Cardiovascular Fluid Mechanics
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Lecture – 12
Flow in Elastic Tubes

In this lecture, we will look at flow in elastic tubes. Until now, we have been looking at different variations of flow in rigid tubes. So, as we know that the cardiovascular system is a network of flexible tubes which have different sizes. So, we have looked at flow in rigid tubes because as an approximation the flow in rigid tube can give quite significant amount of information about flow in the cardiovascular system. We have also looked at pulsatile flow in rigid tubes.

Now, in this lecture, we will relax the assumption of rigidity and try to understand the flow in elastic or flexibility. So, there is a difference between flexible and elastic tubes, because elastic tubes are the ones which strictly follow Hooke's law which is not necessarily true for the blood vessels. But as an approximation, in this lecture to make our equations and mathematics simpler, we will primarily look at the flow in elastic tubes and use Hooke's law to understand or relationship between stress and strain for the solid tubes.

So, until now is the courses on cardiovascular fluid mechanics, we have been focused on the flow part of it, and did not bother about the mechanics of the solids or mechanics of the channel walls. But in this lecture, as the flexible nature of the channel comes into picture, then we also need to take into account the forces that are being exerted on the walls of the channel. And the resultant deformation on the channel walls because of or as a result of the forces that are being employed or that are being exerted on the walls of the channel.

So, to understand this lecture, we need to have a background or we need to have a little bit of understanding of solid mechanics as well. The primary understanding that we will be requiring or we will you will need to go back to your solid mechanics lecture in first year under graduate or second year under graduate course, the primary understanding is required is in terms of stresses or hoop stress or circumferential stress in the circular or cylindrical channels that is what we will be using in this. And these channels or walls of the channels are thin wall tubes. So, hoop stress or circumferential stress in the thin wall tubes that is the solid mechanics topic that will be used here. We will be deriving the relationship, but for the clarity sake, I would request you to look at those chapters.

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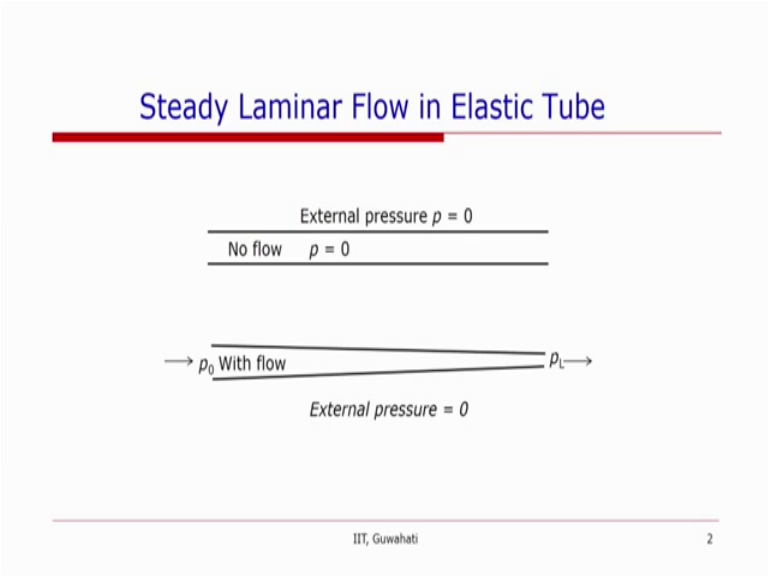

So, in this lecture we will consider two cases the first one where the flow is steady that means, the flow does not change with time. A special case of this for fully developed flow is Poiseuille flow, but Poiseuille flow is flow in a rigid tube. So, we will first look at the steady flow in a rigid tube, and then the next extension will be that we will relax the assumption of steadiness and consider pulsatile flow in a rigid tube. But for pulsatile flow, we will relax or we will have another assumption that we will assume the flow to be inviscid or the flow to be non-viscous that means if you remember the definition of Womersley number this is true for large Womersley number.

So, let us look at the first case when the flow is steady in and it is laminar in elastic tubes. So, as you can see here initially these are the boundaries of the channels. So, if the channel is cylindrical and there is no flow, then there is no deformation in the walls of the channel the pressure inside the channel and outside the channel both are same. So, let us say that the values of the reference pressure is 0 at the external pressure, and zero internal pressure.

When the flow happens the external pressure is not going to change, but because the flow will be pressure driven. So, there will be a gradient setup in the channel. And pressure at the upstream or at the inlet will be higher, let us say this pressure is p 0 and at the exit or after length l this pressure is p l external pressure is still 0. So, what will happen as the pressure will decrease downstream, the forces or the total pressure being exerted on the channel wall will vary.

The external because the external pressure is same and the internal pressure is decreasing, so as we move downstream the pressure on the tube walls will be more. And because of that the tube diameter will change. And when the tube diameter changes that it keeps reducing. So, we have two problems as I said we have fluid mechanics problem and solid mechanics problem. So, for this simpler case, we will take an iterative approach and solve the fluid mechanics problem and solid mechanics problem sequentially iteratively until we reach a converge solution.

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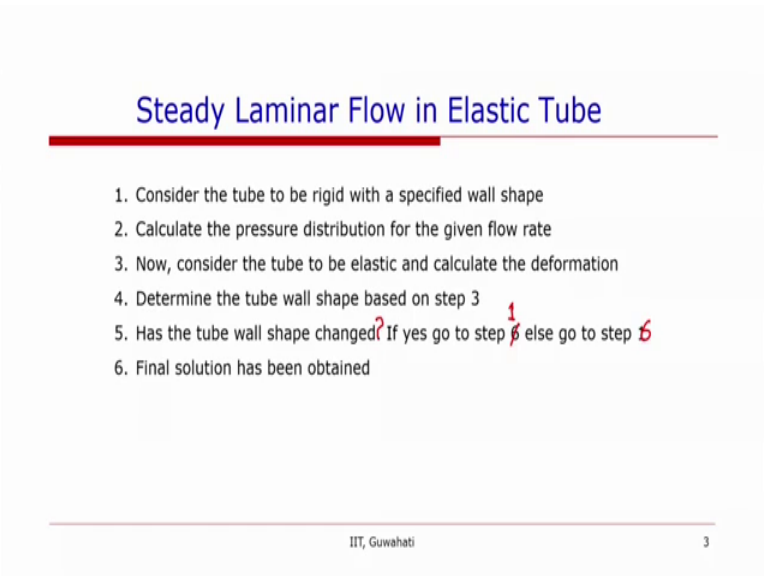

So, first we will solve the fluid mechanics problem. We consider that this tube is rigid tube and the wall shape is specified. And at the first iteration, at the very first iteration, we can consider that the channel radius is same everywhere, and it is a cylindrical channel. So, for this tube rigid tube with the specified wall shape, we calculate the pressure distribution for the given flow rate. So, the simplest problem, that we have considered which can be valid for small deformations in the channel. Where, the radial direction velocity, velocity in the radial direction can be neglected. And the assumption of del u by del x that is the gradient of velocity in the axial direction can be neglected.

