Cardiovascular Fluid Mechanics
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Lecture – 03
Solid Mechanics: A Review

Hello. So, in this lecture we will review some of the basic concepts of solid mechanics, that you would have learned in your first year of a undergraduate engineering course. So, even though this course is on fluid mechanics as you can see here; however, when we talk about flow in the cardiovascular system, the pipes or the tubes or the channels the arteries the veins in which the flow happen, they are not the rigid tube as we encountered in our day to day life, or in the industrial applications they are rather a flexible tubes..

So, in this flexible tube the flow happens and the stresses that are applied on the channel wall, because of that the tube is deformed or the channel wall have deformation and because of those deformation the shape of the channel is changed, consequently the flow behaviour the velocities the pressure inside the channel will change. So, it is important to understand the stress stand relationship in the solid walls. So, we will briefly review the basic concepts which are relevant to cardio of vascular, cardiovascular fluid mechanics in this lecture.

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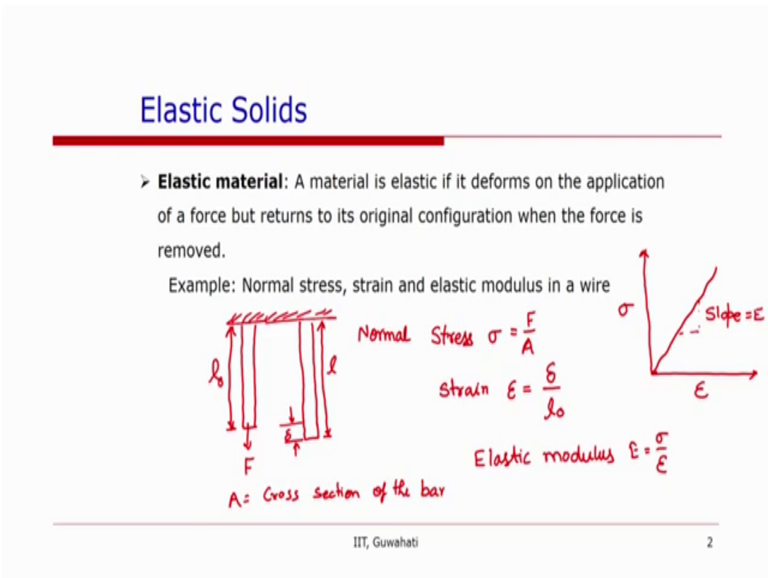

So, just briefly a let us look at solids which are elastic. So, the elastic solids or elastic materials are those, when a force is applied they deform, but after the force is removed the material comes back to it is original configuration, original position then such materials are called elastic materials. So, for example, if we have a elastic bar suspended from a surface, let us say this bar is of length l, and it is being pulled by a force say F, then as a result of this the bar has the length of the bar initially was l naught now it has become l, the change in the length of the bar is delta and A is cross section of the bar..

So, as a result of this there will be the stress which will be acting in the normal direction. So, we can say the normal stress, and this bar is sigma is equal to F over A where A is the area of cross section and strain epsilon is equal to delta over l naught. So, the elastic modulus E is defined as sigma over epsilon, for a elastic material you might remember that the relationship between stress and strain sigma and epsilon is a linear relationship. And so, from this we can say that the slope is equal to the elastic modulus E, the assumption that we had here that the material is homogeneous.

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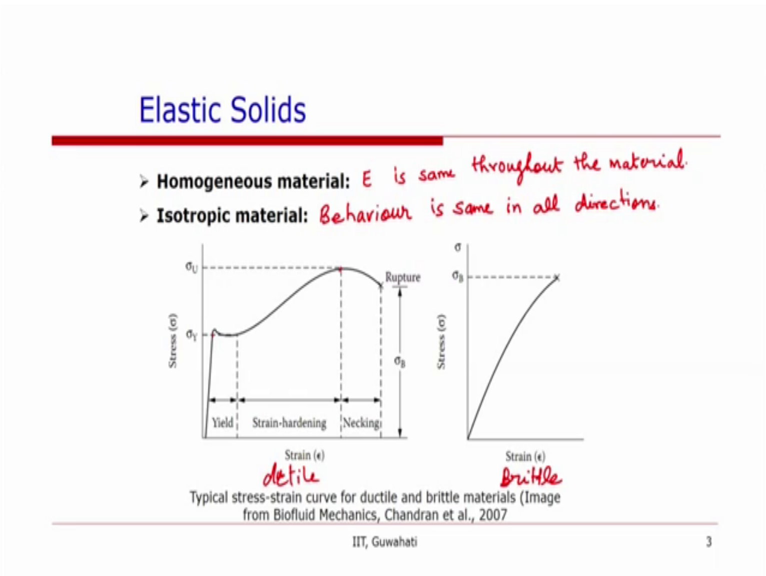

So, homogeneous material means that E is same throughout the bar, or throughout the material that we are considering isotropic material mean that the behaviour or the elastic behaviour is same in all directions, that is if the stress strain relationship is same if we apply is stress in the x direction or we apply a stress in the y direction or the (Refer Time: 06:24) z direction.

