Cardiovascular Fluid Mechanics

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Lecture – 06
Blood Flow in a Channel

So, in this lecture we will look at flow of blood in a channel or flow of fluid in a channel. As we know that in our cardiovascular system. There are number of channels of different diameters. So, it is important to understand flow in a channel problem. So, in this lecture what we will look at is fully developed flow of a Newtonian fluid and fully developed flow of a Casson fluid, which can represent or which can model the rheological behaviour, or the viscosity, or the viscous behaviour of the blood in a channel.

So, you must have studied in your basic fluid mechanics course that the flow of fluid can be modelled by the conservation equations and if we want to model the full three dimensional flow of a fluid, we can represent these conservation equations in the differential form and these are the conservation of mass and momentum, if the fluid properties are dependent on the temperature then energy equation also needs to be solved, but since in our case the flow is isothermal and the fluid is incompressible, that is blood is incompressible.

So, the properties are independent of temperature. So, we just need to solve two conservation equation, conservation of mass and conservation of momentum. So, let us look at the general form of these conservation equations first.

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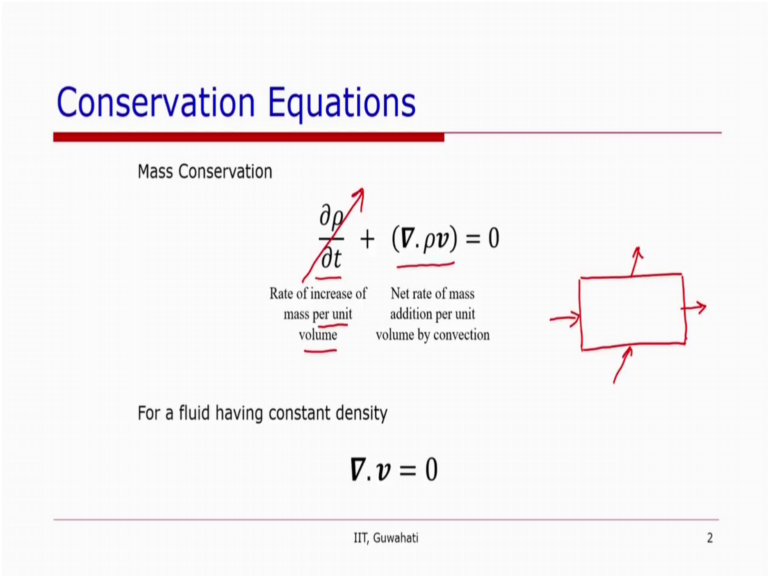

The mass conservation equation is written in this form. The first term in this equation is rate of increase of mass per unit volume, the first term and the second term is the mass that is being convicted or need net rate of mass addition per unit volume by convection. So, if you consider a control volume which might be a part of the channel or a channel. So, the first part represents, you must see here that these equations are in terms of per unit volume.

So, that is why it is rate of increase of mass per unit volume, which will be 0 in our case, because the density of the blood is a constant and the second term, it represents the net rate of mass addition that is provided by the convection. So, that is flow coming in and the flow going out from the boundaries of the domain. So, that is mass conservation equation and for an incompressible fluid which has a constant density this equation this term becomes; so the first term becomes 0. So, this equation can be given by divergence of velocity, vector is 0.

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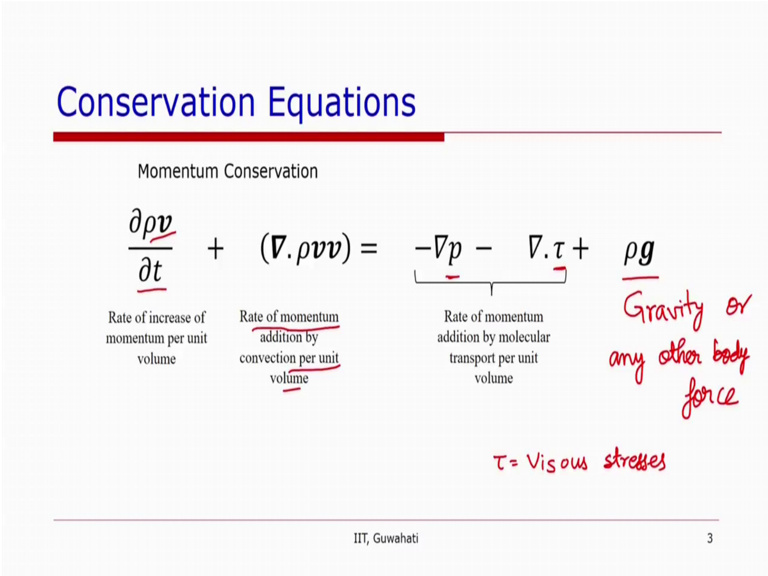

The second conservation that we consider is momentum conservation and in the momentum conservation, the general momentum conservation equation looks like this. The first term represents the rate of increase of momentum per unit volume. So, again this is in terms of per unit volume, rho v is the momentum per unit volume. So, rate of increase of momentum per unit volume, first term. The second term is rate of momentum addition by convection per unit volume here, the first term on the right hand side is the rate of momentum addition by molecular transport per unit volume and the last term is body forces of the gravitational force or any other force. So, this can be gravitational force or any other body force.

This term includes the molecular transport per unit volume, where p is the thermodynamic pressure which is normal stress and t is the tau is the viscous stresses, it can have it, will have normal as well as shear stress components.

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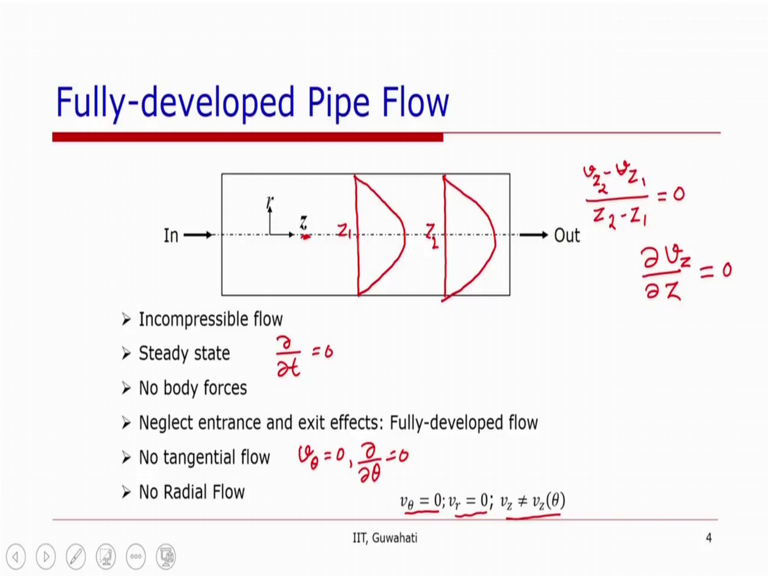

So, for a fully developed pipe flow, let us consider a circular cylinder, a cylinder which has circular cross section and flow has happening in the axial direction, which we represent a z direction here and the radial coordinate is r and angular coordinate is theta and, because there is no flow in the angular direction so, that velocity will be 0. So, let us try to understand the assumptions that will be involved in it, the flow is incompressible so; that means, the density is constant t z steady state so; that means, the del by del t term will be 0 things, do not change with time.