Then we can calculate the pressure distribution for the given flow rate very easily using Poiseuille law because that will still be valid. So, we can calculate using fluid mechanics what is the pressure distribution for the given flow rate. But for the cases where these assumptions are not valid for that also it might take some more mathematics or use numerical methods to solve the flow. But we can solve the pressure distribution for the given flow rate and can also get the velocity distribution.

Now, we will come to the solid mechanics part and we consider the tube to be elastic and calculate the deformation. So, we can for the given flow rate or for the given average velocity, we have calculated pressure distribution from here along the length of the channel. Now, from this pressure distribution because we said that the external pressure is constant, now we will consider that tube is elastic and calculate the deformation. And from this deformation calculation, we can calculate the shape of the wall because of this force distribution or because of the pressure distribution as a result of the flow. And from this deformation, what is the shape of the wall.

Now, once we have calculated the wall shape we need to see that has the shape of the wall changed from step one. If it has changed, then there is a bit of mistake here has the tube wall shape changed; if yes, then we need to go to again step one, and recalculate the pressure distribution and go through step two, three, four and come to five again. If it has not changed, if the wall shape has not changed, then we need to go to next step where the final solution has been obtained that the solution from the solid mechanics and the fluid mechanics they are consistent. So, let us try to do it for a simpler case where we assume the Poiseuille law to be valid. So, we will need to make some assumptions to do that first.

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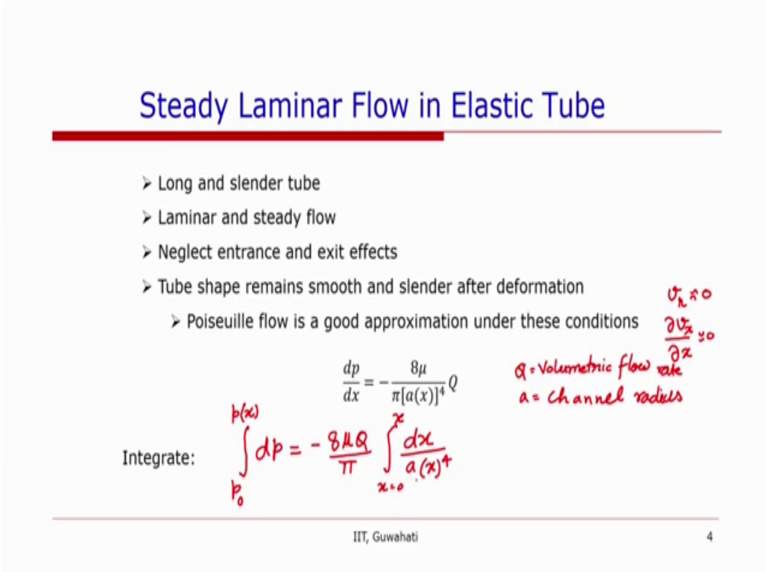

We consider that the tube is very long and the entry and exit effect can be neglected in the tube is slender. Now, the flow is laminar and steady that is anyway the assumption in this part of the problem. Then we have neglected entrance and exit effects. We also assume that with the deformation there are changes in the shape of the tube, but the inner surface of the tube, it remains smooth and it remains slender even after the deformation. So, in such cases Poiseuille flow is a good approximation what are the assumptions that we will need to make mathematically here we are saying when we use Poiseuille flow that v r is negligible, then v x is almost 0 in this case.

So, from our previous lectures, you might remember that Poiseuille flow is the pressure gradient is equal to minus 8 mu by pi a to the power 4 Q, where Q is the volumetric flow rate and a is the channel radius. Now, a is not a constant here, so that is why we have said that a is a x which is a function of x. So, if we know the relationship between p and a then we can integrate it. So, let us say d p is equal to minus 8 mu Q over pi because flow rate is a given quantity, so we will take it to be constant. And this will be d x over a x to be power 4 and that can be at x is equal to 0 to some x, and similarly pressure can be at p 0 and p x. So, the requirement for this is that we need to know what is a as a function of x.

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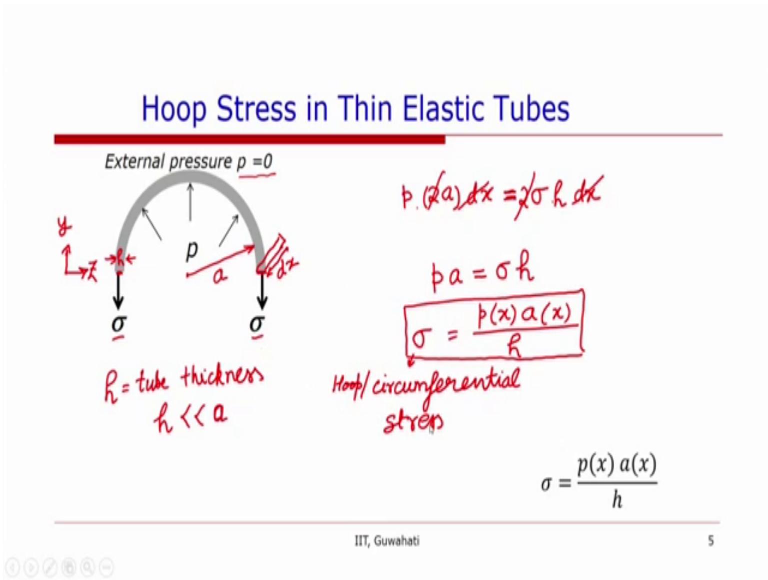

Now, let us look at as I said at the start of this lecture that we need to look at a small part of solid mechanics where we are trying to find the relationship between hoop stress and the pressure for a thin wall tube, hoop stress or circumferential stress. So, the stress that is being exerted in a thin wall tube. We assume that these stresses are or this stress is sigma. For simplicity sake, we consider a semi circular portion of the tube, but one can also consider a small arc of the tube, but the involved algebra there might be a bit a cumbersome.