So, most of the engineering material behaviour can be classified in these 2 different material behaviour, ductile behaviour as we can see here this is for the ductile material, and this graph is for the brittle material so, in the brittle material the relationship is almost linear and then breaks off, wherever it goes through the first to heal distress and then a at a certain value of the highest value the nuking a cursor, and then material ruptures and the nuking the area changes significantly. So, these materials behave as an elastic material up to certain limit, which is called deal stress and after that their behaviour changes.

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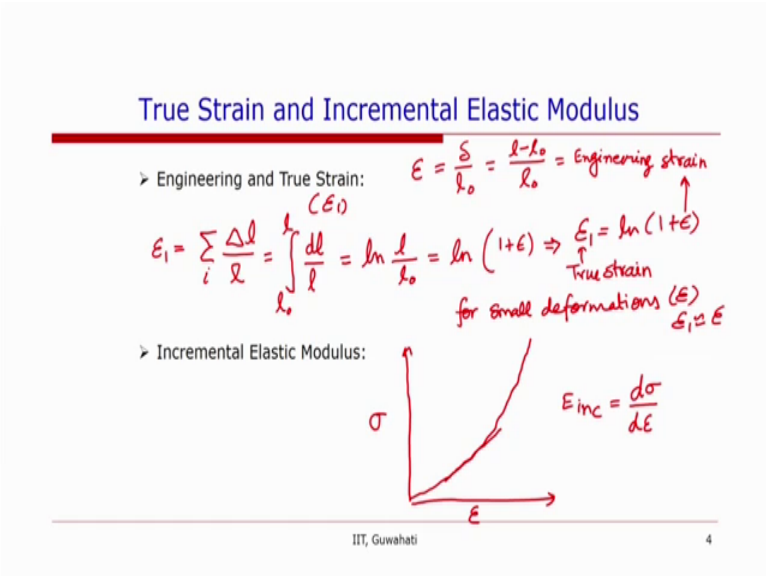

So, when we define a strain, we consider epsilon is equal to delta over l, or l naught, and the that delta is the we consider delta epsilon is equal to delta over l, l minus l naught over l naught; however, if we consider the strain truly the engineer. So, this is called engineering strain, which we have defined just now in the previous slide, now true strain if we represent it by epsilon then the true strain will be sum of the incremental strains, let us say a small change in the length divided by the instantaneous length. So, if we do that continuously, then we will have this as integral d l over l and the limit from l naught the initial length to l the final length.

So, that will be equal to l n l over l naught, where l is equal to l naught into 1 plus epsilon. So, this is l n 1 plus epsilon. So, this gives a relationship to us that epsilon 1 is equal to l n 1 plus epsilon, where epsilon is the engineering strain, and epsilon 1 is the true strain, and we can say that for small deformations that is epsilon is small epsilon 1 is almost equal to epsilon, the other point is that the materials which do not follow a linear relationship between sigma and epsilon for them, there is not a constant value of elastic modulus, any number of biological model behave materials do not behave elastically do not have elastic behaviour.

So, for such a materials, one can define incremental elastic modulus. So, the at one particular point the E incremental or the incremental elastic modulus is defined as d sigma over d epsilon. So, which is the slope at that particular point where the elastic modulus is being sought.

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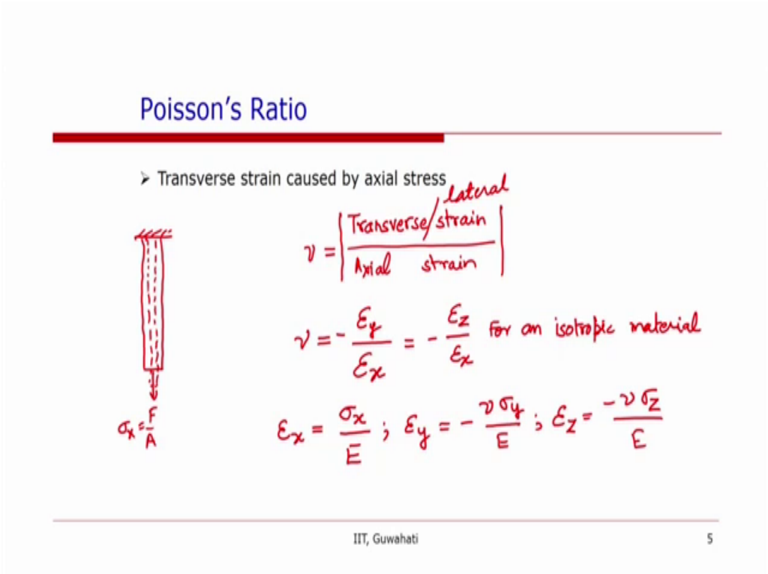

Now, poissons ratio. So, when a rod that first example that we considered when a rod it subjected to a stress a normal stress, then the rod not only it elongates, but to conservate volume the area of the rod also changes. So, there is because of the normal stress, let us say sigma x F over is equal to F over A, there is not only the strain in the axial direction, but there is strain in the transverse direction also.

So, the Poisson's ratio is defined as, transverse strain divided by the normal strain, and because it has the sign with it. So, the poissons ratio is mod of or the magnitude of transverse ratio and or the transverse can also be said as lateral strain divided by the or normal not in place of normal axial probably is a better word. So, the transverse or lateral strain divided by the axial strain. So, for example, nu is equal to epsilon y divided by epsilon x, and if we consider the sign because epsilon y is going to be negative, similarly this will be also equal to minus epsilon z over epsilon x for an isotropic material.

