Cardiovascular Fluid Mechanics
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Lecture – 11
Pulsatile Flow Continued...

In this lecture we are going to continue pulsatile flow. In the previous lecture we discussed about pulsatile flow in rigid tubes. So, we suggested there that as the flow is pulsatile in the cardiovascular system, and any pulsatile the flow is pulsatile as well as periodic. So, as the flow is periodic and from the Fourier series, any periodic function can be decomposed in a infinite number of sinusoidal functions or sinusoidal harmonics in infinite series of a constant term plus sine and cosine terms and we need to find out the coefficients of this sine and cosine terms and the constant term for which there are standard methods using Fourier transform.

So, the objective then reduce to understand the pulsatile flow, when the pressure gradient is sinusoidal in nature and we consider the rigid tube for a fully developed flow we derived the solution or the relationship between velocity and pressure gradient for a sinusoidal pressure gradient. And we did it in terms of when the time dependence is exponential in terms of complex numbers ok. So, in this lecture we are going to continue where we left in the previous lecture and discuss few more characteristics of pulsatile flow in rigid tubes.

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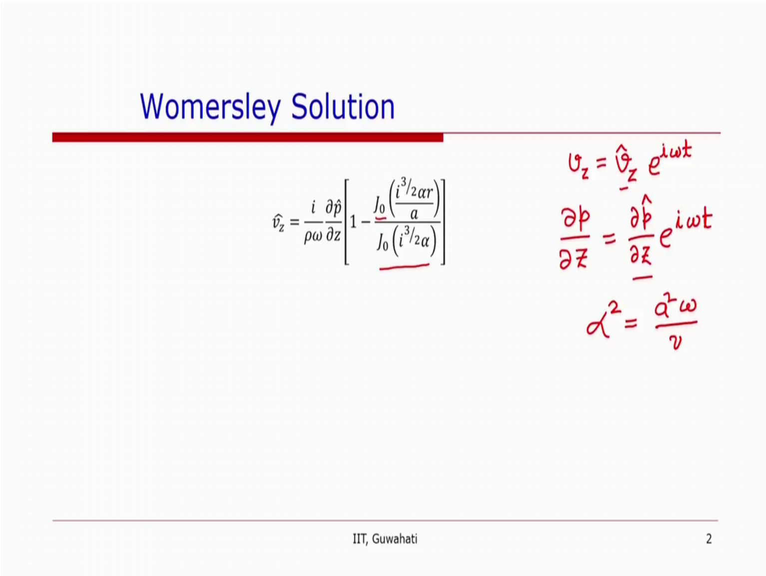

The solution that we obtained we called it Womersley solution after doctor Womersley who looked at the pulsatile flow in rigid tubes and derived a number of relationships for it.

So, the relationship what we see here is for we said that the velocity v z will be a function of v z cap, which is a function of r and the time dependent per term e to the power i omega t now the v z cap and similarly the relationship between the pressure gradient the time dependence of pressure gradient was del p by del Z is equal to del p cap over del Z e to the power i omega t.

Now this equation gives the relationship between the v z cap and del p over del z for all values of Womersley number. So, just to recap Womersley number is equal to alpha square is equal to a square omega over nu and it is the ratio of transitional or transient inertial force and the viscous force. So, in this equation what we see is these are functions of these J 0 and J 0 is Bessel function of first kind. So, we need to a recap or we need to just remind ourselves what Bessel function is what is the series representation of Bessel function how does the graph of a Bessel function of first kind look like, and then some properties which we might require to obtain the shear stress or to obtain the flow arte in pulsatile flows from the Womersley solution

So, let us look at the Bessel function of first kind.

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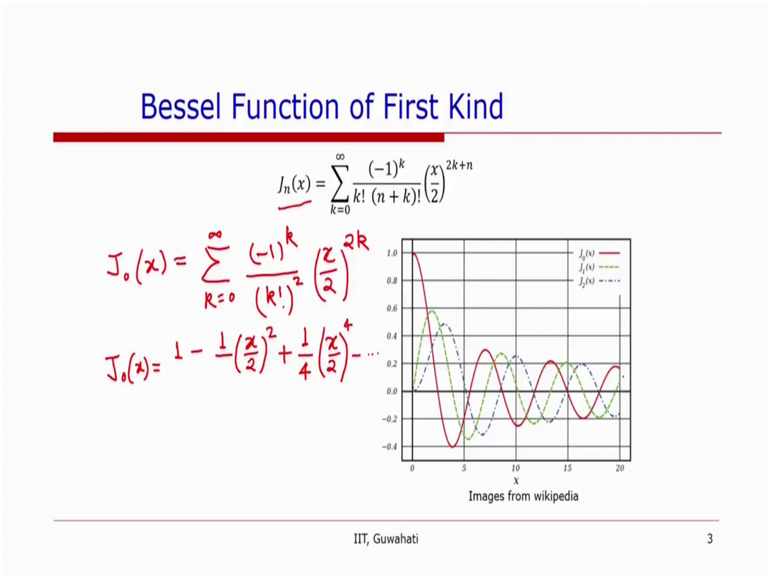

And the series representation is that you have J n x where n is for any number n can be any harmonic and k is equal to 0 to infinity minus 1 to the power k. So, it is a infinite series and k is a representing each term minus 1 or to the power minus 1 to the power k factorial k of factorial and plus k x by 2 to the power 2 k plus n. So, what we are concerned is or we are we are concerned about is J 0. So, we will write that J 0 x is equal to sigma k is equal to 0 to infinity, minus 1 to the power k and factorial k will be square x by 2 exponent 2 k because n is equal to 0 now this is J 0 x.