We do not consider any body forces including gravity, in this analysis and we also neglect the entrance and exit effects which means the flow is fully developed. So, what fully developed flow means that with increase in distance or with distance the velocity profile do not change; so, if I plot velocity profile at z is equal to z 1 and if I plot a velocity profile at z is equal to z 2, it remains unchanged. So, what does it essentially means that z velocity at z 1 and velocity at z 2, they are same. So, if we take a first approximation to the gradient, then this will be 0 so; that means, del v over del v z over del z is equal to 0 for a fully developed flow.

There is no tangential flow; that means, there is no flow in the angular direction and there is no radial flow, as you can see from here, that if there is flow in the radial direction then the velocity profile in the z direction will not remain same. So, v theta tangential flow is 0 so; that means, v theta is equal to 0, there is no radial flow, so; that means, v r is equal to 0 and the gradients in theta direction are also 0, so; that means, v z or z component of velocity or the axial component of velocity is independent of theta. So, v theta is 0 and del by del theta is also 0 for the tangent, the no tangential flow.

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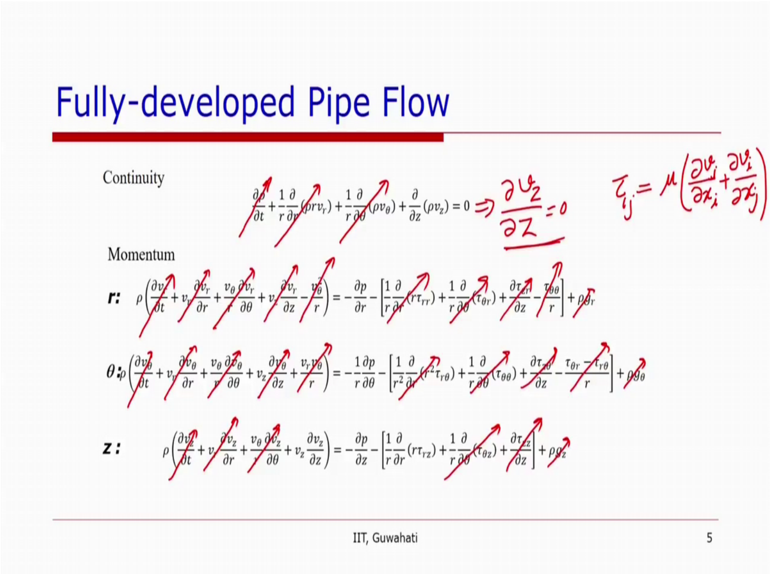

So, let us look at the conservation equation again in the expanded form in cylindrical coordinate. So, what we see here is the continuity and three components of momentum equation r theta and z coordinates and let us try to see different terms in these equations, because flow is a the steady state. So, all the transient term will go to 0 v r and v theta r 0. So, all the term having v r and v theta will go to 0 we neglect body forces now the shear stress is a function of velocity gradient.

So, if v z is non zero, this term gives us that del v z over del z is equal to 0 from here we can see that all the stresses except tau or tau r z will be 0, because tau r z have v z over v r all other stresses will be 0 in this case. So, tau r r is 0 tau theta r is 0 tau theta. Theta is 0 tau r, theta is 0, this is 0, this is 0, this is also 0 tau z, z is 0, because it will have v z over del z. So, this we have seen then this is also 0.

So, this component is also 0 tau theta z is 0, because del by del theta is 0. So, only 2 components remaining again this will be again 0, because tau z over del z and the shear stress will not vary, because the flow is fully developed. So, shear stress will not vary in the axial direction. So, this is also 0.

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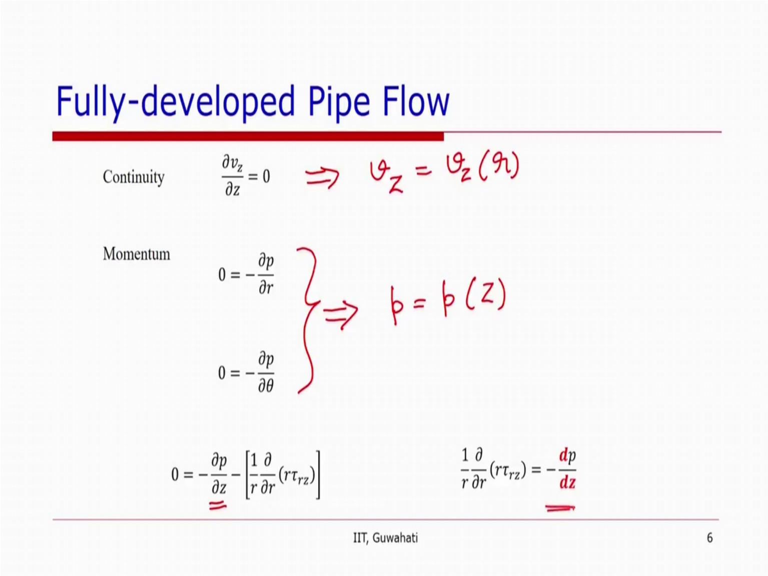

So, if we write down the non 0 terms in all the equations, we will end up this, with this we have already said that if the flow is fully developed then del v z over del z is equal to 0, so; that means, v z is not a function of z. So, from our two assumptions that says that v z is a function of r only, because it was not a function of theta and it is not a function of z.

From these two equations, we see that p is not a function, is constant with respect to r. It does not change along the radial coordinate and p does not change along the theta coordinate. So, that again means that p is a function of p z. So, this partial derivative, we can change it to tangents, total derivative. You can see here in this equation, this has been rearranged. The second term has been brought towards left hand side and we see that one over r del, over del r r tau r z is equal to minus d p by d z.