So, to simplify our algebra, I have considered the semi circular part of the tube. And as earlier we have considered that the external pressure on this is equal to 0, the radius of the channel let us say that this is a, and the small length of this channel that we consider or that we can consider is say dx. Now, because of the pressure, the normal force if we say that this is y and z direction, then the force in y direction will be p into the projected area into the in the y direction. So, that projected area of the tube in y direction will be 2 a which is the diameter of the channel multiplied by the length dx. So, this area the projected area in the y direction of this tube surface is going to be a rectangular area. So, the width of this is 2 a and the length is dx. And on this area the force that is being exerted in the y direction that will be p into 2 a d x.

Now, this pressure will be balanced by the hoop stress which will be exerted on the tube. So, this hoop stress if we say that h is this thickness, and remember that we have also assumed that the tube is thin. So, we need to say that h is very, very small than a under those circumstances only this analysis will be valid. So, this is equal to sigma into h the area of this tube wall on which this is stress is being exerted is h into d x. And we have two such surfaces. So, this will be multiplied by 2. Now, d x d x will cancel out, and we can also cancel out 2. So, we have a relationship that p into a is equal to sigma h. So, we can say that sigma or the hoop stress or hoop or circumferential stress that is equal to p x a x over h, so that is the formula for the hoop stress.

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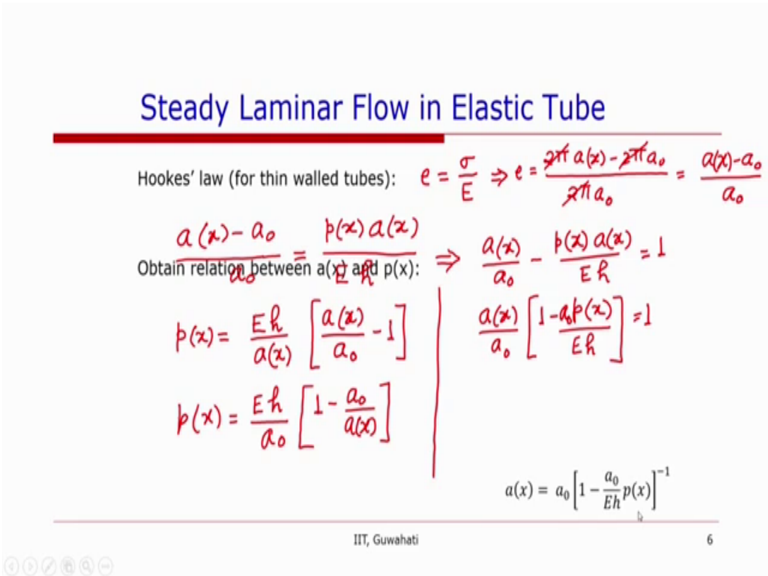

Now, we can use Hooke's law for thin walled tubes. So, Hooke's law you might remember that it is valid for perfectly elastic material which says that the deformation e is equal to sigma over capital E, where E is the capital E is the modulus of elasticity. And now in the circumferential direction what is e, e is equal to the change in the circumference of the thin tube, so that will be 2 pi a x minus 2 pi a naught divided by 2 pi a naught. So, a naught is the tube radius at the inlet or initially and after deformation the tube radius is a x. So, if we cancel out 2 pi then e will be a x minus a naught over a naught.

So, we can substitute from a x minus a naught divided by a naught is equal to sigma over e and we can substitute sigma from our previous slide where we obtained a relationship for hoop stress that is equal to p x a x over h where h is the tube thickness. So, we will have that p x is equal to that means, there will be E also. So, that will be E h over a x into a x by a naught minus 1 or E h over a naught we take a x by a naught out. So, this becomes 1 minus a naught by a x, so that is one relationship for p x or we can also change this into a relationship for a x in terms of p x.

So, we can take both the terms containing a x on one side which will give us a x by a naught minus p x a x over E h is equal to 1. And if we take a x by a naught outside then we will have 1 minus p x a naught by E h is equal to 1 or we will have a x is equal to a naught 1 minus a naught by E h p x. And because it has been taken on this side; so, whole to the power minus 1. So, we will have a relationship between a x and p x.

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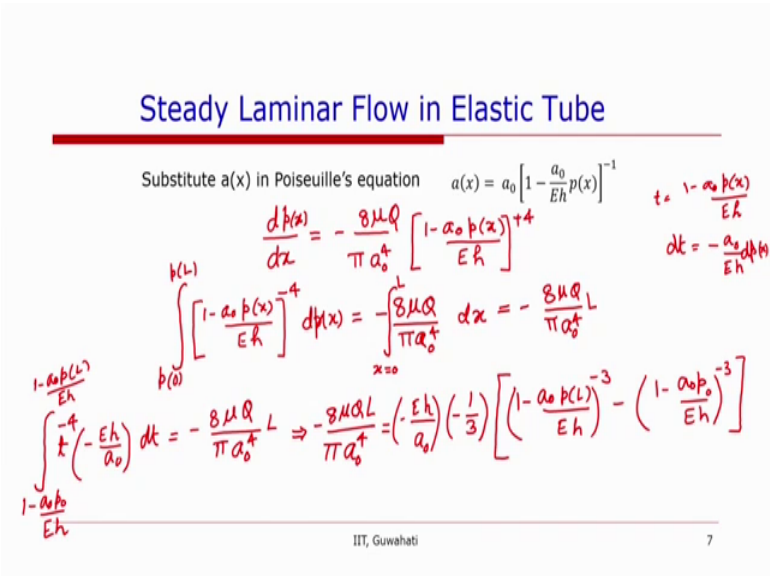

Now, we can try to substitute this relationship in the Poiseuille equation which is the relationship between a x and p x is this. And the Poiseuille equation you might remember was d p by d x is equal to minus 8 mu over pi a x to the power 4. So, we can substitute this relationship here pi a naught to the power 4. And this will go up, so we can have 1 minus a naught p x over E h whole to the power minus 4. So, 8 mu pi a to the power 4 into Q. So, we can bring Q here. Q is the given flow rate.

Now, this differential equation, this is a differential equation in p x, we have eliminated a successfully from here. So, we have only two variables here p and x and we can integrate it. So, this is sorry this is going to be plus because we have taken this up in the numerator. So, this becomes now we have to do a bit of algebra 1 minus a naught p x over E h d p x is equal to minus 8 mu Q by pi a naught to the power 4 d x. And integrated from x is equal to 0 to we can say L and similarly this will be p 0 and p l. So, we need to know the difference between these two points.

