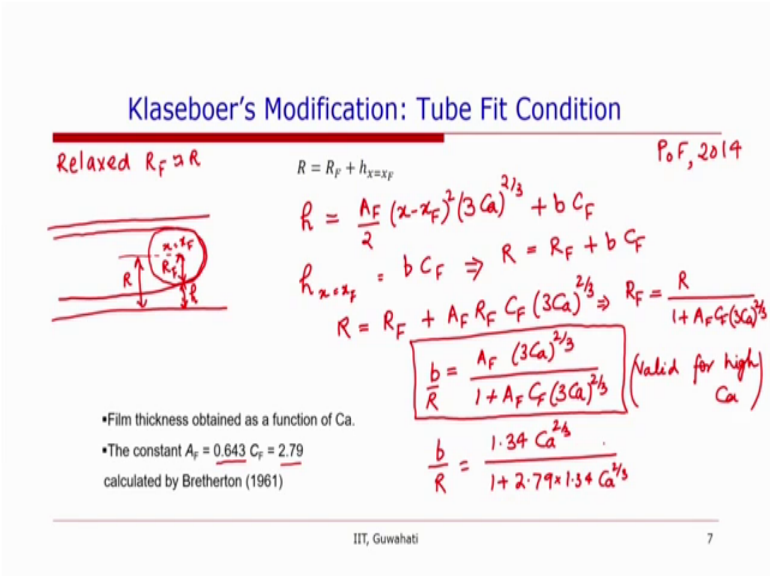

So, recently a Klaseboers et al in their physics of fluid paper in 2014. They have relax this assumption of that one need to assume RF is equal to R. What they rather said that, in the channel, near the bubble, where the this fear is being fitted, the channel radius R, at this point where x is equal to xF is equal to RF, plus the film thickness at this location x is equal to xF. So, recalling the h, we have h is equal to AF by 2. x minus xF square 3 Ca raised to the power 2 by 3 plus b CF.

So, at x is equal to xF, we will have h is equal to bCF. So, that means, R is equal to RF plus b CF, and from the previous relationship, we have b is equal to AF RF 3 Ca to the power 2 by 3. So, we can substitute that here; we will have R is equal to RF plus AF, RF, CF into 3. Ca raise to the power 2 by 3. Or that gives us RF is equal to R over 1 plus AF CF into 3 Ca raised to the power 2 by 3. So, we can write b for the channel the same thickness now becomes b is equal to AF into RF. So, RF we substitute by R. Or we can write as b by R is equal to AF into 3 Ca raised to the power 2 by 3 into 1 plus AF, CF 3 Ca raised to the power 2 by 3.

So, this film thickness has an additional term in the numerator. You might check that the numerator term is same. AF 3 Ca to the power 2 by 3 as obtained by the Brotherton, but there is an additional term; which is 1 plus AF CF 3 Ca to the power 2 by 3 and which will be valid solve high capillary number also. So, that is the film thickness correlation that can be obtained from there.

Now from their analysis the constants AF and CF were obtained by Brotherton by the numerical integration and AF was 0.643. And CF was 2.79; so, one can get b by R is equal to after substituting the values one will get 1.34 Ca raised to the power 2 by 3, over 1 plus 2.79 into 1.34 capillary number raised to the power 2 by 3. So, one can see from here, that the film thickness is only a function of capillary number.

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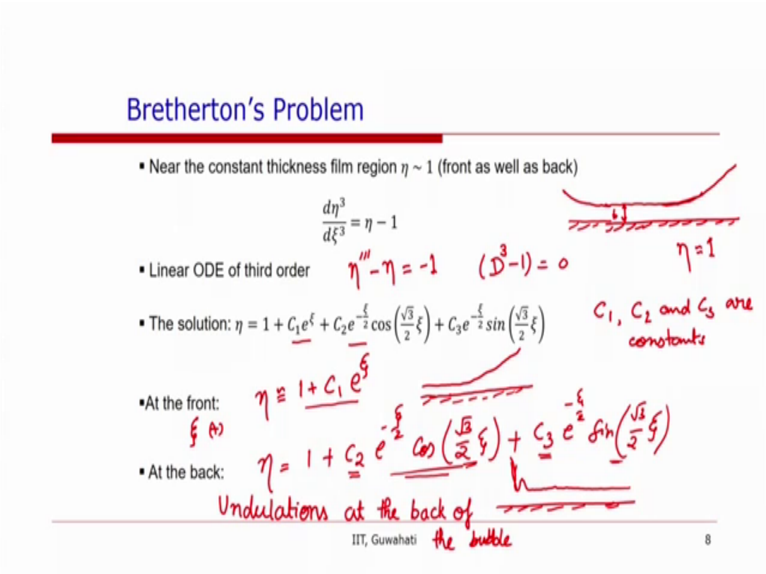