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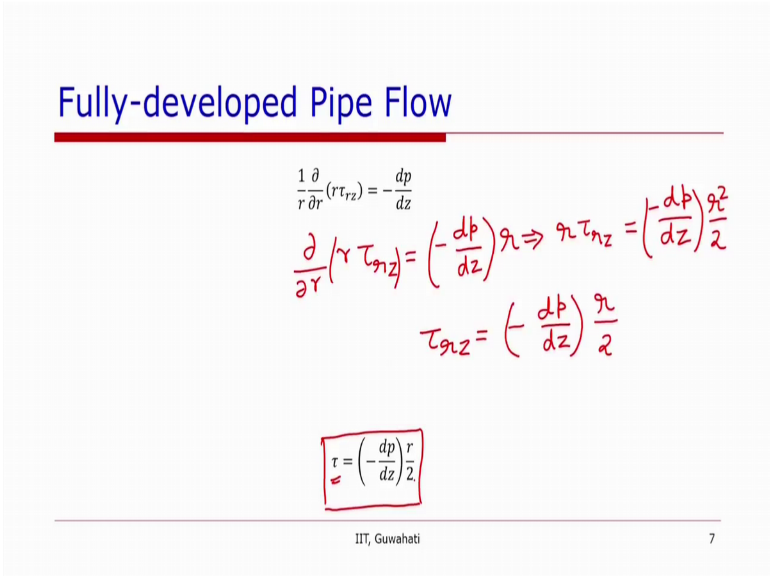

So, let us try to integrate this and if you integrate this, what we will end up with r tau r z is equal to minus d p by d z del by del r of r tau r z is minus d p by d z into r so; that means, r tau r z is equal to minus d p by d z into r square by 2 or tau r z is equal to minus d p by d z r by 2. So, this is a general equation and we get relationship between the shear stress and pressure gradient and the radius of the channel or radial coordinate.

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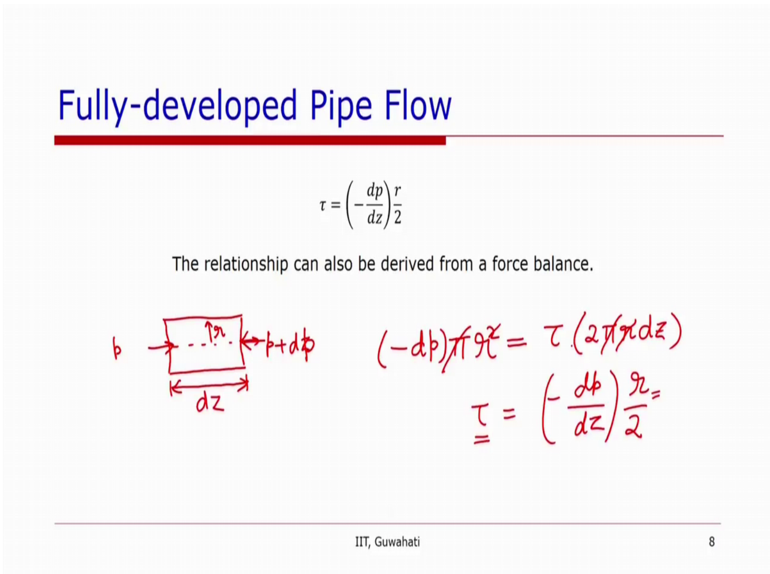

So, this relationship we can also look at a force balance in a channel. So, let us see if you have a fluid element, which has a length of d z and the pressure at the two ends is p and p plus d p the radius of this fluid element, is cylindrical fluid element. The radius is r and if sorry. This is p plus d p and we write the force balance that, this will be acting in this direction in the element.

So, minus d p into pi r square that is equal to tau into 2 pi r d z. So, pi cancels out r cancels out and tau is equal to minus d p by d z r by 2 and this is an important relationship between the shear stress and the radial coordinate and this is independent of the constitutive equation of the fluid. It is true for a Newtonian as well as non Newtonian fluid.

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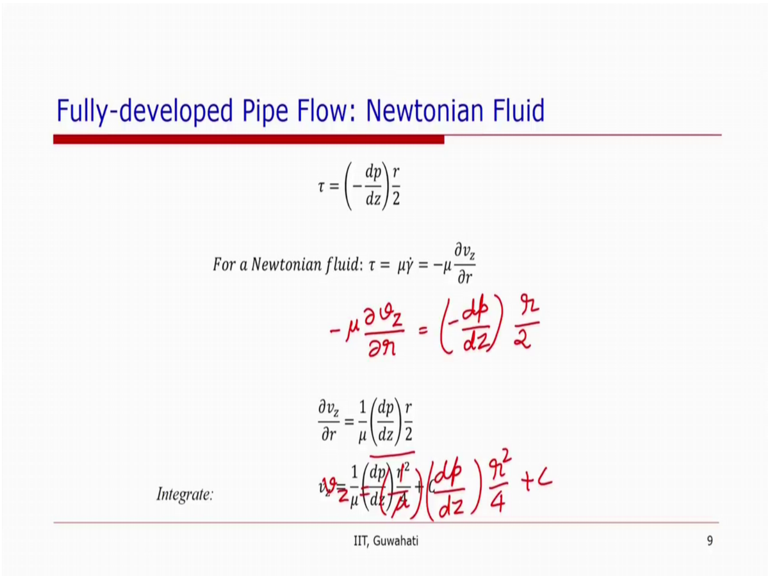

So, let us look at now, for the fully developed flow of a Newtonian fluid and for a Newtonian fluid, the shear stress is proportional to the shear rate. So, that tau is equal to minus mu del v z by v r. So, we substitute this here, we will end up with minus mu del v z by del r is equal to minus d p by d z into r by 2 and if we bring mu on the other side, then it will be 1 over mu d p by d z r by 2. So, let us integrate it and we will get v z is equal to 1 over mu d p by d z r square by 4 plus c, where C is a constant. So, this has already been integrated for us. So, we here, end up with this equation for v z.

So, having obtained the velocity profile in a fully developed pipe flow for a Newtonian fluid now, we can also derive the flow rate in the channels.

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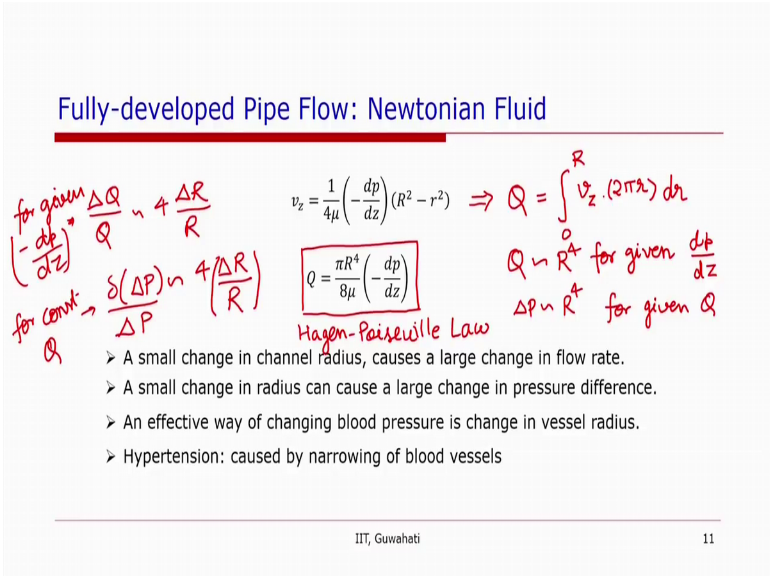

So, flow rate using this profile we can say the volumetric flow rate, which is q is equal to integral 0 to r v z into 2 pi r, which is the area of a differential element that we will take on a cross section into d r. So, after the integration, we will obtain this relationship between flow rate and the pressure gradient. This relationship is known as Hagen Poiseuille law, which says that for constant pressure gradient, Q is proportional to r to the power 4 for given pressure gradient, for given d p by d z q is proportional to r to the power 4 or for given q delta p is proportional to or delta p y l for that matter is proportional to r to the power 4, because r is a, because l for a channel.

