Multiphase Microfluidics
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Lecture - 02
Tylor Flow 1

Hello in today's lecture, we will be looking at Taylor or slug flow regime in particular. So, first of all let us look at the characteristics of Taylor flow.

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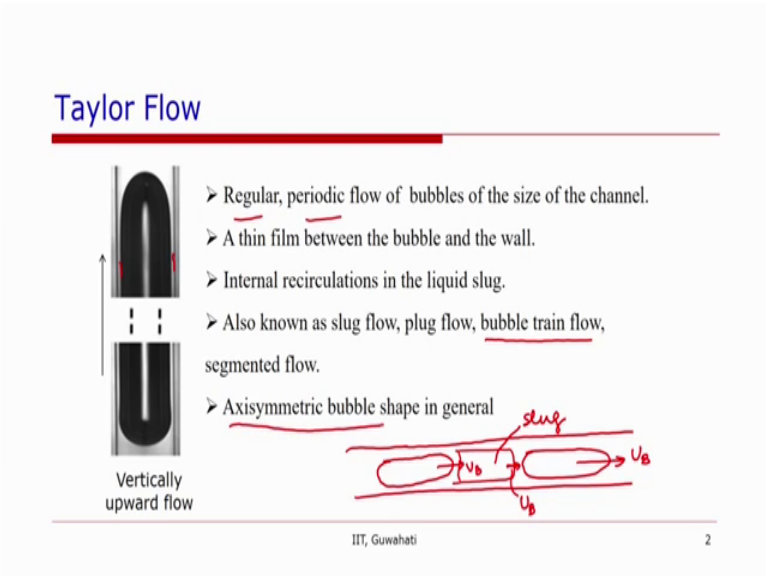

Taylor flow or slug flow regime is characterized by the regular and periodic flow as I have written here regular and periodic flow of bubbles which are of the size of the channel. The picture shown here is for a vertically apart flow, and the bubble is in the black color, and you can see that there is a film on both sides of the bubble which separates it from the wall. So, there is a thin film that separates it from the wall. And the flow of bubble is periodic. So, you have a number of bubbles. And that is why it is also called a train bubbles or bubble train flow. And in between 2 conjugative bubbles so, if we look at the estimate a schematic for a bubble train flow we will have something like this a picture something like this.

That one have bubbles moving with velocity, let us say UB is the velocity, and this bubble is moving with a velocity UB. And in between the bubble we have what we call liquid slug ok because these slugs are trapped between 2 bubbles. So, they will also move with a velocity which is equal to UB, because if they do not, then eventually what will happen that the 2 conjugative bubbles they with come close and eventually make violence or if the slug. or if or the distance between them keep increasing which does not happen and if that that happened and it flow is not periodic. So, the slug velocity is equal to the velocity of the bubbles.

In micro channels, where the gravity effect is not dominant; so, we the bubble shape is generally axisymmetric ok.

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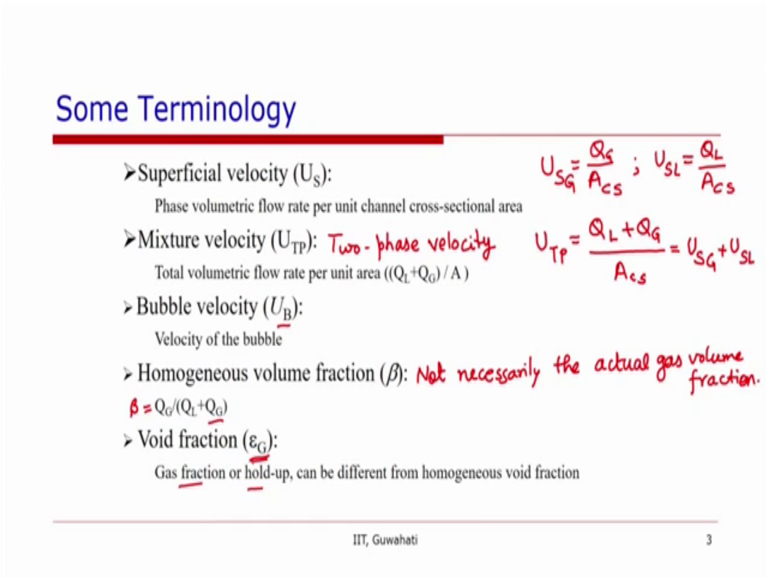

So, before we go on to do anything else, let us look at some of the terminologies, that might be required to understand Taylor flow. So, superficial velocity, it can be the superficial velocity of the gas phase, superficial velocity of the liquid phase. And it is basically US is equal to Q over area of cross section of the channel. now if we are looking at gas then it is USG or superficial velocity of gas is equal to Q of gas, divided by channel cross section area. Similarly, superficial velocity of liquid will be Q of liquid divided by the channel cross sectional area ok.

Mixture velocity is, it is also called 2 phase velocity. And the term TP comes from 2 phase. So, U TP is defined as Q L plus Q G. So, the total flow rate divided by the channel cross sectional area. So, from this we can see this is also equal to U TP is equal to U, superficial velocity of gas plus U superficial velocity of liquid .

Now, bubble velocity, the bubble will generally be moving faster, then the average velocity of liquid in the channel. So, the velocity of the bubble we generally turn it as or represented using UB. Then homogeneous volume fraction and white fraction, or gas traction or gas hold up; there are 2 terms here, which represent the volume fraction of gas. So, the first term is not necessarily not necessarily the actual gas volume fraction.

So, this term represents the ratio of the gas flow rate to the total flow rate, when it is assumed or when the gas and liquid are homogeneously mixed. So, that means, beta is defined here as gas flow rate over Q L and plus Q G. But that is not generally the case in Taylor flow because the bubble move faster. So, the gas velocity will be more than the liquid velocity. So, this will also change the volume fraction of or the average volume fraction of the gas in a channel. So, this is there is actual void fraction or the actual volume fraction is gas is called epsilon G, or void fraction that is the actual volume fraction that you see in the fully developed Tylor flow in a channel ok.

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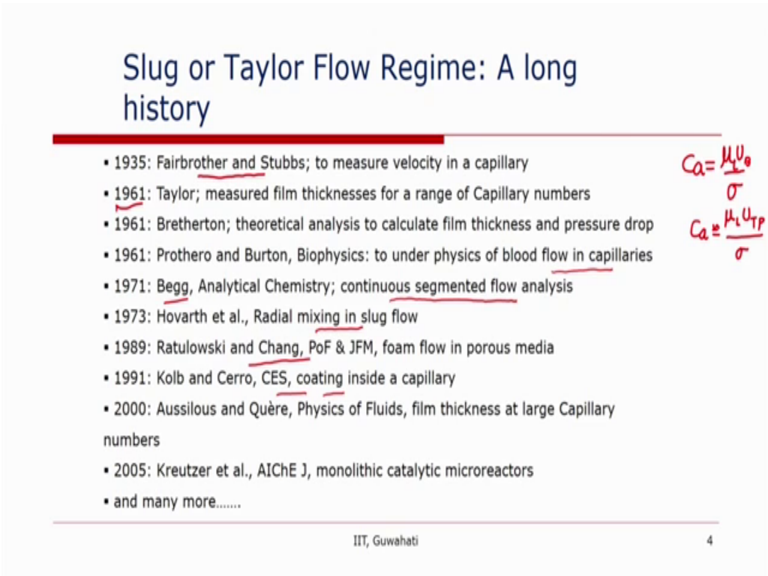

So, little bit of history. It is almost 80 years ago at least that is there I could find that Fairbrother and Stubbs they looked at understanding of the velocity of bubbles in the capillary. The idea was that they thought that a Taylor bubble can be used as an indicator, to measure the velocity of the liquid in the channel. And eventually they found that the bubble velocity is more than average liquid velocity in the channel.

