Cardiovascular Fluid Mechanics

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Lecture – 09
Flow Bifurcation

In this lecture, we will be talking about flow bifurcation flow; bifurcation is a very important problem in cardiovascular fluid mechanics. As we know that when the blood starts from the heart, it comes out from aorta and then the aorta bifurcates, trifurcates into smaller arteries, and these first generation of arteries further bifurcates or further divides themselves into smaller arteries, and then to arterioles which supply blood to different organs and arterioles.

Further bifurcates to capillaries and then capillary join together to become venules and then, venules join together to become veins and then veins finally, bring the blood to superior or inferior vena cava which bring blood back to the heart. So, it is very important to understand the flow bifurcation, additionally there has been strong evidence that, the plate formation which is a cause for atherosclerosis, which causes heart attack it the plate formation happens at the bifurcations and the fluid mechanics, is responsible for plate formation in a way. Because the unequal shear stresses distribution, causes the plate formation at certain locations at the bifurcations and other places in channels. So, it is important to understand the flow bifurcation in this lecture, we will look at some general features of the bifurcation.

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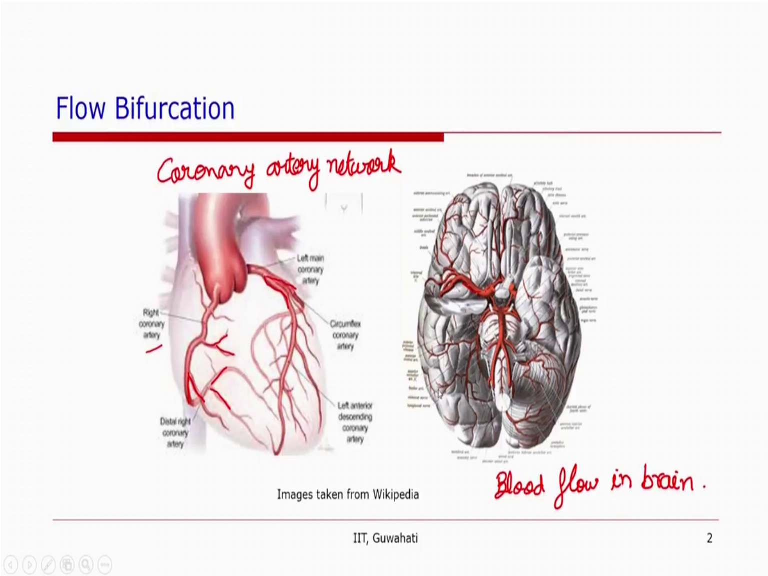

So, these 2 images show typical examples of, flow bifurcations in the cardiovascular systems. So, the image on the left, as you see this is for the coronary artery network, you might remember that coronary artery is the artery that supplies blood to the heart itself. So, you can see that, there are different bifurcations from the right coronary artery here or you can see the bifurcations and, from the left coronary artery. And many reasons for heart attack or poor functioning of the heart; are related with the plate formation at different places, at generally at the junctions or at the bifurcations in this coronary artery. Another example, as you might have guessed by now is, blood flow in the brain.

This, here we can see that the red lines or these lines represent the arteries in the brain. So, there are further several bifurcations in blood flow, in the brain. So, these are 2 typical examples where artery bifurcation takes place, and it is important to understand and to analyse the flow bifurcation at the artery. So, before we look at the flow some of the flow features of the artery, one question that has been bothering or that that, has been discussed or has been studied by the fluid dynamists biologists for about a century.

Now, in 1926 Murray gave a law and he wanted to understand, is there a physical law that can govern the bifurcations in different a natural systems and it was not only cardiovascular system, the interest was that all the biological systems including animals, plants, human being's the cardiovascular system in human beings or the pulmonary system or the lung airflow in the lung. Do they follow a particular law, when it comes to flow bifurcation is there a particular design principle, by which nature has designed the circulatory system including that of sap carrying vessels, in the trees and he postulated this that the design principle is based on minimisation of cost function, and what is the cost? The cost here is the energy, as you might know that the circulatory system at rest conditions it requires about one-sixth of the metabolic rates.

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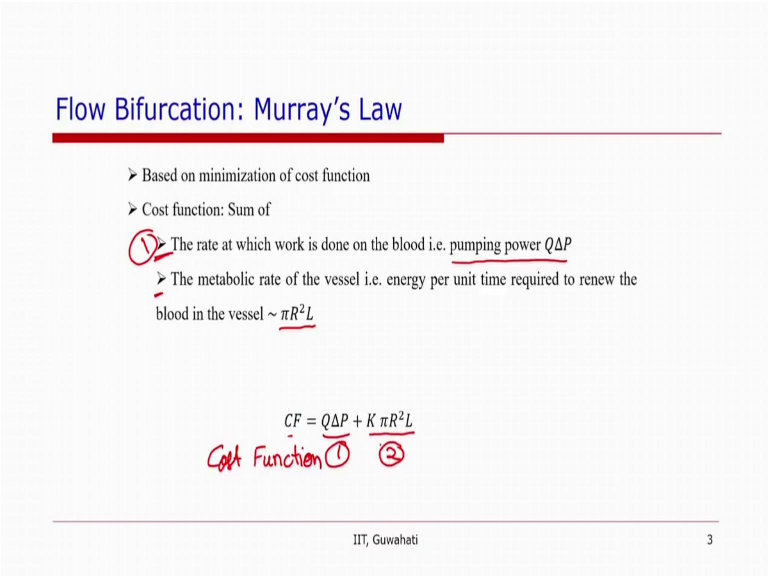

So, one-sixth of the energy is required for the circulatory system. So, is significant about 15 to 16 percent of the energy that the body generates is required for the blood flow in the circulatory system. So, it is important that, when the system is designed, that the cost of this circulation blood circulation is minimised. So, he considered 2 factors which cost the energy, the energy may be required for the operation maintenance or generation in the circulatory system so, generation of the blood vessels or generation of the blood.