This will simply be minus 8 mu Q by pi a naught to the power 4 into L. Whereas, if we want to integrate this we can simply integrate it let us say that t is equal to 1 minus a naught p x over E h and that will give us d t is equal to 1 minus that one will go away. And we will have simply minus a naught by E h d p. So, let us substitute this here and what we will obtain is 1 minus a naught p x E h becomes t and d p x will be d t multiplied by minus E h by a naught into d t. This is equal to minus 8 mu Q over pi a naught to the power 4 l. And this is p naught; so at p naught, this becomes 1 minus a naught p naught over E h whereas at p L it becomes 1 minus a naught p at L over E h. And that there is a mistake of power that we have left here that was power to the power minus 4 that was supposed to be. So, this is t raised to the power minus 4.

And now we can substitute this and say this is minus 8 mu Q L over pi a naught raised to the power 4 is equal to minus E h by a naught then t to the power minus 5 divided by minus 5. So, we will have 1 over or minus 4 plus sorry minus 4 plus 1. So, t to the power minus 3 divided by minus 3. So, minus 1 over 3 multiplied by t to the power minus 3, and we will have 1 minus a naught p L over E h this raised to the power minus 3 minus 1 minus a naught p naught over E h raised to the power minus 3. So, this equation does not come out to be a simple one, but still we have managed to solve this equation successfully for a elastic tube.

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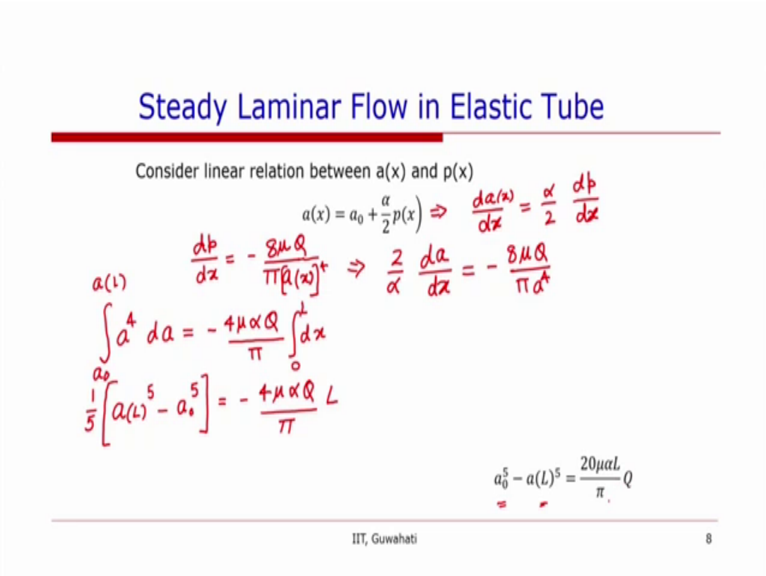

And if we go back and look at what is the relationship between a and p this is not a linear relationship; and a x p x is about not exactly, but about in inversely proportional to p x. So, let us go and look at now. So, we have looked at what will be relationship for steady laminar flow in an perfectly elastic tube. Now, as I said at the beginning that the blood vessels they are not exactly they do not follow the Hooke's law. So, they are not perfectly elastic tube, they have some amount of viscoelasticity, but we will not go in detail of the properties of the elastic or I mean the properties of the vessels of the channel we will not go in great detail of that. We will just assume another relationship between p and a, and this time simpler relationship which is linear.

So, this form is a x is equal to a naught plus alpha by 2 p x, where alpha is the compliance coefficient. So, we can consider this linear relationship between a x and p x. Now, in this linear relationship is generally valid for certain category of arteries, which is pulmonary arteries, the arteries which are in the pulmonary system for that this relationship is valid. So, we will try to look at the relationship and this is the final expression that one can derive.

So, we had this relationship d p by d x is equal to minus 8 mu over pi a to the power 4 where a x so we will write this as a x to the power 4 into Q. Now, from this we can say that d a by d x or d a x by d x is equal to alpha by 2 d p by d x. So, we can substitute this d a by d x from here. So, we will get 2 by alpha d a by d x; for simplicity sake, we will write only d a by d x and we will not write a and x in the bracket. So, 2 by alpha d a by d x minus 8 mu Q over pi a to the power 4. And if we do a bit of algebra a to the power 4 d a is equal to minus 4 mu alpha Q over 2, so that is 2 (Refer Time: 33:30) taken care of divided by pi and d x.

So, let us now integrate it from 0 to L all these terms are constant. So, we can take them out of the integrals integral sign and this is from 0 to L and this is from a 0 to a L, so that will give us a to the power 5 divided by 5. So, we can take that a on the other side or we can let us say we can put that as 1 over 5 a L raised to the power 5 minus a naught raised to the power 5 is equal to minus 4 mu alpha Q over pi into L. So, if we take this here we will get this relationship minus can be brought here. So, a naught 5 minus a L to the power 5 is equal to 20 mu alpha L over pi Q. So, what we can see from here that a naught to the power 5 minus a L to the power 5 is proportional to the flow rate.

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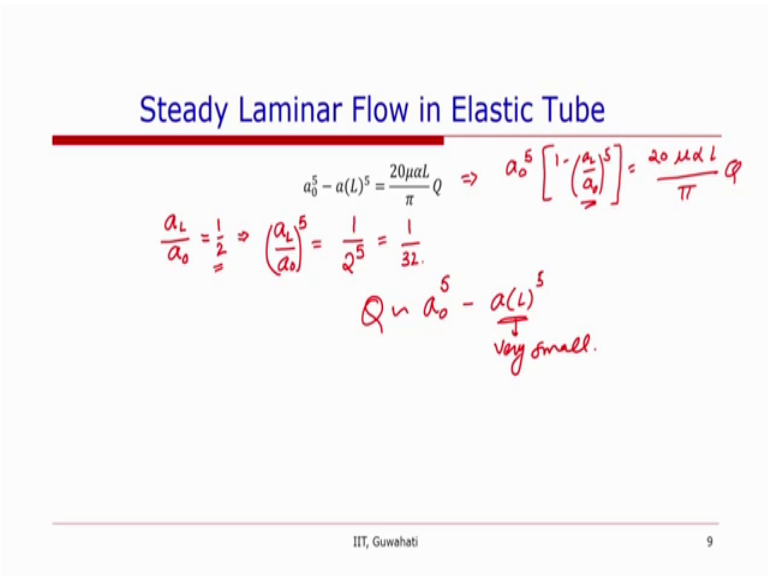

Now, if this is the case let us say if the change in radius has become a L by a naught is equal to half. Then as a consequence of this mu is a constant, alpha is a constant, compliance coefficient L is constant, pi is constant. So, the flow rate is going to change. So, if we look at this and a L by a naught is 5, then a L by a naught to the power 5 we can just write this in terms of the ratios probably a naught to the power 5 1 minus a L by a naught to the power 5 is equal to 20 mu alpha L over pi Q.