Now, right now we have looked at the solution of the equation d cube xi over d cube eta over d xi cube is equal to eta minus 1 over eta cube of for this equation. We have obtained the solution in the region where eta is sufficiently large than 1. Now if we consider another region, which is near the film.

So, the region which is near the constant thickness film from where the film starts going, it might be a different or it might be considered even at the back off the bubble. So, they, but they can be considered independently. So, in this region eta is close to 1; so, eta cube can be approximated as 1 so that the equation can be linearized and one get the linearized equation in this form; which can also be written as eta tripled as minus eta is equal to minus 1. And one can solve the characteristic equation of a this for homogeneous equation that we get d cube minus 1 and one we get the characteristic roots.

And from that one can obtain the solution in this form eta is equal to 1 plus C 1 is exponential xi plus C 2, exponential minus xi by 2 and so on. So, C 1, C 2 and C 3 are constants here and when xi is increasing we are at the front. So, when xi is positive, then one can see that eta will be approximately equal to 1 plus C 1; e to the power xi, and this solution was obtained by Brotherton at the back the solution that will be valid is eta is equal to 1 plus C 2; e to the power minus xi by 2 cos root 3 by 2 xi plus C 3 e to the power minus xi by 2 sin root 3 by 2 xi.

So, by the nature of the equation one can see from this equation that the interface will be at the front will be increasing exponentially away from the constant fluid. Whereas, at the back there are some undulations in the film the show; the film becomes on the constant. So, these 2 terms sine and cosine term will so, the oscillatory behavior or the undulations at the back of the bubble. And this has been observed at the experiment in the experiments. So, oscillatory or not say the undulations at the back of the bubble. They can be explained by these 2 terms.

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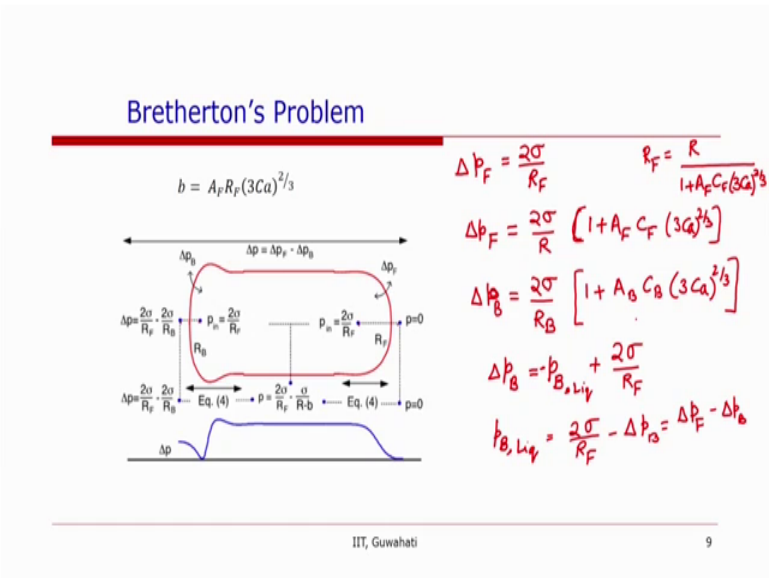

Now, so, we have obtained the asymptotic solution. But in this solution note that, what we have not obtained or what we have not or we do not know what are these constants C 2 and C 3. So, this is only a qualitative representative how representation; however, one can find out the wave length of this undulations in from there in the experiments. Further if we look at the start of the Brotherton problem we discussed the pressure jump at different regions near the bubble.