So, he considered 2 factors here, the first factor is for the operation of the cardiovascular system. So, which is the rate at which work is done on the blood, or the work that is required or the energy that is required to pump the blood which is equal to Q into delta P? So, delta P is the pressure loss and Q is the flow rate, volumetric Flow rate the other component which is not so explicit, is metabolic rate of vessel. It can include the maintenance or generation and this is proportional to the volume of the vessel.

So, that the metabolic rate of the vessel or the energy per unit time, that is required to renew or the blood in the vessel or maintenance of the blood in the vessel, that is proportional to the volume of the vessel, which is pi R square L, L is the length of the vessel and R is the radius of the vessel. So, the C F C F is nothing but cost function. So, the cost function has 2 component number one Q delta P, which is the energy required for the operation of the system and K pi R square L, which is required for the maintenance of the system ok.

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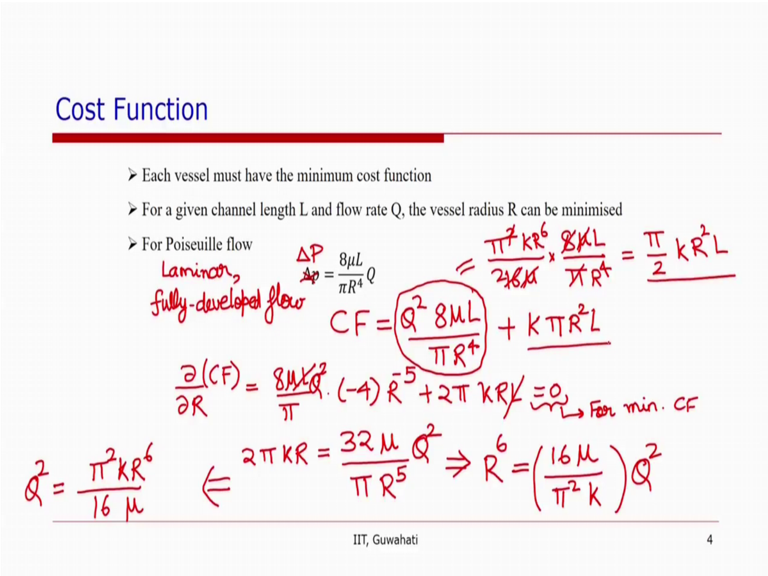

So now, we would like to minimise this energy, because the circulatory system is made up of different these different vessels. So, the total cost will be the sum of the cost function for each vessel. So, it is the minimising of the cost function, is basically it comes down to minimising the cost function for each vessel.

So, let us say for a vessel of radius are, if the channel length or the vessel length is L and the flow rate is Q, then the cost function can be minimised and if we assume that, the flow is Poiseuille flow. So, if you remember the Poiseuille flow is laminar and fully developed flow, for which the pressure drop delta P is given as 8 mu L by pi R to the power 4 Q. So, the cost function is Q delta P.

So, we will replace delta P by 8 mu L by pi R to the power 4 Q, this delta P you can write as delta capital P Q square 8 mu L by pi R to the power 4. So, this is the first term and the second term will be K pi R square L. Now, if we want to minimise this with respect to the channel radius then, delta by delta r of cost function, that will be equal to because Q and L are constant. So, we can write this as 8 mu L by pi Q square minus 4 R to the power minus 5 plus 2 pi K R L is equal to 0 and this is for minimum cost function. So, when we do this, we will arrive at that 2 pi K R L actually L can be cancelled. So, we can write 2 pi K R is equal to 8 mu not again a height into 4 is 32. So, we can write 32 mu divided by pi Q square and R to the power 5.

So, if we want find out relationship between Q and R, then we can write that R square is equal to 32 by 2 is 16 mu by pi square, this pi comes here and this sorry this will be R to the power 6. So, R to the power 6 is equal to 16 mu by pi square K into Q square.

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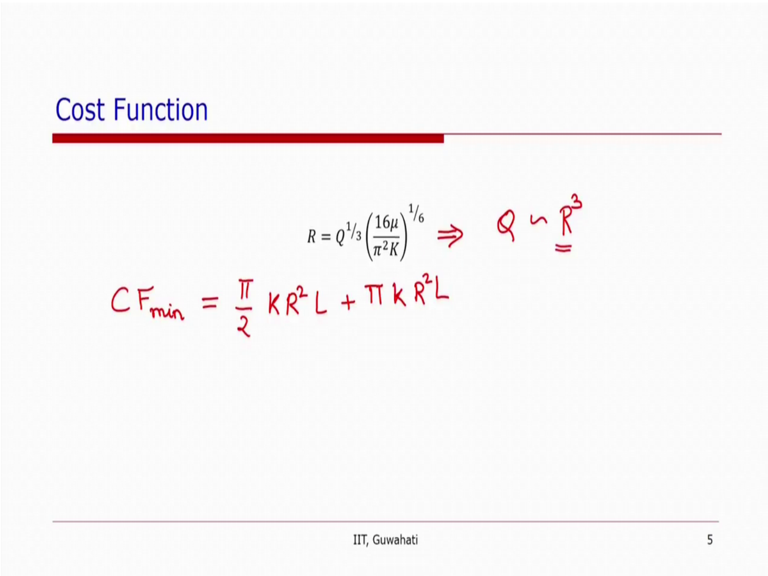

So, this will give us the relationship, which is R is equal to Q to the power 1 by 3 16 mu by pi square K whole to the power 1 by 6. So, that gives a relationship that, Q is proportional to R cube. So, this is the cost function. Now, if we want to find out what is the value of this cost minimum cost function that will be we can substitute the value of Q here. So, if we go back and look at that Q square what is the value of Q square that is 2 pi square K R to the power 6.