So, one can write that epsilon x is equal to in case of a axial stress, sigma x over E, and one can also find epsilon y is equal to minus nu sigma y over E, and epsilon z is equal to minus nu sigma z over e. So, this is poissons ratio.

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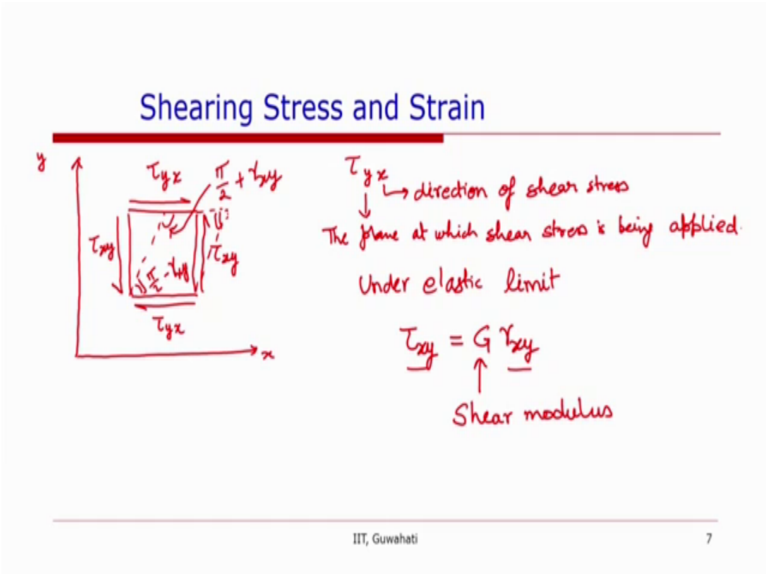

Now, we will talk about shearing stress and shearing strain so, if we consider a planar surface in the x, y coordinate system, let us consider a planar surface. So, in this it is subjected to a shear stress, which is tangential stress tau y x, tau x y, tau x y. So, as we have discussed already in the fluid mechanics review that tau y x has to subscript. So, y is the direction of the shear stress, whereas y is the plane at which shear stress is being applied..

So, as a result of this shear stress, the surface deforms by an angle. And so, the 2 angles the earlier angle as we saw that as pi by 2 the 2 angles are reduced by say gamma x y, and the 2 angles so, this and these angles are reduced by gamma x y, and the 2 angles are increased by the same value so, pi by 2 plus gamma x y. So, under elastic limits, the relationship between the shear stress is tau x y is equal to G gamma x y, where G is called shear modulus. So, that is the relationship between shear stress and shear strain, and like normal stress and normal strain are relationship or a graph can be plotted between the shear stress and shear strain and the slope of it is the shear modulus..

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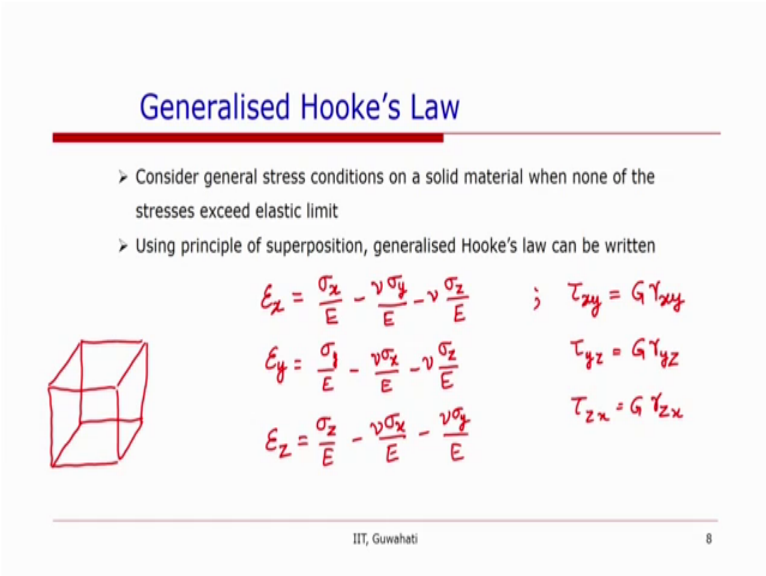

Now, based on this a now we can to write a generalised Hookes law. So, if we have a cubic material or a general material which have a cubic shape and subjected to a 3 kind of a stresses, or 3 stresses in 3 different directions sigma x, sigma y, and sigma z, and a different shear stresses then we can write using the principles of super position, that is the material subjected to different stuffisted sigma 2s. So, we can write sigma x is equal to sorry, epsilon x is equal to sigma x over E minus nu sigma y over E minus nu sigma z over E.

So, the first deformation in the x direction is coming because of the stress in the x direction whereas, the other 2 components are because of the 2 stresses in the y and z directions respectively, similarly one can write sigma y is equal to sigma sorry epsilon y is equal to sigma y over E, minus nu sigma x over E minus nu, sigma z over E, similarly epsilon z is equal to sigma z over E, minus nu sigma x over E, minus nu sigma y over E, one can also write the relationships between tau x y is equal to G gamma x y, tau y z is equal to G gamma y z, tau z x is equal to G gamma z x. So, one can remember in this case that the material is isotropic. So, the E and G they are same in all the directions.