So, we will see that can this pressure jump defined found out. So, if we look at the bubble strength the pressure jump at the front was 2 sigma over RF. And now what was not known is RF. So, if you remember, we have obtained RF in terms of R mCa on the constant. So, R is equal to RF is equal to R over 1 plus AF, CF 3 Ca raised to the power 2 by 3. So, one can write the delta pF is equal to 2 sigma by R into 1 plus AF CF into 3 Ca raised to the power 2 by 3.

Similarly, one can write the pressure difference at the back, that will be equal to 2 sigma over RB, though we have not done the analysis, or we have not done the analysis for the back separately. But from the similar arguments or from the similar analysis one will get 2 constants for the back which will not be necessarily same as at the front. So, we will have those as subscripted as AB and CB. So, 1 plus AB CB 3 Ca raised to the power 2 by 3. That will be the pressure difference inside the bubble and outside the bubble near the back in liquid. So, these are the pressure jumps that one can find out. And from this, one can also calculate the delta pB is equal to p at the back of the bubble in liquid minus p in the bubble which is 2 sigma over RF. So, one can find out what is pB in the liquid, that will be 2 sigma sorry this will be because the pressure in the bubble will be higher. So, this will be positive and this will be negative.

So, we will have pB in the liquid is equal to 2 sigma over RF minus delta pB. So, that is basically delta pF. So, that will be equal to delta pF minus delta pB, and that will be the overall pressure drop from the back of the bubble to the front of the bubble. And one can find out knowing AF, A, AB, AF, CF and AB CB; AF CF we already know AB CB can also be found out and looked at by the paper by Cherumuki.

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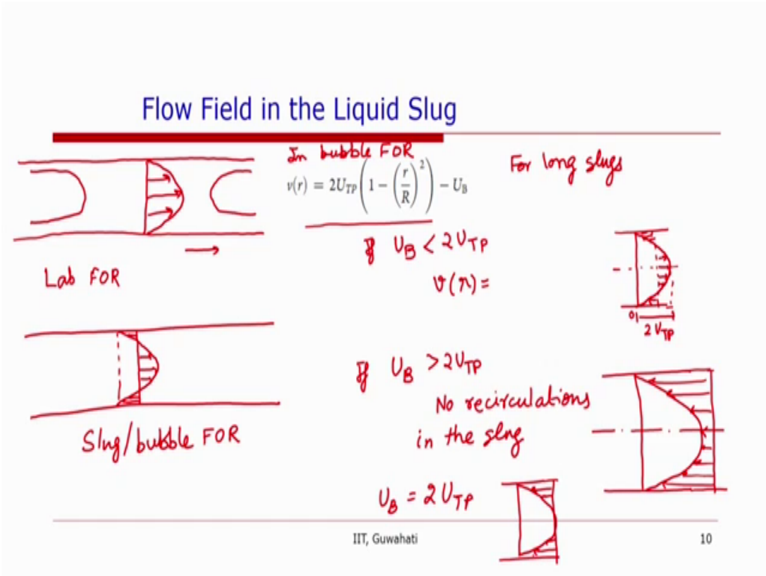

So, till now we have looked at the Brotherton problem and in the Brotherton's problems, we have analyzed the shape of the bubble and from that we were able to find out the important and useful relation for the film thickness. And once we have the film thickness, we can also calculate the bubble velocity, we can calculate the void fraction. We can also calculate what is the total pressure drop across the front of the bubble, and if we know the constants for the back of the bubble, then we can also calculate the total pressure drop across the bubble.

Now, we will have a look at the flow field in the liquid slug. So, if you have this picture of the liquid slug, where there are 2 adjacent bubbles at the front and the back and in a laboratory frame of reference, if the slug is long enough, for long slugs. If the slug is long enough then one will get the profile to be parabolic, or at least near parabolic in the channel. When the profile is parabolic in the channel, then we can approximate this profile as 2 p 1 minus r by capital R squared in laboratory frame of reference.