So, from here what we are going to obtain 2 pi square K R to the power 6 divided by 32 mu, or we can write this as pi square K R to the power 6 by 16 mu and if we want to get the first term in the cost function. So, we can get this first term is equal to pi square K R to the power 6 divided by 16 mu into 8 mu L divided by pi R to the power 4. So, this will reduce to 2, mu and mu will cancelled out, pi will cancelled out with 1 pi and what will we get is, pi by 2 K R square note that R to the power 6 divided by R to the power 4 R to the power 4 that will be R square L. So, pi by 2 K R square L and this is k pi R square L ok. So, we will be ending up with pi by 2 K R square L plus pi K R square l ok.

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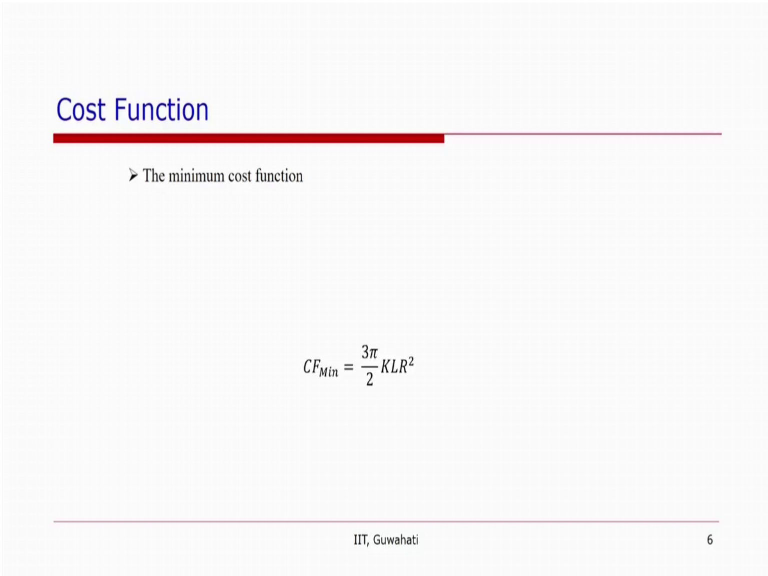

So, the minimum C F will be the minimum cost function will be, 3 pi by 2 K L R square out of which, the operating cost is pi by 2 K L R square and the remaining is the maintenance and generation cost ok.

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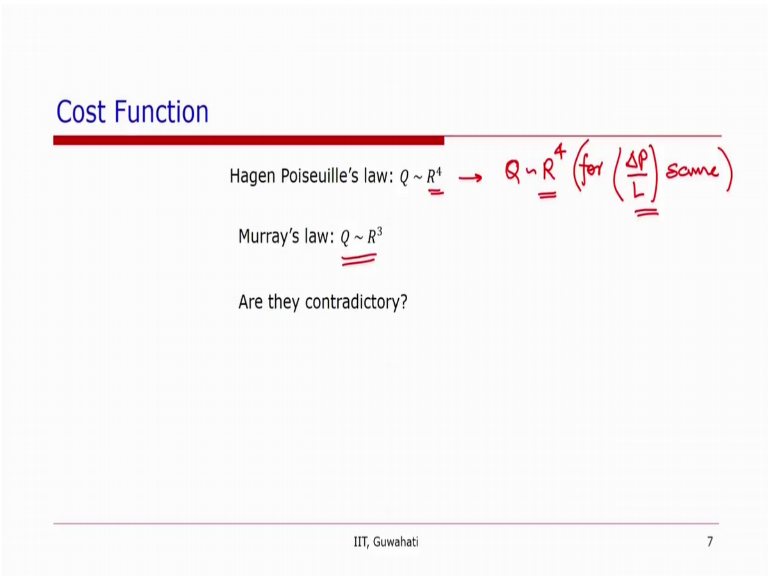

So, what one might wonder that, just now we have used Hagen Poiseuille law, which says that Q is proportional to R to the power 4, but Murrays law, it says that Q is proportional to R to the power 3. So, which one is correct are they, contradictory are they not in synchronisation with each other. So, we must remember that, when we derive the Murrays law. In Murrays law Q is proportional to R to the power 3, but everything else is constant K is a constant and V is a constant if the fluid properties are assume to be constant, which we did assume.

So, Q is proportional to r to the power Q whereas, when we say that Q is proportional to R to the power 4 for Hagen Poiseuille law, we had this Q is proportional to R to the power 4 for delta P by L same right? Because in the expression; we also have delta P by L which is pressure drop per unit length, which is not going to be same. If the radius of the channel changes because, Hagen Poiseuille law is based on the assumption that, the channel size is same.

So, if we consider that the channel size is changing, then there will be another factors coming into plate. So, they are not contradictory, what we need to understand is that Q is proportional to R to the power 4, given the fact that delta P by L is same for the 2 cases that we are comparing ok.