So, if we want to find out say value of J 0 at 0 or let us just expand it first a bit let us write few terms. So, we will see that J 0 x is equal to for k is equal to 0 this is 1 this will also be 1 everything is 1 minus 1 to the power 1. So, that is 1. So, 1 over and k is 1. So, 1 into x by 2 square plus when k is equal to 2 we will get term 1 divided by 2 square; that means, 4 x by 2 4 and so, on. So, we can see that the value of J 0 at x is equal to 0 will be 1 as we can see from this graph, and then the graph goes into the negative direction and then it oscillates in the magnitude of this oscillation we can see that it comes down ok.

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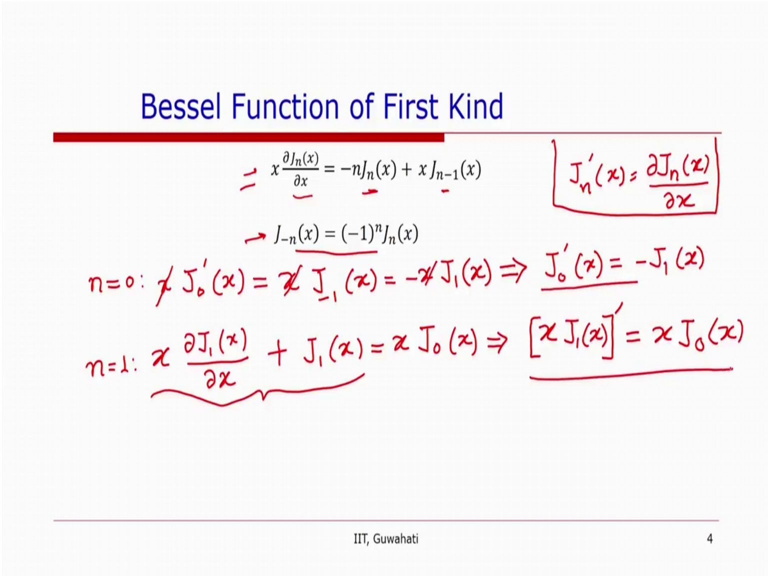

So, 2 important properties of Bessel functions we can obtain these properties by looking at the properties of the Bessel functions these 2 equations. So, J minus n x is equal to minus 1 exponent or minus 1 to the power n J n x, and the other property between relationship between J n x the derivative of J n x and J n x and J n minus 1 x. So, let us take if n is equal to 0, then what we will have is x J 0 dash x is equal to this term is going to be this term is going to be 0 because n is equal to 0. So, we will have x J minus 1 x, but from this relationship we have that J n, J minus 1 x is equal to minus J 1 x. So, we will have x minus x J 1 x. So, this basically give us the relationship between we can cancel out x from all and we have that J 0 dash x or the derivative of J 0 with respect to x is equal to minus J 1 x.

Do we might require this later on, now for n is equal to 1 if we substitute 1 here in the first equation then we will get x del J 1 x divided by del x is equal to minus J 1 x plus x J 0 x. Now we do a bit of recifal of terms. So, we can put a plus here and this becomes equal. So, from this let us look at this, this is a derivative of x J 1 x that is equal to x J 0 x. So, you can say the derivative of x, x the second term is differentiated J 1 dash x plus the differentiation of x is 1 and then J 1 x. So, we might want to say this explicitly here that J dash x means del J n x over del x. So, we have obtained 2 important relationships here which we might require to use further ok.

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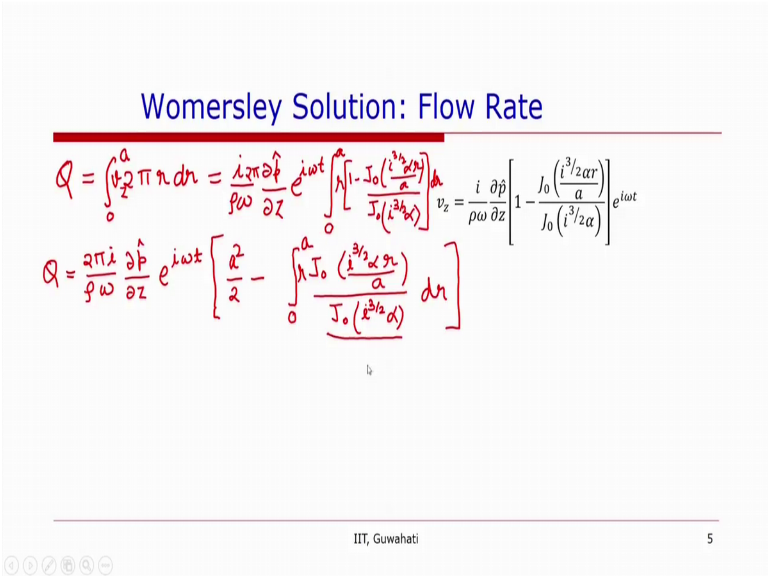

So, we have obtained a relationship between the velocity and pressure gradient. We have obtained the velocity profile, but often when we are looking at flow in the pipes we need to obtain the characteristics of the average velocity or the average velocity or flow rate. So, let us obtain the flow rate. The flow rate is Q is equal to integral 0 to a, where a is the radius of the channel 2 pi r d r multiplied by v z ok. And you might we have the velocity profile here. So, if we substitute that velocity profile then we will require to integrate it the terms which are independent of r, they can come out of the integral sign.