Then in 1961, Taylor did the experiments. So, for a range of capillary numbers. So, to remind you the capillary number is equal to mu U over sigma. So, sigma is surface tension between gas and liquid phases. Mu is the liquid viscosity. And U should be UB bubble velocity,but because in many cases, we might not know the bubble velocity beforehand. And there is not much difference at least I means the difference between the bubble velocity and the 2-phase velocity case about 10 to 15 or 20 percent at next.

So, this can often be written or this is often written as mu L U TP over sigma ok. So, he measured Taylor measured the film thickness for a range of capillary numbers experimentally. And after him this bubble shape has been known as Taylor bubbles. then in 1961 the same year Brotherton in his classic paper, he did the theoretical analysis to analyze, the shape of the bubble calculate the thickness of the films surrounding it and from their find out the bubble velocity and the pressure drop across the bubble.

Then because this particular configuration that a bubble flowing a capillary, this also can mimic the flow of red blood cells in capillaries, the mind you that the red blood cells they are not exactly like bubble, but they have a membrane and they are flexible particle; however, as a first approximation one could understand us relatively simpler problem of bubble flow in micro channels. So, they looked to understand with the motivation to understand the blood flow in capillaries, they looked at Taylor flow. then in 1971 there are the begg he used it to form a continuous segmented flow analysis, or for a segmented flow in the liquid region, and in the in 1973 forward we looked at radial mixing.

In the slug flow, and then there have been a number of studies some of which I have written the Ratulowski and chang they looked at the shape of the bubble numerically and Kolb and Cerro they had a quite a few papers for application in porting inside a capillary. So, the liquid coating in the capillaries, and when you have the liquid filled with the material or the capillary filled with the material with which it is to be coated. And then the air is blown through the capillary and the front shape of the meniscus or the interface that forms that is like in a bubble and that gives you what is the film thickness.

So, they looked at in the problem from this perspective, and in 2,000 Aussilous and Quere. They looked at the film thickness at large capillary number, extending the Brotherton’s correlation who developed a correlation for the film thickness. And after that in the past 2 or 2 decades, there have been a number of studies on Taylor flow in micro channel to obtain film thickness to obtain heat transfer for mass transfer applications, or even simple validation of the numerical course, because when one can do the experiments one can digitize the flow very accurately and it can be used to further validation of a of a say volume of fluid method where the interface is captured.

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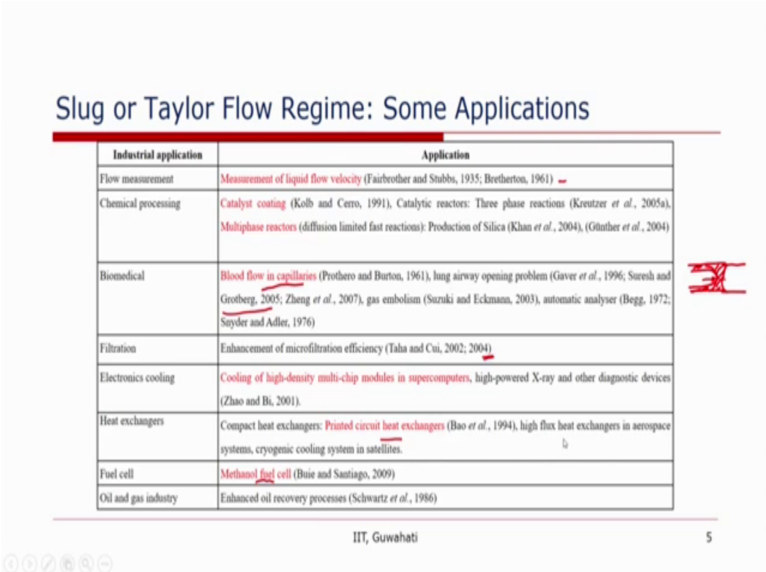

Slug or Taylor Flow Regime: Some Applications

Industrial application	Application
Flow measurement	Measurement of liquid flow velocity (Fairbrother and Stubbs, 1935; Bretherton, 1961)
Chemical processing	Catalyst coating (Kolb and Cerro, 1991), Catalytic reactors: Three phase reactions (Kreutzer *et al.*, 2005a), Multiphase reactors (diffusion limited fast reactions): Production of Silica (Khan *et al.*, 2004), (Günther *et al.*, 2004)
Biomedical	Blood flow in capillaries (Prothero and Burton, 1961), lung airway opening problem (Gaver *et al.*, 1996; Suresh and Grotberg, 2005; Zheng *et al.*, 2007), gas embolism (Suzuki and Eckmann, 2003), automatic analyser (Begg, 1972; Snyder and Adler, 1976)
Filtration	Enhancement of microfiltration efficiency (Taha and Cui, 2002; 2004)
Electronics cooling	Cooling of high-density multi-chip modules in supercomputers, high-powered X-ray and other diagnostic devices (Zhao and Bi, 2001).
Heat exchangers	Compact heat exchangers: Printed circuit heat exchangers (Bao *et al.*, 1994), high flux heat exchangers in aerospace systems, cryogenic cooling system in satellites.
Fuel cell	Methanol fuel cell (Buie and Santiago, 2009)
Oil and gas industry	Enhanced oil recovery processes (Schwartz *et al.*, 1986)

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So, some of the applications as I said it was earlier described to use for measurement of liquid velocity. it the concept which is still trying to be used or people are trying to develop multiphase flow meters, based on this concept. then in chemical processing for example, catalyst coating multiphase micro reactors. If you look at the gas liquid reactions in microfluidics or in micro channels or chemical micro processing, then about 80 to 90 percent of the work that you will observe is being done, it might be a gas liquid flow all liquid liquid flow or sometimes even gas liquid solid flow most of the work is being done in the Taylor flow regimes.

It is very important to understand the hydrodynamics integral of flow regime in biomedical applications. The blood flow in capillaries as I said this can mimic the same problem of R B c flow in capillary or in micro capillaries.

Then the lung airway opening problems. So, in the lungs there is mucous surrounding the walls of the alveoli, and then in between those there is sometimes the mucus forms a liquid plug. And this interface looks like the Taylor bubble. So, for example, professor grotberg, his group has looked at a number of effects on say the characteristics of this structure, and of what force is required to break this liquid plug, because it is not it can be harmful for the for the kids in which this problem generally attack.

Then in last 2 decades, the gases parching has been used for removal of fouling in the membrane. and it has been shown that this fouling is most efficient if the gas liquid flow occurs in the Taylor flow regime. So, this enhanced the micro filtration efficiency in micro filtration Nano filtration.

So, in the membrane fouling removal then in the electronics cooling we are one want to have the advantage of the high thermal or heat capacity of liquids. And further enhance the heat transfer. It has been it is being used and it is application is being explored in electronics cooling, and heat exchangers. in fuel cells one have gas liquid flow. So, one have this flow regime, and in the oil and gas industry, where the channels might not be the straight channel. But there is flow of oil and gas and the displacement of one fluid by another they are one will observe Taylor flow regime might be in gas liquid or liquid liquid flow.