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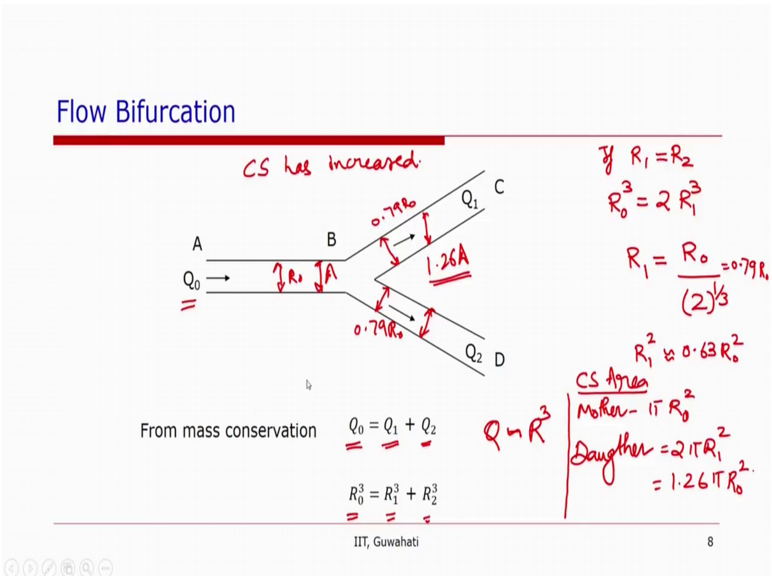

Now, if we take another look at Murray's law, or if we look at it from a different view point and assume or apply continuity at a bifurcation of channels. So, here is a mother channel or a parent channel which we call say, A B and it bifurcates into 2 channels B C and B D the flow rate distribute in such a manner, that the flow in the mother channel is

Q naught and it distribute itself as Q 1 in channel B C, and Q 2 in channel B D. From the mass conservation, we need that the flow from the mother or from the parent channel is equal to the sum of the flow rates in the 2 channels. So, Q 0 is equal to Q 1 plus Q 2. Now we have just learned from Murrays law, that Q is proportional to R is to the power 3. So, if we substitute that knowledge here, and we get that R naught Q is equal to R 1 to the power 3 plus R 2 to the power 3.

If we say that R 1 is equal to R 2; that means, the size of the 2 channels are same, then what we will get R naught cube is equal to 2 R 1 is to the power 3; that means, R 1 is equal to R naught divided by 2 race to the power 1 by 3 or about 0.79 R naught. So, the radius of the daughter tube is about, if this is R naught then in case when they are equal this is 0.79 R naught and this is 0.79 R naught. So, the area will be about R 1 square will be about 0.63 R naught square ok.

So; that means, the area in the mother channel was pi R 1 square or the cross-sectional area for the mother channel, is pi R 1 square whereas, the area in the daughter channels is sorry it is not pi R pi R 0 square and this is 2 pi R 1 square, which is 1.26 pi R naught square so; that means, the cross-sectional area has increased. So, bifurcation cross sectional area has increased, remember this relationship we have got based on the Murrays law; which says, that the energy or cost is minimised and from this relationship.

We see that the cross-sectional area has increased which means, the flow rate in the daughter tube is bound to decrease or not the sorry the not the flow rate, but the velocity because the flow area, which was say A here combined both the channels it has become 1.26 A. So, the velocity of the flow is bound to decrease.

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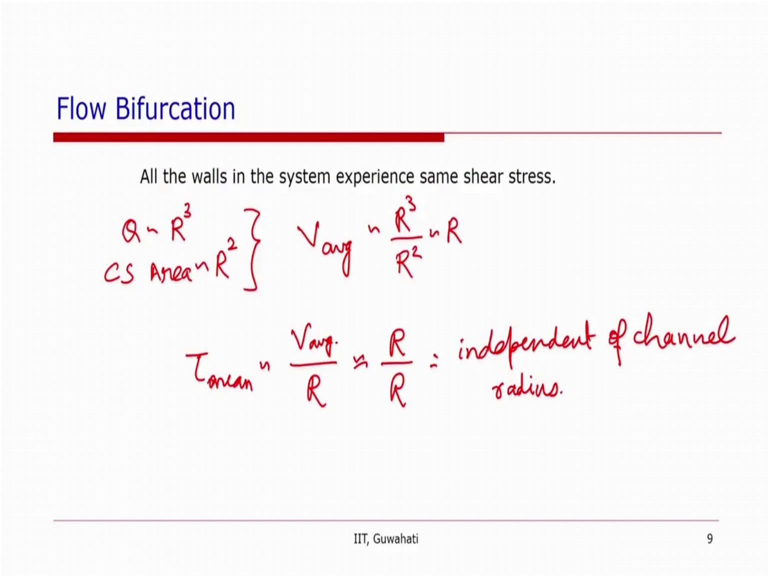

Now, as we have seen that the Q is proportional to R cube and cross-sectional area is proportional to R square. So, that gives us that average velocity will be proportional to R cube by R square or will be proportional to R. Now, the shear stress, if we talk about a means shear stress, then that will be about V average over radius.

So, that is going to be proportional R by R so; that means, the shear stress in such case is going to be independent of channel radius; that means, if the Murray law is being followed in a network of channels, then the shear stress is same everywhere in the channel or in the velocity profile, will also be parabolic in the entire channel network ok. So, this is an important result, that following Murrays law the shear stress is going to be same in the entire network of channels ok.