If it is a constant, then we can say the pressure drop is proportional to r to the power 4 for given flow rate. Now, this has very interesting implications in blood flow. We can show using this relationship that delta q over q is proportional to 4 delta r over r; that means, a very small change, only 1 percent change in flow rate will cause 4 percent change sorry, 1 percent change in the radius will cause 4 percent change in the flow rate.

Similarly, if the flow rate is constant then very small change in the radius will cause significant change in the pressure drop. So, this is for given pressure drop, whereas, this is for, given of a constant flow rate. So, this is an effective way in the cardiovascular system to change the blood pressure. So, the hypertension is generally caused by narrowing of the blood vessels. Blood vessels become narrow and the pressure increases significantly. Similarly, the muscle tension or the pressure hypertension can be reduced by smoothing the muscles. So, Hagen Poiseuille flow has a number of interesting implications in the cardiovascular system.

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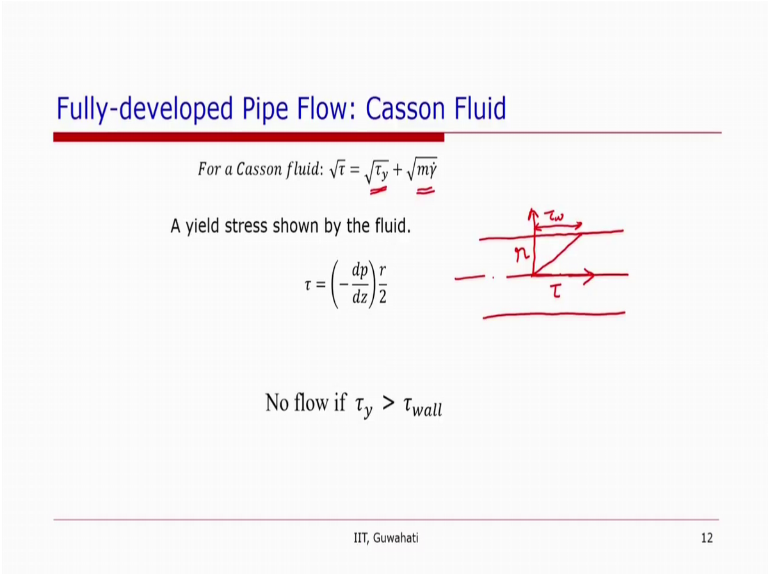

Now, having looked at the flow, in a of flow, of a Newtonian fluid in a circular channel. Now, let us consider the case of a non Newtonian fluid. So, we will consider the flow of a Casson fluid. Now, why Casson fluid? Because the first and foremost reason is, because blood is a non Newtonian fluid, which follows the Casson model or which has shown that Casson model can be used to model the flow of blood accurately, the another reason is, because as you can see in this relationship Casson model includes a yield stress behaviour as well as a power law kind of behaviour.

So, by understanding how we can model flow of a Casson fluid. We can also use the analysis or some parts of the analysis to model of the fluid, which show yield stress behaviour. For example, Bingham plastic fluid or we can model the flow of a power law fluid, using the similar methodology as we will use for Casson fluid. So, the fluid shows a yield stress and as we have seen in the previous slide that for all the fluids tau is equal to pressure gradient into r by 2; that means, tau is proportional to the shear stress.

So, if I plot this on a x y graph let us say on the y axis, I show radius as shown in this figure and tau. So, from this relationship tau will be 0 at the centre and then it will increase and become maximum at the wall. So, let us call this value as tau wall. Now, yield stress means that below yield stress, there will be no flow or the shear rate will be

0, if there is no flow. So, if because as we can see from this plot that the shear stress is maximum at the wall. So, if the value of the yield stress is more than the wall shear stress then there will be no flow in the channel, even if some shear stress less than the yield stress are being applied.

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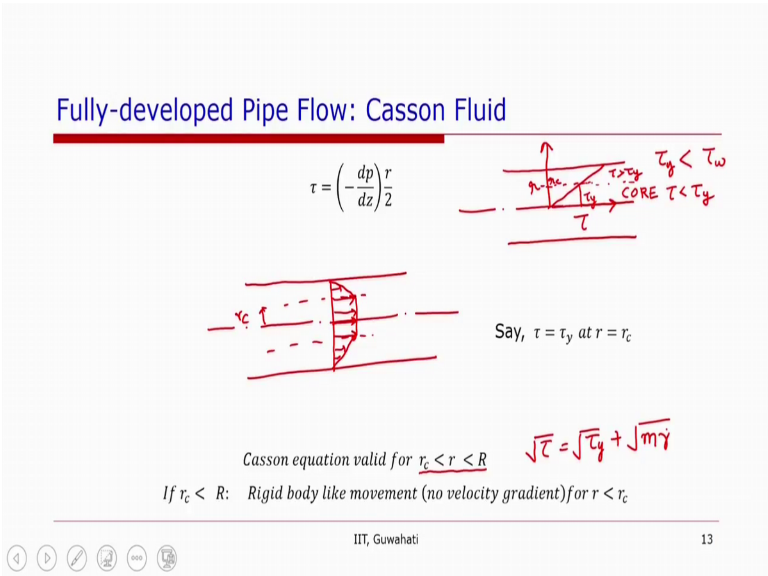

So, now if the yield stress is more than stress on the wall; so, let us draw this plot again, where we plotted yield, the stress verses r curve if tau y, which is yield stress is less than tau wall, then it will fall somewhere in between. Let us say, that at tau is equal to tau y at this point where the shear stress is tau y. Let us say the radius, we call it r c or the core radius, why do we call it core, we will be looking at it in a minute if the radius; that means, that below r c or when the radius is less than r c in that region, the stress is less than. So, we can have two regions, this is core and in this region tau is less than tau y whereas, in this region tau is greater than tau y.

So, when the tau is greater than tau y; that means, when the radius is between r c and r then fluid will follow, root tau is equal to root tau y plus root m gamma dot m gamma, where gamma dot is the shear rate. Now, if the radius is less than r then the shear rate will be 0; however, that does not mean that there will be no flow at the wall. So, let us draw this picture of the channel again and this is r c.

So, we will have a velocity profile something like this, in the region where r is greater than r c. now, at this point at the interface or at r is equal to r c from this equation. We will be able to find out what is the velocity and the velocity in this region, in the core region will be the same. So, the fluid will have two kinds of regions; one is the core region, where the fluid moves like a rigid body, move like a plug or a piston and there will be gradients in the near wall annular region.