Now, for the polar coordinates because in the cardiovascular system, what we are going to encounter are the arteries or the cylindrical tubes, and these cylindrical tubes it is easier to work in polar coordinates.

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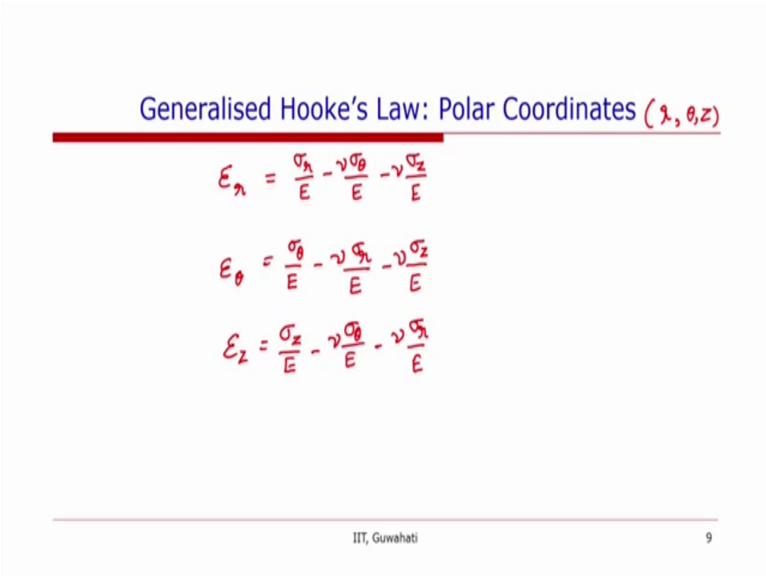

So, in polar coordinates we have r theta and z. So, like cylindrical like cartesian coordinate, the relationship in the polar coordinate can be written as epsilon r, epsilon theta, and epsilon z, they are equal to sigma r over E minus nu sigma theta over E minus nu sigma z over E, similarly sigma theta over E minus, nu sigma r over E minus, nu sigma z over E equal to, sigma z over E minus, nu sigma theta over E minus, nu sigma r over E, similarly one can write the relationships for the shear stresses in the r theta coordinates.

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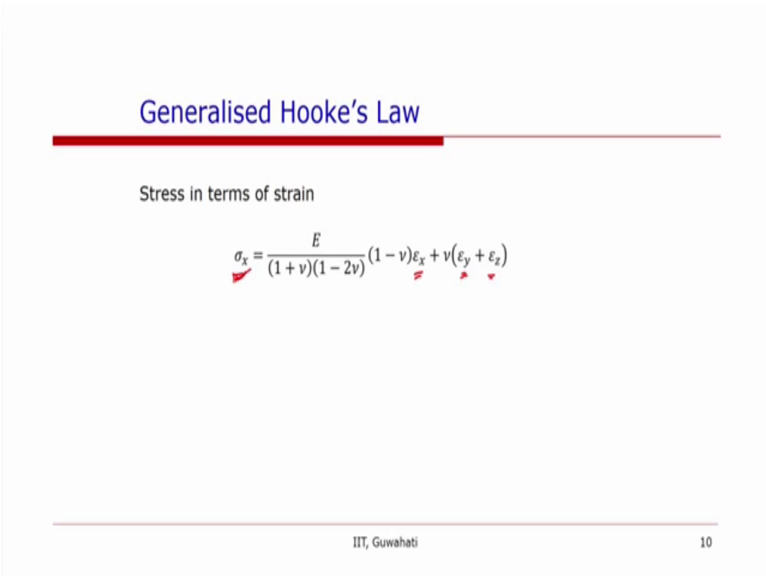

Now, one can also re cast these equations. So, that one can write the stresses in terms of the 3 strains epsilon x, epsilon y, and epsilon z. So, that is (Refer Time: 22:22) algebra and one need to reconstitute or a reframe these equations so, as to obtain shear stress or not the shear stress, but the extra stress normal stress in terms of the 3 strains. So, that is left for you as an exercise.

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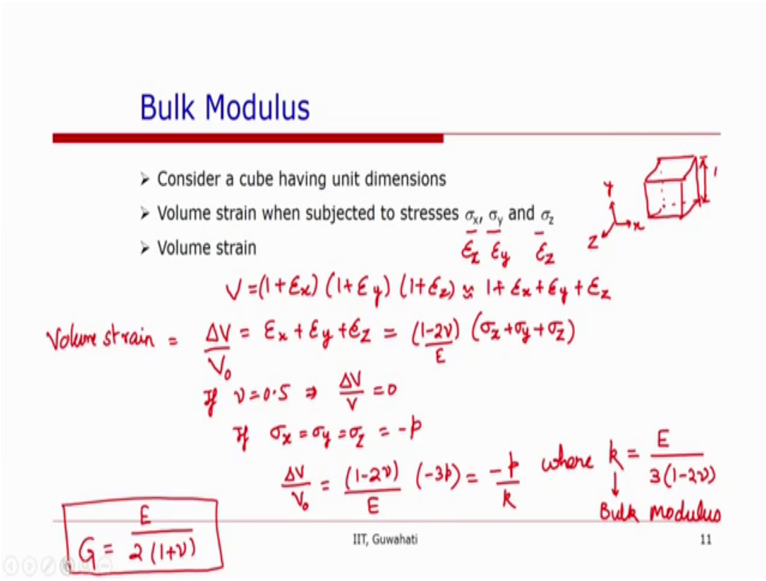

Now, we will look at Bulk modulus. So, Bulk modulus. So, when we consider let us say a cubic volume, and this volume has unit dimension the dimension is in direction is 1. So, the V is 1, now after the stresses in the 3 directions, epsilon x, epsilon y, and epsilon z, the strains are sorry the stresses are sigma x, sigma y, and sigma z. So, the resultant strains are epsilon x, epsilon y, and epsilon z.