So, we will have i over rho omega del p over del Z, p cap and the exponential term time dependence e to the power i omega t plus we also have 2 pi here all of this is independent of r. So, it can come out of the integral sign then we have integral 0 to a 1 minus J 0 i to the power 3 by 2 alpha r over a divided by J 0 i to the power 3 by 2 alpha into d r remember that i have included the r term here. Now if we write this down again in integrate Q is equal to 2 pi i over rho omega del p cap over del z, exponential i omega t the first term when we integrate multiply with r. So, r the integration will be r square by 2 and when we put the limits we will end up with a square by 2 minus integral of limits from 0 to a J 0 i to the power 3 by 2 alpha r over a divided by J 0 i to the power 3 by 2 alpha d r.

Now, i the again there is a r here, you might notice that this term is independent of r. So, it can. So, also be taken out of the integral sign. So, now, the objective is to find out this integral.

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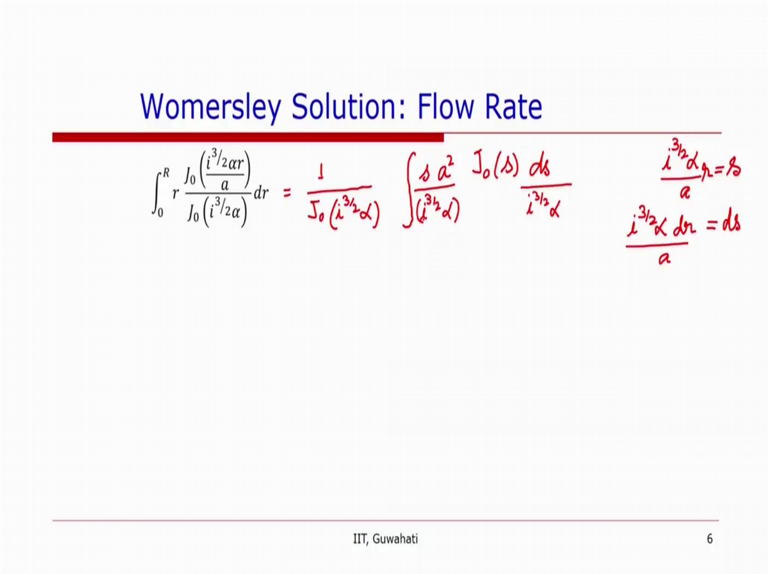

So, let us try to find out this integral and we will see that integral 0 to r. So, we can take the denominator term outside, because it is independent of r 1 over J 0 i to the power 3 by 2 alpha. Now what we can do, we can assume because this if you remember what we had assumed earlier that i to the power 3 by 2 alpha over a into r is equal to s. So, we can do a change of variable here. If we do that then we also need to do the change of variable for d or differential terms. So, d s is equal to i to the power 3 by 2 alpha d r over a. Now if we substitute in place of r. So, we have s over i is to the power 3 by 2 alpha over a which is for r, now J 0 and this entire term is s and again d r is d s over i to the power 3 by 2 alpha and this becomes a square. So, we can say that we can get rid of all this here and just a square here now all this is independent of s.

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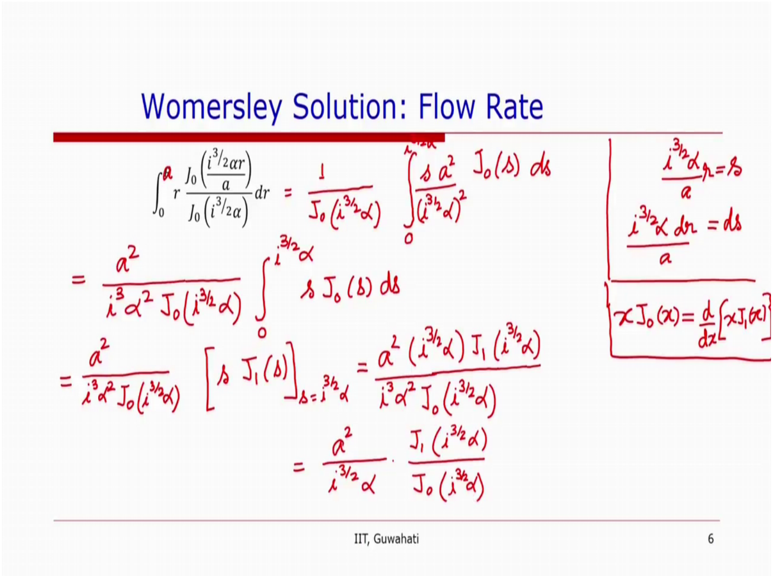

So, we can bring this here and this will be a square over i cube alpha square J 0, i is to the power 3 by 2 alpha we need to also work out the limits. So, integral 0 to r is equal to it should be small a, because we are using a for the radius. So, i to the power 3 by 2 alpha a over a. So, that will be 1 these are the limits integral 0 to i to the power 3 by 2 alpha s J 0 s d s.

Now, we had just seen the relationship in the previous slides that x J 0 x is equal to d by d x of x J 1 x. So, what we are going to do? When we integrate it this equation we will get this a square i cube alpha square J 0 i to the power 3 by 2 alpha and s J 1 s at s is equal to i is to the power 3 by 2 alpha. So, let us substitute this and what we are going to get is a square divided by i raised to the power 3, alpha square J naught i raised to the power 3 by 2 alpha and for s if we substitute i raised to the power 3 by 2 alpha J 1 i raised to the power 3 by 2 alpha and we will get a square divided by i raised to the power 3 by 2 alpha, because this will cancel out and the ratio of J 1 of i raised to the power 3 by 2 alpha divided by J 0 i raised to the power 3 by 2 alpha remember this was an integral in a bigger expression.