So, we can analyse the flow field for a Casson fluid by considering these two regions, separately a similar approach is required while considering the flow of any non Newtonian fluid, which shows the yield stress behaviour. For example, a Bingham plastic fluid or a Herschel Burkley fluid; so, let us consider the region, where r is greater than r c it will follow this law.

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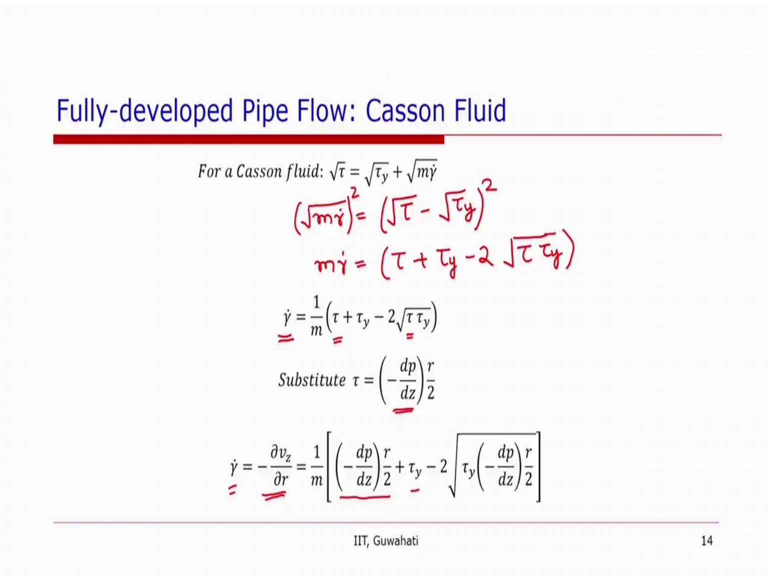

So, if we rearrange this, we will obtain this relationship that is work out the step in between. So, we will have root m gamma dot is equal to root of tau minus root of tau y and if we square on both the sides, then we will get m gamma dot is equal to tau plus tau y minus 2 root tau tau y.

So, we have obtain a relationship between the shear rate and the shear stress, we have shear stress in two terms on the right hand side and one is the shear stress only in the other term has root of tau there. So, now, because our objective is to find out the velocity profile; so, we will substitute the shear stress as a function of radius, which also include d p by d z for all these cases d p by d z is a constant.

So, the pressure is, we are varying linearly or the pressure gradient is a constant. So, if we substitute these here, what we obtain is this? We can substitute that shear rate is equal to minus del v z over del r is equal to one over m as it is in place of tau we substitute minus d p by d z r by 2 plus tau y. We will not touch it as of now, 2 again we substitute the value of shear stress as minus d p by d z r by 2.

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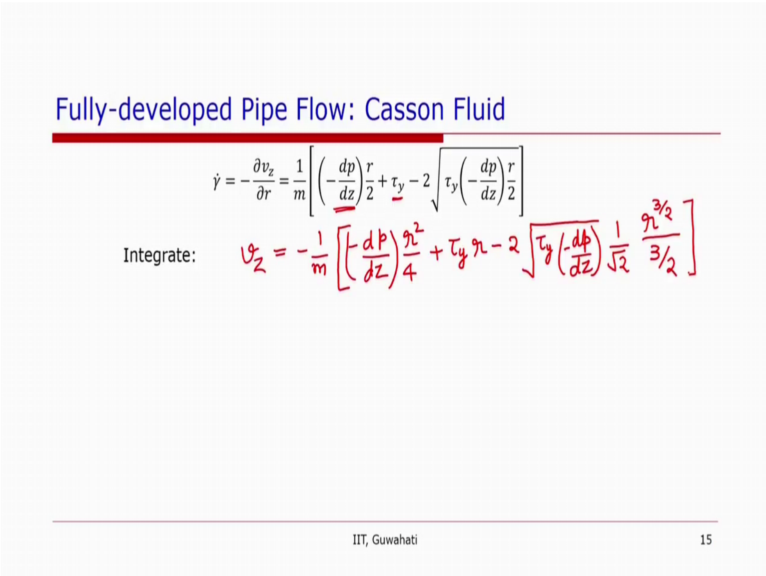

So, let us now integrate this equation, because the right hand side have only r or constants, because yield stress and the pressure gradient both are constant in independent of the radius. So, we can integrate this, to obtain v z is equal to minus 1 over m within bracket minus d p by d z r square by 4 plus tau y r minus 2 root of tau y minus d p by d z. Now, one over root 2 r to the power 3 by 2 divided by 3 by 2. So, if we write this down this is what we get and then integration constant.

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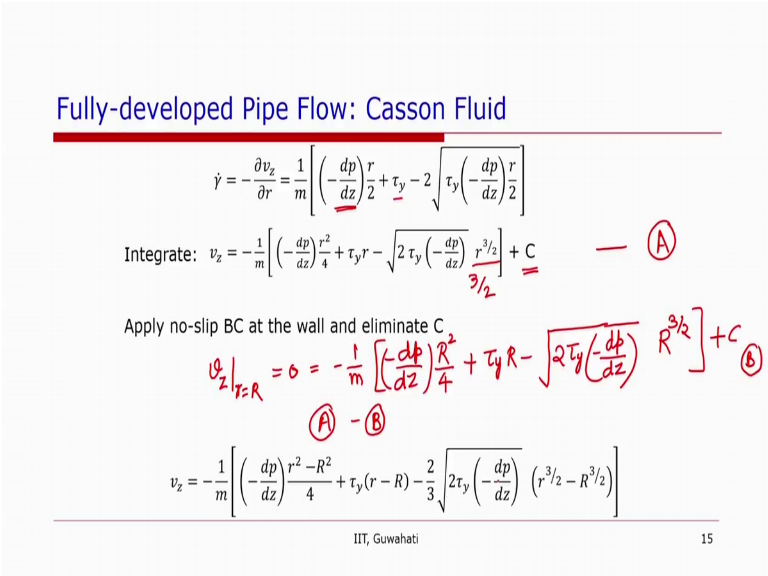

Now, we apply a no slip boundary condition on the wall, so that we can find out this integration constant c. So, we can substitute and get the velocity profile. So, we can write that v z at r is equal to capital R is equal to 0, because of the no slip boundary condition and we substitute r is equal to capital R in the equation. So, minus d p by d z capital R square by 4 plus tau y capital R minus root of 2 tau y minus d p by d z capital R to the power 3 by 2 plus integral constant.