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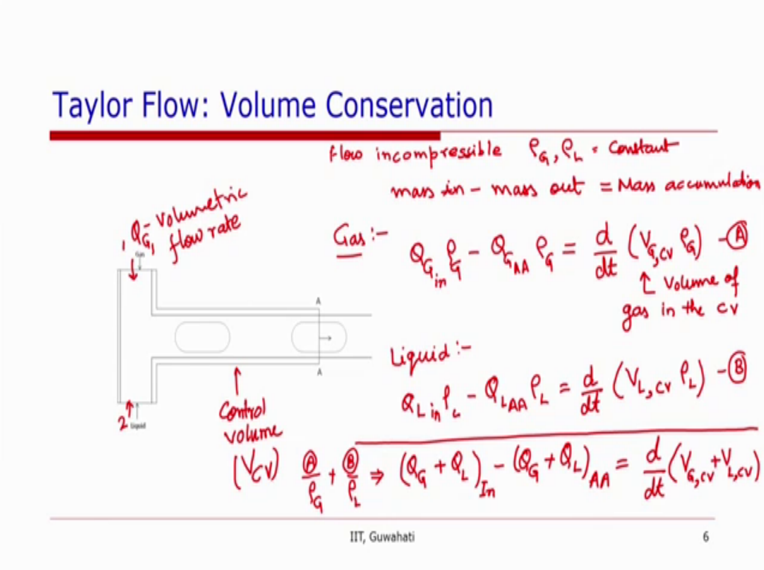

So, it has a number of applications. Now let us look at some of the physics of it. So, what we have drawn here is a channel in which the gas is coming from one inlet, let us call it inlet 1, and the liquid comes from another inlet inlet 2. And we consider a control volume that is represented by the dotted line and outlet of this control volume or the the other end of this control volume. We take an arbitrary cross section in the channel A A. In this figure we have taken it where both the phases are present, but one can take it there it is only liquid phase.

So, let us look at and write the mass conservation equation here. One of the assumptions that we make here is that the flow is incompressible. so, that means, rho G and rho L are constants ok mass conservation we apply then we can write mass in the control volume minus mass out of the control volume, and that will be equal to mass accumulation. We are not considering any reaction. And so, no generation or some symptoms will be there.

So, if we look at the gas phase, then if the gas flow rate is Q G 1 and the inlet. So, we write it as Q G 1, and then this density of gas is rho G. Minus, let us say the gas flow rate at A A. So, Q is volumetric flow rate. So, Q G 1 rho G Minus Q G A A rho G is equal to d by d t of gas volume in the control volume and V represented by V gas in the control volume into rho G ok. So, the control volume V G or V C V is the total volume of the control volume. And V G V is the volume of gas in the control volume .

Similarly, if we write down the same equation for liquid, then we have Q L1 into rho L minus Q L at A A into rho L is equal to d by d t, the volume of liquid in the control volume into rho L. Let us say this is equation A and this is equation B, and this we add them, but after dividing them by density. So, what we do is a by rho G equation A divided by rho G. Plus, equation B divided by rho L.

You can see that rho G rho G we cancel out from there um anyway. So, if you need to do that then what we get is Q G 1 plus Q L1. All this is at the inlet. So, we can actually remove the one, and that understanding that can be at the inlet minus Q G plus. So, actually we should have written this earlier itself as at inlet ok, minus Q Q G plus Q L at cross section A A, that will be equal to d by d t of gas volume in the control volume plus liquid volume in the control volume ok.

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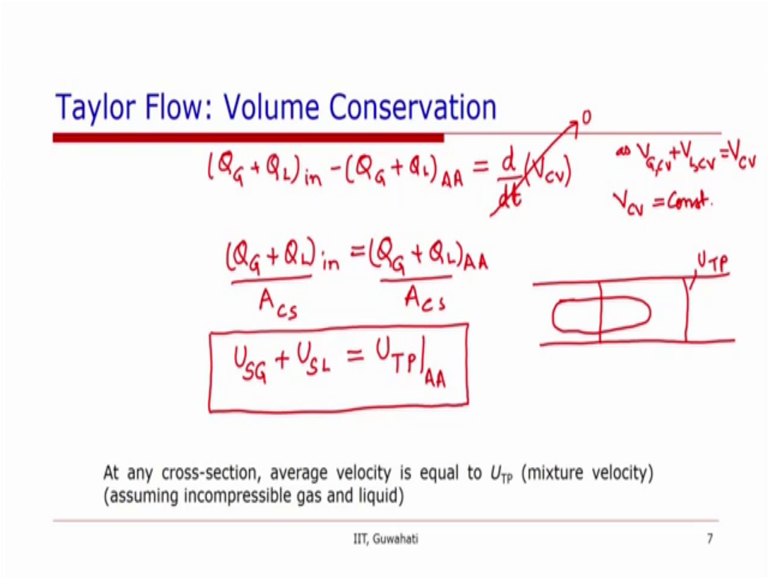

So, what we end up with is this equation that Q G plus Q L at the inlet minus Q G plus Q L at cross section A A is equal to d by d t of volume of control volume. Remember, that volume of gas in the control volume plus volume of liquid in the control volume, because there are only 2 phases. So, that will be equal to volume of control volume. And volume of control volume as a our control volume is fixed. So, that is a constant it does not vary with time. So, this term is 0. So, that means, that Q G plus Q L at the inlet is equal to Q G plus Q L at any cross-section A A.

Now, if we divide by the channel cross section A here. And the A cross section A cross section. So, this is U superficial velocity of gas plus, U superficial velocity of liquid. And we had defined this Q G plus Q L over channel cross section. This is equal to U TP, or mixture velocity at A A. So, this gives a important result; that at any place in the channel the 2-phase velocity is equal to the gas and liquid velocities. And dimension that we have made in this derivation is that the flow is incompressible.

So, this is valid not only for the capillary or the slug flow or the Taylor flow, but for any other flow also.. So, if you have a flow in a channel, then if you measure the average velocity at this cross section; this is equal to U TP. You major the average velocity at this cross section this is also U TP ok.

So, this is an important result that can be quite handy and useful. as I said, that the Taylor flow is periodic in nature. So, that means, it if you are doing the experiments and you are in the laboratory or a stationary frame of reference. So, then you will observe if you have a channel, and if you are looking at this point.

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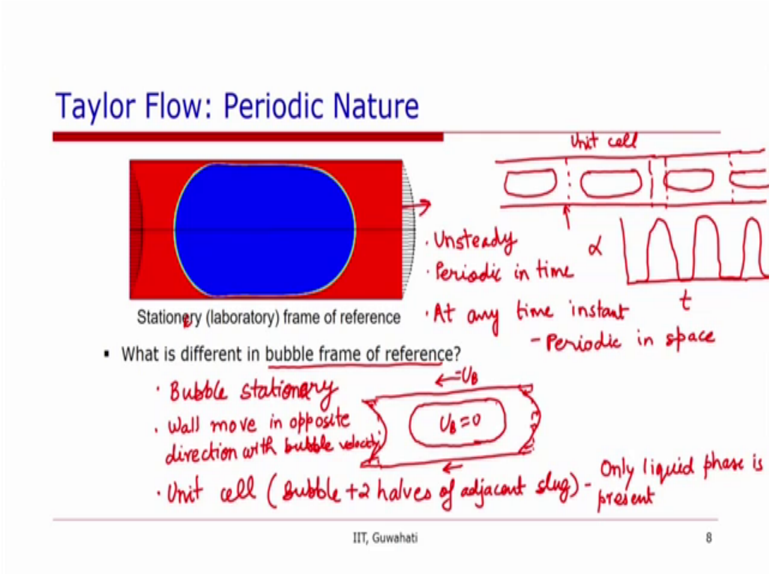

Then you will observe and what I am going to plot is let us say with time the volume fraction. So, one will observed that the volume fraction keeps changing. So, and it is periodic. So, one bubble comes and then another bubble comes then another bubble comes. So, the flow is unsteady of course, in the bubble frame of difference flow is unsteady, periodic in time at any time instant this is also periodic in space. So, if you take any snapshot of Taylor flow in a channel, then you will have the bubble shapes are supposed to be same in periodic pardon me for my poor drawing skills. So, there is periodic flow of bubbles and this is also periodic in space.