So, if this is 1, then this number will be 1 over 2 to the power 5. So, 2 to the power 3 is 8 and that will be about 8 into 4 32. So, only about 1 over 33, so only about 3 percent, so the radius will remain or the radius of the channel will be only 3 percent of the inlet radius, so which can be neglected in those terms. So, basically Q for channel radius half of the initial radius or more, Q will be almost proportional to a to the power 5 rather than because a L to the power 5 is going to be very very small. So, this number is going to be very small, so that is directly proportional to a to the power this is when there is linear relationship between the a and p, where a is the channel radius and p is the pressure variation along the channel.

So, what we have looked at is at the first problem steady flow in a channel first we had our relationship between pressure and flow rate from fluid mechanics. And we simply assume Poiseuille's flow under certain conditions. And then based on that pressure distribution we calculated what is the deformation in the channel or what is the change in the channel radius. And then we substituted this channel radius in the Poiseuille's flow and calculated the flow rate. So, this problem in this problem we had to consider both the fluid mechanics problem which is Poiseuille flow and the solid mechanics problem where we looked at the deformation for a thin walled tube.

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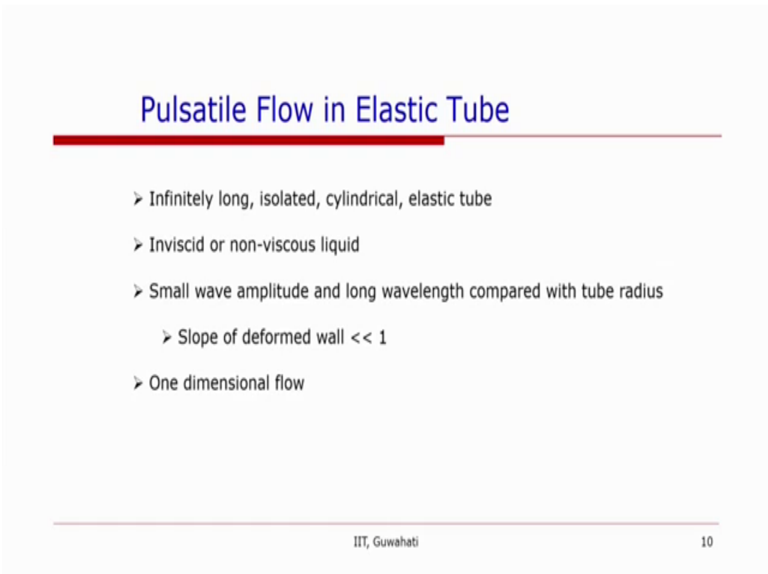

Now, we will look at the second problem which is pulsatile flow in the elastic tube. So, for pulsatile flow in the elastic tube, we will make certain assumptions and look at only at only a very ideal case. We assume that the channel is infinitely long. So, no entrance and exit effect are being considered here. The channel is or the tube is isolated with other system and the external pressure remains constant throughout. We are not considering the changes in the external pressure. The system is isolated right. And the cross section is circular, and the channel is cylindrical, and it is an elastic tube. So, we will consider that the tube follow Hooke's law. And again we will use the relationship between sigma and p or between a and p for a elastic tube that we just obtained in the previous problem.

And then we assume that the fluid that we are considering unlike the previous problem where we considered the flow to be laminar here we will consider that the flow is inviscid that means, the viscosity is 0. So, the velocity profile is going to be same, I mean it will be uniform flow or plug flow. Another assumption is that the wave amplitude is small. So, the amplitude of the pressure wave that will be generated will be small. And the wavelength of this wave is very very large or sufficiently large then channel radius of the wavelength is large or the channel radius is very very small than the wavelength.

This is got long wave approximation. And this make sure that the slope a of this deformed wall is less than one or very very less than one. So, there are only very small changes under these conditions. We will be able to linearize the equations, because in this case we will have non-linear equations which is difficult to solve, you solve try to linearize the equations that is why we are trying to make this assumptions. And the flow is one-dimensional. So, we consider that when the flow is one-dimensional v r is equal to 0. Now, under these assumptions when there is a deformation at one point of the tube there will be a pressure wave generated on the tube, and this pressure wave will travel on the channel wall. So, we need to get the speed of this pressure wave or pressure pulse.

So, the objective here now is to obtain the speed of this pressure wave, and then eventually we will obtain a wave equation which can be solved to find out the pressure profile as well as the velocity profile or pressure as a function of x and t, and velocity as a function of x and t. But in this problem that we are going to deal with here right now we just want to solve or we do not want to solve the equations we will write down the equations for this simpler case and try to obtain a relationship for the wave speed or the pressure pulse speed and the equation that govern the pressures.

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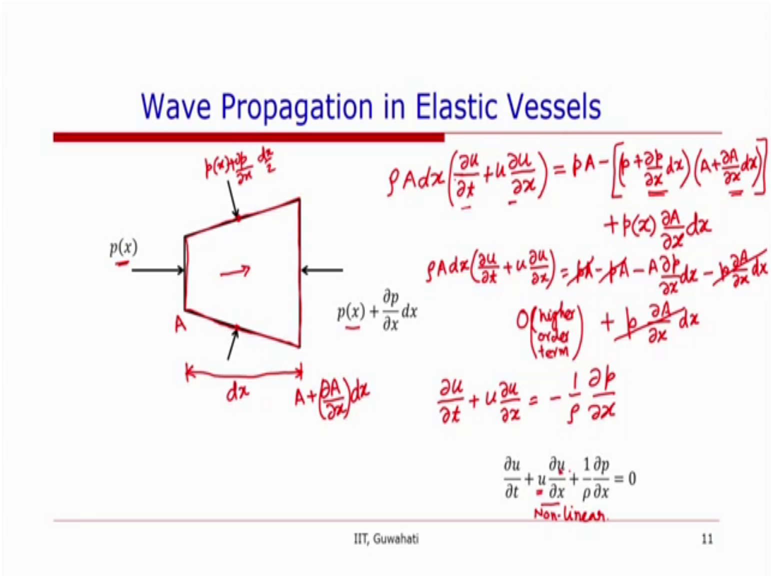

So, as always we will start with the conservation equation. So, let us look at the force balance or the momentum conservation equation. So, we can write m into a where we have considered a small part of the fluid element here. You can see here. And in this case, the area is changing. And the length of this fluid element let us say this is d x. The area at starting point is A, and at end point is A plus del A by del x d x. Pressure at the left end is p x, and then at the right end it becomes p x plus del p by del x d x. And the at the two ends you will have pressure as p x plus del p by del x. If you want to consider this at these points, then you can consider as a d x by 2; and similarly on the other side.