So, if we say this equation as a and this equation as B or we can do to eliminate C is a minus B , if we do that v z minus 0 is v z minus 1 over m and now, we can subtract the terms 1 by 1. So, minus d p by d z minus d p by d z multiplied by r square minus capital R square by 4 plus tau y r minus capital R minus this all root 2 tau y minus d p by g d z r power 3 by 2. What we have missed here is, this is divided by 3 by 2 here. So, this will be 2 by 3 and this all inside.

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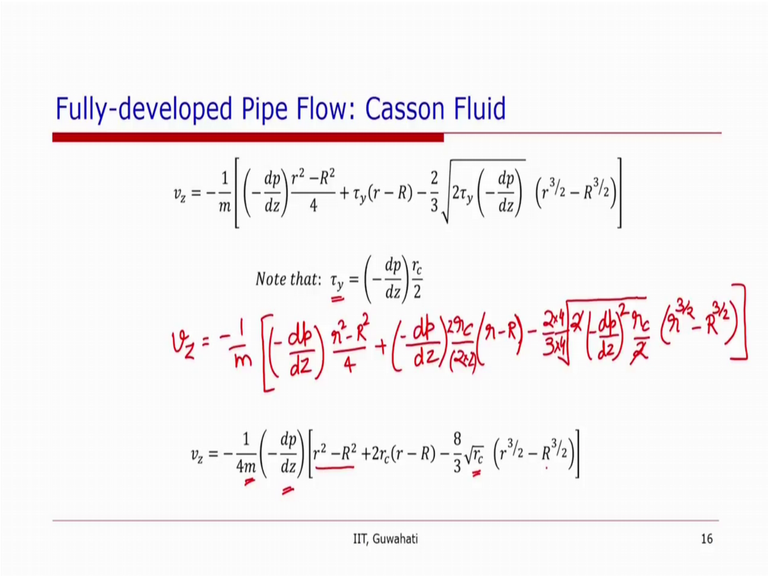

Now, we look at this further these equations. So, v z is equal to the same equation and now, we can substitute this in terms of r c tau y is minus d p by d z r c by 2. So, we substitute and obtain this in terms of r c the radius of the core. So, we will get velocity profile v z is equal to minus 1 over m minus d p by d z and this will be r square minus capital R square by 4 plus minus d p by d z into r c by 2 multiplied by r minus capital R minus 2 by 3 root 2 minus d p by d z, we have 2 d p by d z.

So, its square into r c by 2 here and this is multiplied by r power 3 by 2 minus capital R power 3 by 2. So, if we take 1 by 4 minus d p by d z out from this bracket minus 1 by 4 m minus d p by d z .What we are left with is minus r square minus capital R square here, and this will be, because this is force.

So, we can multiply by 2 and 2 here. So, that will be 2 r c r minus capital R and again, if we multiply this by this 2 will cancel out and you may multiply this by 4 by 4. So, we have 8 by 3 and this 4 will go out and d p by d z has gone out, already what we are left inside the bracket is root r c r to the power 3 2 minus capital R by 3 by 2. So, that is a velocity profile in terms of the m constant d p by d z and r c and r.

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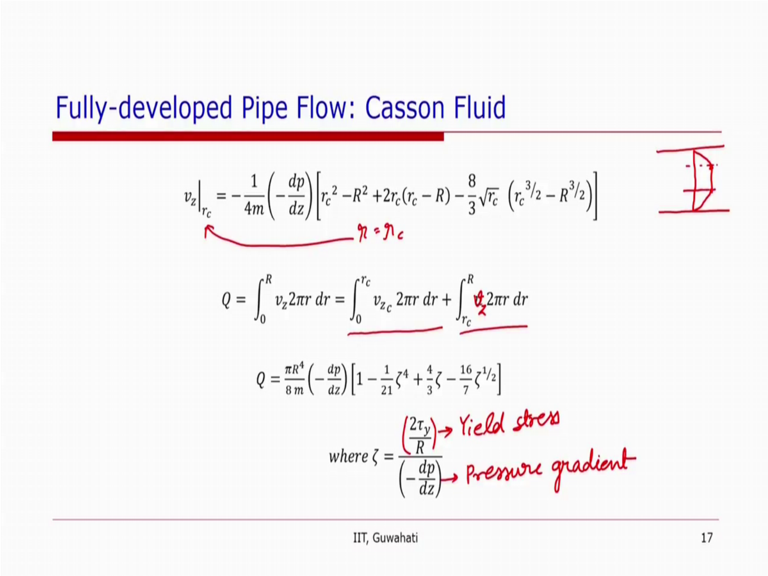

So, having obtained this velocity profile now remember that, the velocities that we obtained r is a general velocity profile and this is only in the annular region. So, if you want to obtain or if you want to obtain a flow rate, we need to obtain the velocity in the entire region. So, at this point where r is equal to r c by continuity of velocity, we can obtain the velocity of the solid core, if we substitute in the velocity profile obtained r is equal to r c that was the primary region, why we were interested to substitute tau y in terms of r c, because we will know tau y anyway, but we have substituted this in terms of r c.

So, that this equation we can also find out from this equation, we also find out the velocity of the core. So, after substituting r is equal to r c small r is equal to r c that is the velocity profile. Now, to obtain flow rate we need to integrate it 0 to r v z 2 pi r d r and because the velocity profile is different in the core region and in the annular region.

So, we can divide into two parts; first we can integrate 0 to r c v z c 2 pi r d r and then we can integrate r c 2 r sorry. This is v z 2 pi r d r. So, after substitution we will get after the integration and the substitution we will get this flow rate, which is pi r to the power 4 by 8 m minus d p by d z into this entire sector, where this factor the s i z i, in this factor is equal to 2 tau y y r divided by minus d p by d z. This you might notice, this is a yield stress and this is the pressure gradient.

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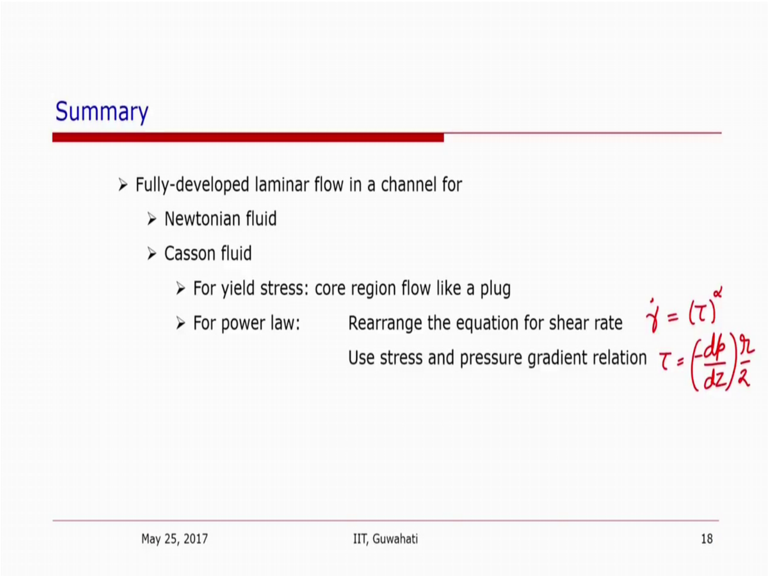

So, in this lecture what we have looked at is flow fully developed flow in a channel for the laminar flow of a Newtonian fluid and of a Casson fluid. In the Casson fluid, there are two things that we should remember that, if the there is an yield stress, then we need to think about that the fluid in the channel will have two region in one region, where the shear rate will be varying across the cross section whereas, in the centre the fluid will flow as a core.