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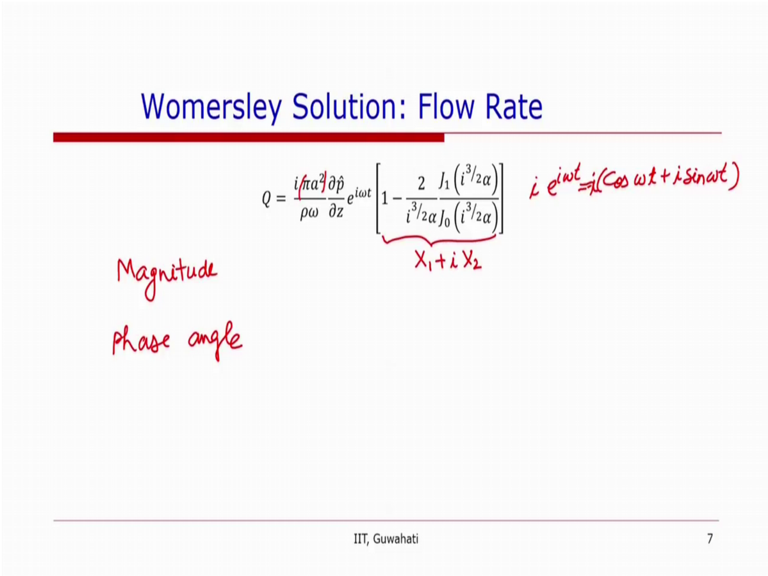

So, if we substitute that expression we are going to get the flow rate to be i pi a square which is the cross sectional area of the channel, divided by rho omega this is the pressure gradient and this term inside the bracket. So, as you might notice that this is a complex term and we need to obtain its magnitude and phase angle. So, if rep[resent]- we represent this term as x 1 plus i x 2 and remember that e to the power i omega t is equal to cos omega t plus i sin omega t and we have another i here. So, we can substitute that i here and from that we can reduce this into one. So, we can have multiply x 1 i x 2 into this.

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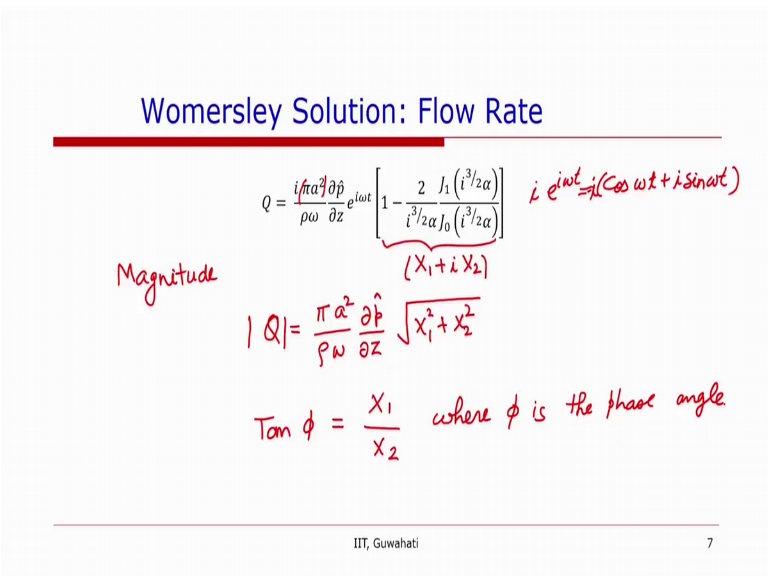

So, we can get what is the magnitude of Q and that will be pi a square by rho omega del p over del z root of x 1 square plus x 2 square, and if i is the angle phase angle; then we will get tan phi is equal to x 1 over x 2. So, this is the magnitude of the flow rate and this is the phase angle so where phi is the phase angle between the pressure gradient and the flow rate?

You might recall from yesterday's analysis that at small Reynolds number the flow rate is same as in a poiseuille flow, and the phase angle or the between the phase angle between the 2 is 180 degree whereas, for large Reynolds number the phase angle is about 90 degree between the 2.

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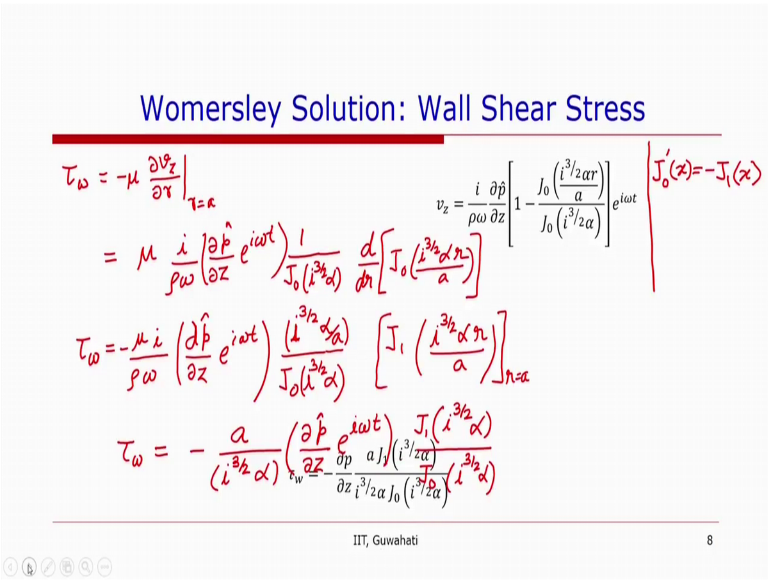

So, what we have looked at based on the velocity profile till now we have obtained Q or the flow rate now we will try to obtain tau, which is the wall shear stress tau w and the formula for this is equal to minus mu del v z over del r at r is equal to a. So, again we have the velocity profile here, from this velocity profile we want to obtain wall shear stress which is the derivative of this. So, to obtain wall shear stress we will differentiate the velocity term.