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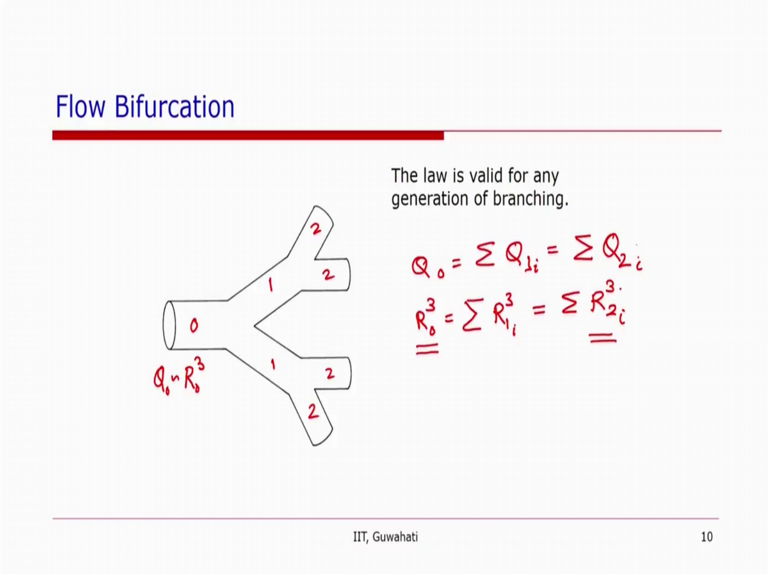

Another important corollary or another important result from Murrays law, that for the first generation we can write that Q 0 is proportional to R 0 cube, similarly in the first generation which is 0 generation let us say, and this is generation 1 these are generation 2. So, what we can see from here that Q 0 is equal to sigma Q generation 1 is equal to sigma in all generations. So, you can say 1 i and generation Q in generation 2 sigma in all i th generation of channels that are there in the second generation.

So, R 0 cube is equal to sigma R 1 i cube is equal to sigma R 2 i cube so; that means, the Murrays law is valid for any generation of branching that, we can directly say that R 0 cube is equal to sum of the flow in the second generation and the cube of the R 0 cube is equal to sum of R 2 cube where R 2 is the radius in the second generation. So, sum of all the channels, right?

So, this means that we can say, that flow in an aorta or the flow rate in an aorta is equal to sum of the flow in the capillaries, say in very downstream of the channel. And so, if we know the number of capillaries then, flow rate in the aorta or the radius of the cube of the radius of aorta is equal to number of capillaries, at one particular generation multiplied by the cube of the radius of the capillary.

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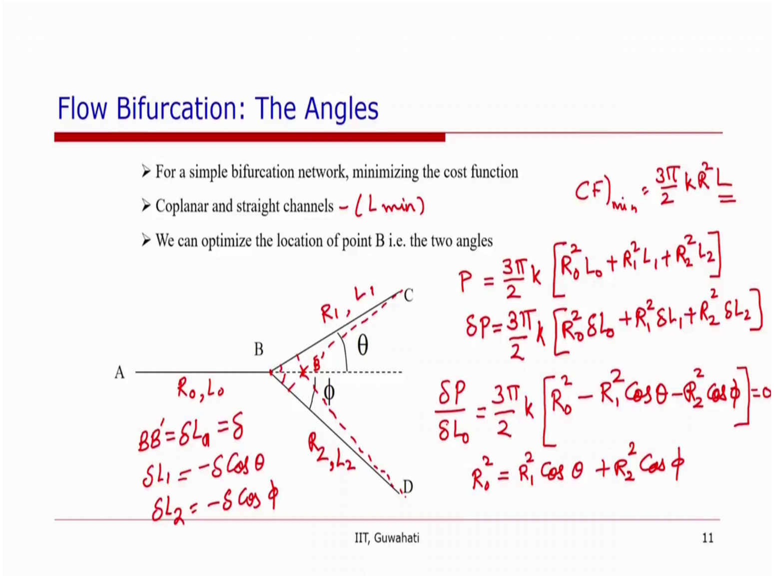

Another corollary to it is the angles. So, if we have this might not be valid for all the cases, because and this that will be the region for that, we may say in a minute that, for a simple bifurcation network the cost function can be minimised and the angle of the network can also be found out, and the radius based on the radius. So, if we have the freedom of deciding the plane, in which the bifurcation is going to happen, then we will always like to have the channels to be coplanar and straight.

So, that the cost functions, if you remember the cost function minimum for a channel is, 3 pi by 2 K R square L. So, L will be minimum if the channels are coplanar and channels are straight. So, one would like to have that, now if a bifurcation is like this, where we can optimise what should be the location of point V and correspondingly what should be the lengths of A B B C and C D ah. So, that can be done if we write the cost function P, let us say the P is the total cost function.

So, that will be 3 pi by 2 K R, let us say A B has radius of R 0 B C has a radius of R 1 and B D has a radius of R 2 and similarly lengths L 0 L 1 and L 2. So, we can write this is equal to R 0 square L 0 plus R one square L 1 plus R 2 square L 2. Now, if we want to find out delta P, that will be equal to 3 pi by 2 K R 0 square delta L 0 plus R 1 square delta L 1 plus R 2 square delta L 2. Now let us assume that, there is a small change in L 1 and the point B is brought to point B dash as marked here. So, then the lengths will change accordingly and B D dash will be delta L 1 and let us call this delta.

So, this distance if this angle is theta and this angle is phi then, we can say that delta L 2 will be delta cos theta, but it is going to reduce. So, we can put a minus sign there. Similarly, delta see we sorry we should say this is delta L 0, because this is change in length of A B delta L 1 and similarly delta L 2 is going to be minus delta cos phi. So, if we substitute these here, we will get delta P by delta L is equal to 3 pi by 2 K R 0 square and we have taken this is delta L 0.