Now, the another lesson for power law of fluids is that we can rearrange the equation as gamma dot, is equal to shear rate to the power alpha, where alpha can be any number and once we have rearrange this, what we need to know is, we can substitute tau is equal to minus d p by d z r by 2 and when we substitute this for constant pressure gradient, we can get the velocity gradient in terms of radius and then we integrate it to obtain the velocity itself.

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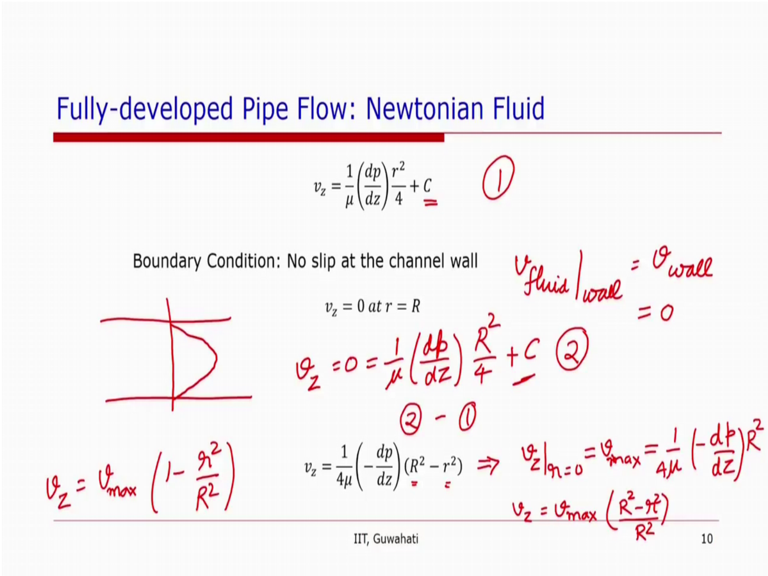

And now, our task is to find this constant in the integration and this can be found by the use of a boundary condition, the no slip boundary condition at the channel wall, which essentially say that when a liquid is in contact with a solid wall then there is no slip between the fluid molecule, that is in contact with the wall. So, v fluid at the wall is equal to v wall, whatever the velocity of the solid wall will be the velocity of the fluid that is in contact with the wall and in this case, because the pipe is a stationary. So, this is 0. So, we will use this boundary condition that at r is equal to capital R, where r is the channel radius the velocity v z is 0.

So, if we substitute that then we will end up with v z is equal to 0 at 1 over mu d p by d z capital R square by 4 plus c and if we subtract then we will get 1 over mu minus d p by d z capital R square minus small r square. So, you might notice that if you substitute from this equation. Let us say, this is equation 1 and equation 2 and if you subtract 2 minus 1 you will get this.

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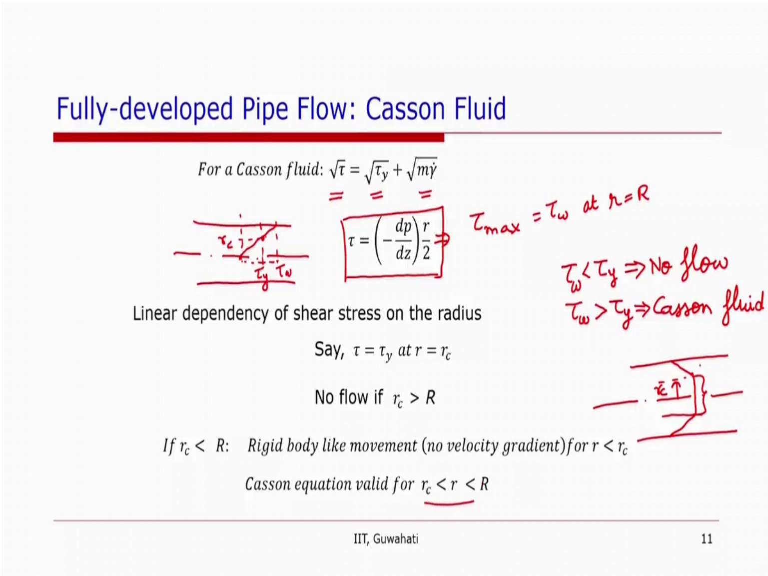

So, what we have done here, we have obtained the velocity profile for the Newtonian fluid in a channel. Now, if I want to obtain this in terms of v maximum, then I can say that at v z at r is equal to 0 that will be v maximum. So, v maximum is equal to 1 over mu 1 over 4 mu minus d p by d z. So, we can write v z is equal to v max into capital R square minus small sorry, this is r square. So, if we divide v by v max and we will get this is equal to divided by r square.

Now, v z is equal to; so, what we will get? V z is equal to v max into 1 minus small r square by capital R square. So, this is a parabolic velocity profile in a channel, which is also known as Hagen Poiseuille flow, you can also calculate the flow rate and average velocity for this pipe flow. Now, we will look at the flow of a Casson fluid. For a Casson fluid, you might remember from the previous lecture that the constitutive equation, that is the relationship between shear stress and a strain rate is given by this formula, that is this square root of shear stress is equal to square root of yield stress plus square root of m gamma dot where gamma dot is shear rate and m is a consistency index.

So, again for this fluid also the relationship between tau and r is a still valid and this shows that tau is proportional to r by 2, for this fluid Casson fluid, shows yield stress behaviour; that means, if the applied stress tau is less than tau y; that means, flow is there is no gradient right. So, from this what we can see or what we can observe that tau in a channel will be maximum at the wall at r is equal to capital R, if stress shear stress at the wall, is less than the yield stress; that means, there is no flow in channel; however, if shear stress is greater than the yield stress, then it will follow Casson fluid flow or it will follow the Casson fluid model. So, from this equation, it is clear that the shear stress will in a channel, the shear stress will be 0 at the centre and it will vary to a value tau w here. Now, somewhere in between, this will be equal to tau y.

Let us say this radius is r c so; that means, that in the channel the condition will be such that there is yield stress, the stress is less than yield stress in the centre. So, there will be no gradients or 0 gradients there whereas, near the wall above r c, there will be gradients. So, this core region will flow as a rigid body and the gradients will be present in the annular kind of region there. So, for r c is rigid body like movement will be in the core and the Casson equation will be valid in this region.