Now, what if we are sitting on the bubble? So, if we change the frame of reference to the bubble frame of reference, then the first thing will be that you will have bubble stationery. So, when the bubble is stationary, or then what will happen that? You are looking at a different picture. In this case you have a velocity if the slug is long enough maybe the velocity profile is almost parabolic if not then, you will have some gradients in the liquid. But all in the positive direction that is what it has been shown here [noise].

But if you are in the bubble frame of difference and the bubble is stationary. So, fluid in this direction in the earlier case, but now the bubble is stationary, let us say there is no. And before we describe the periodic flow let us say, because the flow is periodic. So, we often take one-unit cell, and this unit cell is one bubble and one slug, it can be any choice you can have the boundaries like this, or for convenience one can have the boundaries, that the bubble is in the middle and one are from the slug in the back and one are from the slug at the front.

So, in the bubble frame of reference UB is equal to 0. So, that means, you will have the walls, the actual flow is in this direction, or in the laboratory frame of reference to fluids in this direction. So, when you are sitting on the bubble you would see, the walls moving with minus UB velocity in the opposite direction. And as a consequence, you will also see the velocity profile to be something like this. Bubble is stationary, wall move or in opposite direction with bubble velocity, and for a unit cell, which is like this; that we have unit cell which has a bubble plus 2 halves of adjacent slug only liquid phase is present. and it has half in half the channel when it half the cross section it is coming in and half the challenge is going out ok.

So, if you have a if you have noticed in the previous slides, there was another statement, that one can see internal recirculations in the liquid slug. So, those internal recirculation are basically this which are which can be seen in the bubble frame of reference. So, it is sometimes quite useful to look at or study the bubble flow or the Taylor flow, analyze it in the bubble frame of difference, because the bubble becomes a stationary and then we can analyze number of things on this along this stationary bubble.

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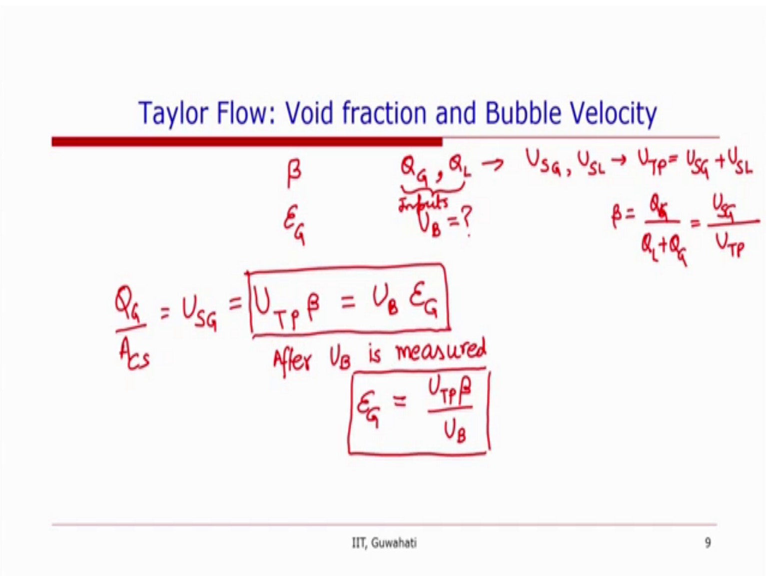

So, we had define 2 void fractions beta and epsilon G. And generally, what we have at our disposal is flow rate of gas, and flow rate of liquid from the experiment. so, that means, that we have used superficial velocity of the gas, and use superficial velocity of liquid. And that gives us U TP; which is equal to U superficial velocity of gas plus U superficial velocity of liquid. But if you want to find out bubble velocity that is not known a priory. we can also find out beta very easily from the flow rates. This can be the inputs.. So, beta is equal to Q G over Q L plus Q G, or we can also write this sorry Q G over Q L plus Q G, and this is U superficial velocity of gas over U TP.

Now, we can have a relationship from the mass conservation of gas, that U TP into beta; which is also equal to U superficial velocity of gas or Q G or channel cross sectional area; that is equal to UB into epsilon G. So, if we are able to measure the bubble velocity. So, after UB is measured, for example, visually then one can find what is epsilon G. So, epsilon G is equal to U TP beta over UB. So, this is another important relationship, which can be used for analysis of Taylor flow.

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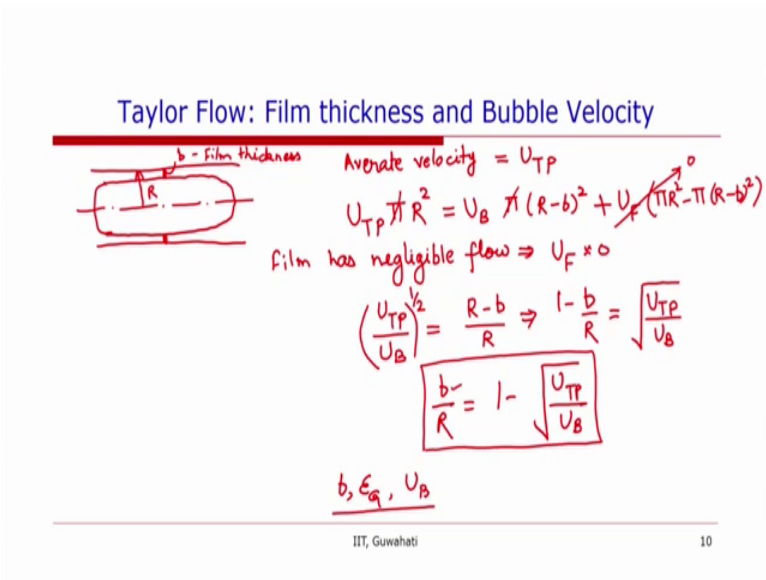

Now, moving on further, if we look at the bubble shape, the bubble has a front and the tail, and this is a constant thickness film region. So, bubble shape is almost cylindrical here. And this film region is we have to say the film thickness is B. So, b is the film thickness. We apply mass balance of a, then we can write that the also assume that not mass balance, let us say. We have just found that average velocity equal to U TP. So, if we use that relationship here. And let us say the channel radius is R, then we have U TP into pi R squared is equal to U of bubble into pi R minus B squared, plus U f which is flow in the film into pi R square minus pi R minus B squared.

Now, film is generally film has negligible slow. When we talk about the film this is, I am talking about the constant thickness film region.. So, that implies that U f can be neglected. So, this term goes away. Then what we have is pi and pi also cancelling out. So, we have U TP over U V is equal to R minus B over R square or R min R minus B over R is equal to U TP by UB raised to the power 1 by 2. So, 1 minus B by R is equal to root of U TP over UB or B by R is equal to 1 minus root of U TP over UB. You might note that the bubble moves faster than the average velocity in the channel. So, that means, this number is always going to be less than 1.

So, this film this is the relationship for film thickness. So, we have seen that if we can find out the bubble velocity, then we can know what is the film thickness. If we find out the bubble velocity, then we can also know what is epsilon G, or in among B epsilon G and UB. Out of these 3 if we can find one parameter, then we can further analyze and find other 2 parameters by the relationships that we have just out of (Refer Time: 40:13) ok.

So, let us look at a bit in the flow field in the liquid slug.