So, when we write this in velocity profile in bubbles frame of reference. So, remember we said that, because the bubbles move the velocity UB, the slug that is trapped between the 2 bubbles will also need to move with the bubble velocity. So, in the bubble frame of reference the slug will also be stationary and we will see the internal recirculations in the liquid. So, the velocity profile in the slug; in the laboratory frame of reference will look like this, but if we plot the velocity profile in slug or bubbles frame of reference, then we will see that the profile will look like this.

So, that means, there is recirculation in the slug, and the profile that we have drawn is in the middle of the slug. You may also note that the velocity profile that we are talking about will be observed in the middle of the slug not near the 2 ends at the front or back of the slug; where they have a bubble front and bubble slugs close by. So now, if we do that then we can have 3 conditions for UB and based on the relationship between UB and UTP.

So, let us say if UB is less than 2 UTP, then we will have v r is equal to, it will be the profile look like something like this. This is the profile for a, this is the parabolic profile. And this is 2 UTP; this value is 2 UTP; so U and this is where we have 0. So, UB will be somewhere here; so we will have a recirculating flow in the slug.

If UB is greater than 2 UTP; that means, the velocity profile. So, this will be something like this; in this case, because UB is more than UTP. So, the negative value will be everything will be moving in the negative direction and one will not see any recirculations. But it is unlikely to be; such condition to be present for air water flow for example.

So, in this case there is no; recirculations in the slug and if we have the third case, where there will be equality UB is equal to 2 UTP, then one will have the profile to be abjectly at the same point.

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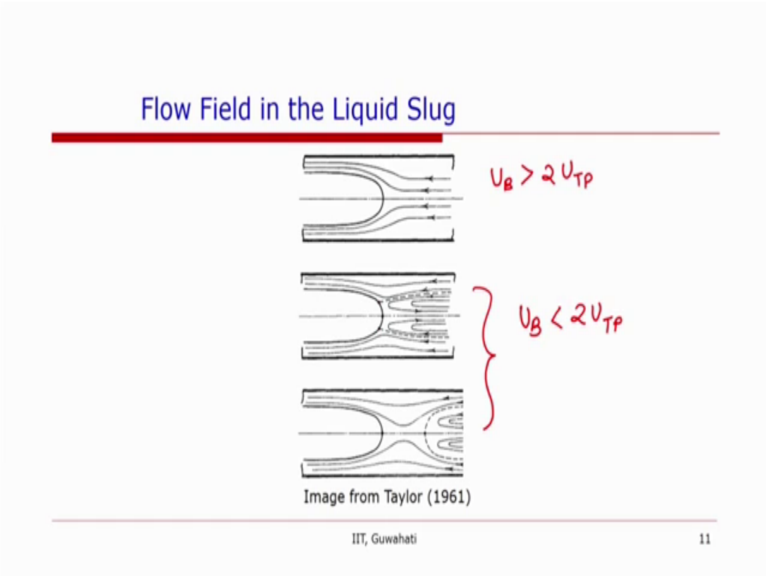

So Taylor in his paper on the film thickness in the capillaries; he also drawn 3 different profiles and this was for when UB is greater than twice the average velocity or mean velocity. And these 2 versus are UB is less than 2 UTP; so from our analysis what we have done in the previous slide both of these have re-circulating flow in the channel; however, this does not say anything about that, will there be 2 stagnation points on the bubble or will there be only one stagnation point; so the recirculation will not be just close to the bubble from it. So, these are the 3 hypothetical flow profile that were shown by Taylor or possible 2 h 2.