So, delta L 0 minus R 1 square cos theta minus R 2 square cos phi and for minimisation this should be equal to 0. So, the relationship will be getting that R 0 square is equal to R 1 square cos theta plus R 2 square cos phi. We can do the similar exercise, for if there is a small change in made in L 1 and what are the corresponding changes in L 0 and L 2? So, we can get what is delta P over delta 1? What is the minimum consistency?; And similarly, for L 2.

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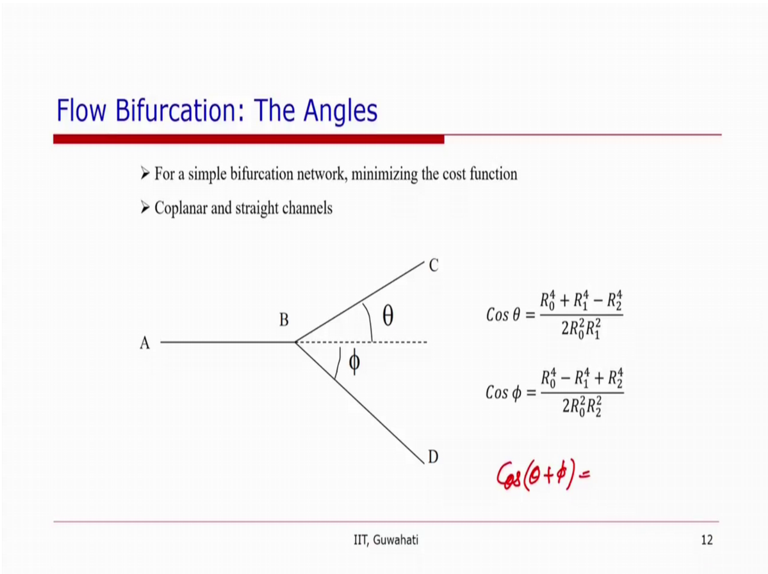

So, based on that we can get the relationship, for cos theta cos phi and another third relationship, for cos theta plus phi and this will be in terms of R 0 R 1 and R 2. So, we can get the optimum angles for a channel network, for different channel radius.

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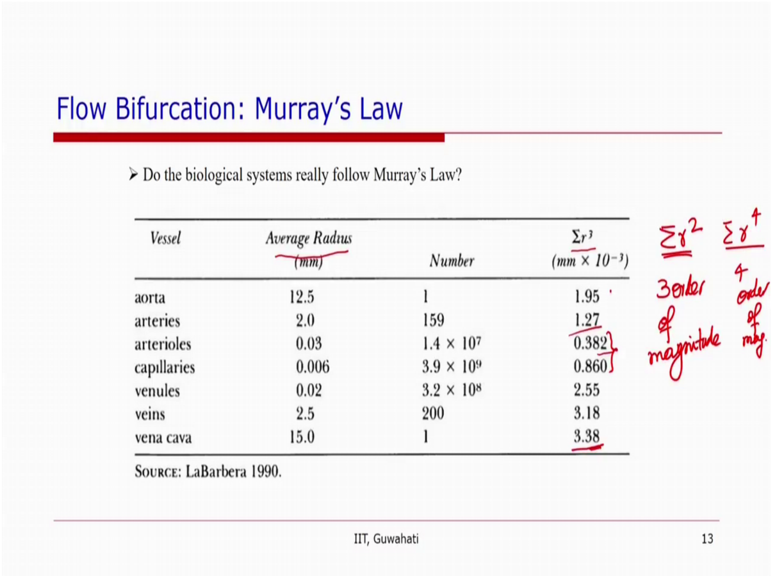

Vessel	Average Radius (mm)	Number	Σr^3 ($mm \times 10^{-3}$)
aorta	12.5	1	1.95
arteries	2.0	159	1.27
arterioles	0.03	1.4×10^7	0.382
capillaries	0.006	3.9×10^9	0.860
venules	0.02	3.2×10^8	2.55
veins	2.5	200	3.18
vena cava	15.0	1	3.38

SOURCE: LaBarbera 1990.

So, have you looked at this design principle of Murray, this looks very rosy, but is it really true? The question that one would be asking that, the nature has such complex network of channels and do any of these network of channels do they follow, such a simple law and many people have looked at the looked and they have tried to verify the establish the veracity of the Murrays law.

So, one such gentleman was LaBarbera, who looked at the average radius for different vessels and the number of vessels at different labels. So, aorta the average radius and arteries average radius and the number of arteries and so on and the last column shows, the sigma R cube. So, at each generation the sum of the cubes of the radii, multiplied by numbers and this turned out that these are the values and, they vary from minimum value is 0.382 and maximum is 3.38. So, the very they vary about one order of magnitude, if Murrays law is followed then they all should be same.

Now, if we discard these 2, which are arterioles and capillaries where the velocity profile is not necessarily parabolic, then they are of the same order, the value vary from 1.27 to 3.38 ok. Again, one might think that there is quite significant variation, but consider this if one does the summation of sigma R square then 1 find that 3 order of magnitude variation and similarly, sigma R to the power 4, 4 order of magnitude.

So, when you compare with this for such a complex system of network or the Murrays law gives, quite good validity or it appears that, the cardiovascular system do really follow the Murrays law. The author LaBarberas, they have done experiments on rats and some other animals and where the average where they have not taken the average radii, but the actual radii and they found that, if they do not take the average radii if they take the actual radii of all the vessels, then this number is about exact the exponent is about 3. So, when they have done the exact experiments or when the exact radius has been taken, for the rats then they found this to be even more accurate.