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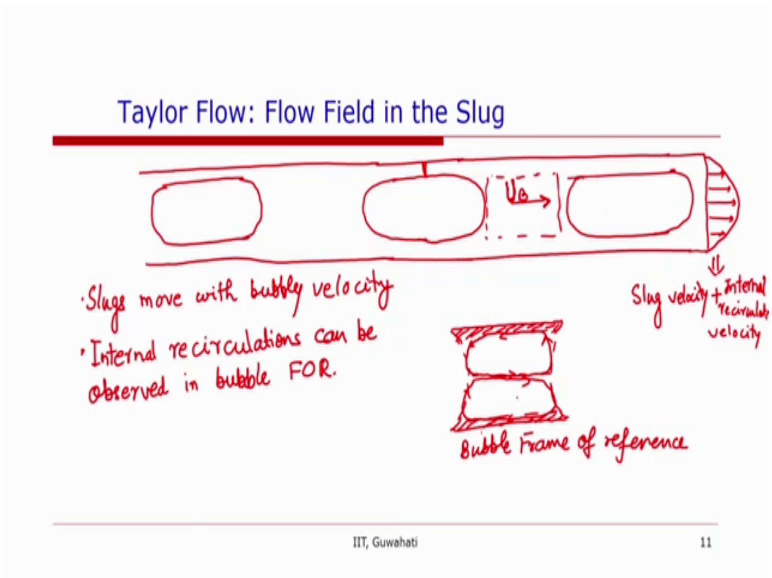

So, when we have flow field in the liquid slug, and I briefly talked about that we have the slug, the velocity profiles in the slug also we have plotted. So, let us plot few bubbles here. So, if we look at a velocity field in the slug. it will look something like this now the question comes or people are often confuse that it has been written in the literature that it has internal recirculation with them.

So, the internal recirculation is so, this can be decomposed actually, that it can be the slug velocity the velocity profile that one see is; slug velocity plus internal recirculation velocity ; that means, the velocity in this because the slug on an average, it has to move with the bubble velocity in the front direction ok.

So, it is total velocities such that that it move with the bubble velocity. If we look at a slug, let us say, if we have a slug something like this. Then the picture in the slug will be something like this. So, there is internal recirculation and that can be seen in bubble frame of reference. So, you see that, and in along this there is also a small film surrounding the slugs.

So, this film may or may not be of the same thickness; that is, the film surrounding the bubbles. So, what we have is that the slug few things that we can note about the slag that the slug move with bubble velocity. Internal recirculations can be observed in bubble frame of reference. And the slug velocity is the sum of to the internally circulations as well as the slug velocity.

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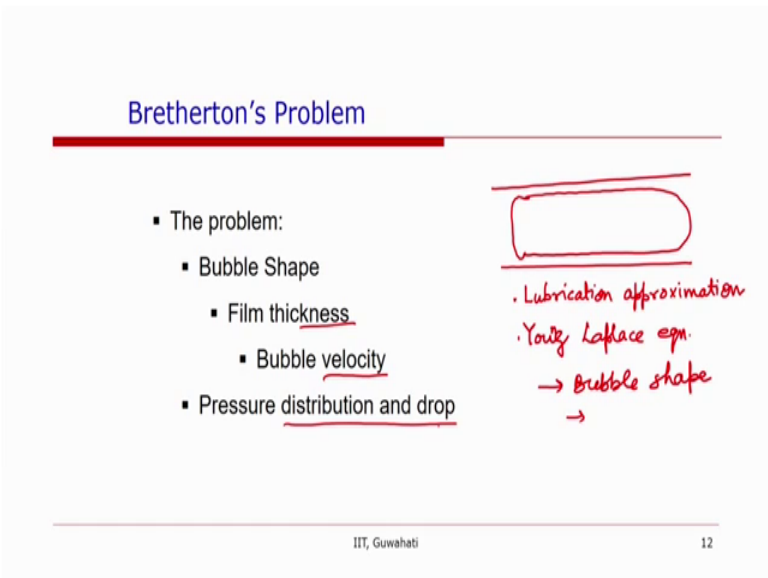

Now, next we will talk about Brotherton's problem. So, in 1961, in his classical j f m paper general fluid mechanics paper Bretherton and has analyze the flow of a Taylor bubble in a capillary. And what he was able to do that using lubrication approximation he analyzed the flow behavior around the bubble, and used young Laplace equation for the pressure jump across the 2 phases near the interface.

And from those 2 equations, you are able to find out the bubble shape. And more useful from that he could find out the film thickness. And we as we have seen that some thickness, we can also find out bubble velocity, and the volume fraction and he could also find out the of course, the pressure distribution and the total pressure drop across the bubble ok. So, we will look at this problem we will it in detail now.

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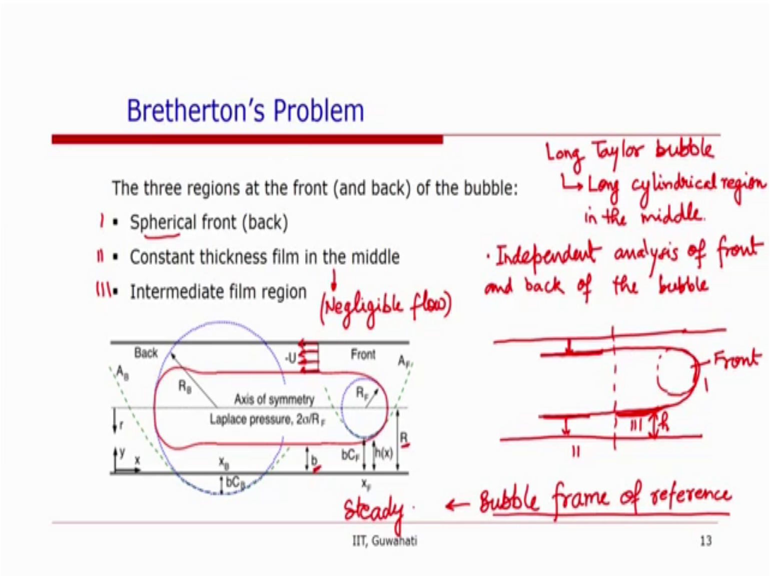

So, first thing he looked at, he considered a long Taylor bubble, and by that what he meant is long cylindrical region in the middle. If you look at the shape of the Taylor bubble, you can fit the front part which is part of the circle and that comes because of a young Laplace equation at the front. And at the back at very low velocities this might be almost same as at the front, but a high velocity you will see that the bubble tail becomes flatter and flatter. So, at the back once you will have the back of the bubble also you can fit part of a sphere and in between the film thickness. So, what he was able to do or what he said or he suggested that we can analyze the 2 parts of the bubble front and back independently.

Now he considered the front. So, for convenience, we will analyze the front part and then use the result for the analysis of the back part. So, if we considered the front of the bubble, then we will look at 3 regions in the bubble for example. So, one is the front; which is a spherical. Then this region I have not been able to draw the films to be same, but they have same here.

Or we are considering axisymmetric bubble. So, the constant thickness film region. So, this is they say this is 1 2 and let us say 1 2 and 3. So, this is 1, this is 2, and the main region of interest is 3. see, as we have said that the flow in the film is negligible. flow and it is cylindrical in nature. So, we know that if we know, the film thickness then we know the shape of the bubble front it is spherical. So, we need to find what we need to find with the radius of with this is sphere.

Now, we look at this intermediate region or intermediate film; which is between the 2. We also consider this in bubble frame of reference. So, when we are looking at the problem in bubbles frame of reference, then the wall the bubble will be a stationary, and the wall will be moving with a velocity minus U. So, that is why you see these uniformed vectors in the constant thickness film region the channel radius is R the film thickness is B and the thickness of the film in the intermediate region in this region, it is considered to be as and it is varying along the axial direction. So, axial direction is x, and considering the origin at the wall, we have x and y the coordinates near the wall. And this the coordinates are in the film region.