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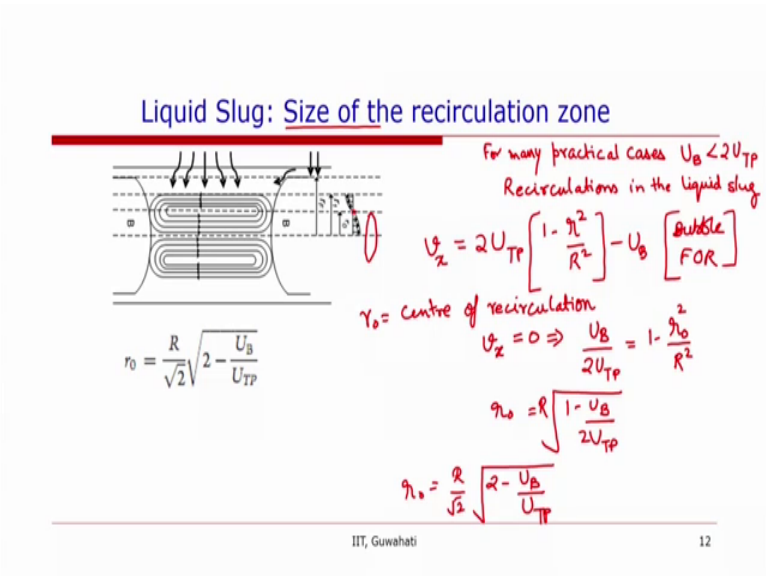

So, another thing that we will discuss here that in quite a few cases, we have UB is equal to or UB is less than 2 UTP. So, that means, we will have recirculations for many practical cases, we have UB is less than 2 UTP; so they have a recirculations in the liquid slug. Or one have recirculations in the liquid slug; then of course, one would like to know what is the size of this recirculation? So in this; we will talk about the size of the recirculation. So, we had written the velocity profile in the liquid slug, in bubble frame of reference 2 UTP; 1 minus r square by capital R square minus UB.

So, if the profile is recirculating also, there will be a point, at which you will have the velocity to be 0. So, the center of the recirculation; so the ring that will form, so the radius of the so, what we call r 0, let us say center of recirculation. So, at this point v x will be equal to 0 and just remember that this is in bubble frame of reference.

So, when v x is equal to 0; that means, we have UB over 2 UTP is equal to 1 minus r 0, square r 0 is the location where the center of recirculation is located divided by R square. So, we will have r 0 is equal to 1 minus UB over 2 UTP square root, multiplied by R. Or one can write in this form; r 0 is equal to R over root 2 into 2 minus UB over UTP.

So, that is the center of recirculation. The other parameter of interest will be the radius of this the entire circulating zones.

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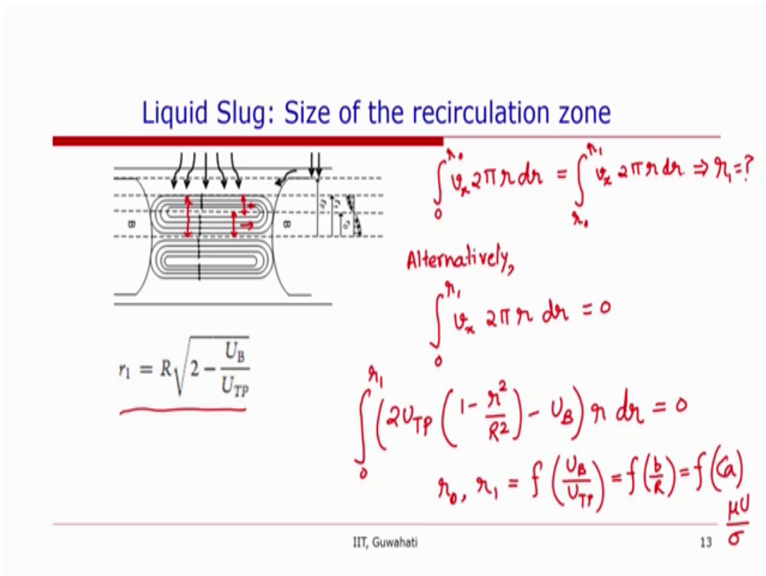

So, this r 1 how can we find r 1 here? So, there are two ways to look at it; if you look at the flow that is happening from here to here. See if you look at the flow; Q or you can write this Q in the integral form. So, if we write integral 0 to r 0 ;v x 2 pi r; d r. That will be the flow that is happening in this direction. And the same flow will move during the recirculation; so that should be equal to integral r 0 to r 1; v x 2 pi r; d r; so one can find out from this what is r 1.