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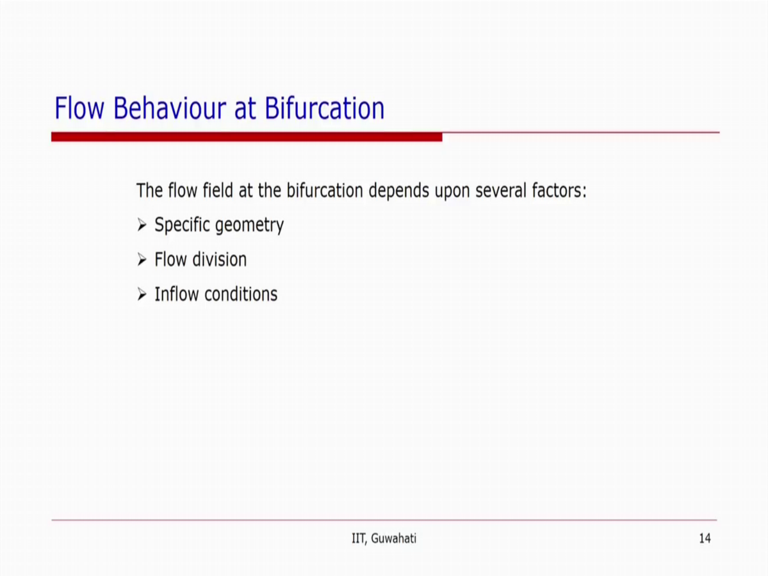

So, now let us look at a bit about the general flow behaviour, when we look at the bifurcations, when we zoom into the bifurcations and try to understand the flow field there; but unfortunately because the geometry of the channels or geometry of the different arteries at different bifurcations are very different. The radii of the mother and daughter tubes will be different, the angles will be different the planes in which these different, channels mother and daughter tubes are there they are different. The flow division, how much flow is going to the one channel?

How much flow is going to another channel? Or the downstream boundary conditions, they will be different and the inflow conditions, it is pulsatile or steady state and what is the flow rate? They all will be different. So, they to a large extent they determine, how the flow field is going to look like at the bifurcations? However, some general flow features can be or has been identified at bifurcations.

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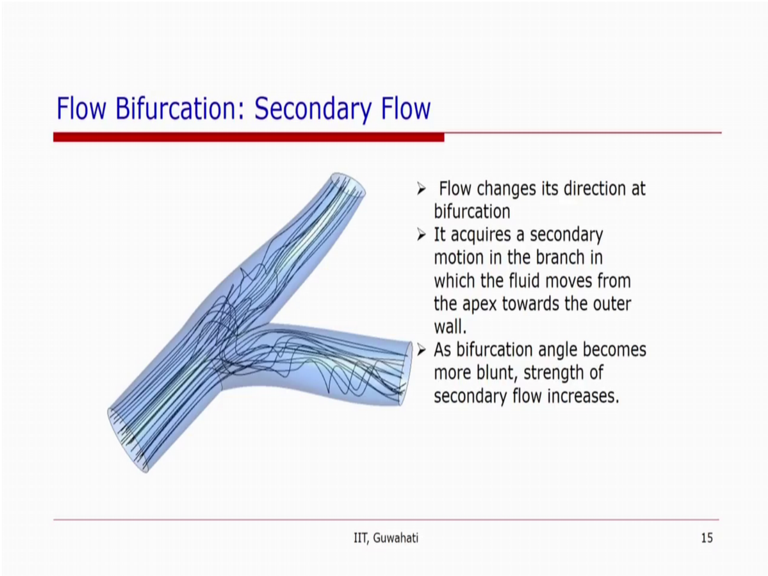

So, one such thing is, you might remember or if you have heard about dean flow, that in a curved channel say for example, aorta because of the geometry of the channel, flow turns and when the flow turns there is a secondary flow. So, if you look at normal in a direction normal to the flow shape, my hand is in direction normal to the flow you will get some velocity vectors, and which is termed as secondary flow or after dean, who look after the who looked at this phenomena, in characterised this phenomena this flow is also called dean flow.

So, in bifurcations the flow essentially turns it is direction, there is always some amount of curvature at the bifurcation as you might see here. So, the flow changes it direction at the bifurcation, as a result it also has secondary motion, just downstream of the bifurcation, as you can see by the stream lines here, that there is a helical nature of the flow, which will die down downstream of the channel, if the channel is sufficiently long.

So, it is important that if you study flow in bifurcations, because this flow is 3 dimensionally it is important to model this flow. If you are doing a C F D analysis it is important to model full 3-dimensional flow at the bifurcations, otherwise you are going to miss very important flow features and it will be a cross assumption, if you do a 2-dimensional flow. The second observation or the second result that has come from the literature that, as bifurcation angle becomes more blunt, it becomes say it move towards tease and some then the strength of this secondary flow increases and that comes, by the dean number as the radius of curvature increases the strength of this secondary flow is going to increase.

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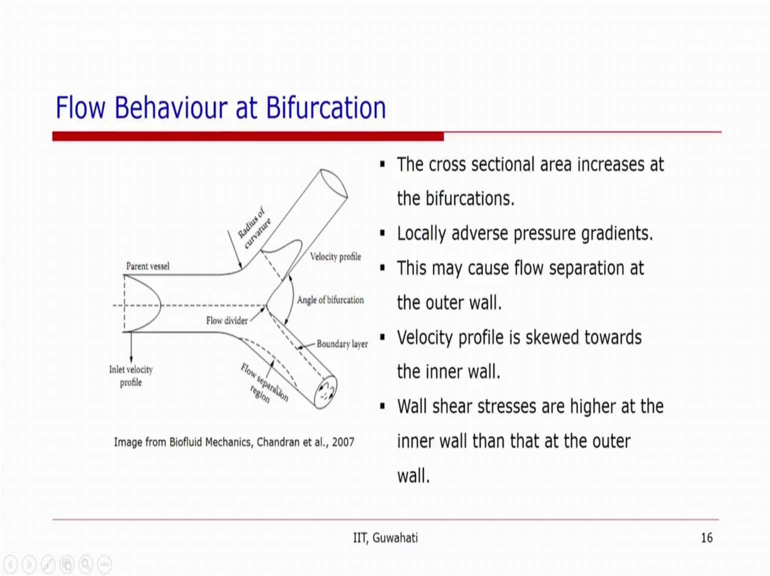