Now, we can apply lubrication approximation. So, lubrication approximation is just scaling, that if one of the approximation say it is because the problem is axisymmetric.

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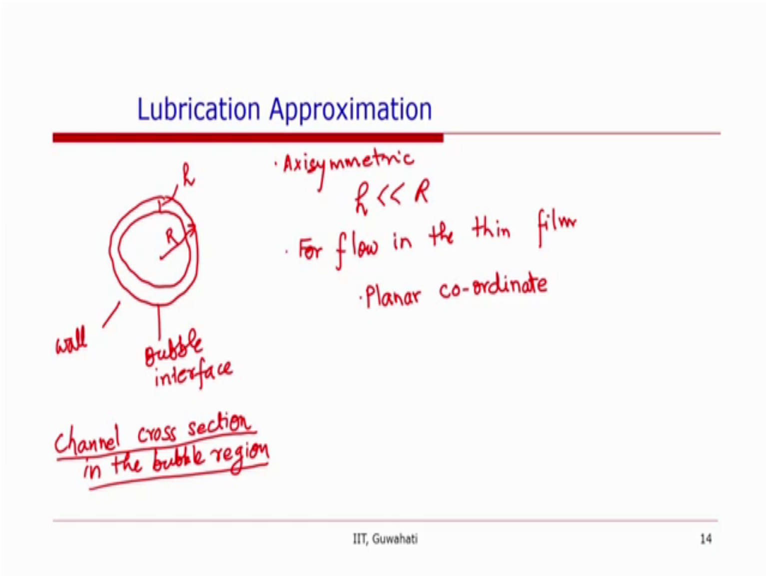

So, we look at A cross section in the bubble, we have this channel wall, and this is bubble interface wall bubble interface and this is channel cross section in the bubble region ok.

So, the film thickness is B, and the channel radius is R, or if we are in the intermediate region then this channel thickness is h in the channel radius size R. So, then h is very, very less than R, then for flow in the thin film we can use planner coordinate system. Also remember that when we have looked at the problem in the bubble frame of reference then we have been able to convert it to a steady problem. The fluid in the film the coordinates are planar and now we look at the some of the further let us look at the pressure jump at different points.

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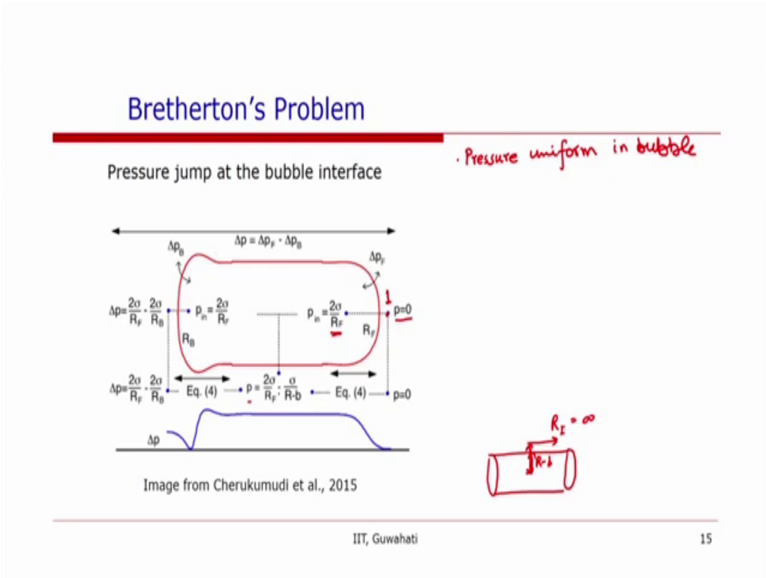

So, if we assume the pressure just at the front of the bubble in the liquid region at the pressure at this point is 0, at the front of the bubble in liquid, if the pressure at this point let us call this point as 1, at this point the pressure is 0. Then by young Laplace equation, we have a sphere will have a radius of curvature or the sphere will have a radius R. And the pressure difference across the sphere will be 2 sigma by R. So, if the radius of curvature at the front is R F, then we have pressure just near the bubble front, but in the bubble, it is 2 sigma over R F.

Now, note that pressure is almost uniform in the bubble, when you compare the, that in the liquid region. So, when the pressure uniform in the bubble; so, with respect to this point we have the pressure difference 2 sigma over R F, and this and the bubble front, and that same pressure will be everywhere in the middle and back of the bubble.

So now look at the pressure jump, in the middle of the bubble, and if you analyze the pressure jump in the middle of the bubble, then you have p in the liquid film in the cylindrical region. So, this is this is cylindrical portion, and the pressure jump here will be equal to 2 sigma by R F is the pressure inside. And that is equal to the the difference between the pressures is equal to sigma over R minus B.

So, p bubble minus p liquid is equal to sigma R minus B. So, that is why we have pressure in the film region is equal to 2 sigma by R F minus sigma over R minus B, because the radius this is cylinder. So, the radius of curvature for the cylinder in one direction it is infinite, and in other direction normal to it it is radius of cylinder which is R minus B. So, 1 over R 1 plus 1 over R 2 for a cylinder is 1 over R minus B.

So, we have pressure jump and from that the pressure in the liquid constant thickness liquid film region, 2 sigma by R F minus sigma R minus B. Similarly, at the back of the bubble, just near the back we will have 2 sigma over R B is the pressure difference, and pressure in the bubble is to 2 sigma over R F. So, the pressure difference is to sigma over R F minus 2 sigma over R b. So, that will also be the pressure difference there delta p b

So, this is the pressure distribution at different points in the bubble and some of it might be used in later.

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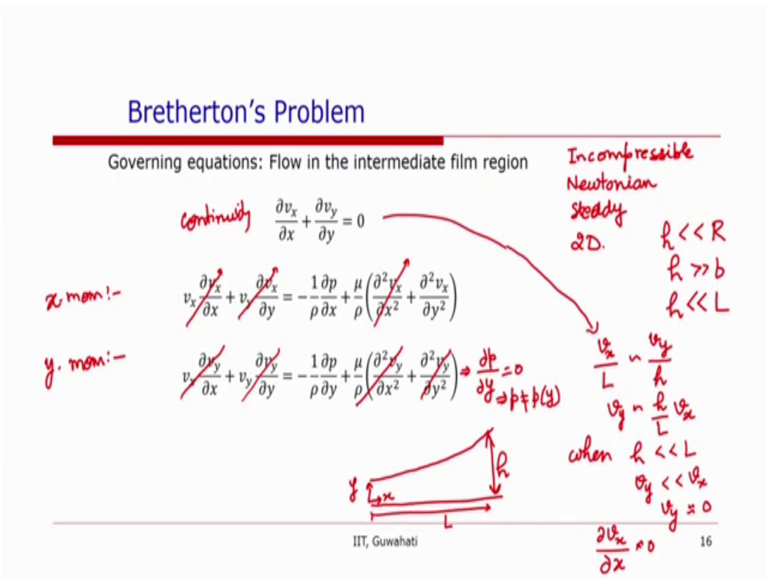

(Refer Time: 58:00). So, using the lubrication approximation equations, as we said that we can analyze this problem using the planer coordinate system, here I have written the navier stokes equation for an incompressible Newtonian fluid, and that steady state, this is continuity, and this is of course, in 2D and this is momentum x momentum another one is y momentum.