So, the new volume will be 1 plus epsilon x, 1 plus epsilon y, into 1 plus epsilon z, and if one neglect the higher order terms, and then this will be equal to 1 plus epsilon x, plus epsilon y, plus epsilon z. So, the change in volume delta V is equal to because the initial volume is 1. So, the change in volume is epsilon x, plus epsilon y, and epsilon z, and because the initial volume is 1. So, delta V over V is equal to epsilon x plus epsilon y and V is the initial volume. So, V naught epsilon x, plus epsilon y, plus epsilon z. Now we can substitute the values of epsilon x, epsilon y, and epsilon z. So, we will have a this is equal to 1 minus 2 nu, when nu is the poissons ratio into sigma x, plus sigma y, plus sigma z.

So, the volume is strain, which is the ratio of change in volume to the original volume or the initial volume, is equal to 1 minus 2 nu sigma x, plus sigma y, plus sigma z. So, one can see here that if the poissons ratio is 0.5 then delta V over V, is equal to 0. If the material is subjected to uniform loading and sigma x and said sigma y and sigma z are same. So, let us say this is a in case of a hydrostatic pressure, and material subjected to minus p then delta V over V naught, is equal to 1 minus 2 V, we missed an E here 1 minus 2 nu over E, minus 3 p or minus p over k, where k can be defined as E over 3 into 1 minus 2 nu and it is also called Bulk modulus.

We might also want to remember here, that the relationship between the shear modulus G is equal to E over 2 into 1 plus nu. So, this is the relationship between the shear modulus and the elastic modulus.

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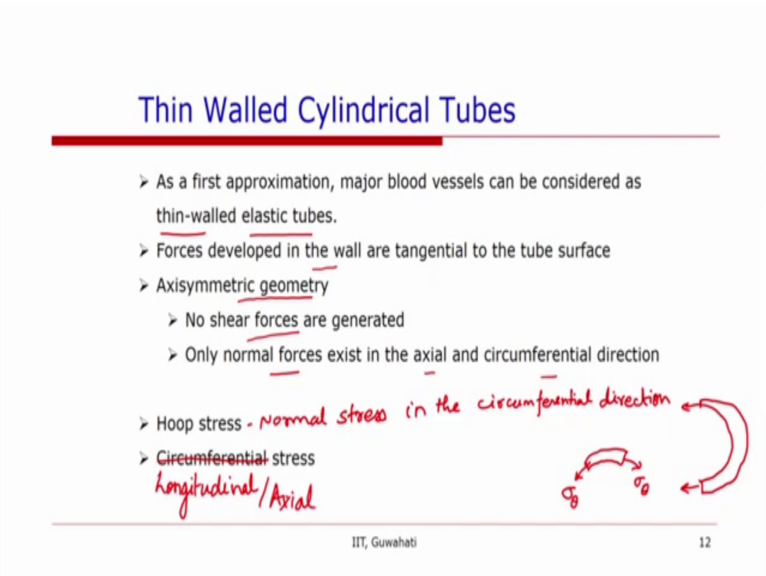

And you have the poisson ratio which relates to so, until now what we have been doing is defining a different modulus what we have done is defined the elastic material, and then the elastic modulus, the elastic modulus for the material in which the stress, stress strain relationship is not linear, then we have looked at the generalised hookes law and the strains in terms of the 3 different stresses, and then we have looked at the bulk modulus and the shear modulus. So now, with this information we would like to apply this to a cylindrical tube. So, for simplicity let us think or let us assume that this cylindrical tube is of thin wall. So, our analysis becomes simpler and as a first approximation let us assume that the arteries, which we will encounter in the cardiovascular system they are elastic tubes and they are thin walled. So, that we can apply this analysis to them, because the walls of or the tubes are can be bend easily. So, the forces that develop in these tubes they are generally tangential in nature, because the vessels they offer very little resistance to the bending, and we consider because the geometry where cylindrical. So, it is a axisymmetric geometry as a result no shear forces are generated, consequently only normal forces exist in the axial, which is the axial direction or the in a cylindrical tube. So, the axial direction the forces will be in this direction and in the angular direction or circumferential direction.

So, the stress in the circumferential direction is also known as hoop stress, hoop stress is normal stress in the circumferential direction. So, if we consider a half part of the tube, then the stresses on these portions in this direction is the circumferential, or if we take just a small angular portion of the tube then these stress or this stress is known as sigma theta, and it is called hoop stress. The other stress that will be important is longitudinal stress, in the axial direction. So, you also call it axial stress.