Now, as we have seen just few slides before, that when the bifurcation happens the flow area is going to increase. So, when the cross-sectional area will increase; that means, there will be less or the flow velocity will decrease or there will be locally adverse pressure gradients.

So, when there is adverse pressure gradient, then flow separation might happen and this generally happens at the outer walls. So, what this may happen at near the outer wall. So, you might see a typical velocity profile, at a bifurcation that near the or towards the inner wall, near the apex or near the flow divider the velocity profile will be skewed, towards the inner wall, or the maximum velocity will shift towards the inner wall and you will see a flow separation region towards the outer wall.

Consequently, the wall shear stress will be higher at the inner wall, and lower at the outer wall. So, this has implications for the plate formation, when the shear stress is lower, than the chances of formation of plates are higher, at the outer walls because the shearing at that place is lower.

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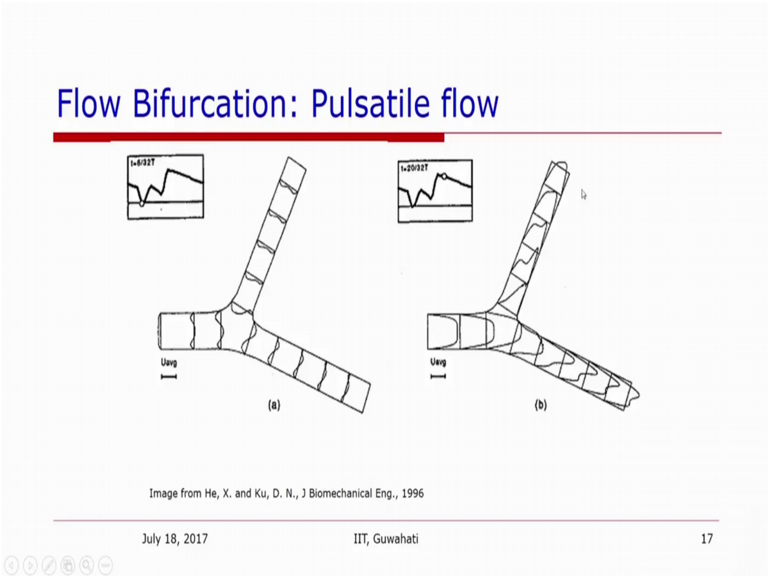

Another characteristic of the flow in circulatory system is that, it is pulsatile. So, if we look at the flow at different time instants. So, for example, here the velocity profiles have been plotted for the smallest flow rate, and just after the diastole. So, the velocity profiles are very different. So, at the same bifurcation the flow profile may look very different, but in both the cases you might look at that the velocity profile is skewed. The velocity is maximum or velocity is skewed towards the inner wall and you might the flow separation near the outer wall.

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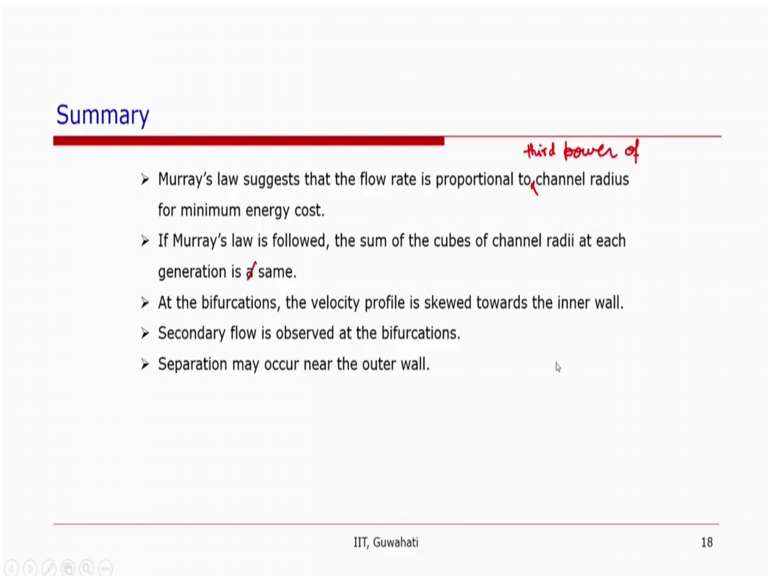

So, in summary what we have learned about flow bifurcation is, Murrays law that suggest that the nature has designed the bifurcation systems, such that the energy for the operation maintenance and generation is minimum, and this gives us that the flow rate is proportional to the cube read cube root or sorry this is wrong. So, you can write proportional to third power of channel radius for minimum energy cost.

If Murray law is followed then sum of the cube of channel radii at each generation is same, at the bifurcations the velocity profile is skewed towards the inner wall. So, the maximum velocity is towards the inner wall, the secondary flow which is that the helical nature of the flow is observed at the bifurcations, because of the curvature there and the separation may also occur near the outer wall.

So, this is all for today, in the next lecture we will look at the Wimberley solution which is for the pulsatile flow in a channel.

Thank you.