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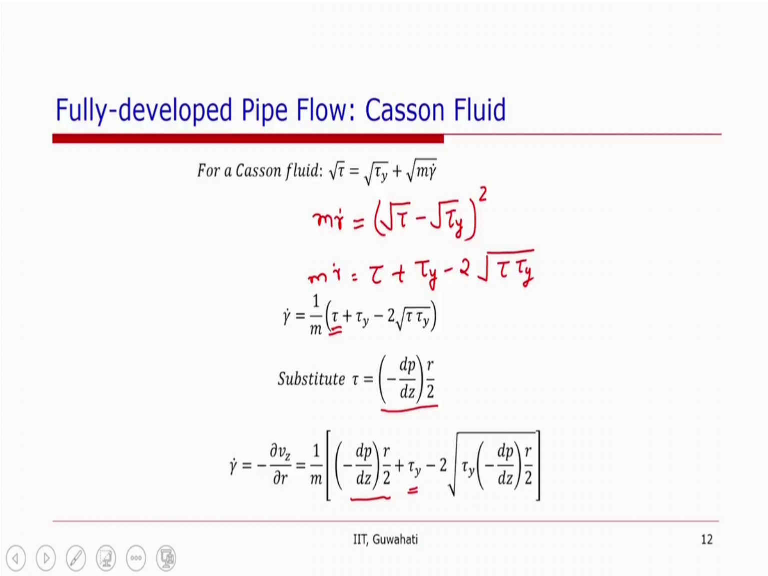

Let us now look at this equation. So, again write down this Casson equation and let us rearrange this. So, if we want to write down m gamma dot, this will be equal to root of tau minus root of tau y square; that means, m gamma dot is equal to tau plus tau y minus 2 root tau y. Now, we substitute tau is equal to minus d p by d z r by 2 in this and what we will end up, with this equation that shear rate minus d v z by d v r, which is equal to to 1 over m tau is replaced by this value minus d p by d z r by 2 plus tau y which is yield stress minus 2 root of tau y into again, we have substituted tau here.

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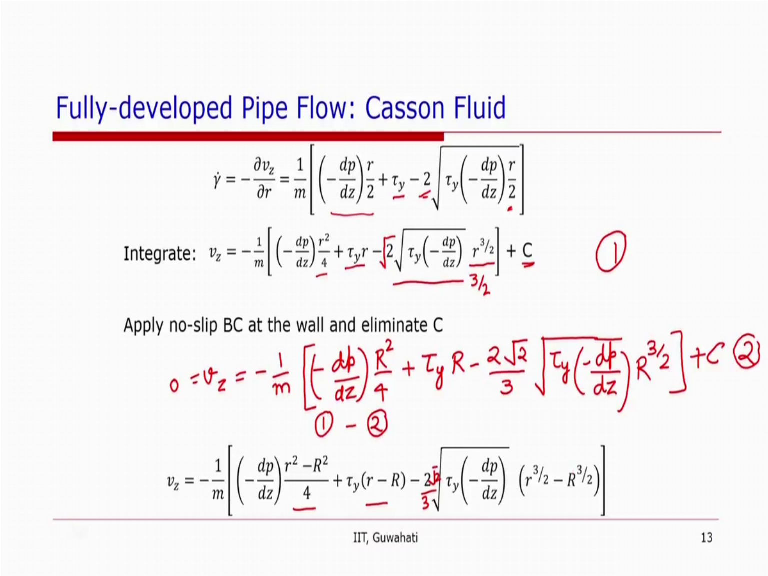

Further, let us move on to take this equation and we integrate and get the velocity from this. So, if we take minus sign on the right hand side it becomes v z is equal to minus 1 over m, the first term minus d p by d z r square by 4, the second term tau y is a constant. So, tau y r minus 2 root of tau y minus d p by d z is a constant 2 by 2 is become a root 2 here and there is r. So, this is r to the power 3 by 2. So, that will be divided by 3 by 2.

So, now we apply there is again an integration constant here. So, now, we apply no slip boundary condition at the wall and we can eliminate c. So, if we do that then we will end up with v z is equal to minus 1 over m minus d p by d z r square. So, in this place now, we are writing it for the walls. So, capital R square by 4 plus tau y capital R minus 2 root 2 by 3 root of tau y minus d p by d z r to the power 3 by 2 plus C and we subtract equation 2 from equation 1.

So, this will be 0 at r is equal to capital R. So, we get from 1 minus 2 v z is equal to minus 1 over m minus d p by d z r square minus capital R square by 4 tau y r minus capital R and this term sorry, for this mistake.

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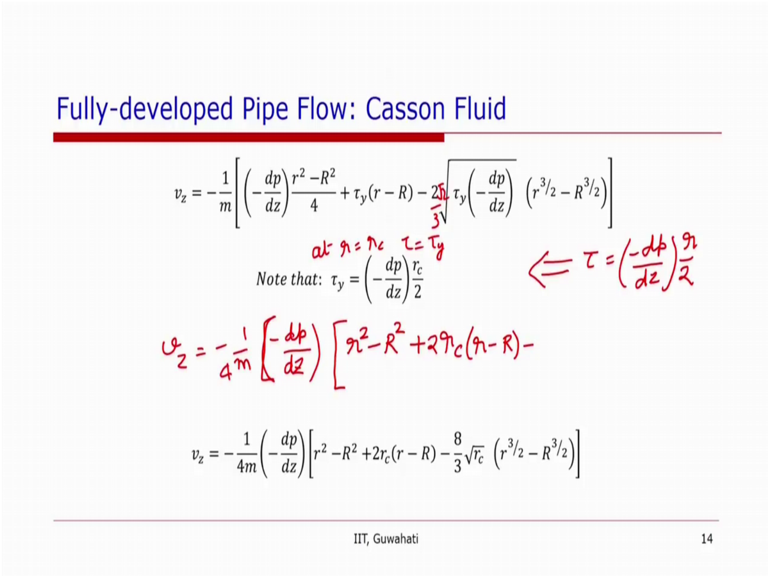

So, 2 2 by 3 tau y minus d p by d z r to the power 3 by 2 minus capital R to the power 3 by 2 and now bring this here again and in this, that is noticed that tau is proportional to or tau is equal to minus d p by d z at r by 2 and if we do that at r is equal to r c, where tau is equal to tau y. So, we substitute this, in this equation then we get this value that tau y is equal to minus d p by d z r c by 2 and we get this velocity profile.

So, after substituting we will get v z is equal to minus 1 over m, we can take minus d p by d z out of this bracket 4 also can be taken out. So, this will be small r square minus capital R square plus tau y we replace by minus d p by d z. So, minus d p by d z has gone out, there is a 2 there and we have taken 4 out. So, we will multiply by 2 here, 2 r c small r minus capital R minus tau y, is equal to minus d p by d z. So, minus d p by d z has gone out and within bracket what remains is r c 2 and 2 is cancel out.