So, if we write down the force balance m into A, so that will be m is rho into v, v is the volume of this fluid elements. So, A into d x the area of cross section into d x this into the acceleration del u by del t plus u del u by del x. So, we are not going into detail of derivation of this acceleration, we can directly consider this. This is the unsteadiness or the acceleration, and this is the convective acceleration.

Now, we will consider the forces. So, the force because the flow is happening from left to right. So, we will consider the force p x into A. Remember that A is also a function of x, but we are not going to write that here. And we will write p also simply as p not as p x. So, p A from left to right minus p plus del p by del x d x into A plus del A by del x d x. So, these are the forces that are acting on these two ends. And the third force that will be also because of pressure that will come in the total sum of these two pressures acting on these walls, so that will be the projected area multiplied by the pressure. So, neglecting the higher order term, we will have this as p x into del A by del x into d x the entire using these two.

So, now let us expand this, and neglect the second order terms which are there. We can also maybe we can retain this as of now rho A d x del u over del t plus u del u over del x is equal to p A minus p A minus A del p over del x d x minus p del A over del x d x. The multiplication of these two terms is going to be a second order term which we are going to neglect anyway, so plus, so I am not going to write that plus p x del A by del x d x. So, p A, p A will cancel out; I will just write here order higher order term. I have only limited space. So, I need to save some space here. And this is p, you should have while I am writing p x here. So, we will just write this as plus p and this will cancel out.

So, now we see in these two equations A d x will cancel out. So, what will obtain is del u by del t plus u del u by del x is equal to A d x over rho A d x. So, we will cancel out this A d x and A d x, and what we will have is minus 1 over rho del p over del x. So, this is our force balance or momentum equation del u by del t plus u del u by del x plus 1 over rho del p by del x. So, remember that this is the term which have non-linearity. Here we have multiplication of u with u. So, this is the non-linear term in this equation.

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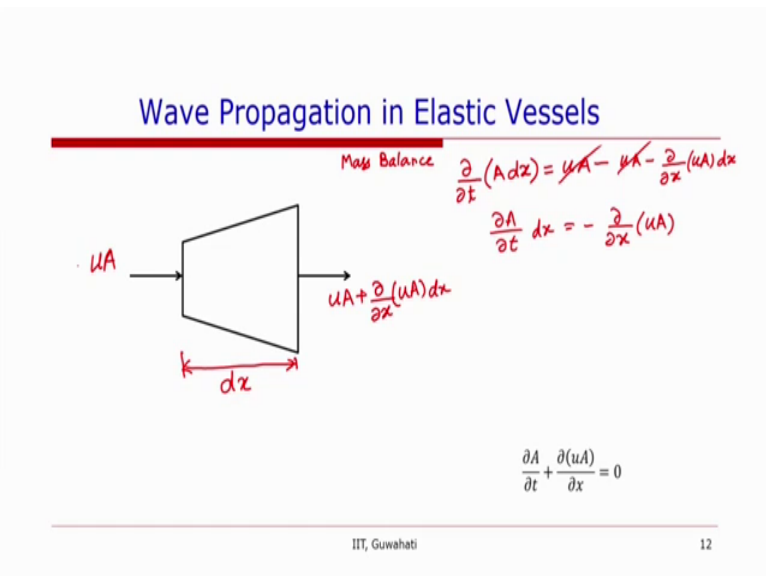

Now, next we consider mass balance here. So, again for mass balance as earlier this length is d x, and the flow that is coming from here is say volumetric flow rate is u A because this is an incompressible flow. So, the density will cancelled out, density will not have any effect. So, if the mass flow is conserved then rho because del rho over del t will be 0. So, we can take rho out if you want we may take it out and here what will have is u A plus del by del x of u A. So, mass balance is the accumulation. And the important thing here to note is that volume is also changing. So, we can say that del over del t of A d x right, the change in volume is equal to u A minus this will be del u by u A minus del by del x u A d x. So, this u A, u A will cancel out. And what we are left with is del A by del t d x is equal to minus del by del x u A or after substitution del A by del t plus del by del x u A. Now, u and A both are functions of x. So, this term might have some linearity into it, we will look into now.

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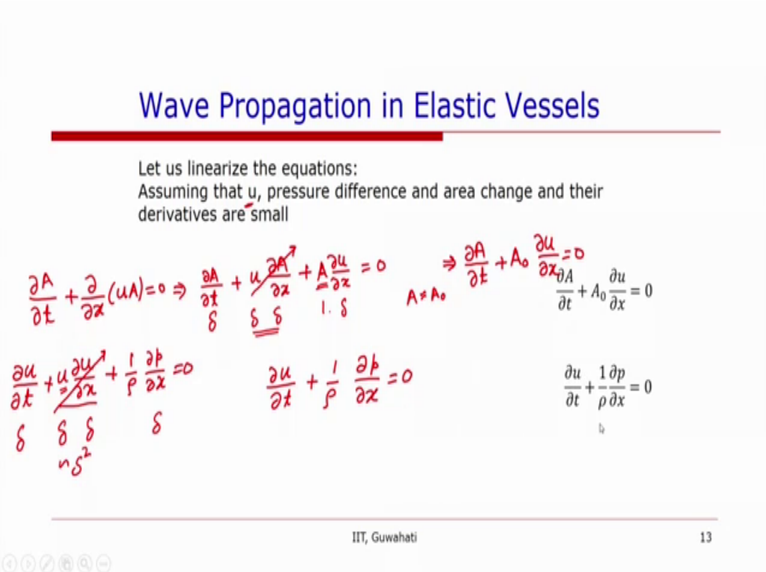

So, now let us linearize the two equations. To linearize we will need to make certain assumptions under which these linearized equations, or these equations can be linearized or the linearized equations will be valid. So, we assume that the blood velocity u is small the pressure difference the pressure difference between internal and a external that is small. And area change the change in area if the area is A naught at the at the entrance of the tube at the start and A is the area which is changing with the distance then this area change is small.