So, it will be minus mu multiplied by i over rho omega del p over del Z exponential i omega t, which is the time dependent pressure gradient. The first term in this bracket because it is a constant it is not dependent on r. So, that will be 0 and there will be a minus sign. So, that minus and minus they will make plus. So, we can just get rid of this sign here and then 1 over J naught i to the power 3 by 2 alpha, this is also independent of r and what we are left with is d by d r of J naught i to the power 3 by 2 alpha r over a.

And we had derived earlier that the derivative of J naught x is equal to minus J 1 x. So, the derivative of J naught is minus J 1 x. So, if we can substitute that here and tau w is equal to mu i over rho omega del p over del z we can write this in terms of capital e to the power i omega t, 1 over J naught i to the power 3 by 2 alpha.

Now, that the derivative of this will be minus J 1 i to the power 3 by 2 alpha r over a, and i to the power 3 by 2 alpha over a. So, the wall shear stress is we have alpha over a. So, finally, when we do the algebraic calculations, the final relationship that we will obtain is minus a over i to the power 3 by 2 alpha del p over del z we will write this in terms of exponential term.

So, del p over del z e to the power i omega t, J 1 remember that this will be at r is equal to a. So, this a and a will cancel out. So, that will be sorry J 1 i raised to the power 3 by 2 alpha over J 0, i to the power 3 by 2 alpha this is the final value what we have just written and this del p by del z is written in terms of time dependent it is.

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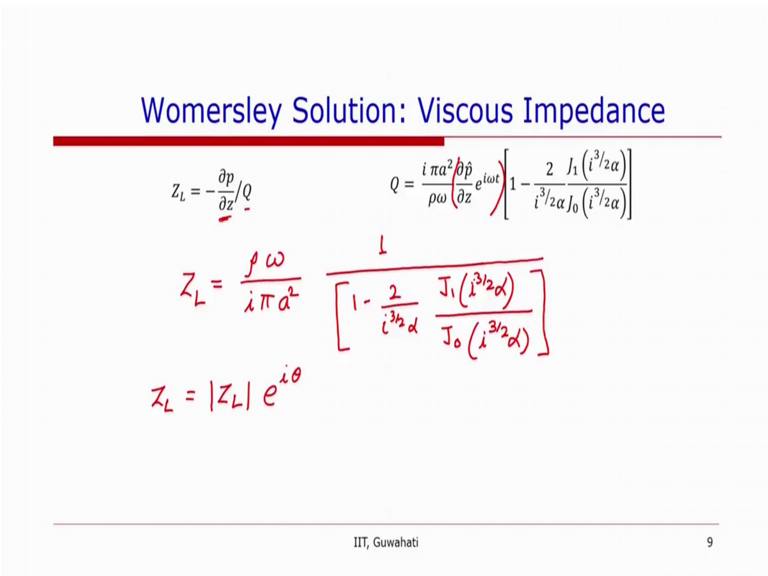

Another important relationship is viscous impedance, which is the ratio of del p over del z which is the pressure gradient and the flow rate. So, the viscous impedance is equal to pressure gradient and the, and the ratio of flow rate. So, if we look at this relationship we can easily find out that Z L is equal to del p over del z which is this term. So, this is Q and what we want to do is del p over del z divided by Q. So, that will be rho omega by i pi a square into 1 over 1 minus 2 over i to the power 3 by 2 alpha J 1, i raised to the power 3 by 2 alpha divided by J naught i raised to the power 3 by 2 alpha.

So, this is the viscous impedance term and again we will have the magnitude of J l here Z L here and that will be and there, will be a theta term which will be the phase difference between the pressure gradient and flow rate, and this is quite handy in calculations of the work that is done in the system for the flow or for the pulsatile flow to happen.

So, in the previous class or in the previous look lecture we looked at what is Womersley number.

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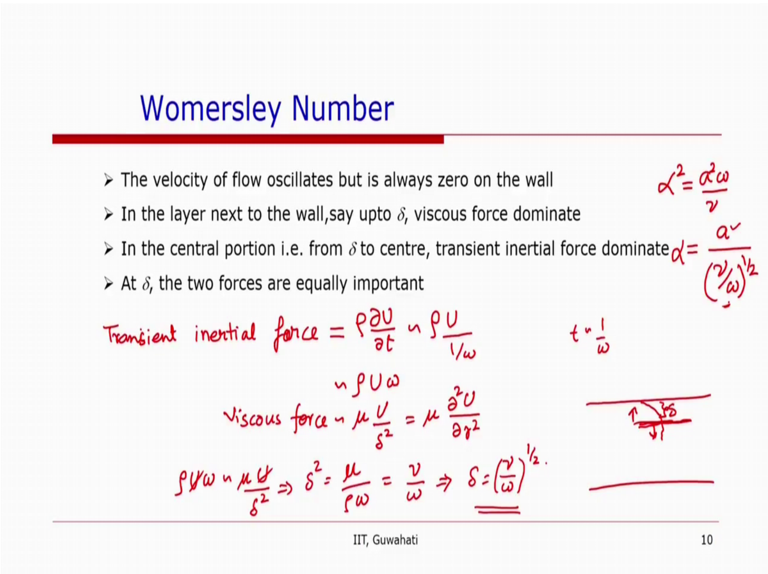

And Womersley number is the ratio of transient inertial force and the viscous forces. We also said in the previous class that Womersley number which is. So, it is ratio of alpha square is equal to a square omega over nu or it can be represented as a over nu over omega raised to the power 1 by 2. So, the velocity of the flow oscillates, but is always 0 on the wall. Even though when the flow is pulsatile there is always no slip boundary condition at the wall; that means, for a viscous flow the velocity near the wall is going to be 0 and there are going to be velocity gradients near the wall.