So, what we want to look at is we want analyzed is the film in the intermediate region, where it is the film thickness is h and this direction is L. So, h is such that h is very, very less than R, but it is bigger than B. h will also be smaller than the L, where L is the length along the axial direction. So, this is x direction this is y direction.

Now, we look at the continuity equation and an order of magnitude analysis. So, the V x over x which is L, the comparable with V y over h, and if we write that then V y is equal to h by L V x, when h is smaller than L, then V y can be negligible with respect to V x. So, we will not consider. So, we will consider on neglect the V y component and this. So, if that is the case, we will also have the flow to be fully developed in such case. so, that means, that del V x over del x will also be approximately 0. this stage from we have got from continuity equation.

Now, let us look at y momentum equation. So, this term will be 0 because V y is 0 on this is going to be 0, the only thing that we get from here is del p over del y is equal to 0; that means, p is not a function of y or only a function of x ok. in this equation, this term is 0, because V y is equal to 0. This term is 0 because del V x over del x is negligible. And this is also 0, because del V x over del x is equal to 0. So, what we have is del p over del x 1 over rho is equal to mu over rho del 2 V x over del y 2. So, we have a simplified.

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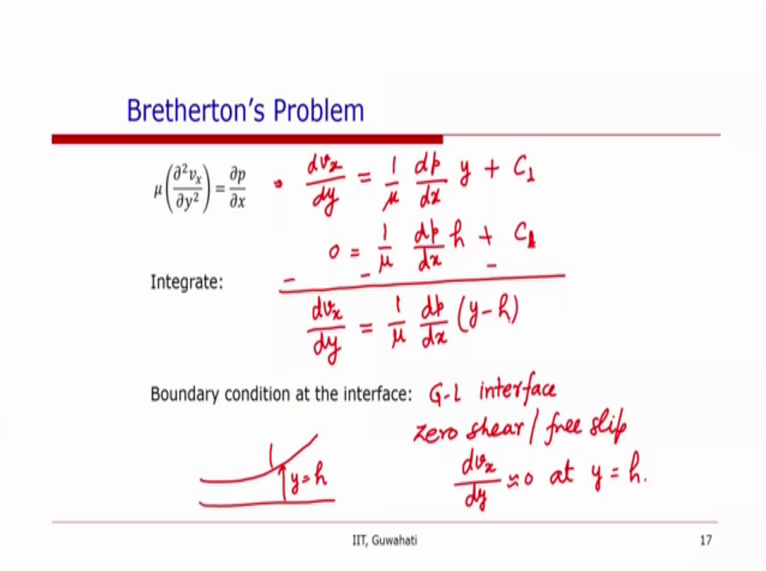

equation mu del 2 V x over del y 2 is equal to del p over del x. Let us integrate this. So, what we will get is; we can actually write this as total derivative. So, we will right now this has total derivative, and d V x over d y is equal to 1 over mu d p over d x y plus C 1.

Now, the boundary condition at the interface will be this is gas liquid interface. So, if it is gas liquid interface, then we will have 0 shear boundary condition, what we call free slip boundary condition; which means that the shear off the gas at the liquid phase is negligible. So, d V x over d y is almost equal to 0, at the interface. So, the interface is this. So, this is y is equal to h at y is equal to h. So, we will have d V x over d y y is equal to 0 at y over h 1 over mu d p over d x h plus C 1. So, this gives us, if we subtract the 2 equations, we will get d V x over d y is equal to 1 over mu d p by d x into y minus h.

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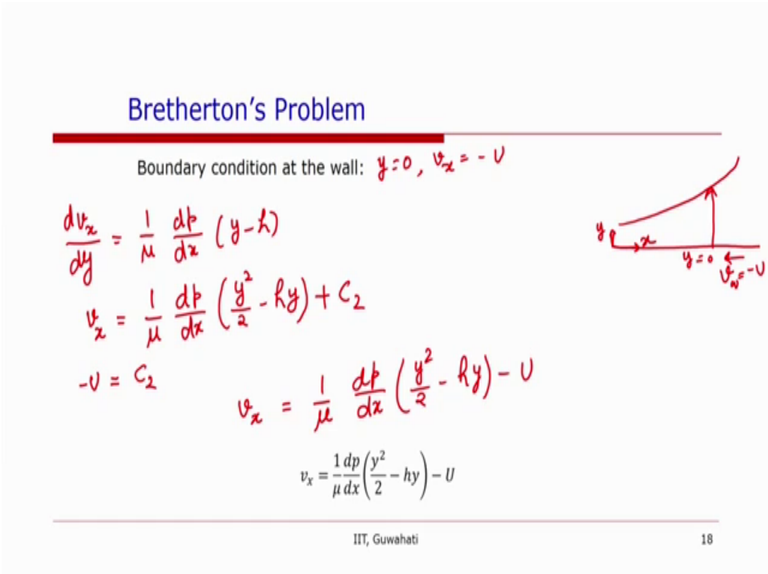

Let us integrated further, and apply the boundary condition at the wall. So, we had this d V x over d y is equal to 1 over mu d p by d x into y minus h. let us just confirm it. 1 over mu d p by d x y minus h and we can integrate it to find out V x is equal to, 1 over mu d p over d x y squared over 2 minus h y. plus, another constant c 2. For that we can use the boundary condition at the wall. So, if we again draw the problem where we at y is equal to 0. and it is in bubble frame of reference. So, we will have wall moving V w equal to minus u. So, V at y is equal to 0 V x is equal to minus U. So, we substitute that here, then minus U is equal to c 2. So, that gives us V x is equal to 1 over mu d p over d x y squared over 2 minus h y minus U.

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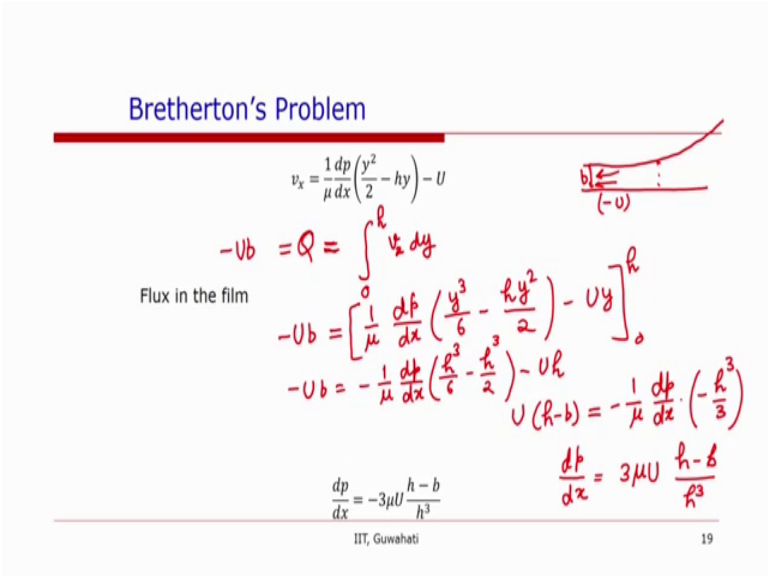

So, having obtained this relationship. Let us now calculate the flux in the film. So, if we calculate flux in the film still in the planar coordinate system we will get Q, film is equal to integral 0 to h V x d y. And remember, this is also equal to. So, when the constant thickness film region, the film thickness is B. And the velocity is minus u. So, this Q is equal to minus U B. So, we substitute this here.

So, we got minus UB is equal to 1 over mu d p over d x, y cube over 6 minus h y square divided by 2 minus UB. And the integration limits from 0 to h. on simplification we will get minus UB is equal to 1 over, and this is sorry this is minus U into y here. 1 over mu d p over d x this is h cube over 6 minus 0. So, this will be h cube minus 6 minus h cube over 2 minus U h.