Alternatively, one can see that the net flow in this region integral 0 to r 1; v x 2 pi r; d r. That is basically the sum of these 2, but the other one in will be in the different direction. So, the directions will change that will be equal to 0 and once we substitute the values 0 to r 1; 2 UTP into 1 minus r squared by R squared, minus UB into r d r is equal to 0 2 pi can be eliminated, because it will can be divided by divided to 0 that will have no effect on the result.

So, after the substitution, one will get the value of r 1; which is the radius of circulation. So, you can notice here that r 0 and r 1. The size of the recirculation they are function of UB by UTP only. And UB by UTP is a function of b over R; which is the film thickness, which is a function of capillary numbers. So, you can say that the size of the recirculation zone also depends on the capillary number, which is mu U over sigma. So, if the liquid is viscous, it means viscosity is high, then one might have a lower size of the re-circulations.

So, in somebody in this section what we have analyzed is we look at the Brotherton's problem.

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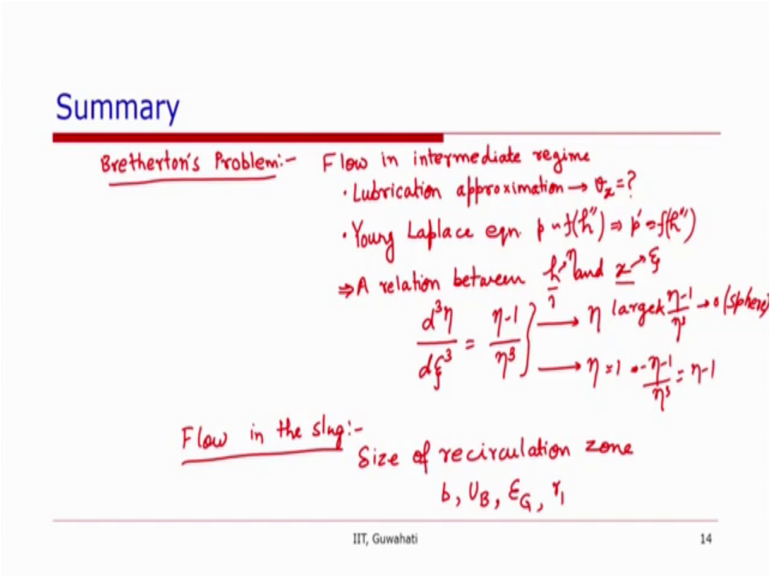

And in the Brotherton's problem, we looked at the flow in intermediate region. First, we divided it into 3 regions; where in the other 2 regions near the front and in the film thickness in the constant thickness film cylindrical region, we had some information about the flow; flow was film are is stagnant.

So, we looked at flow in the intermediate region, use the lubrication approximation and obtained what is the expression for v x. Once we have obtained this was in terms of dp by dx and we obtained I case on from Young Laplace equation. So, Young Laplace equation had pressure and h correlating or h double dash are correlated. Because the one of the curvature much can be approximated as double derivative of the film.

So, from that we can substitute this pressure, and then we can find out what is p dash. And that will be as a and then we got a relationship between. So, from then we got relation between h and x; which are basically the transverse and streamline coordinates of the film in the intermediate region. And when we non dimensionalized it was h was replaced by eta, and x was replaced by xi and we got it will associate d cube xi over d eta cube is equal to eta minus 1 over eta cube.

And then we looked at because this equation is non-linear then we looked at the or the solution of it for the 2 regions one is where eta large, and eta minus 1 over eta cube can be approximated to 0. And the other region near the film, and this was approximated by a sphere. And in other region we looked at where eta is about equal to 1. So, we had this as approximated as eta minus 1 over eta cube is equal to eta minus 1.

So, when it was sphere then we compared the 2-film thickness are 2 principal radius of curvature, because it can be fitted in a sphere and these 2 curvatures will be collision. We got what is the front radius of the front spherical portion and from that we could calculate the film thickness. The other thing that we looked at is the flow behavior or the flow in the slug; specially, the size of recirculation zone in the slug.

And from all this analysis we have got expressions for film thickness, for UB, for epsilon G or void fraction, for r 1 size of the recirculation zone and so on. So, we will continue our discussion about the Taylor flow and transport processes in it in the next lecture.

Thank you.