So, where the viscous force dominate whereas, in the layer just next to the wall. So, if you have the channel here and after some distance from the wall where you have gradient and then, beyond that distance let us say that this distance is delta where viscous force is dominate. And after this the velocity is oscillating and where the transient inertial forces are dominating and that is where you see the more change of transient effects ok.

So, as we move from this distance towards the wall, the viscous force will be dominating and as and as move away from here, the transient force becomes more important. So, at this distance what we call the transient boundary layer the 2 forces are equally important. So, what is the transient inertial force, that is we can say that it is equal to rho del U by del t. So, a magnitude will be rho U divided by say t si proportional to 1 over omega. So, 1 over omega. So, this force will be rho U omega.

And the viscous force will be mu U by delta square because it is it comes from mu del 2 U by del r 2. So, if we substitute that a from a from dimensional analysis and at delta we will see that rho U omega is of the same order of magnitude as rho mu U by del square.

So, that gives us u and u cancel out and they gives that delta square is equal to mu over rho omega or nu over omega. So, from this we get the relationship that delta is equal to nu over omega to the power 1 by 2. So, what we are trying to say here that, the thickness of this oscillating boundary layer can be obtained by the comparison of the transient inertial force and the viscous force at the boundary layer thickness or at the at the just the extend where the boundary layer is. So, from this we can obtain the boundary layer and we can show that alpha is the ratio of a, which is the cube radius and transient boundary layer thickness.

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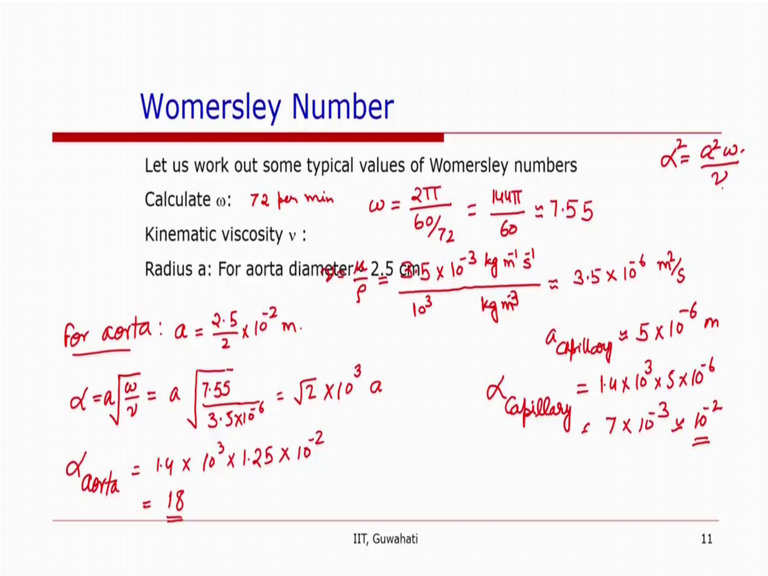

Now, we will look at some of the typical values of alpha square or Womersley number that we encounter in general. So, we know that alpha is equal to a alpha square is equal to a square omega over nu. So, let us find out what are typical a omega and nu as the radius of the channel vary in a system. So, we will have different radii, but omega and nu are going to be same for one system.

So, the typical value of omega we know that 72 the heart beats 72 times per minute. So, from that we can calculate omega is equal to 2 pi over time period and time period is for 1 b at it is 60 divided by 72. So, that will come out 144 pi divided by 60 and that number will be about 7.55 ok

Next we will calculate the kinematic viscosity, which will be omega sorry mu over rho and the typical value of the blood viscosity is about 3.5 centipoise. So, if we change that centipoise into s i units 3.5 into 10 to the power minus 3 k g per meter per second divided by rho, which we will take about 10 to the power 3 k g per meter sorry this is k g per meter cube and so, that will be of the order of 3.5 into 10 to the power minus 6 meter square per second.

Now we will look at for say aorta the radius is the diameter of aorta is about 2.5 centimetre. So, radius is about 2.5 divided by 2 into 10 to the power minus 2 meters. So, alpha is equal to root of omega over nu into a. So, let us work out this number a root of omega is 7.55 divided by nu which is about 3.5 into 10 to the power minus 6. So, what we will have roughly root 2 this number is will be more than 2, but for the simplicity sake let us say that this is root 2 into 10 to the power 6 that the square root of 10 to the power 6 will be 10 to the power 3 into a. So, a is in meters for this to be true. So, for aorta this will be 1.4 which is about root 2 value into 10 to the power 3 into 1.25 that is 2.5 divided by 2 into 10 to the power minus 2. So, what we will get is this number is about 17 or 18. So, the order of magnitude of alpha in aorta is about 18 and because omega and nu are going to became in the cardiovascular system. So, that is the about the maximum value of alpha, that will be there in the circulatory system.