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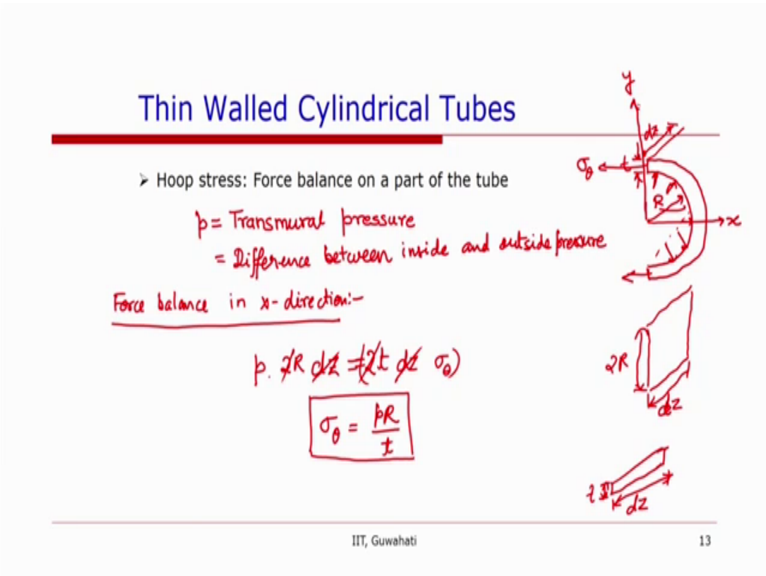

Now, hoop stress we can find by a force balance on a part of the tubes, if we consider a let us consider a semi-circular tube, the thickness of the tube which say t, and the radius is R and we consider a depth of the tube or length of the tube say d z, the internal pressure or p is what we call transmural pressure, which is the difference between inside and outside pressure. So, the pressure if the 2 sides are pressure p 1 and p 2, then what we consider the transferral pressure p x along this direction normal to it, will act the pressure acts normal to the surface everywhere correct and the outside pressure the difference we have considered. So, the outside pressure is 0 here.

Now, if we take this as x direction, and this as y direction, and do the force balance in let us say x direction, then what we will have is p into the area on which it is being applied. So, the internal area is this area. So, if we draw this area edge this area which is this distance is 2 R and the other distance is delta z. So, you will have p 2 R d z, where R is the radius of the channel, this is equal to t and the hoop stress in this is sigma theta.

So, sigma theta into the area of a these 2 so, there are 2 parts of the tube here, and these parts are the small area and this is small area this distance is t and this the length is d z. So, 2 t d z sigma theta d z, d z will cancel out and what we have is 2, and 2 will also cancel out sigma theta is equal to p R over t, which is the hoop stress. So, you find out the relationship for the hoop stress, in terms of the transmural pressure, the radius of the vessel and thickness of the tube for a thin walled cylindrical tube.

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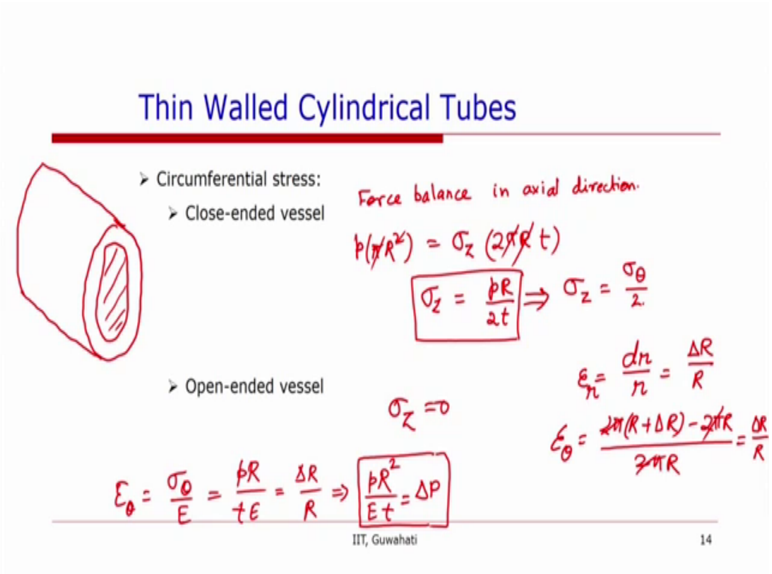

For the circumferential stress if the vessel is close ended. So, for the close ended vessel if we can draw a small schematic of the close ended vessel, then we can do a force equilibrium or force balance, in axial direction then we will have p into pi R square. So, what will be the area the in inner area of the tube that will be a putting a pressure that will be applying a force, and then this will be balanced by the axial stress sigma z into 2 pi R into t. So, pi and pi will cancel out, and R and R will cancel out. So, we will have sigma z is equal to p R by 2 t; however, if the vessel is open ended then there is no axial stress and sigma z is equal to 0, from there we can also see that sigma z is equal to sigma theta by 2 pi for a close ended vessel.