So, if we substitute here, we will get you into h minus b equal to minus 1 over mu d p over d x into minus h cube over 3. So, what we will get is d p by d x is equal to 3 mu U h minus b over h cube.

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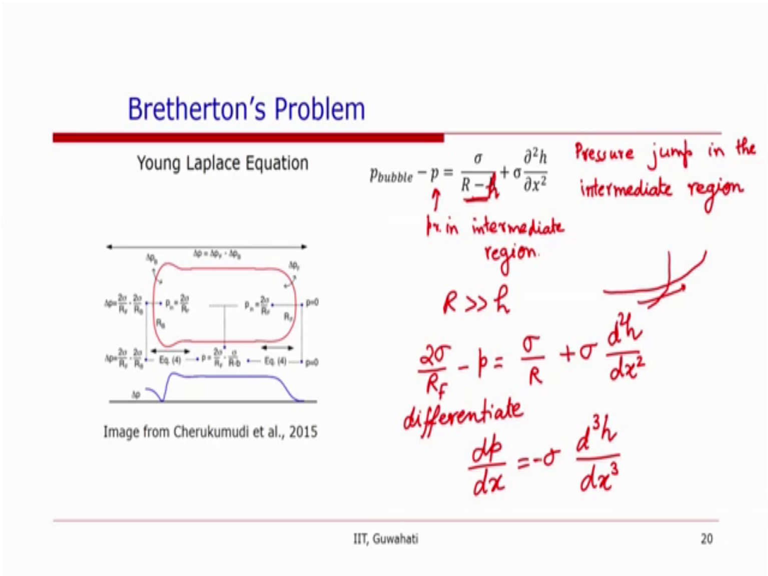

So, using the young Laplace equation, we can calculate the pressure difference across the. So, what this equation is? Pressure jump in the intermediate region. So, in this intermediate region p bubble minus p, p is the pressure in the intermediate region or the region; where we are looking at the flow, or where we are analyzing the flow is equal to. So, this will be sigma over R minus B. So, this will look something like this.

So, it has 2 radii of curvature, one is in this direction, and another one is normal to it, which is in the say cylindrical region. So, this is R minus h. So, this have been approximated approximated h like B is h is very small. So, this has to be sigma over R minus h. But then because R is very small as we say that R is large than h. So, we can write that p bubble, or here p bubble is 2 sigma over R F minus p equal to sigma over R plus sigma d 2 h over d x 2 the other radius of curvature. So, this is Kemrock approximated this radius of curvature can be approximated as d 2 h by dx 2. And the other radius of curvature normal to it is sigma over R minus h which we have approximated as sigma over R.

So, if we differentiate it. We get d p over d x is equal to sigma d cube h over d x cube minus here.

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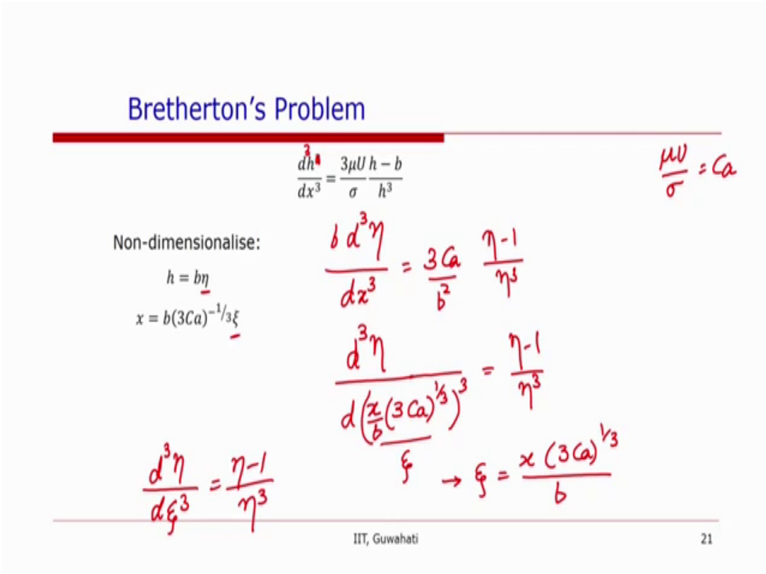

So, if we substitute that in the film region then we will get this equation this is d cube h over d x cube is equal to 3 mu U into h minus b over h cube. And if we non dimensionalise it by having 2 non-dimensional coordinate eta and psi. So, h h is approximated as b into eta. So, let us substitute that here we will at get d cube eta b over d x cube is equal to 3 mu U over sigma is capillary number. So, you have 3 c a into h minus b. So, we can write this as eta minus 1 over eta cube.

And here you will have be square. So, if we further write this they will have d cube c eta over d over B into 3 c a power 1 by 3 cube is equal to eta minus 1 over the cube. So, what we have done is approximated this this is x over B. So, if we look at x over B, over 3 c a to the power x, this is psi. So, what we have defined as psi is equal to x into 3 c a power 1 by 3 divided by b or x is equal to b 3 c a power minus 1 by 3 psi.

So, we have got the equation the final equation d cube eta, over d psi cube is equal to eta minus 1 over eta cube; which is very useful equation, and we will look at it solution in the forward front, and bubble back region in the letter classes. So, what we have looked at today is looked at what is the basic flow characteristic or flow behavior in the Taylor flow. And the useful relationships between film thickness bubble velocity, and epsilon G.

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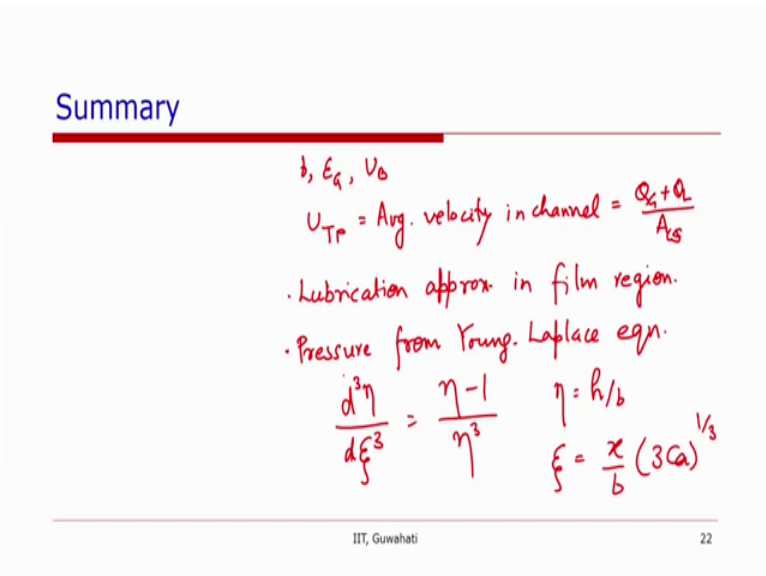

So, b epsilon G and UB relationship. We have also looked at that U TP is the average velocity in channel; which is equal to Q G over plus Q and over channel cross sectional area

Then we will have looked at Brotherton's problems in the Brotherton's problem what we did is applied lubrication approximation in film region. And substitute pressure from Young Laplace equation. and then obtain a relationship between d Q eta over d psi cube is equal to eta minus 1 over eta cube where eta is h by b. So, this is the radial coordinate or the or the coordinate in the transverse direction. And eta is and psi is x over b 3 c a power 1 by 3. So, this is a coordinate non-dimensional coordinate in the axial direction. So, the solution of this equation will give us the bubble interface of course, we will need to apply the boundary conditions to it ok.

Thank you.