So, A minus A naught or A naught minus A is small. And the derivative of all of these del u by del x del p by del x and del a by del x del capital A by del x they are all small. So, we assume that and under these assumptions, we will try to simplify the equations. So, the force balance equation we will just go back, and look at the equations, the force balance equation was del u by del t plus u del u by del x plus 1 over rho del p by del x is equal to 0. The mass balance equation was del A over del t plus del over del x of u A is equal to 0. So, these are the two equations, and now we need to linearize this.

The force balance equation if we look at this term is the non-linear term, and it is multiplication of u which we said which is small, it is value is of the order of delta. And del u by del x which is again small, so this term is going to be of the order of delta square. Whereas, other terms will be of the order of delta, for example, del u by del t and del p by del x. So, they are of the order of delta. So, this term is can be neglected. So, we will end up with del u by del t plus 1 over rho del p over del x is equal to 0.

Now, we look at the mass balance equation which is del A by del t plus let us expand this u del A by del x plus A del u by del x is equal to 0. Again this term will be of the order of delta, this will be delta into delta. And A is large anyway, so we cannot say that what is this going to be if A is say of the order of one and del u by del x is again delta. So, term is going to be small in this case and we will have to neglect this.

Another thing here is that because this term is again it bring non-linearity because A is a function of x, and u is a function of x. So, what we can assume here that A the change in A is small. So, we can assume A is almost equal to A naught, so that will give us del A over del t plus A naught del u by del x is equal to 0. So, we have our linearized equation del A by del t plus A naught del u by del x is equal to 0, and del u by del t plus 1 over rho del p over del x is equal to 0.

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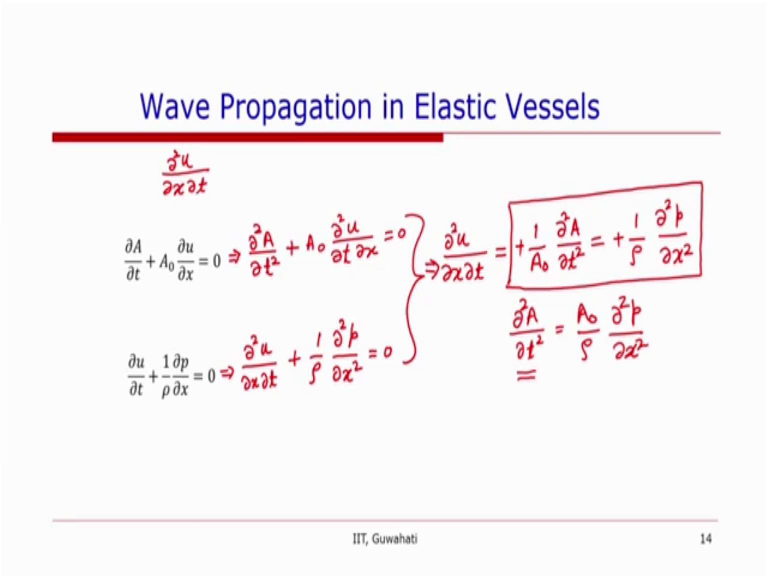

So, now what we will do is obtain a relationship between second derivative of del 2 u by del x del t from both of these equations and equate the two. So, if we do that from here we will get del 2 A over del t 2 plus A naught del 2 u over del x del t is equal to 0. Similarly, from here we will have del 2 u over del x del t, you might remember that you might can write this as del x del t or del t del x, it does not make any difference. But just for the clarity sake I have written in this manner here del 2 u by del x del t plus 1 over rho del 2 p over del x 2 is equal to 0.

So, if we from these two equation, if we equate the two values of del 2 u over del x del t then what we will get is minus 1 over a naught del 2 A over del t 2 is equal to minus 1 over rho del 2 p over del x 2. Then compare these two terms and what we get is these two equations from this we get del 2 A over del t 2 is equal to A naught over rho del 2 p over del x 2. So, remember what we have been able to get is a relationship between the double derivative of area with respect to time and double derivative of pressure with respect to distance, and this is based on the flow equations only.

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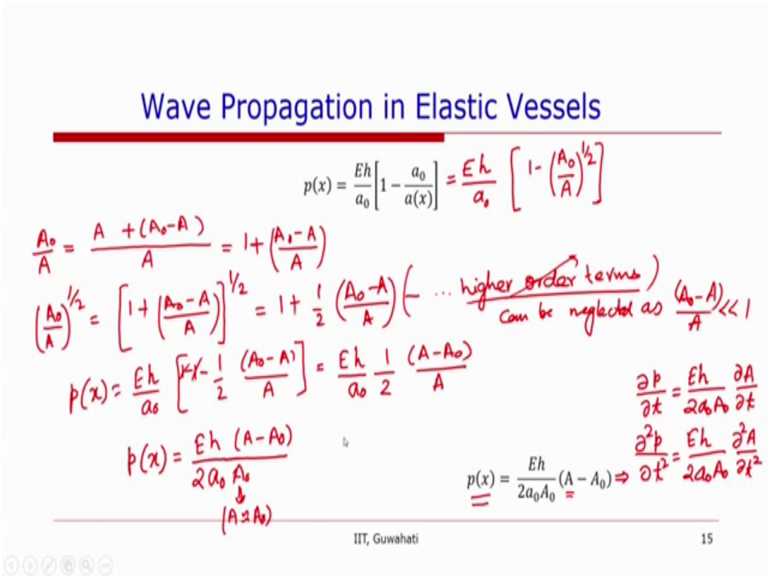

Now, we will go back to the solid mechanics equation the relationship that we obtained between pressure and the radius for a thin tube. So, if we write this in terms of the area then it will be E h over a naught into 1 minus A naught over A raised to the power 1 by 2 we will keep a naught intact here and this ratio is A naught over A raised to the power 1 by 2. If we now linearize this equation then we will need to use A naught by A is equal to we can write A naught plus A minus A naught A naught by A.