The epsilon for this tube will be equal to epsilon, the change will be in the radius of the tubes. So, the epsilon r will be d r over r, or delta R over R and epsilon theta will be 2 pi R plus delta R minus, 2 pi R over 2 pi R so, 2 pi 2 pi will cancel out and that will also equal to delta R over R that will be epsilon theta. So, we will have a relationship between epsilon theta is equal to sigma theta over E, which is p R by t E and that is equal to delta R, over R because this is what epsilon R is so, we can there find a relationship that p R square, over E t is equal to the deformation of the tube which is subjected to a internal pressure of P, for a thin walled tube.

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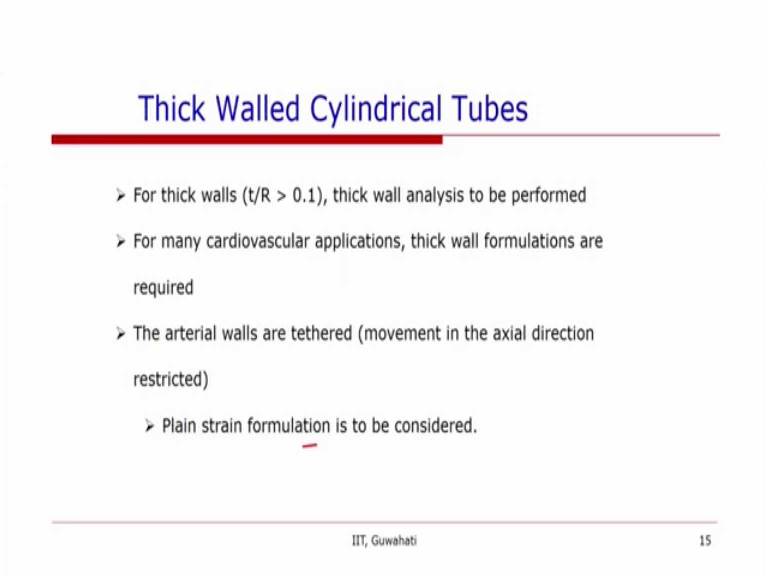

Now, the assumption of thin walled tube is good enough, when the ratio of the thickness and the channel radius is less than 0.1 or so; however, for the thicker a walls, the thick wall analysis needs to be performed and one need to take into account the stress variation in the walls of the tube, for many cardiovascular application this might be the case. So, we will briefly look at the formulation that can be developed for the thick wall tubes, and as the arterial walls are tethered; that means, their movement in the axial direction is restricted.

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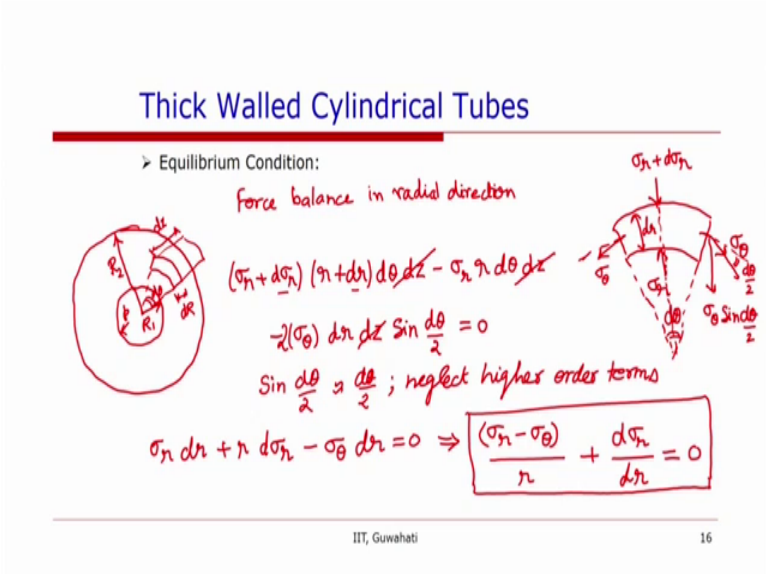

So, the only plain strain formulation or 2-dimensional strain formulation in the R, and theta direction need to be considered so, for the equilibrium condition let us consider a part of the tube. So, we have a tube, which have a thick wall and the inner radius is R 1 and the outer radius of the tube is R 2, and they consider a small angular portion, which has thickness R, and it subtends a angle of d theta at the centre, the length of the tube that we consider is d z, the internal pressure is p and we consider only transmural pressure here.

So, because of that we will have say we can draw this element here, for clarity and this subtends then angle of d theta at the centre, the stresses in this r sigma r, and sigma r plus, d sigma r, and the hoop stress or the angular or stress is sigma theta, now let us balance the force in the radial direction.

So, if we have force, then if we consider this direction. So, along this direction the force will be at this surface the forces, the force stress on this direction the stress will be there prime in this direction (Refer Time: 42:12). So, sigma r plus, d sigma r, into the area of this surface which is r plus, d r, d theta into d z minus, r sorry sigma r, when this is multiplied by the r, r is the radius and this place and d d r in this distance. So, this radius is r, and sigma r r d theta d z.

Now, the force sigma theta will have a component in the radial direction. So, if you look at this angle this is d theta by 2. So, the force component on this direction is sigma theta sin d theta by 2. So, we will have another force, sigma theta into the area which is d r into d z, multiplied by sin d theta by 2, and there are 2 components we will multiply this by 2, this is equal to 0. Now we can straight away cancel out d z from this, and we also assume that sin d theta by 2 is equal to d theta by 2, because d theta is a small angle, and we also will neglect higher order terms.