If we look at the capillary system, which will be the say minimum size of the channel. So, in the capillaries let us say alpha capillary and this will be 1.4 into 2 to the power 3 and a capillary the radius of capillary will be about say 5 microns. So, 5 into 10 to the power minus 6 meters into 1.4 into 10 to the power 3 into 5 into 10 to the power minus 6. So, that will come out about 7 into 10 to the power minus 3 or roughly we can say that about 10 to the power minus 2.

So, this Womersley number is going to be very very small; that means, the viscous forces are dominating in this case and the transient inertial force is neglected. So, that is why the transient forces are not very important in the a small channels ok. So, now let us look at some of the velocity profiles in the channels.

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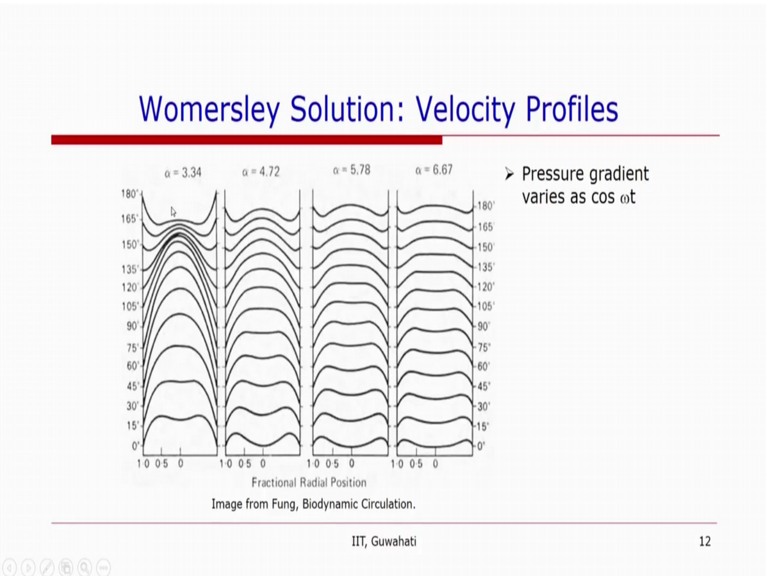

So, these velocity profiles have been plotted for 4 different ha values of alpha 3.34, 4.72 5.78, 6.67 and in these 4 cases the velocity profiles have been plotted for different values of omega t ranging from this the first profile here is for omega t is equal to 0 then omega t 15, omega t 30 and so on so forth up to 180 and you might notice that after 180 the profile has become inverted in all the cases the profile has become inverted. So, 180 to 360 the profiles will be same, but inverted that 180 plus 5 ok.

Ah another thing that you should notice here, that at low values the profile is parabolic there are large gradient in the centre, but as the Reynolds number or as the Womersley number increases, the flow profile becomes flatter in these cases here. So, all these have been plotted for pressure gradient, when the pressure gradient vary as del p over del z cos omega t ok. So, you can see or you can easily make out the thickness of the boundary layer that the thickness of boundary layer reduces as alpha increase ok. Boundary layer is where there is a change in the velocity gradient near the wall and we can easily make out in all these cases decrease ok.

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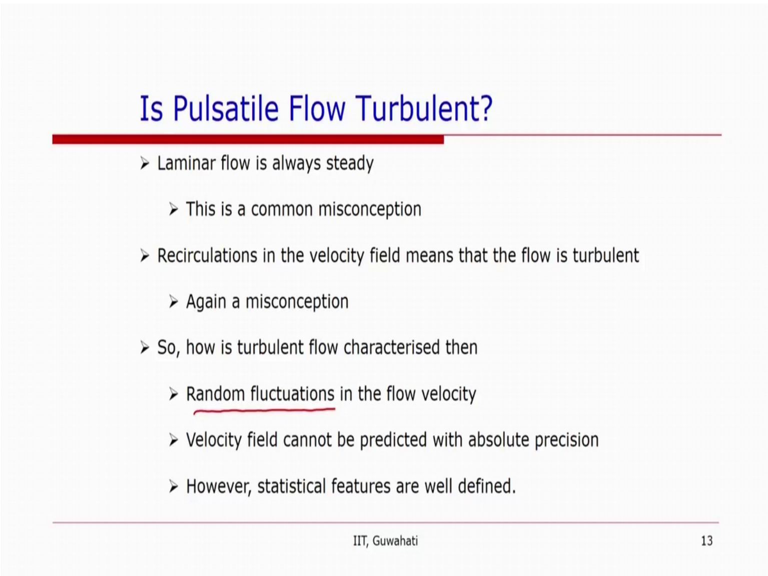

So, another question that we might come across that often sometimes or many students have this perception that, if the flow is transient or if the flow is time dependent then is it laminar or not or being time dependent does it directly mean that the flow is turbulent. In another misconception what people think is that if there is recirculation in the flow; that means, if you have some kind of vertices in the flow then people tend to think that the flow is turbulent in nature, but these are 2 different thing the turbulence have eddies of different length scales starting from very small scale what we call Kolmogorov length scale to the largest possible length scale in the system.

However just the presence of eddies does not mean that the flow is turbulent. So, the question comes how one can define or how one can think about or how one can determine that the flow is turbulent. So, the turbulence is defined by random fluctuations in the flow velocity. So, the key here is the fluctuations that are random, they are they do not follow any pattern the fluctuations in the velocity and consequently in the pressure they are they are random fluctuations in the flow velocity.

So, one cannot determine with precise absolute precision it is not possible to determine the velocity field in turbulent flows; however, there is an order in this randomness also, and the statistical feature of the turbulent flow can be well defined ok. So, the question that we are trying to look at here is that is the pulsatile flow that we have is turbulent or it is laminar or something in between.