So, you can write A plus A naught minus A divided by A or this will be equal to 1 plus A naught minus A over A. So, using binomial expansion, we can write this A naught by A power 1 by 2 is equal to 1 minus A naught minus A divided by A raised to the power 1 by 2, so that will be 1 minus 1 by 2 A naught minus A by A. And higher order terms we know that A naught minus A is can be neglected as A naught minus A over A is very very small than 1. So, we can neglect the higher order terms.

So, we substitute that here we will end up with p x is equal to E h over a naught into 1 minus 1 by 2 1 minus 1 plus 1 by 2 A naught minus A over sorry I have done a mistake here, this should have been plus this should have been plus and this should have been plus. So, if we do that then this would have been minus 1 and 1 cancel out, and we have p x is equal to E h by a naught into half A minus A naught over A. So, this is A, but as A is very that the difference between A and A naught is very small and we want the equation to linearize, so only linear relationship between p and a. So, we will have this a is equal to a naught we will just write here that A is almost equal to A naught, so that is why this has come up E h by 2 a naught capital A naught A minus A naught. So, p x is equal to E h over 2 a naught capital A naught A minus A naught that is the relationship between pressure and the area that is coming from the solid mechanics.

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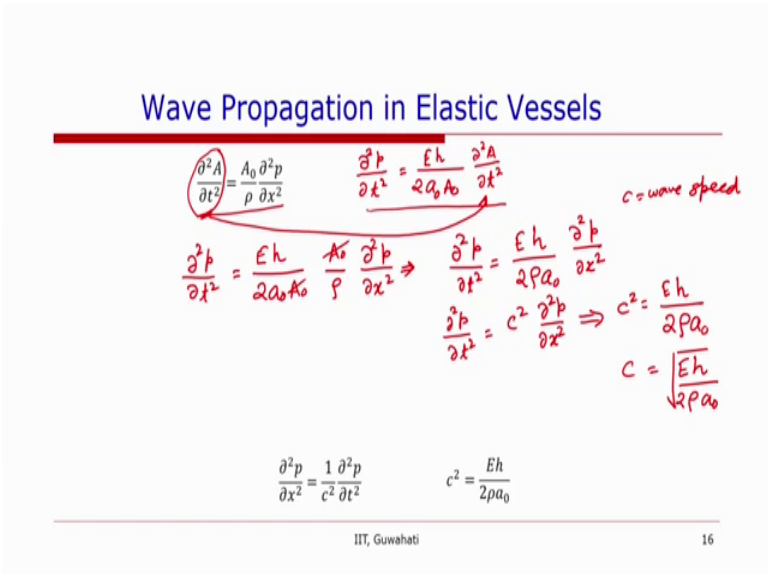

And we can double differentiate this equation actually the equation p x is equal to E h by 2 a naught A naught. So, what we can get from here del p over del t is equal to E h over 2 a naught capital A naught del A over del t. And the second derivative will be del p 2 del 2 p over del t 2 is equal to E h over 2 a naught capital A naught del 2 A over del t 2. So, E h by 2 a naught capital A naught that is what we need to remember.

So, we will just write the equation del p 2 over del t 2 is equal to E h over 2 small a naught capital A naught del 2 A over del t 2, this is the equation that we obtain from the solid mechanics. And this is the equation that we obtain from the two mass and momentum conservation equations. And if we equate del 2 A by del t 2 in both the terms then we will have A naught by rho or we can substitute directly del 2 A by del t 2 from this equation we can substitute in this equations. So, we will get del 2 p over del t 2 is equal to E h over 2 a naught capital A naught small a is the radius and capital A is the area remember, and this will be A naught over rho del 2 p over del x 2. So, A naught and A naught cancelled out, and we will have our equation as we will just write them and cut them. So, this will be del 2 p over del t 2 is equal to E h over 2 rho a naught del 2 p over del x 2.

If we compare this with the standard wave equation, we will have say in terms of del 2 p over del t 2 is equal to c square del 2 p over del x 2. And from that we will have c square is equal to E h over 2 rho a naught, where c is equal to pulse speed or the wave speed. So, the speed of the pulse propagation is square root of E h over 2 rho a naught, this is what we have obtained. And then we know from our mathematics course is that how to solve a wave equation and a solution can be found out. A similar equation can be found for u also for the fluid velocity u.

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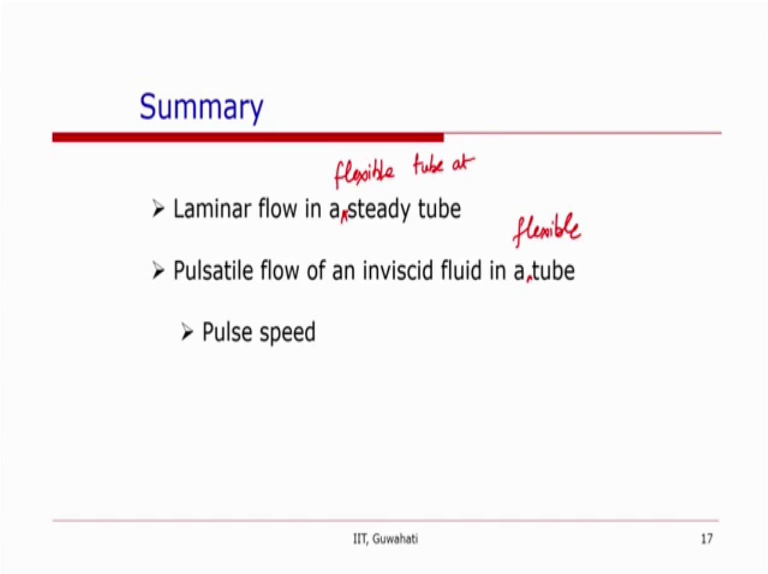

So, let us summarise what we have seen today is that laminar flow in a flexible tube flexible tube at steady state and pulsatile flow of an inviscid fluid in a again flexible tube. So, for the laminar flow, our objective was to obtain the steady state solution and we could do we can do that iteratively, but here because we had the relationships simplified, so we could obtain a relationship between a and p from solid mechanics and substitute in the Poiseuille flow equation. And could obtain a relationship between pressure and a that was obtained from solid mechanics and we also obtained relationship between the channel radius and the flow rate. So, what is the effect of the tube flexible tube on the channel radius or on the flow rate through the channel. For the pulsatile flow, we had assumed that the flow is or the fluid is going to be inviscid and neglected the viscous terms in the conservation equation momentum conservation equation; and from these equations, we obtain the pulse speed. So, we will stop here.

Thank you.