So, we will neglect the multiplication of d sigma r and d r, because that will be a smaller in magnitude term. So, we will have sigma r and sigma r, which will basically cancel out. So, we will have sigma r d r plus, r d sigma r, minus sigma this term is already cancelled out minus 2 2 will cancel out. So, we will have sigma theta d r is equal to 0, that will give us sigma r minus sigma theta over r plus d sigma r, over d r is equal to 0.

So, this is the first equation that we will get as a result of the equilibrium condition. So, that says the derivative of radial stress plus the difference between the radial and hoop stress divided by r, the sum of these 2 is equal to 0. So, this is the equilibrium condition, now, we substitute the compatibility conditions in this.

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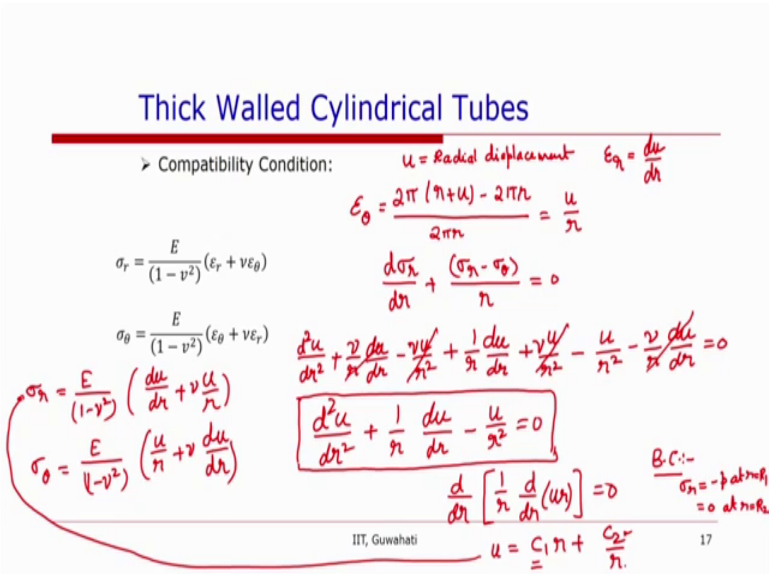

So, if we assume that u is the, radial displacement of the in the tube, instantaneously the local displacement in the tube. So, epsilon r is equal to d u by d r, and epsilon theta will be or the angular strain will be equal to 2 pi r plus u, minus 2 pi r divided by 2 pi r. So, that will be equal to u by r, now from the previous slide we had the equilibrium condition d sigma r over d r plus sigma r minus sigma theta over r, is equal to 0.

If we substitute those values here, and also substitute the values of epsilon r and epsilon theta, then we will have so, first let us find out this at the substitute sigma r, there then we will have sigma r is equal to E over 1 minus nu square into epsilon r is d u by d r, plus nu u by r and epsilon sigma theta will be E by 1 minus nu square epsilon theta is u by r plus, nu d u by d r, we substitute this in here we also need to find out d sigma over d r. So, let us do that E by 1 minus nu square will be everywhere. So, that can be cancelled. So, we can write d 2 u by d r 2 plus nu by r, d u by d r minus, nu u by r square, which is differentiation of 1 over r, plus now we substitute sigma r and sigma theta here.

So, we will have sigma r is 1 over r d u by d r, plus nu u by r square, minus sigma theta 1 r. So, u by r square minus, nu over r d u by d r is equal to 0. Now what we will have is d 2 u over d r 2, nu by r d u by d r is cancelled we have plus 1 over r d u by d r, that is the only term and a first order this a differentiator and then these 2 terms will cancel out and we will have minus u by r square is equal to 0.

So, this is a relationship for the displacement and say strain. So, if we reconstitute or recast it, it can be recast or read it in as d by d r is equal to 1 over r, d by d r of u r is equal to 0. So, one can find out this equation or one can integrate this equation, and get u is equal to c 1 r plus, c 2 r in this form, one can get an expression for u, for the displacement and if we substitute back this in substitute this in back into sigma r, and then we can have boundary conditions at sigma r is equal to minus p, at r is equal to R 1 and h 0 at r is equal to R 2. So, one can obtain the both 2 constants c 1 and c 2 and from that one can obtain sigma r sigma theta and so on. And so, forth.

So, that is the analysis for thick walled cylindrical tubes and from this analysis, one can obtain relationship for the displacement or for the deformation in the thick-walled tubes as a result of a transmural pressure p. It this relationship has been used to measure the properties of the material rather. So, one can measure the displacement for a known transmural pressure and from that displacement one calculates the properties of the material for example, poissons ratio and the elastic modulus. And so, so in summary what we have looked at is that how the hoop stress or how the deformation of a thin walled and thick walled tube can be measured or can be found out can be calculated for a given transmural pressure, and for that we also looked at some of the basics or we have reviewed the basics of solid mechanics, and the relationship between a stress and a strain for normal stresses and shear stresses and the bulk modulus and so on. And so, forth we have looked at the general hookes law. So, hope you have all you can remember now the basics of the solid mechanics, and if you find any difficulty, you can discuss while the course is on, or you can review your solid mechanics notes or books that you have studied in a first-year undergraduate courses.

Thank you.