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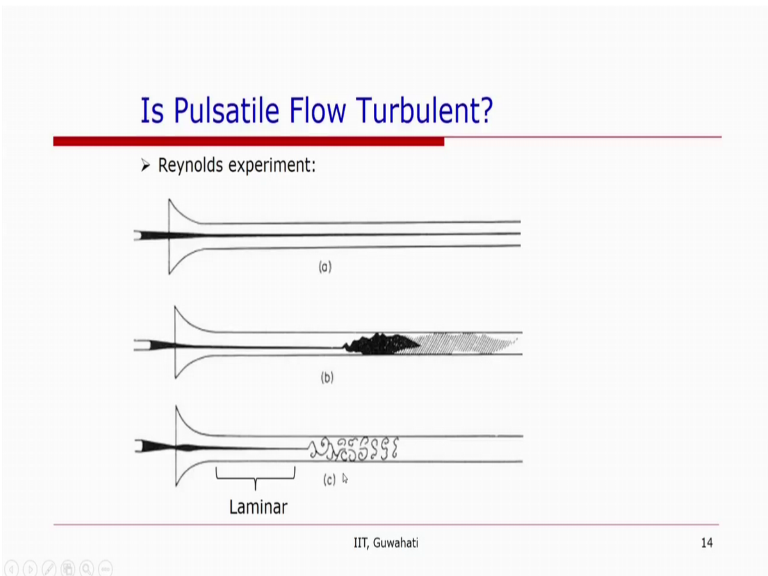

So, let us look at this experiment first, this is the first famous Reynolds experiment image and it has been taken from funks book. So, when the flow is laminar at low velocities the dye that was injected at the inlet, it moves just in that line whereas, when the flow become turbulent or from there is transition, the dye is start to defuse in the entire channel and this is fully turbulent flow, where you see that dye is defused, but what I want you to notice here that, even in the turbulent flow the flow does not become turbulent just at the entrance.

It takes some times for the eddies to develop, for the eddies to grow, and for the flow to become turbulent and this has a consequence in the pulsatile flow.

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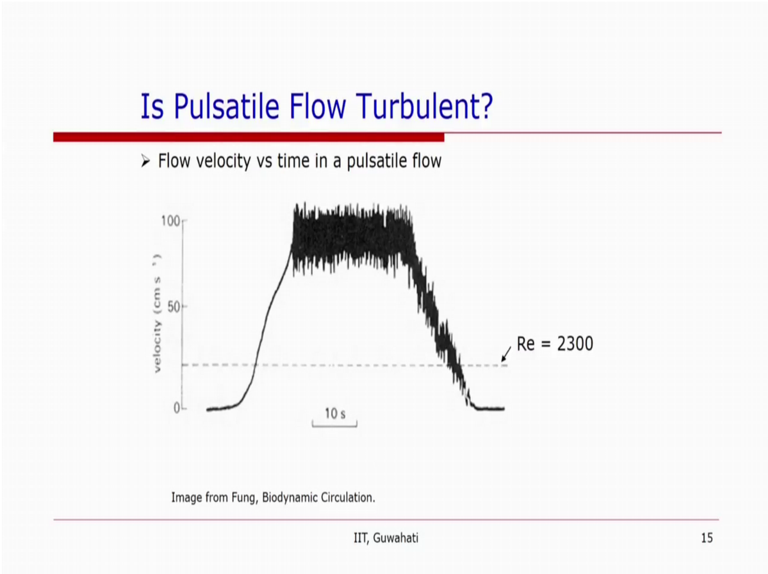

So, this image is velocity with respect to time for a pulsatile flow. So, as an this dotted line is the this line corresponds to the Reynolds number and this Reynolds number is based on the velocity scale which is the average velocity in the channel cross section. So, this dotted line corresponds to Reynolds number 2300, which is considered as the critical Reynolds number at which in a smooth pipe the steady laminar flow transits or the laminar flow starts becoming turbulent or the transition starts happening.

So, you might see here from this that as the velocity grow and the flow remains laminar for sufficiently higher value of Reynolds number, before it becomes turbulent; that means, before the fluctuations in the velocity become random. So, what does this suggest that, in an accelerating flow where the velocity of the flow is growing because it takes some time for the fluctuations or the eddies to grow or turbulence to set in the accelerating flow is more stable than the steady flow at the corresponding Reynolds number. So, the transition Reynolds number for an accelerating flow where the flow velocity is increasing is higher.

Similarly, if you look at this end where the flow is decelerating flow velocity decreases. So, because the turbulent eddies they have to die down and they do not die down just then and they are it take some time for the turbulent eddies to down the die down. So, the flow will half well below through Reynolds number of 2300, the flow remains turbulent and then slowly die down ok.

So, this is one important conclusion in terms of turbulence in pulsatile flows. So, in summary what we have looked at in including both the lectures in pulsatile flow that one of the critical Reynolds number or the critical parameter which is important for the pulsatile flow in rigid tubes is Womersley parameter or you can also call it a transient Reynolds number or the square of Womersley number can be called transient Reynolds number. So, this is ratio of transient inertial force and viscous force then the Womersley solution.

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So, we looked at the flow of a periodic flow time dependent periodic flow in a rigid channel and obtain the solution and from that we also obtain the flow rate and pressure gradient and it is a function of it because it is a function of it has Womersley solution Bessel's function. So, we also looked at the Bessel function of first kind and its properties. And we briefly discussed about the pulsatile flow and turbulence that is the flow always turbulent, when the flow is pulsatile or not ok. So, with that we will end this lecture.

Thank you.